TIMERS: Error-Bounded SVD Restart on Dynamic Networks

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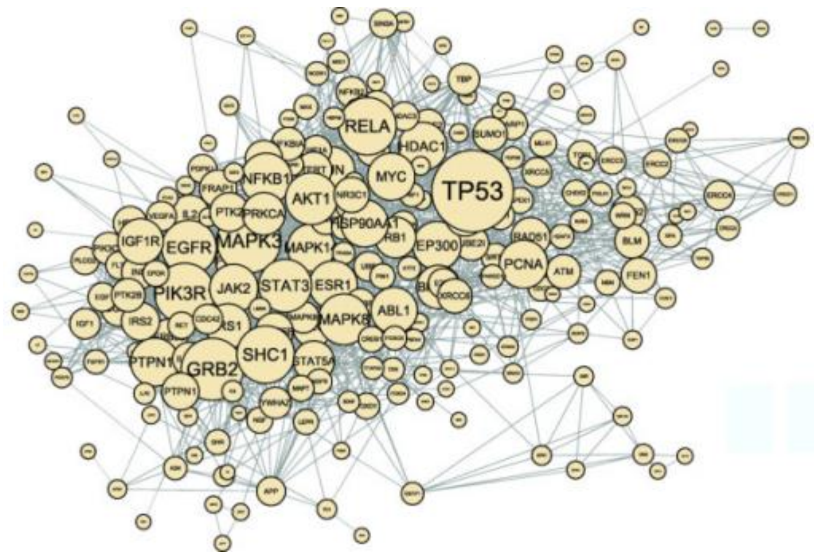
Networks

Social Network



Biology Network

Traffic Network

Matrix Representation

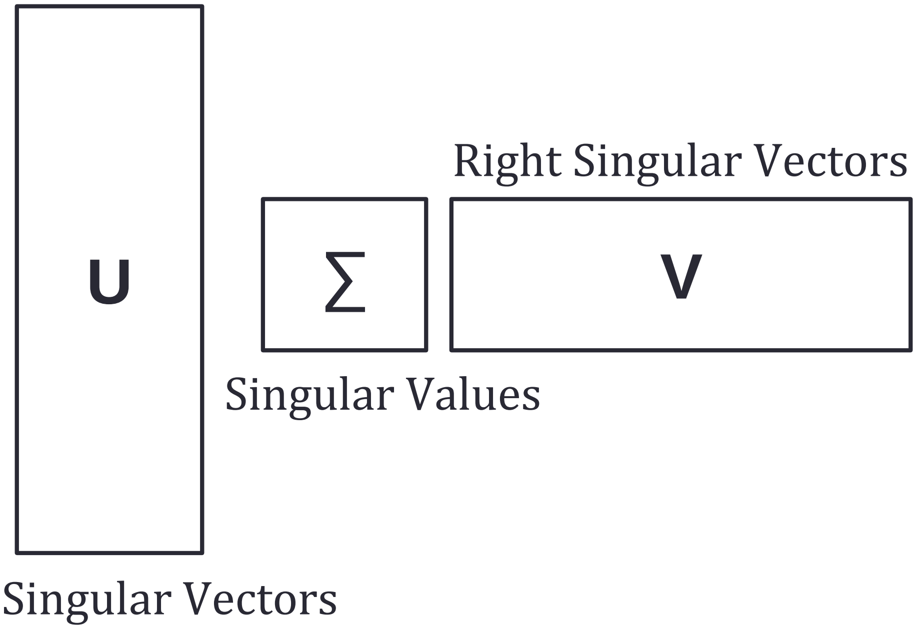
$$
\begin{array}{|ccccccc|}
\hline
0 & 0 & 0 & 3 & & 0 & 1 \\
1 & 0 & 2 & 0 & \cdots & -1 & 0 \\
-1 & 0 & 0 & 1 & & 0 & -2 \\
 & & \cdots & & & \cdots & \\
0 & 1 & 0 & 0 & & 1 & 0 \\
0 & 0 & -2 & 8 & \cdots & 0 & 0 \\
0 & 2 & 0 & 1 & & 0 & 0 \\
\hline
\end{array}
$$

Adjacency Matrix

- Other variants: high-order similarity matrix, transition matrix, Laplacian matrix, etc.

Network Analysis: SVD

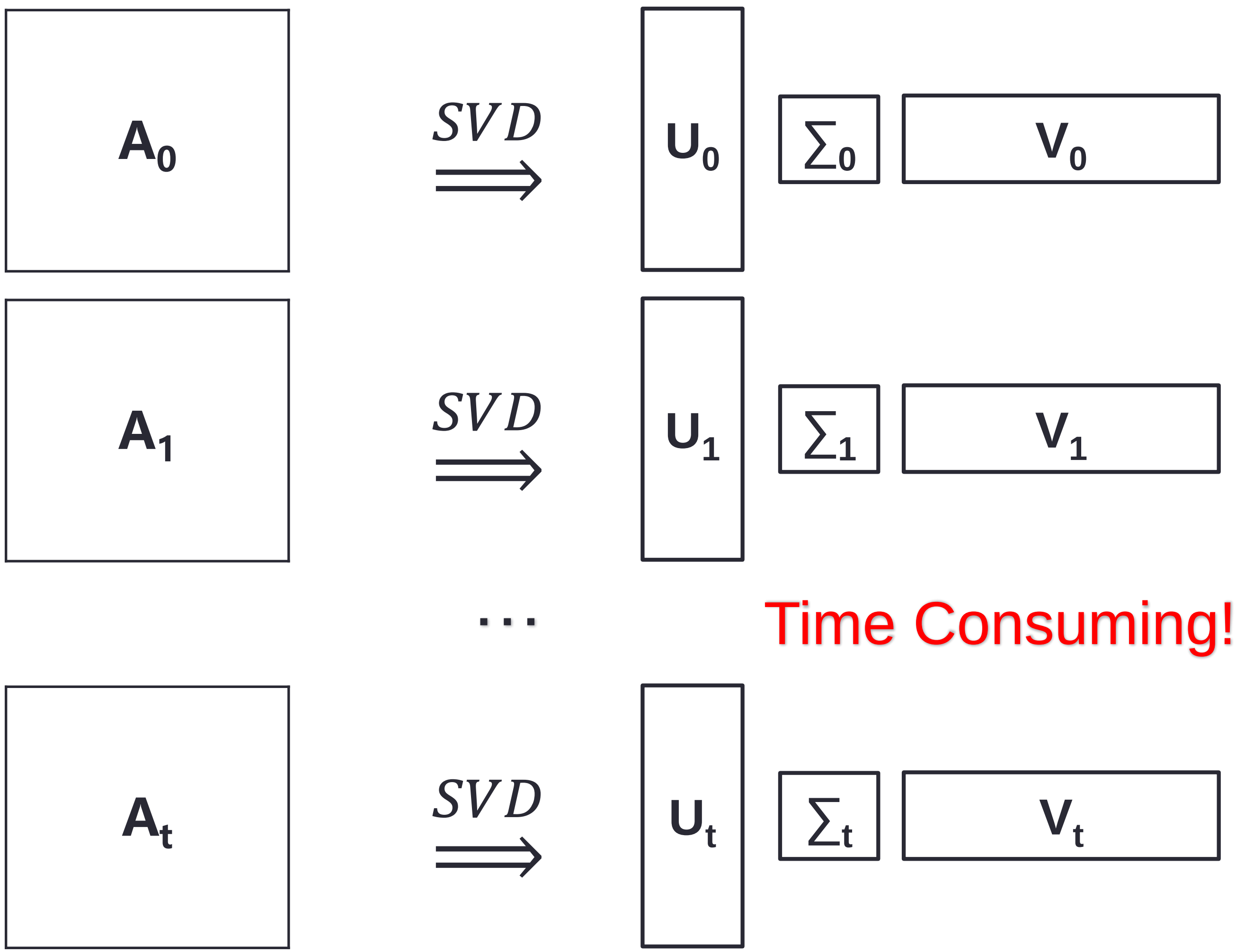
$$
\begin{array}{cccccc}
0 & 0 & 0 & 3 & & 0 & 1 \\
1 & 0 & 2 & 0 & \dots & -1 & 0 \\
-1 & 0 & 0 & 1 & & 0 & -2 \\
 & & \dots & & & \dots & \\
0 & 1 & 0 & 0 & & 1 & 0 \\
0 & 0 & -2 & 8 & \dots & 0 & 0 \\
0 & 2 & 0 & 1 & & 0 & 0
\end{array}
\overset{SVD}{\Longrightarrow}
$$



- Applications: network embedding, link prediction, characterizing network parameters, etc.
- Computationally expensive procedure

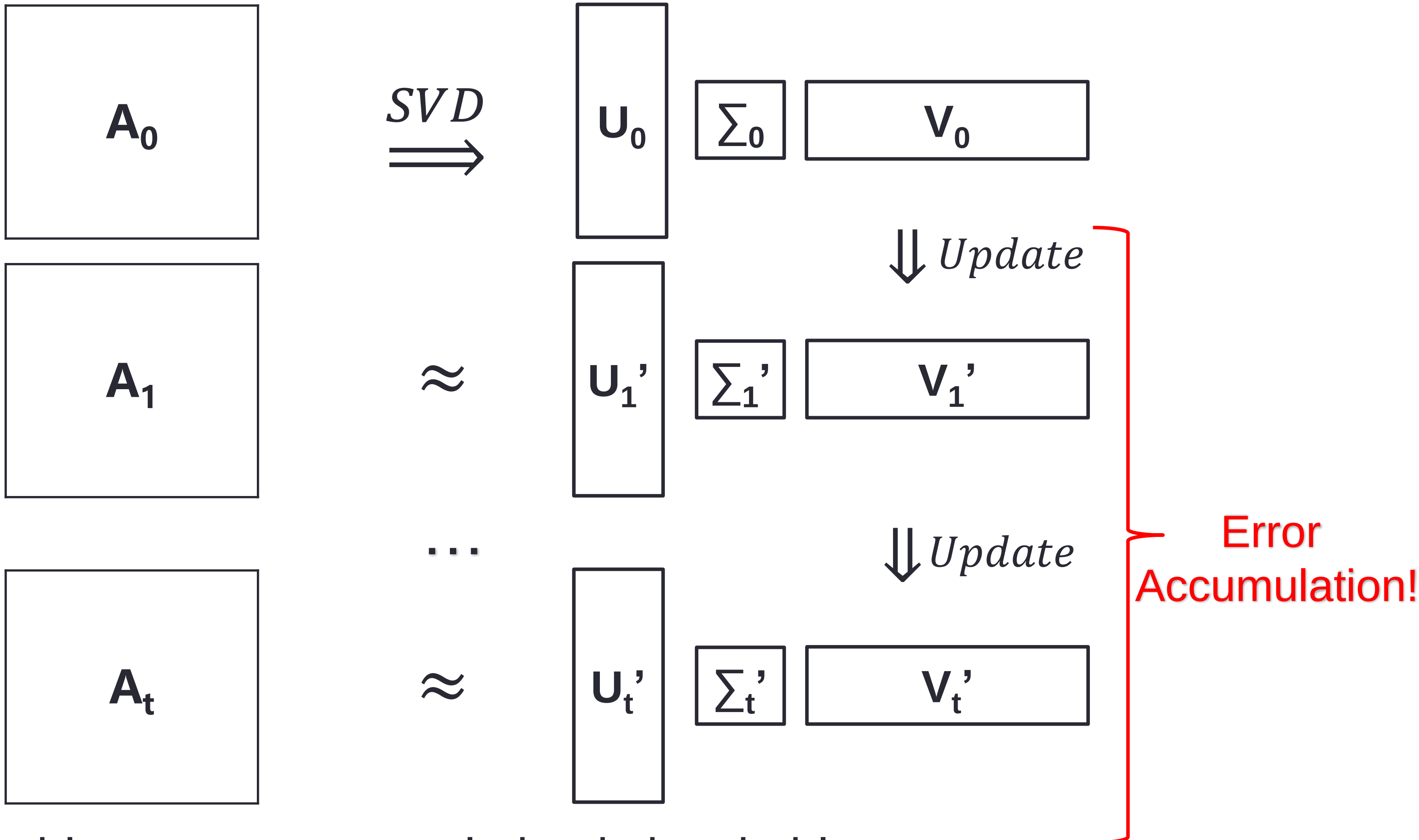
Dynamic Networks

- Networks are dynamic in nature
  - SVD results need to be updated accordingly



Incremental SVD

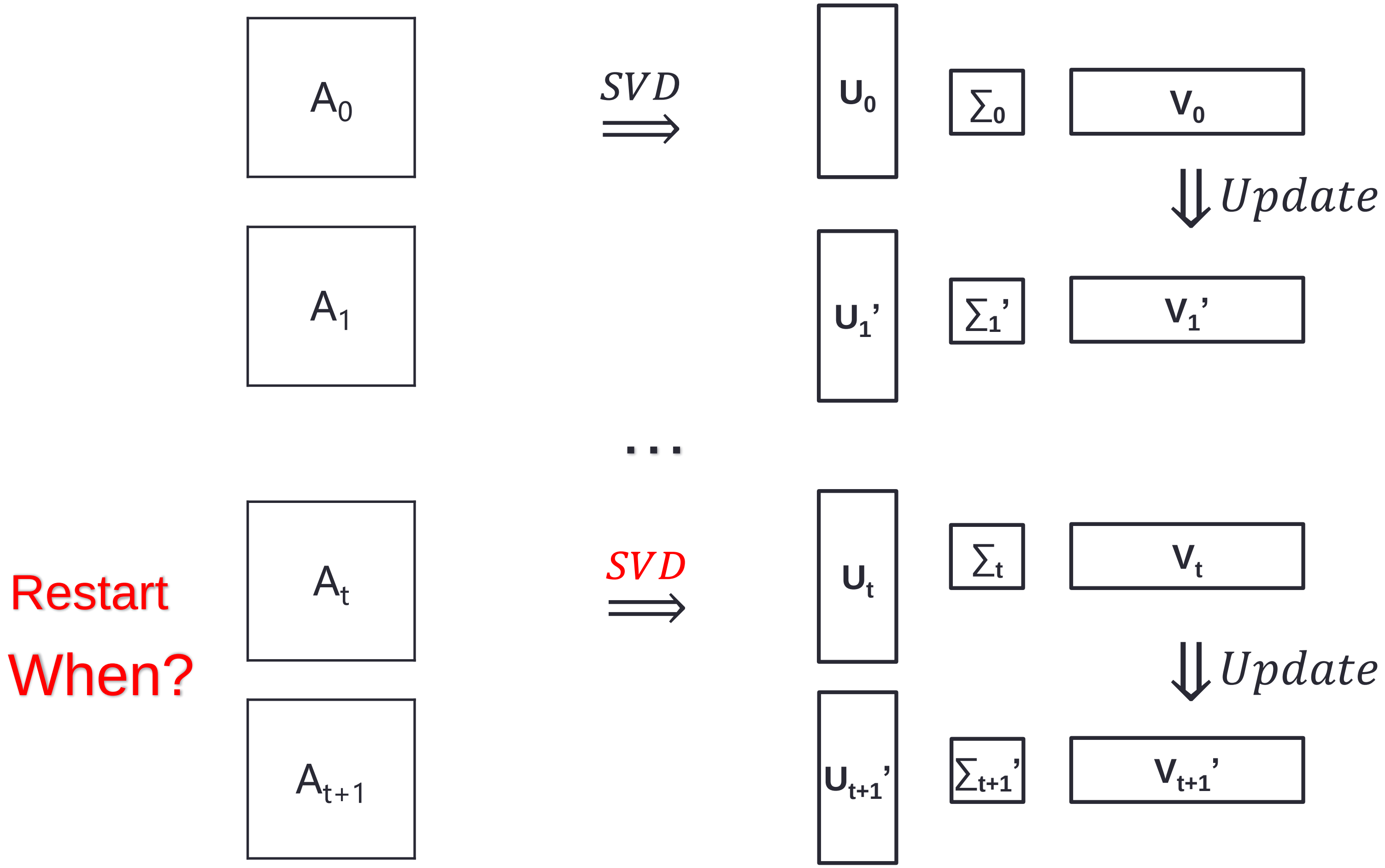
- Incremental SVD is proposed by inducing approximation



- Problem: error accumulation is inevitable

SVD Restarts

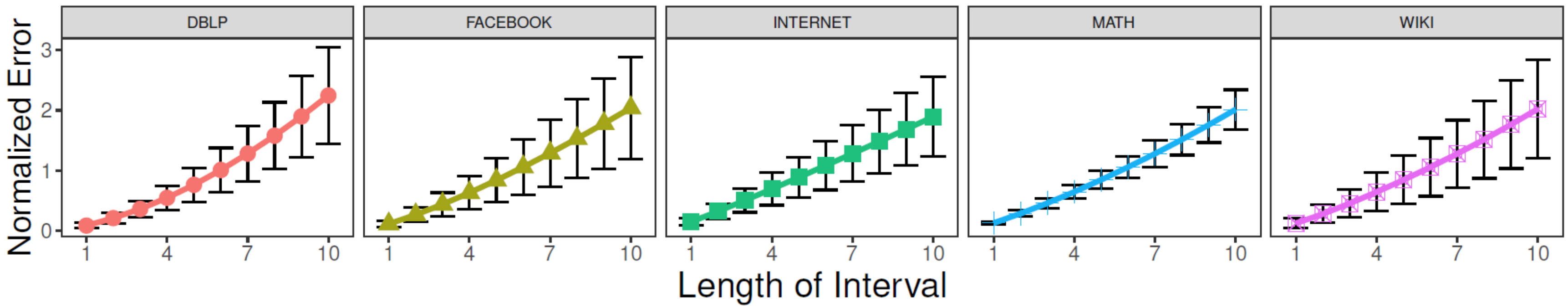
- Solution: restart SVD occasionally

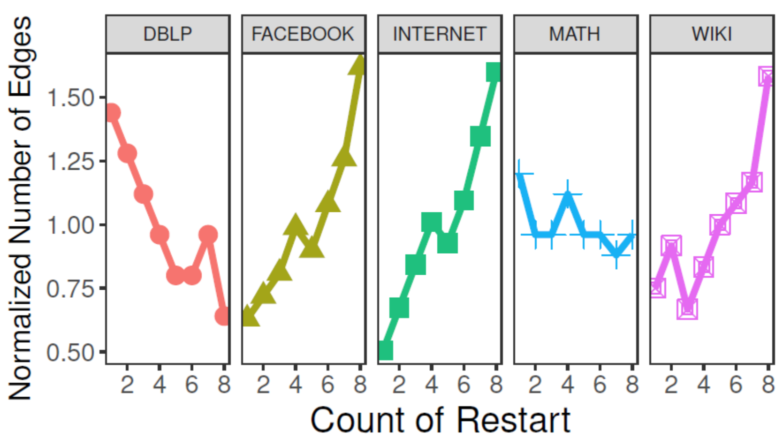


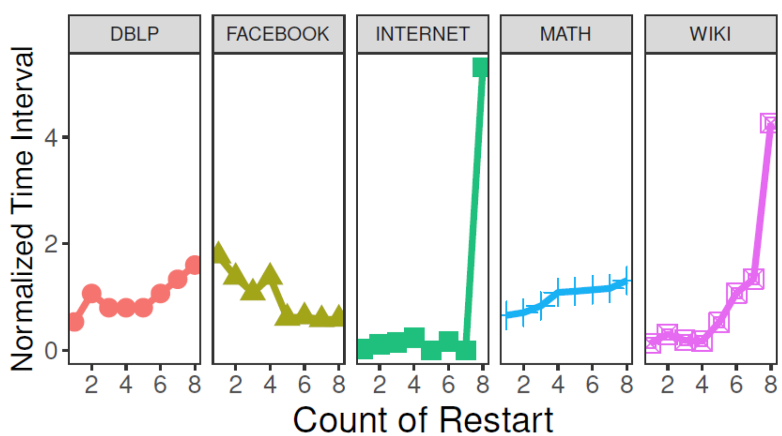
- What are the appropriate time points?

Naïve Solution

- Naïve solution: fixed time interval or fixed number of changes
- Difficulty: error accumulation is not uniform
  - Validated by preliminary studies







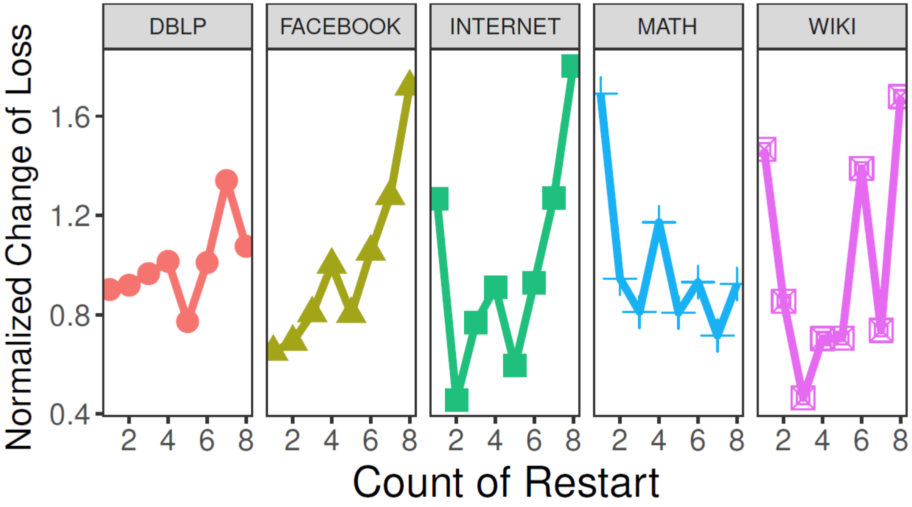
Existing Method

- Existing method: monitor loss (Chen and Candan, KDD 2014)
- Loss in SVD:

$$\mathcal{J} = \|S - U\Sigma V^T\|_F^2$$

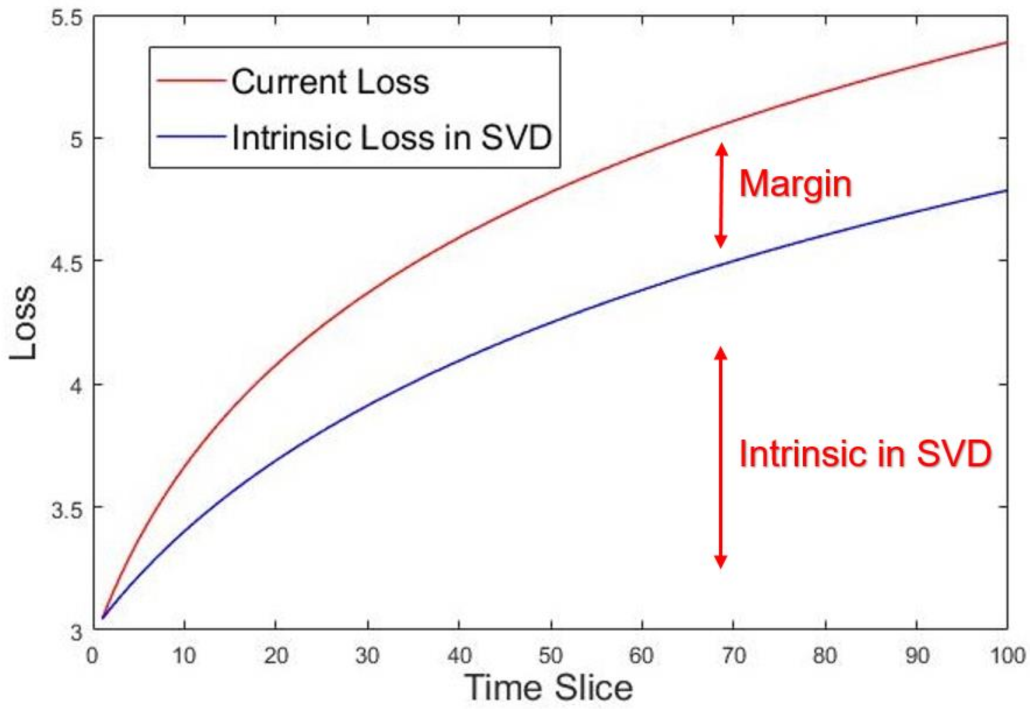
S: target matrix, $[U, \Sigma, V]$: results of SVD

- Problem: loss includes approximation error and intrinsic loss in SVD
- Preliminary study results:



Framework: Monitor Margin

- Observation: the margin between the current loss and intrinsic loss in SVD is the actual accumulated error
  - Current loss: $\mathcal{J} = \|S - U\Sigma V^T\|_F^2$
  - Intrinsic loss: $\mathcal{L}(S,k) = \min_{U^*,\Sigma^*,V^*} \left\|S - U^*\Sigma^* {V^*}^T\right\|_F^2, k: dimensionality$



Framework: Monitor Margin

- Framework: monitor the maximum margin
- Formulation: constrained optimization

$$\min_{c_1,\ldots,c_T} \sum_{t=1}^{T} c_t$$

$$s.t.\ \ \mathcal{G}\left(\mathbf{S}_0 \ldots \mathbf{S}_T, \left[\mathbf{U}_t, \mathbf{\Sigma}_t, \mathbf{V}_t\right], 1 \leq t \leq T\right) \leq \Theta$$

  - c_t: whether to restart; $\mathcal{G}$: evaluating the margin; Θ: threshold

$$\mathcal{G} = \max_{1 \leq t \leq T} \frac{\mathcal{J}(t) - \mathcal{L}(\mathbf{S}_t, k)}{\mathcal{L}(\mathbf{S}_t, k)}$$

  - Intuition: keep the maximum margin within a threshold while reducing the number of restarts

Solution: Lazy Restarts

- Lazy restarts: restart only when the margin exceeds the threshold
- Problem: intrinsic loss is hard to compute
  - Direct calculation has the same time complexity as SVD
- Relaxation: an upper bound on margin
  - A lower bound on intrinsic loss $\mathcal{L}(S, k)$

$$\mathcal{L}(\mathbf{S}_t, k) \geq B(t) \Rightarrow \frac{\mathcal{J}(t) - \mathcal{L}(\mathbf{S}_t, k)}{\mathcal{L}(\mathbf{S}_t, k)} \leq \frac{\mathcal{J}(t) - B(t)}{B(t)}.$$

  - $\mathcal{J}(t)$: current loss; $\mathcal{L}(S_t, k)$: intrinsic loss; $B(t)$: bound of intrinsic loss

A Lower Bound of SVD Intrinsic Loss

- Idea: use matrix perturbation

Theorem 1 (A Lower Bound of SVD Intrinsic Loss). *If* $\mathbf{S}$ *and* $\Delta\mathbf{S}$ *are symmetric matrices, then:*

$$\mathcal{L}(\mathbf{S}+\Delta\mathbf{S}, k) \geq \mathcal{L}(\mathbf{S}, k) + \Delta tr^2(\mathbf{S}+\Delta\mathbf{S}, \mathbf{S}) - \sum_{l=1}^{k} \lambda_l, \quad (9)$$

where $\lambda_1 \geq \lambda_2 ... \geq \lambda_k$ *are the top-*k *eigenvalues of* $\nabla_{S^2} = \mathbf{S}\cdot\Delta\mathbf{S} + \Delta\mathbf{S}\cdot\mathbf{S} + \Delta\mathbf{S}\cdot\Delta\mathbf{S}$, *and*

$$\Delta tr^2(\mathbf{S}+\Delta\mathbf{S}, \mathbf{S}) = tr\left((\mathbf{S}+\Delta\mathbf{S})\cdot(\mathbf{S}+\Delta\mathbf{S})\right) - tr(\mathbf{S}\cdot\mathbf{S}).$$

  - Intuition: treat changes as a perturbation to the original network

- Results: need to calculate top-k eigenvalues of $S \cdot \Delta S + \Delta S \cdot S + \Delta S \cdot \Delta S$

Time Complexity Analysis

Theorem 2. *The time complexity of calculating $B(t)$ in Eqn* (13) *is* $O(M_S + M_L k + N_L k^2)$, *where M_S is the number of the non-zero elements in* $\Delta\mathbf{S}$, *and N_L, M_L are the number of the non-zero rows and elements in ∇_{S^2} respectively.*

- If every node has a equal probability of adding new edges, we have: $M_L \approx 2d_{avg}M_S$, where d_{avg} is the average degree of the network .
- For Barabasi Albert model (Barabási and Albert 1999), a typical example of preferential attachment networks, we have: $M_L \approx \frac{12}{\pi^2}\left[log(d_{max}) + \gamma\right] M_S$, where d_{max} is the maximum degree of the network and $\gamma \approx 0.58$ is a constant.

- Conclusion: the complexity is only linear to the local dynamic changes

TIMERS: Theoretically Instructed Maximum-Error-bounded Restart of SVD

Algorithm 1 TIMERS: Theoretically Instructed Maximum-Error-bounded Restart of SVD

Require: Static Adjacency Matrix $\mathbf{A}_0$, Dynamic Changes $\Delta\mathbf{A}_1 ... \Delta\mathbf{A}_T$, Similarity Function $\mathcal{S}(\cdot)$, Dimensionality k, Error Threshold Θ, Incremental SVD method $\mathcal{F}(\cdot)$

Ensure: SVD results in each time slice $[\mathbf{U}_t, \mathbf{\Sigma}_t, \mathbf{V}_t]$

```
Calculate the initial similarity S_0 = S(A_0)
Calculate SVD on S_0 to obtain [U_0, Σ_0, V_0]
for t in 1:T do
    Calculate the similarity change in that time slice ΔS_t
    Use F(·) to get updated SVD results [U_t, Σ_t, V_t]
    Compute loss J(t) and bound B(t) using (13)
    if (J(t) − B(t)) / B(t) > Θ then
        Calculate SVD on S_t to obtain new [U_t, Σ_t, V_t]
    end if
    Return [U_t, Σ_t, V_t]
end for
```

- General across different types of networks and dynamic scenarios
  - Weighted/unweighted, signed/unsigned networks
  - Add/delete nodes/edges, adjust edge weights
- Flexible to cooperate with any incremental SVD method

Experimental Setting

- Five real dynamic networks:
  - FACEBOOK, MATH, WIKI: social networks
  - DBLP: citation network
  - INTERNET: autonomous system connections
- Types: weighted/unweighted, signed/unsigned

Table 1: The Statistics of Datasets

Dataset	Nodes	Static E	Evolving E	Type
FACEBOOK	52804	962654	632188	SN
MATH	13586	600000	138408	W
WIKI	28223	1000000	375270	W,S
DBLP	28331	200000	79436	W
INTERNET	32077	111644	196270	W,SN

W = Weighted, S = Signed, SN = Static Network marked

Experimental Setting

- Baselines:
  - Heu-FL: restart SVD after a fixed number of edges changed
  - Heu-FT: restart SVD after a fixed amount of time passed
  - LWI2 (Chen and Candan, KDD 2014): restart SVD whenever the reconstruction loss exceeds a preset threshold
- Tasks:
  - Approximation error
    - Fixed number of restarts
    - Fixed maximum error
  - Applications
    - Link prediction
    - Eigenvalue tracking
  - Analysis

Experimental Results: Approximation Error

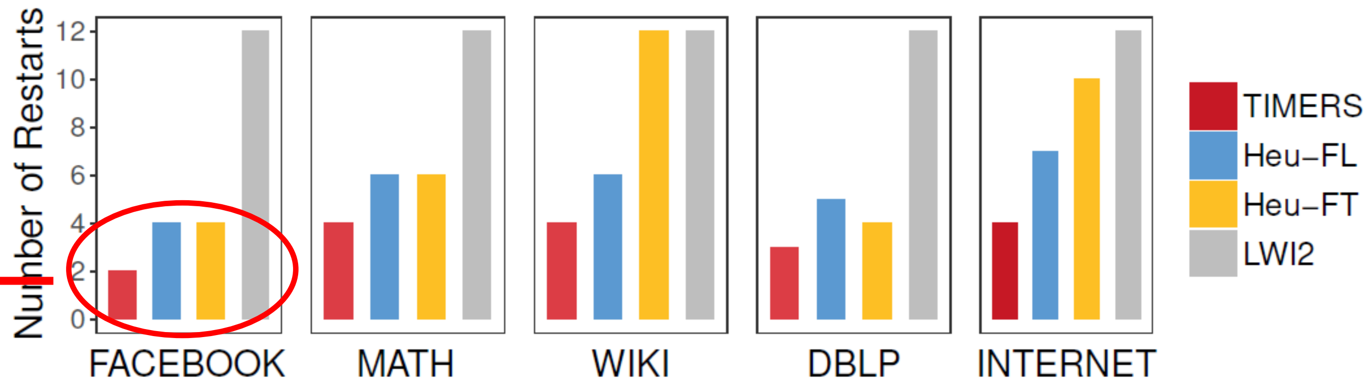
- Fixing number of restarts

Dataset	avg(r)				max(r)			
	TIMERS	LWI2	Heu-FL	Heu-FT	TIMERS	LWI2	Heu-FL	Heu-FT
FACEBOOK	**0.005**	0.020	0.009	0.011	**0.014**	0.038	0.025	0.023
MATH	**0.037**	0.057	0.044	0.051	**0.085**	0.226	0.117	0.179
WIKI	**0.053**	0.086	0.071	0.281	**0.139**	0.332	0.240	0.825
DBLP	**0.042**	0.110	0.053	0.064	**0.121**	0.386	0.198	0.238
INTERNET	**0.152**	0.218	0.196	0.961	**0.385**	0.806	0.647	1.897

27%~42% Improvement

- Fixing maximum error

-50%



Experimental Results: Applications

- Link prediction:

-30%

Dataset	TIMERS	LWI2	Heu-FL	Heu-FT
FACEBOOK	**1.54***	4.27	2.21	2.81
MATH	**1.14***	2.68	1.31	1.29
WIKI	**1.06***	3.44	1.63	4.13
DBLP	**0.18***	0.32	0.22	0.27
INTERNET	**11.27***	23.36	16.98	34.30

*: outperform other methods at 0.005 level paired t-test in 10 runs.

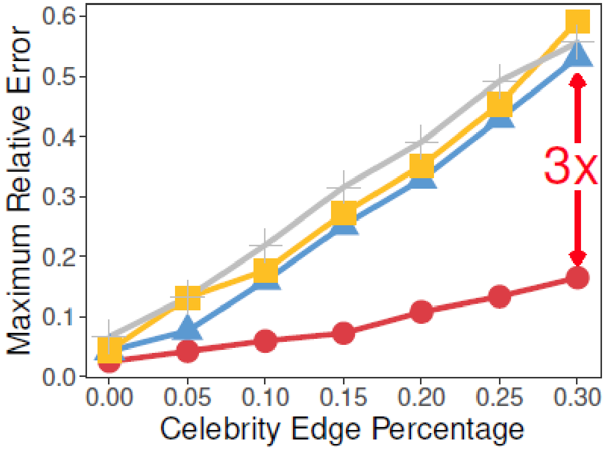
- First eigenvalue tracking:

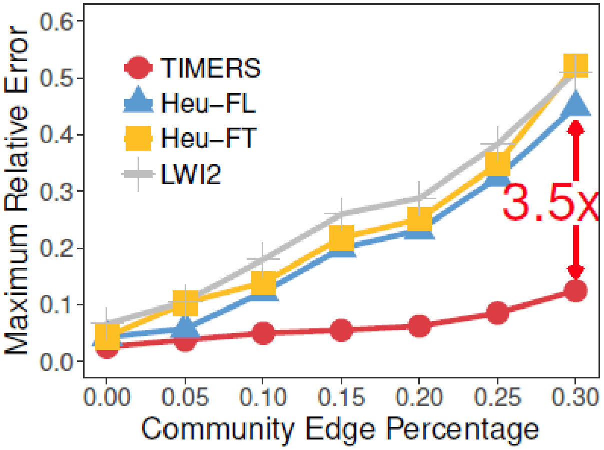
-50%

Dataset	TIMERS	LWI2	Heu-FL	Heu-FT
FACEBOOK	**0.66**	0.94	1.11	1.53
MATH	**2.27**	5.03	4.92	4.80
WIKI	**15.45**	18.42	17.73	97.08
DBLP	**8.31**	11.42	13.92	24.66
INTERNET	**4.18**	12.56	7.95	57.38

Experimental Results: Analysis

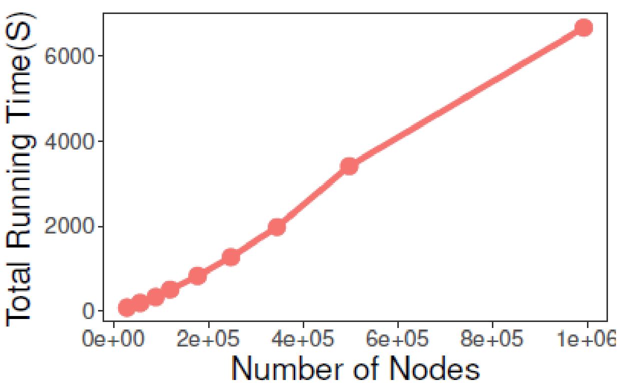
- Syntactic networks: simulate drastic changes in the network structure

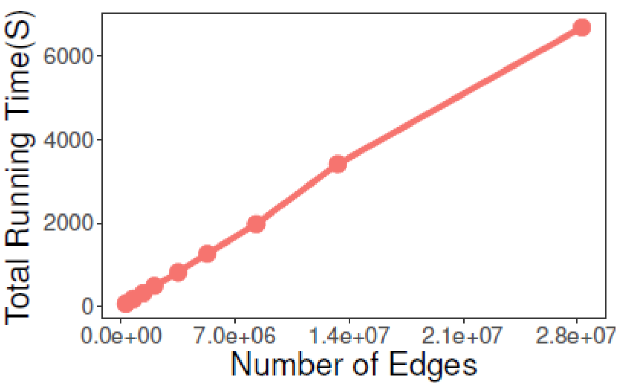




  - Robust to sudden changes

- Linear scalability





Conclusion

- When to restart SVD for dynamic networks
  - Key to prevent error accumulation for incremental SVD
- Framework: monitor margin
  - Constrained optimization and lazy restarts
  - A lower bound of SVD Intrinsic Loss to reduce time complexity
    - Linear time complexity to the local dynamic changes
- Extensive experiments on several dynamic networks
  - 50% improvement in approximation error
  - Robust to sudden changes in the network structure
- Future direction: generalize to directed networks and non-square matrices (e.g. user-product)

Thanks!

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Reference:

[1] Zhang, Z; Cui, P; Pei, J; Wang, X; Zhu, W. TIMERS: Error-Bounded SVD Restart on Dynamic Networks. AAAI, 2018.