a.). Σ_x corresponding to X is positive semidefinite, In order to be invertible, Σ_x must be positive definite. Thus, 0 can't be a eigenvalue for Σ_x. Since $\Sigma_x = US^2U^T$ where S^2 is a diag matrix with eigenvalue as entery, U is basis eigen vector. When Σ_x is invertible. If, some eigenvalue is 0. S^2 is not invertible. Also the basis eigenvectors need to be independent to each other to make U invertible.

Thus, if some X_i are deterministic. Then all $Cov(X_i, X_j) = 0$ $\forall j \in \{1, \cdots, n\}$. all that column and row will be zero.

Then, Σ has an eigenvalue of 0. Σ_x will be not invertible.

We can change X into X' by remove the deterministic item.

Then, the new $\Sigma_{x'}$ convariance matrix will be invertible and. without Loss information

b.). $\Sigma = US^2U^T$, where S^2 is diagnal matrix with eigen value as entries and U be the eigenvectors (normalized). U is orthogonal normal. $(U^{-1} = U^T)$

$$\Sigma^{-1} = US^{-2}U^T$$

$$x^T\Sigma^{-1}x = x^T A^T \cdot A \cdot x \Rightarrow US^{-1} \cdot S^{-1}U^T = A^T \cdot A \Rightarrow A = S^{-1}U^T$$

Thus, exist matrix $A = S^{-1}U^T$, such that $x^T\Sigma^{-1}x = \|Ax\|_2^2$.

c.) With $x^T\Sigma^{-1}x$, ~~its~~ it is not obvious see the meanings of it.

However with $\|Ax\|$, we know $A = S^{-1}U^T$, where U is matrix with eigen vectors, S^{-1} is diagonal matrix with square root value of eigenvalue. Thus, it look like map X to new basis. And calulate the distance to O_N in the new bases. And square it for $\|Ax\|_2^2$.

d.) ① $\|x\|_2 = 1$, $\|U^T x\|_2^2 = 1$ as well U is normal matrix

let $U^T x = \langle U_1, \cdots, U_n \rangle$.

$\|AX\|_2^2 = \sum_{i=1}^{N} \frac{U_i^2}{\lambda_i}$, and we have $U_1^2 + U_2^2 + \cdots + U_N^2 = 1$

$0 \leq |U_i^2| \leq 1$

Thus, in order to maximum $\|AX\|_2^2$, we need $|U_i^2| = 1$ with respect to maximum $\frac{1}{\lambda_i}$, for $i \in \{1, \cdots, n\}$

and in order to minimum $\|AX\|_2^2$, we need $|U_j^2| = 1$ with respect to the minimum value of $\frac{1}{\lambda_j}$ for $j \in \{1, \cdots, n\}$

Thus, the max value is $\frac{1}{\lambda_i}$ with minimum eigenvalue λ_i

the min value is $\frac{1}{\lambda_j}$ with maximum eigenvalue λ_j.

② when $X_i \perp X_j$ $\forall i, j$, $\Sigma_{i,j}^{-1} = \begin{cases} \frac{1}{\mathrm{cov}(X_i, X_j)} & \text{if } i = j \text{ for } i,j \text{ entry} \\ 0 & \text{otherwise.} \end{cases}$

similar as above. The maximum value will be $\frac{1}{\mathrm{cov}(X_i, X_i)}$ for X_i such that give minimum $\mathrm{cov}(X_i, X_i)$. and The minimum value will be $\frac{1}{\mathrm{cov}(X_j, X_j)}$ with X_j which given the maximum value $\mathrm{cov}(X_j, X_j)$

Sin X is a ~~whole~~ unit circle shape in original shape, the max and min value represent the max and min square distance to the new mean. Such that we can do projection in the new bases space.

③ To maximize $f(x)$, we need minimize $\|AX\|_2^2$.

we want choose X that corresponding to the eigenvector that has maximum eigenvalue λ. or to the vector has maximum covariance for itself.