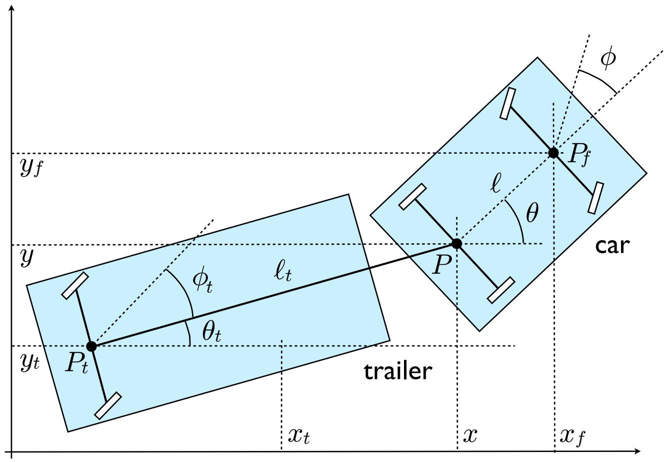
Solution of Problem 1

A convenient choice of generalized coordinates is $\boldsymbol{q} = (x \; y \; \theta \; \phi \; \theta_t \; \phi_t)^T$ (see figure), i.e., a set of generalized coordinates for the car plus two additional coordinates (orientation and steering angle) for the trailer. Hence, the dimension of the configuration space is $n = 6$. In the following, all two-wheel axles are assimilated to a single wheel located at the axle midpoint. The robot has then three wheels: the car front wheel, the car rear wheel, and the trailer wheel.



The $k = 3$ kinematic constraints acting on the robot are therefore (one "pure rolling" condition for each wheel):

$$\begin{aligned}
\dot{x}_f \, \sin(\theta + \phi) - \dot{y}_f \, \cos(\theta + \phi) &= 0 \\
\dot{x} \, \sin\theta - \dot{y} \, \cos\theta &= 0 \\
\dot{x}_t \, \sin(\theta_t + \phi_t) - \dot{y}_t \, \cos(\theta_t + \phi_t) &= 0,
\end{aligned}$$

where (x_f, y_f) and (x_t, y_t) are the Cartesian coordinates of P_f (the centre of the tricycle front wheel) and P_t (the trailer axle midpoint), respectively. Being

$$\begin{aligned}
x_f &= x + \ell \cos\theta \\
y_f &= y + \ell \sin\theta
\end{aligned}$$

and

$$\begin{aligned}
x_t &= x - \ell_t \cos\theta_t \\
y_t &= y - \ell_t \sin\theta_t
\end{aligned}$$

it is easy to obtain the following expression for the kinematic constraints

$$\begin{aligned}
\dot{x} \, \sin(\theta + \phi) - \dot{y} \, \cos(\theta + \phi) - \dot{\theta} \, \ell \cos\phi &= 0 \\
\dot{x} \, \sin\theta - \dot{y} \, \cos\theta &= 0 \\
\dot{x} \, \sin(\theta_t + \phi_t) - \dot{y} \, \cos(\theta_t + \phi_t) + \ell_t \, \dot{\theta}_t \cos\phi_t &= 0,
\end{aligned}$$

or, in Pfaffian form

$$\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 & 0 & 0 \\ \sin(\theta_t+\phi_t) & -\cos(\theta_t+\phi_t) & 0 & 0 & \ell_t\cos\phi_t & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\theta}_t \\ \dot{\phi}_t \end{pmatrix} = \boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}.$$

Since $\boldsymbol{A}^T$ is a 3×6 ($k \times n$) matrix, its null-space has dimension $6 - 3 = 3$. A basis for this null space must therefore consist of three linearly independent vectors. Note also that the submatrix consisting of the first two rows and the first four columns of $\boldsymbol{A}^T$ coincides with the constraint matrix for the bicycle. A basis of $\mathcal{N}(\boldsymbol{A}^T)$ can then be easily written by suitably "extending" (from dimension 4 to dimension 6) the two vectors that provide a basis for the rear-wheel drive bicycle , and adding a third linearly independent vector.

One easily obtains

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{pmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \\ \tan\phi/\ell & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{\sin(\theta_t-\theta+\phi_t)}{\ell_t\cos\phi_t} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{g}_1(\boldsymbol{q}) & \boldsymbol{g}_2(\boldsymbol{q}) & \boldsymbol{g}_3(\boldsymbol{q}) \end{pmatrix}.$$

The kinematic control system is then

$$\dot{\boldsymbol{q}} = \boldsymbol{g}_1(\boldsymbol{q})\, v + \boldsymbol{g}_2(\boldsymbol{q})\, \omega + \boldsymbol{g}_3(\boldsymbol{q})\, \omega_t,$$

where v, ω and ω_t are respectively the driving and steering velocity of the car and the steering velocity of the trailer.

Solution of Problem 2

Denote by $P_c = (x_c, y_c)$ the contact point between the caster and the ground. To write the velocity of P_c as a function of the velocity inputs ω_R, ω_L, one can first consider the robot as a unicycle and find the velocity inputs v, ω which would result in the required V_c; and then transform v, ω in the equivalent velocity inputs ω_R, ω_L of the original differential-drive robot.

We have

$$\begin{aligned} x_c &= x + L\,\cos\theta \\ y_c &= y + L\,\sin\theta \end{aligned}$$

so that

$$V_c = \begin{pmatrix} \dot{x}_c \\ \dot{y}_c \end{pmatrix} = \begin{pmatrix} \cos\theta & -L\,\sin\theta \\ \sin\theta & L\,\cos\theta \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix} = \boldsymbol{T}(\theta) \begin{pmatrix} v \\ \omega \end{pmatrix}.$$

Note that matrix $\boldsymbol{T}(\theta)$ has determinant L and is therefore always invertible. Therefore, the required unicycle inputs are

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \boldsymbol{T}^{-1}(\theta)\, V_c = \begin{pmatrix} \cos\theta & \sin\theta \\ \dfrac{-\sin\theta}{L} & \dfrac{\cos\theta}{L} \end{pmatrix} \begin{pmatrix} ||V_c||\cos(\theta-\alpha) \\ ||V_c||\sin(\theta-\alpha) \end{pmatrix} = ||V_c|| \begin{pmatrix} \cos\alpha \\ \dfrac{-\sin\alpha}{L} \end{pmatrix}.$$

Obviously, these inputs do not depend on the configuration of the robot (in fact, one could have let $\theta = 0$ from the beginning to simplify the computations).

The corresponding inputs for the differential-drive robot can be computed by inverting the well-known formulas

$$\begin{aligned} v &= \frac{r(\omega_R + \omega_L)}{2} \\ \omega &= \frac{r(\omega_R - \omega_L)}{d}, \end{aligned}$$

obtaining

$$\begin{aligned} \omega_R &= \frac{2v + d\,\omega}{2r} \\ \omega_L &= \frac{2v - d\,\omega}{2r}. \end{aligned}$$

Plugging the required v and ω in these formulas we finally obtain

$$\begin{aligned} \omega_R &= \frac{||V_c||}{2r}(2\cos\alpha - \frac{d}{L}\sin\alpha) = 0.157 \ \text{rad/sec} \\ \omega_L &= \frac{||V_c||}{2r}(2\cos\alpha + \frac{d}{L}\sin\alpha) = 0.785 \ \text{rad/sec}. \end{aligned}$$