Lecture 02 Mathematical Description of Physical Phenomena

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物理现象的数学描述

- 用数学形式的控制方程来表示流动、热传导和相关过程
- 讨论控制方程。

控制微分方程

- 连续性方程 $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$$
- 动量方程 $$\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \mathbf{U}) = \nabla \cdot (\mu \nabla u) + S_\phi$$
- 动量方程 $$\frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v \mathbf{U}) = \nabla \cdot (\mu \nabla v) + S_\phi$$
- 动量方程 $$\frac{\partial \rho w}{\partial t} + \nabla \cdot (\rho w \mathbf{U}) = \nabla \cdot (\mu \nabla w) + S_\phi$$
- 湍动能 $$\frac{\partial \rho k}{\partial t} + \nabla \cdot (\rho k \mathbf{U}) = \nabla \cdot \left(\frac{\mu_t}{\sigma_k} \nabla k \right) + 2\mu_t \mathbf{S_{ij}} \cdot \mathbf{S_{ij}} - \rho \epsilon$$
- 湍耗散 $$\frac{\partial \rho \epsilon}{\partial t} + \nabla \cdot (\rho \epsilon \mathbf{U}) = \nabla \cdot \left(\frac{\mu_t}{\sigma_k} \nabla \epsilon \right) + C_{1\epsilon} \frac{\epsilon}{k} 2\mu_t \mathbf{S_{ij}} \cdot \mathbf{S_{ij}} - C_{2\epsilon} \rho \frac{\epsilon}{k}$$

控制方程的通用微分和积分形式

- 微分形式 $$\frac{\partial \rho\phi}{\partial t} + \nabla \cdot (\rho\phi\mathbf{U}) = \nabla \cdot (\mu\nabla\phi) + S_\phi$$

- 积分形式 $$\int\limits_{CV} \frac{\partial \rho\phi}{\partial t} dV + \int\limits_{CV} \nabla \cdot (\rho\phi\mathbf{U}) dV = \int\limits_{CV} \nabla \cdot (\mu\nabla\phi)\, dV + \int\limits_{CV} S_\phi dV$$

有限体积法初步

- 积分形式 $$\int_{CV} \frac{\partial \rho\phi}{\partial t} dV + \int_{CV} \nabla \cdot (\rho\phi\mathbf{U}) dV = \int_{CV} \nabla \cdot (\mu\nabla\phi)\, dV + \int_{CV} S_\phi dV$$
- 高斯散度定理来分析对流项
- 对流项 $$\int_{CV} \nabla \cdot (\rho\phi\mathbf{U}) dV = \int_{S} \mathbf{n} \cdot (\rho\phi\mathbf{U}) dS$$

有限体积法初步

- 积分形式可以改写为

$$\frac{\partial}{\partial \mathsf{t}}\left[\int_{\mathsf{CV}} \rho\phi \mathsf{dV}\right] + \int_{\mathsf{S}} \mathbf{n}\cdot(\rho\phi\mathbf{U})\mathsf{dS} = \int_{\mathsf{S}} \mathbf{n}\cdot(\mu\nabla\phi)\,\mathsf{dS} + \int_{\mathsf{CV}} \mathsf{S}_\phi \mathsf{dV}$$

有限体积法初步

- 稳态问题

$$\int_{\mathsf{S}} \mathbf{n} \cdot (\rho\phi\mathbf{U})\mathsf{dS} = \int_{\mathsf{S}} \mathbf{n} \cdot (\mu\nabla\phi)\,\mathsf{dS} + \int_{\mathsf{CV}} \mathsf{S}_\phi \mathsf{dV}$$

- 瞬态问题

$$\int_{\Delta t} \frac{\partial}{\partial t}\left[\int_{CV} \rho\phi dV\right] dt + \int_{\Delta t}\int_{S} \mathbf{n} \cdot (\rho\phi\mathbf{U}) dSdt = \int_{\Delta t}\int_{S} \mathbf{n} \cdot (\mu\nabla\phi)\, dSdt + \int_{\Delta t}\int_{CV} S_\phi dVdt$$

物理问题分类

- 平衡问题, Equillbrium problems
- 前进问题, Marching problems

平衡问题

- 平衡问题, Equillbrium problems
- 拉普拉斯方程

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- 椭圆形问题

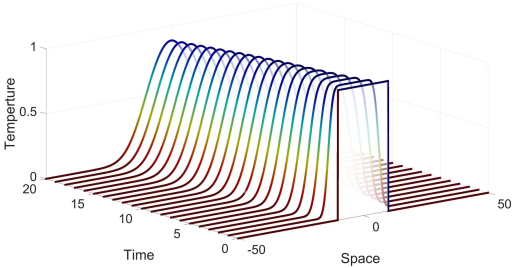
前进问题

- 前进问题
- 热传导方程

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

- 抛物问题

前进问题



前进问题

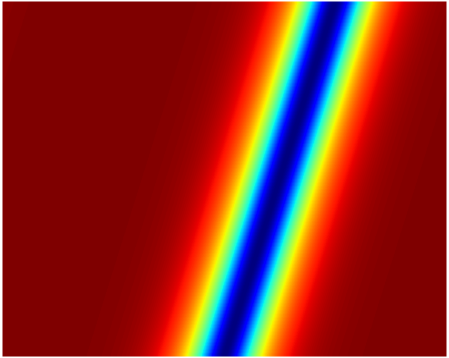
- 前进问题, Marching problems
- 波动方程

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

- 双曲问题

前进问题

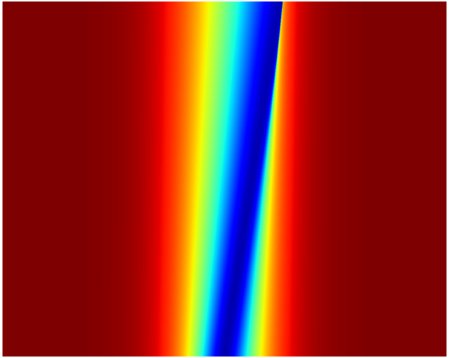
- `one-way equation`
- $u_t + cu_x = 0$



前进问题

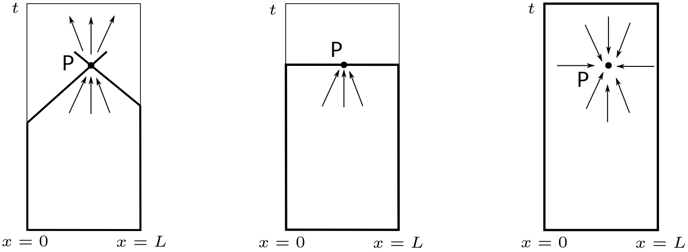
- 一维有粘性 burgers 方程

- $u_t + uu_x = \nu u_{xx}$



依赖区域影响区域

- 影响区域和依赖区域



PDE 方程通用分类

- 判别式

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + \cdots = 0$$

- 椭圆 $B^2 - 4AC < 0$
- 抛物 $B^2 - 4AC = 0$
- 双曲 $B^2 - 4AC > 0$

二维定常扩散方程

- 判断

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

瞬态扩散方程

- 判断

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

边界层方程

- 判断

$$\mathsf{u}\frac{\partial \mathsf{u}}{\partial \mathsf{x}} + \mathsf{v}\frac{\partial \mathsf{u}}{\partial \mathsf{y}} = \nu\frac{\partial^2 \mathsf{u}}{\partial \mathsf{x}^2}$$

定常 N-S 方程

- 判断

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

一维无粘欧拉方程

- 判断

$$\frac{\partial \mathsf{u}}{\partial \mathsf{t}} + \mathsf{u}\frac{\partial \mathsf{u}}{\partial \mathsf{x}} = 0$$