Lecture 10 Convection Term

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对流项计算 Introduction

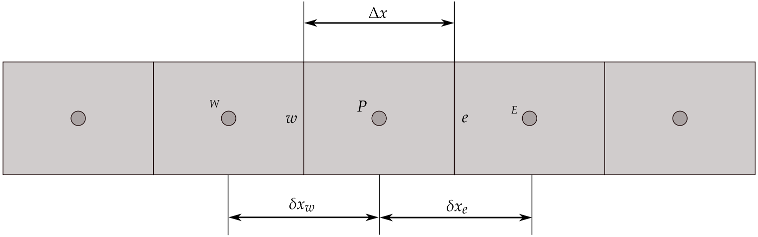
- 对流项的研究超过 30 年
- 对流项内容相当多，包含两章内容 (`11, 12`)
- 本章主要讲解基本对流项格式，从 first order upwind 开始
- 下一章讲解高分辨格式 (High resolution schemes)
- 数值色散、数值扩散和数值稳定性，numerical dispersion, numerical diffusion, numerical stability

对流项计算 Introduction

- first glimpse
- div term

对流项计算 Steady One dimensional convection and diffusion

- 一维定常对流扩散方程



$$\frac{d(\rho u \phi)}{dx} - \frac{d}{dx}\left(\lambda \frac{d\phi}{dx}\right) = 0$$

- 该方程是否有解析解?

对流项计算 Steady One dimensional convection and diffusion

- 一维定常对流扩散方程

$$\frac{d(\rho u\phi)}{dx} - \frac{d}{dx}\left(\lambda\frac{d\phi}{dx}\right) = 0$$

- 解析解

$$\rho u\phi - \left(\lambda\frac{d\phi}{dx}\right) = c$$

对流项计算 Steady One dimensional convection and diffusion

- 解析解

$$\left(\frac{d\varphi}{dx}\right) = \frac{\rho u}{\lambda}\varphi$$

$$\text{其中:}\varphi = \frac{\rho u}{\lambda}\phi - \frac{c}{\lambda}$$

$$\frac{d\varphi}{\varphi} = \frac{\rho u}{\lambda}dx \Rightarrow ln(\varphi) = \frac{\rho u}{\lambda}x + C$$

- ϕ

$$\phi = \frac{C\lambda e^{\frac{\rho u}{\lambda}x} + c}{\rho u}$$

对流项计算 Steady One dimensional convection and diffusion

- 解析解

$$\frac{\phi - \phi_W}{\phi_E - \phi_W} = \frac{e^{\frac{\rho u L}{\lambda} \frac{x - x_W}{L}} - 1}{e^{\frac{\rho u L}{\lambda}} - 1}$$

- Peclet number

$$Pe = \frac{\rho u L}{\lambda}$$

对流项计算 Steady One dimensional convection and diffusion

- 不同 Peclet 计算结果

对流项计算 Steady One dimensional convection and diffusion

- 数值解

$$\int_{V_P} \left(\nabla \cdot (\rho \mathbf{U} \phi) - \nabla \cdot (\lambda \nabla \phi)\right) dV = 0$$

$$\int_{V_P} \left(\nabla \cdot (\rho \mathbf{U} \phi) - \nabla \cdot (\lambda \nabla \phi)\right) dV = \int_{\partial V_P} (\rho \mathbf{U} \phi - \lambda \nabla \phi) \cdot d\mathbf{S} = \int_{\partial V_P} \left[\rho \mathbf{U} \phi \mathbf{i} - \lambda \frac{d\phi}{dx} \mathbf{i}\right] \cdot d\mathbf{S} = 0$$

- 离散后

$$\sum_f \left(\rho u \phi \mathbf{i} - \lambda \frac{d\phi}{dx} \mathbf{i}\right)_f \cdot \mathbf{S}_f = 0$$

对流项计算 Steady One dimensional convection and diffusion

- 中心差分格式，`central difference scheme`
- 公式

$$\phi(x) = k_0 + k_1(x - x_P)$$

- e 处的插值格式

$$\phi_e = \phi_P + \frac{\phi_E - \phi_P}{x_E - x_P}(x_e - x_P)$$

- 对于均匀网格

$$\phi_e = \frac{\phi_P + \phi_E}{2}$$

对流项计算 Steady One dimensional convection and diffusion

- 数值解

$$(\rho u\phi\Delta y)_e - \left(\lambda\frac{d\phi}{dx}\Delta y\right)_e = 0$$

$$(\rho u\Delta y)_e\frac{\phi_E + \phi_P}{2} - \left(\frac{\lambda}{\delta x}\Delta y\right)_e(\phi_E - \phi_P) = 0$$

- 系数矩阵

$$a_P\phi_P + a_E\phi_E + a_W\phi_W = 0$$

对流项计算 Steady One dimensional convection and diffusion

- Peclet 数

$$\frac{\lambda}{\delta x_e}\Delta y_e - \frac{\rho u}{2}\Delta y_e > 0$$

$$Pe = \frac{\rho u \delta x}{\lambda} > 2$$

- 讨论分析

对流项计算 Steady One dimensional convection and diffusion

- 一阶迎风格式 `first order upwind scheme`

$$\phi_e = \phi_P \quad \phi_w = \phi_W$$

- 推导

$$(\rho u \phi \Delta y)_w - \left(\lambda \frac{d\phi}{dx} \Delta y\right)_w = 0$$

$$(\rho u \Delta y)_w \phi_W - \left(\frac{\lambda}{\delta x} \Delta y\right)_w (\phi_P - \phi_W) = 0$$

- 系数矩阵

$$a_P \phi_P + a_E \phi_E + a_W \phi_W = 0$$

对流项计算 Steady One dimensional convection and diffusion

- 一阶迎风格式优缺点
- 稳定
- 有界
- 精度低

对流项计算 Truncation Error: Numerical Diffusion and Anti-Diffusion

- 一阶迎风格式
- 一阶背风格式
- 中心差分格式

对流项计算 Truncation Error: Numerical Diffusion and Anti-Diffusion

- 一阶迎风格式

$$\phi_P = \phi_e + \phi_e'(x_P - x_e) + \frac{1}{2}\phi_e''(x_P - x_e)^2 + \cdots$$

- 均匀网格

$$\phi_P = \phi_e - \phi_e'\frac{\delta x}{2} + \frac{1}{2}\phi''\frac{\delta x}{2} + \cdots$$

对流项计算 Truncation Error: Numerical Diffusion and Anti-Diffusion

- 一阶迎风格式

$$
\begin{aligned}
&(\rho u \Delta y)_e \phi_P - (\rho u \Delta y)_w \phi_W - \left[\left(\lambda \frac{d\phi}{dx}\Delta y\right)_e - \left(\lambda \frac{d\phi}{dx}\Delta y\right)_w\right] \\
&= (\rho u \Delta y)_e \phi_e - (\rho u \Delta y)_w \phi_w - \left[\left(\lambda + \rho u \frac{\delta x}{2}\right)\left(\frac{d\phi}{dx}\Delta y\right)_e - \left(\lambda + \rho u \frac{\delta x}{2}\right)\left(\frac{d\phi}{dx}\Delta y\right)_w\right]
\end{aligned}
$$

- 截断误差

$$
\lambda_{truncation} = \rho u \frac{\delta x}{2}
$$

- 讨论

对流项计算 fist order downwind scheme

- 一阶背风格式

$$
\begin{aligned}
&(\rho u \Delta y)_e \phi_E - (\rho u \Delta y)_w \phi_P - \left[\left(\lambda \frac{d\phi}{dx}\Delta y\right)_e - \left(\lambda \frac{d\phi}{dx}\Delta y\right)_w\right] \\
= &(\rho u \Delta y)_e \phi_e - (\rho u \Delta y)_w \phi_w - \left[\left(\lambda - \rho u \frac{\delta x}{2}\right)\left(\frac{d\phi}{dx}\Delta y\right)_e - \left(\lambda - \rho u \frac{\delta x}{2}\right)\left(\frac{d\phi}{dx}\Delta y\right)_w\right]
\end{aligned}
$$

- 截断误差

$$
\lambda_{truncation} = -\rho u \frac{\delta x}{2}
$$

- anti-diffusion

对流项计算 Central difference scheme

- 中心差分格式

$$\phi_P = \phi_e - \frac{\Delta x}{2}\phi'_e + \frac{\Delta^2 x}{8}\phi''_e + \cdots$$
$$\phi_E = \phi_e + \frac{\Delta x}{2}\phi'_e + \frac{\Delta^2 x}{8}\phi''_e + \cdots$$

- 阶段误差

$$Truncation\ Error = \mathcal{O}(\Delta^2 x)$$

对流项计算 Numerical Stability

- 数值稳定性

对流项计算 High order upwind schemes

- 二阶迎风格式 `second order upwind scheme`
- 阶段误差

$$\phi(x) = k_0 + k_1(x - x_P)$$

- e 处的插值格式

$$\phi(x) = \phi_P + \frac{\phi_P - \phi_U}{x_P - x_U}(x - x_P)$$

- 对于均匀网格

$$\phi_e = \frac{3}{2}\phi_P - \frac{1}{2}\phi_U$$