線性系統期中報告

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Symbol define

- λ : eigenvalue, 特徵值
- $\vec{K}$: eigenvector, 特徵向量
- A : nxn matrix
- M : transition matrix M such that $M^{-1}AM = J$, in this case is also called generalized modal matrix for A
- J : Jordan normal form of A
- $span\{\overrightarrow{K1}、\overrightarrow{K2}...\}$: the linear span of a set of vectors in a vector space is the intersection of all linear subspaces which each contain every vector in that set.

矩陣的動態特徵eigenvalue、eigenvector

Reference: chi-Tsong. ch3.5 p55, Jordan Canonical Form

$A\vec{K} = \lambda\vec{K}$, $\vec{K}$經過A的映射，方向相同，大小不一樣

- 若$\{\lambda_k\}$互不相同 $\Rightarrow |A-\lambda_k I| = 0$
  且$dim(N(A-\lambda_k I)) = 1,\ \exists\overrightarrow{K_k} \neq \vec{0}$, N表示null space, dim表示dimension
  $(A-\lambda_k I)\overrightarrow{K_k} = \vec{0}$, eigenvalue 所形成的次空間
  eigenvector是null space的基底

Question: 矩陣不一定都能夠對角化(Diagonalizable matrix)

- 如果存在一格可逆矩陣M使得$M^{-1}AM$是對角矩陣，則稱A矩陣是可對角化的。
- 特徵化，在域F上的*nxn*矩陣A是可對角化的，若且為若它的特徵空間的維度等於*n*，它為真若且為若存在由A的特徵向量組成的F^n的**基(basis)**。如果找到了這樣的基，可以形成有**基向量**作為縱列的矩陣M，而$M^{-1}AM$將是對角矩陣。這個矩陣的**對角元素是A的特徵值**。
- 線性映射$T: V \to V$是可對角化的，若且為若它的特徵空間的維度等於$dim(V)$，它為真若且為若存在由T的特徵向量組成的V的基。T關於這個基將表示為對角矩陣。這個矩陣的**對角元素是T的特徵值**。

Solution: Gneralized eigenvector (GEV)

- Definition: A vector $\overrightarrow{K_m}$ is a generalized eigenvector of rank m of the matrix A and corresponding to the eigenvalue λ if
  $(A-\lambda I)^m\overrightarrow{K_m} = 0$
  but, $(A-\lambda I)^{m-1}\overrightarrow{K_m} \neq 0$

本次作業、實驗為計算Jordan Form，並討論其特性

Example1 matrix A below

$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 & 0 \\ 0 & 0.3 & 1 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \\ 0 & 0 & 0 & 0 & 0.3 \end{bmatrix}$$

To find eigenvalue, it satisfied $A\vec{K} = \lambda\vec{K}$

$$(A - \lambda I)\vec{K} = \{\vec{0}\}$$

$$det(A - \lambda I) = 0$$

$$det(A - \lambda I) = \left| \begin{bmatrix} 0.3-\lambda & 1 & 0 & 0 & 0 \\ 0 & 0.3-\lambda & 1 & 0 & 0 \\ 0 & 0 & 0.3-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0.3-\lambda & 1 \\ 0 & 0 & 0 & 0 & 0.3-\lambda \end{bmatrix} \right| = 0$$

Reduce dimension, and we can get the equation below

$$-(10 \times \lambda - 3)^5 = 0$$
$$\therefore \lambda = \{0.3,\ 0.3,\ 0.3,\ 0.3,\ 0.3\}$$

Sovle the eigenvector

$$(A - \lambda I)\vec{K} = \{\vec{0}\}$$

Let λ be 0.3

$$(A - \lambda I)\vec{K} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \\ e \end{Bmatrix} = \{\vec{0}\}$$
$$\therefore\ we\ can\ get\ \ b = 0,\ c = 0,\ e = 0$$
$$the\ eigenvectors\ are\ span \left\{ \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} 、 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \right\}$$

To get more eigenvectors, we need to generalized eigenvectors (GEV)

$$(A-\lambda I)^2\vec{K}=\begin{bmatrix}0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{Bmatrix}a\\b\\c\\d\\e\end{Bmatrix}=\{\vec{0}\}$$

$$\therefore\ we\ can\ get\ \ c=0$$

$$the\ eigenvectors\ are\ span\left\{\begin{Bmatrix}0\\1\\0\\0\\0\end{Bmatrix}、\begin{Bmatrix}0\\0\\0\\0\\1\end{Bmatrix}\right\}$$

$$\because\begin{Bmatrix}1\\0\\0\\0\\0\end{Bmatrix}doesn't\ satisfy\ the\ condtion\ (A-\lambda I)^{m-1}\overrightarrow{K_m}\neq 0$$

$$\therefore It\ can't\ be\ one\ of\ new\ eigenvector.$$

So far, we get four eigenvectors, but we need one more, do GEV again.

$$(A-\lambda I)^3\vec{K}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{Bmatrix}a\\b\\c\\d\\e\end{Bmatrix}=\{\vec{0}\}$$

$$\therefore\ a,b,c,d,e\ are\ all\ zero.$$

$$span\left\{\begin{Bmatrix}0\\0\\1\\0\\0\end{Bmatrix}\right\}$$

Now, we have five eigenvectors to construct matrix M

$$M=\begin{bmatrix}\overrightarrow{K1}&\overrightarrow{K2}&\overrightarrow{K3}&\overrightarrow{K4}&\overrightarrow{K5}\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}=I,\ where\ I\ is\ idendity\ matrix$$

$$\therefore\ Jodan\ normal\ form\ J=M^{-1}AM=A=\begin{bmatrix}0.3&1&0&0&0\\0&0.3&1&0&0\\0&0&0.3&0&0\\0&0&0&0.3&1\\0&0&0&0&0.3\end{bmatrix}$$

$$\therefore Jordan\ block\ rank=3$$

Jordan block rank is max size of red rectangle from the following figure.

$$\therefore\ Jodan\ normal\ form\ J = M^{-1}AM = A = \begin{bmatrix} 0.3 & 1 & 0 & 0 & 0 \\ 0 & 0.3 & 1 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \\ 0 & 0 & 0 & 0 & 0.3 \end{bmatrix}$$

This is a special example that A matrix is the same as J matrix.

Example2 matrix A below

This example reference from wiki Jordan normal form
(https://en.wikipedia.org/wiki/Jordan_normal_form)

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

To find eigenvalue, it satisfied $A\vec{K} = \lambda\vec{K}$

$$(A - \lambda I)\vec{K} = \{\vec{0}\}$$

$$det(A - \lambda I) = 0$$

$$det(A - \lambda I) = \left| \begin{bmatrix} 5-\lambda & 4 & 2 & 1 \\ 0 & 1-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & 0 \\ 1 & 1 & -1 & 2-\lambda \end{bmatrix} \right| = 0$$

Reduce dimension, and we can get the equation below

$$(\lambda - 2)(\lambda^3 - 9\lambda^2 + 24\lambda - 16) = 0$$
$$\therefore \lambda = \{1,\ 2,\ 4,\ 4\}$$

Sovle the eigenvectors

$$(A - \lambda I)\vec{K} = \{\vec{0}\}$$

Let λ be 1

$$(A - \lambda I)\vec{K} = \begin{bmatrix} 4 & 4 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \{\vec{0}\}$$

Use Gaussian Elimination

$$\left[\begin{array}{cccc|c} 4 & 4 & 2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore c=0,\ d=0,$$
$$a:b=1:-1$$
$$the\ eigenvectors\ are\ span\left\{\begin{Bmatrix}1\\-1\\0\\0\end{Bmatrix}\right\}$$

Let λ be 2, and do the same method

$$\left[\begin{array}{cccc|c}3&4&2&1&0\\0&-1&-1&-1&0\\-1&-1&1&0&0\\1&1&-1&0&0\end{array}\right]=\left[\begin{array}{cccc|c}0&0&1&0&0\\0&1&1&1&0\\0&0&0&0&0\\1&1&-1&0&0\end{array}\right]$$

$$\therefore c=0,$$
$$a:b:d=-1:1:-1$$
$$the\ eigenvectors\ are\ span\left\{\begin{Bmatrix}-1\\1\\0\\-1\end{Bmatrix}\right\}$$

Let λ be 4, and do the same method

$$\left[\begin{array}{cccc|c}1&4&2&1&0\\0&-3&-1&-1&0\\-1&-1&-1&0&0\\1&1&-1&-2&0\end{array}\right]=\left[\begin{array}{cccc|c}0&1&0&0&0\\0&1&0&0&0\\1&0&1&0&0\\0&0&1&1&0\end{array}\right]$$

$$\therefore b=0,$$
$$a:c:d=1:-1:1$$
$$the\ eigenvectors\ are\ span\left\{\begin{Bmatrix}1\\0\\-1\\1\end{Bmatrix}\right\}$$

So far, we only get three eigenvectors, but we need four. Therefore, we need to use generalized eigenvectors (GEV) to get more eigenvector.

Let λ be 4

$$(A-\lambda I)^2\vec{K}=\begin{bmatrix}0&-9&-5&-5\\0&9&5&5\\0&0&0&0\\0&0&4&4\end{bmatrix}\begin{Bmatrix}a\\b\\c\\d\end{Bmatrix}=\{\vec{0}\}$$

Use Gaussian Elimination

$$\left[\begin{array}{cccc|c}0&-9&-5&-5&0\\0&9&5&5&0\\0&0&0&0&0\\0&0&4&4&0\end{array}\right]=\left[\begin{array}{cccc|c}0&1&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&1&1&0\end{array}\right]$$

$$\therefore b = 0,$$
$$c : d = 1 : -1$$
$$the\ eigenvectors\ are\ span\left\{\begin{Bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{Bmatrix}\right\}$$

Now, we have four eigenvectors to construct matrix M

$$M = \begin{bmatrix} \overrightarrow{K1} & \overrightarrow{K2} & \overrightarrow{K3} & \overrightarrow{K4} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$
$$\therefore\ Jodan\ normal\ form\ J = M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$
$$\therefore Jordan\ block\ rank = 2$$

Jordan block rank is max size of red rectangle from the following figure.

$$\therefore\ Jodan\ normal\ form\ J = M^{-1}AM = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Reference

- Wiki: Generalized eigenvector (https://en.wikipedia.org/wiki/Generalized_eigenvector)
- Wiki: Basis (linear algebra) (https://en.wikipedia.org/wiki/Basis_(linear_algebra))
- Wiki: Diagonalizable matrix (https://en.wikipedia.org/wiki/Diagonalizable_matrix)
- Wiki: Jordan normal form (https://en.wikipedia.org/wiki/Jordan_normal_form)
- Wiki: Linear span (https://en.wikipedia.org/wiki/Linear_span)