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3

Differential Kinematics and Statics

In the previous chapter, direct and inverse kinematics equations establishing the relationship between the joint variables and the end-effector pose were derived. In this chapter, *differential kinematics* is presented which gives the relationship between the joint velocities and the corresponding end-effector linear and angular velocity. This mapping is described by a matrix, termed *geometric Jacobian*, which depends on the manipulator configuration. Alternatively, if the end-effector pose is expressed with reference to a minimal representation in the operational space, it is possible to compute the Jacobian matrix via differentiation of the direct kinematics function with respect to the joint variables. The resulting Jacobian, termed *analytical Jacobian*, in general differs from the geometric one. The Jacobian constitutes one of the most important tools for manipulator characterization; in fact, it is useful for finding *singularities*, analyzing *redundancy*, determining *inverse kinematics algorithms*, describing the mapping between forces applied to the end-effector and resulting torques at the joints (*statics*) and, as will be seen in the following chapters, deriving dynamic equations of motion and designing operational space control schemes. Finally, the *kineto-statics duality* concept is illustrated, which is at the basis of the definition of velocity and force *manipulability ellipsoids*.

3.1 Geometric Jacobian

Consider an n-DOF manipulator. The direct kinematics equation can be written in the form

$$\boldsymbol{T}_e(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}_e(\boldsymbol{q}) & \boldsymbol{p}_e(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{bmatrix} \tag{3.1}$$

where $\boldsymbol{q} = [q_1 \quad \dots \quad q_n]^T$ is the vector of joint variables. Both end-effector position and orientation vary as $\boldsymbol{q}$ varies.

The goal of the differential kinematics is to find the relationship between the joint velocities and the end-effector linear and angular velocities. In other words, it is desired to express the end-effector linear velocity $\dot{\boldsymbol{p}}_e$ and angular velocity $\boldsymbol{\omega}_e$ as a function of the joint velocities $\dot{\boldsymbol{q}}$. As will be seen afterwards, the sought relations are both linear in the joint velocities, i.e.,

$$\dot{\boldsymbol{p}}_e = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3.2}$$

$$\boldsymbol{\omega}_e = \boldsymbol{J}_O(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3.3}$$

In (3.2) $\boldsymbol{J}_P$ is the $(3 \times n)$ matrix relating the contribution of the joint velocities $\dot{\boldsymbol{q}}$ to the end-effector *linear* velocity $\dot{\boldsymbol{p}}_e$, while in (3.3) $\boldsymbol{J}_O$ is the $(3 \times n)$ matrix relating the contribution of the joint velocities $\dot{\boldsymbol{q}}$ to the end-effector *angular* velocity $\boldsymbol{\omega}_e$. In compact form, (3.2), (3.3) can be written as

$$\boldsymbol{v}_e = \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \boldsymbol{\omega}_e \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3.4}$$

which represents the manipulator *differential kinematics equation*. The $(6 \times n)$ matrix $\boldsymbol{J}$ is the manipulator *geometric Jacobian*

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_P \\ \boldsymbol{J}_O \end{bmatrix}, \tag{3.5}$$

which in general is a function of the joint variables.

In order to compute the geometric Jacobian, it is worth recalling a number of properties of rotation matrices and some important results of rigid body kinematics.

3.1.1 Derivative of a Rotation Matrix

The manipulator direct kinematics equation in (3.1) describes the end-effector pose, as a function of the joint variables, in terms of a position vector and a rotation matrix. Since the aim is to characterize the end-effector linear and angular velocities, it is worth considering first the *derivative of a rotation matrix* with respect to time.

Consider a time-varying rotation matrix $\boldsymbol{R} = \boldsymbol{R}(t)$. In view of the orthogonality of $\boldsymbol{R}$, one has the relation

$$\boldsymbol{R}(t)\boldsymbol{R}^T(t) = \boldsymbol{I}$$

which, differentiated with respect to time, gives the identity

$$\dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) + \boldsymbol{R}(t)\dot{\boldsymbol{R}}^T(t) = \boldsymbol{O}.$$

Set

$$\boldsymbol{S}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t); \tag{3.6}$$

the (3×3) matrix $\boldsymbol{S}$ is *skew-symmetric* since

$$\boldsymbol{S}(t) + \boldsymbol{S}^T(t) = \boldsymbol{O}. \tag{3.7}$$

Postmultiplying both sides of (3.6) by $\boldsymbol{R}(t)$ gives

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{S}(t)\boldsymbol{R}(t) \tag{3.8}$$

that allows the time derivative of $\boldsymbol{R}(t)$ to be expressed as a function of $\boldsymbol{R}(t)$ itself.

Equation (3.8) relates the rotation matrix $\boldsymbol{R}$ to its derivative by means of the skew-symmetric operator $\boldsymbol{S}$ and has a meaningful physical interpretation. Consider a constant vector $\boldsymbol{p}'$ and the vector $\boldsymbol{p}(t) = \boldsymbol{R}(t)\boldsymbol{p}'$. The time derivative of $\boldsymbol{p}(t)$ is

$$\dot{\boldsymbol{p}}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{p}',$$

which, in view of (3.8), can be written as

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{S}(t)\boldsymbol{R}(t)\boldsymbol{p}'.$$

If the vector $\boldsymbol{\omega}(t)$ denotes the *angular velocity* of frame $\boldsymbol{R}(t)$ with respect to the reference frame at time t, it is known from mechanics that

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{R}(t)\boldsymbol{p}'.$$

Therefore, the matrix operator $\boldsymbol{S}(t)$ describes the vector product between the vector $\boldsymbol{\omega}$ and the vector $\boldsymbol{R}(t)\boldsymbol{p}'$. The matrix $\boldsymbol{S}(t)$ is so that its symmetric elements with respect to the main diagonal represent the components of the vector $\boldsymbol{\omega}(t) = [\,\omega_x \quad \omega_y \quad \omega_z\,]^T$ in the form

$$\boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \tag{3.9}$$

which justifies the expression $\boldsymbol{S}(t) = \boldsymbol{S}(\boldsymbol{\omega}(t))$. Hence, (3.8) can be rewritten as

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}. \tag{3.10}$$

Furthermore, if $\boldsymbol{R}$ denotes a rotation matrix, it can be shown that the following relation holds:

$$\boldsymbol{R}\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}^T = \boldsymbol{S}(\boldsymbol{R}\boldsymbol{\omega}) \tag{3.11}$$

which will be useful later (see Problem 3.1).

Example 3.1

Consider the elementary rotation matrix about axis z given in (2.6). If α is a function of time, by computing the time derivative of $\boldsymbol{R}_z(\alpha(t))$, (3.6) becomes

$$\boldsymbol{S}(t) = \begin{bmatrix} -\dot{\alpha}\sin\alpha & -\dot{\alpha}\cos\alpha & 0 \\ \dot{\alpha}\cos\alpha & -\dot{\alpha}\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\dot{\alpha} & 0 \\ \dot{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{S}(\boldsymbol{\omega}(t)).$$

According to (3.9), it is

$$\boldsymbol{\omega} = [0 \quad 0 \quad \dot{\alpha}]^T$$

that expresses the angular velocity of the frame about axis z.

With reference to Fig. 2.11, consider the coordinate transformation of a point P from Frame 1 to Frame 0; in view of (2.38), this is given by

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1. \tag{3.12}$$

Differentiating (3.12) with respect to time gives

$$\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \dot{\boldsymbol{R}}_1^0 \boldsymbol{p}^1; \tag{3.13}$$

utilizing the expression of the derivative of a rotation matrix (3.8) and specifying the dependence on the angular velocity gives

$$\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{S}(\boldsymbol{\omega}_1^0) \boldsymbol{R}_1^0 \boldsymbol{p}^1.$$

Further, denoting the vector $\boldsymbol{R}_1^0 \boldsymbol{p}^1$ by $\boldsymbol{r}_1^0$, it is

$$\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0 \tag{3.14}$$

which is the known form of the velocity composition rule.

Notice that, if $\boldsymbol{p}^1$ is *fixed* in Frame 1, then it is

$$\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0 \tag{3.15}$$

since $\dot{\boldsymbol{p}}^1 = \boldsymbol{0}$.

3.1.2 Link Velocities

Consider the generic Link i of a manipulator with an open kinematic chain. According to the Denavit–Hartenberg convention adopted in the previous chapter, Link i connects Joints i and $i+1$; Frame i is attached to Link i

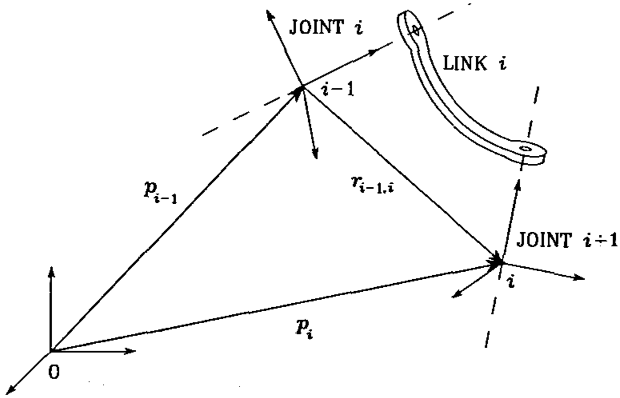

Fig. 3.1. Characterization of generic Link i of a manipulator

and has origin along Joint $i+1$ axis, while Frame $i-1$ has origin along Joint i axis (Fig. 3.1).

Let $\boldsymbol{p}_{i-1}$ and $\boldsymbol{p}_i$ be the position vectors of the origins of Frames $i-1$ and i, respectively. Also, let $\boldsymbol{r}_{i-1,i}^{i-1}$ denote the position of the origin of Frame i with respect to Frame $i-1$ expressed in Frame $i-1$. According to the coordinate transformation (3.10), one can write[1]

$$\boldsymbol{p}_i = \boldsymbol{p}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1}.$$

Then, by virtue of (3.14), it is

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{R}_{i-1}\dot{\boldsymbol{r}}_{i-1,i}^{i-1} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1} = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i} \tag{3.16}$$

which gives the expression of the linear velocity of Link i as a function of the translational and rotational velocities of Link $i-1$. Note that $\boldsymbol{v}_{i-1,i}$ denotes the velocity of the origin of Frame i with respect to the origin of Frame $i-1$.

Concerning link angular velocity, it is worth starting from the rotation composition

$$\boldsymbol{R}_i = \boldsymbol{R}_{i-1}\boldsymbol{R}_i^{i-1};$$

from (3.8), its time derivative can be written as

$$\boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{R}_i = \boldsymbol{S}(\boldsymbol{\omega}_{i-1})\boldsymbol{R}_i + \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1} \tag{3.17}$$

where $\boldsymbol{\omega}_{i-1,i}^{i-1}$ denotes the angular velocity of Frame i with respect to Frame $i-1$ expressed in Frame $i-1$. From (2.4), the second term on the right-hand side of (3.17) can be rewritten as

$$\boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1} = \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_{i-1}^T\boldsymbol{R}_{i-1}\boldsymbol{R}_i^{i-1};$$

[1] Hereafter, the indication of superscript '0' is omitted for quantities referred to Frame 0. Also, without loss of generality, Frame 0 and Frame n are taken as the base frame and the end-effector frame, respectively.

in view of property (3.11), it is

$$R_{i-1}S(\omega_{i-1,i}^{i-1})R_i^{i-1} = S(R_{i-1}\omega_{i-1,i}^{i-1})R_i.$$

Then, (3.17) becomes

$$S(\omega_i)R_i = S(\omega_{i-1})R_i + S(R_{i-1}\omega_{i-1,i}^{i-1})R_i$$

leading to the result

$$\omega_i = \omega_{i-1} + R_{i-1}\omega_{i-1,i}^{i-1} = \omega_{i-1} + \omega_{i-1,i}, \tag{3.18}$$

which gives the expression of the angular velocity of Link i as a function of the angular velocities of Link $i-1$ and of Link i with respect to Link $i-1$.

The relations (3.16), (3.18) attain different expressions depending on the type of Joint i (*prismatic* or *revolute*).

Prismatic joint

Since orientation of Frame i with respect to Frame $i-1$ does not vary by moving Joint i, it is

$$\omega_{i-1,i} = \mathbf{0}. \tag{3.19}$$

Further, the linear velocity is

$$v_{i-1,i} = \dot{d}_i z_{i-1} \tag{3.20}$$

where z_{i-1} is the unit vector of Joint i axis. Hence, the expressions of angular velocity (3.18) and linear velocity (3.16) respectively become

$$\omega_i = \omega_{i-1} \tag{3.21}$$

$$\dot{p}_i = \dot{p}_{i-1} + \dot{d}_i z_{i-1} + \omega_i \times r_{i-1,i}, \tag{3.22}$$

where the relation $\omega_i = \omega_{i-1}$ has been exploited to derive (3.22).

Revolute joint

For the angular velocity it is obviously

$$\omega_{i-1,i} = \dot{\vartheta}_i z_{i-1}, \tag{3.23}$$

while for the linear velocity it is

$$v_{i-1,i} = \omega_{i-1,i} \times r_{i-1,i} \tag{3.24}$$

due to the rotation of Frame i with respect to Frame $i-1$ induced by the motion of Joint i. Hence, the expressions of angular velocity (3.18) and linear velocity (3.16) respectively become

$$\omega_i = \omega_{i-1} + \dot{\vartheta}_i z_{i-1} \tag{3.25}$$

$$\dot{p}_i = \dot{p}_{i-1} + \omega_i \times r_{i-1,i}, \tag{3.26}$$

where (3.18) has been exploited to derive (3.26).

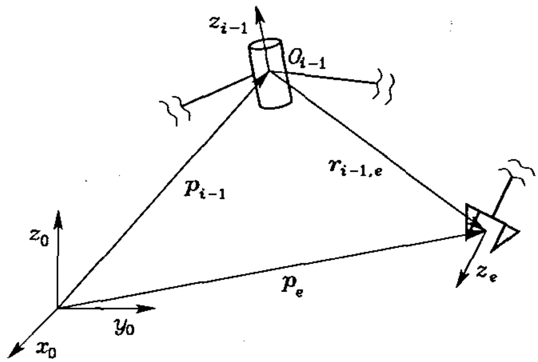

Fig. 3.2. Representation of vectors needed for the computation of the velocity contribution of a revolute joint to the end-effector linear velocity

3.1.3 Jacobian Computation

In order to compute the Jacobian, it is convenient to proceed separately for the linear velocity and the angular velocity.

For the contribution to the *linear velocity*, the time derivative of $\boldsymbol{p}_e(\boldsymbol{q})$ can be written as

$$\dot{\boldsymbol{p}}_e = \sum_{i=1}^{n} \frac{\partial \boldsymbol{p}_e}{\partial q_i} \dot{q}_i = \sum_{i=1}^{n} \boldsymbol{\jmath}_{Pi} \dot{q}_i. \tag{3.27}$$

This expression shows how $\dot{\boldsymbol{p}}_e$ can be obtained as the sum of the terms $\dot{q}_i \boldsymbol{\jmath}_{Pi}$. Each term represents the contribution of the velocity of single Joint i to the end-effector linear velocity when all the other joints are still.

Therefore, by distinguishing the case of a *prismatic* joint ($q_i = d_i$) from the case of a *revolute* joint ($q_i = \vartheta_i$), it is:

- If Joint i is *prismatic*, from (3.20) it is

$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \dot{d}_i \boldsymbol{z}_{i-1}$$

  and then

$$\boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1}.$$

- If Joint i is *revolute*, observing that the contribution to the linear velocity is to be computed with reference to the origin of the end-effector frame (Fig. 3.2), it is

$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,e} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})$$

  and then

$$\boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1}).$$

For the contribution to the *angular velocity*, in view of (3.18), it is

$$\boldsymbol{\omega}_e = \boldsymbol{\omega}_n = \sum_{i=1}^{n} \boldsymbol{\omega}_{i-1,i} = \sum_{i=1}^{n} \boldsymbol{\jmath}_{Oi}\dot{q}_i, \tag{3.28}$$

where (3.19) and (3.23) have been utilized to characterize the terms $\dot{q}_i \boldsymbol{\jmath}_{Oi}$, and thus in detail:

- If Joint i is *prismatic*, from (3.19) it is

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \mathbf{0}$$

  and then

$$\boldsymbol{\jmath}_{Oi} = \mathbf{0}.$$

- If Joint i is *revolute*, from (3.23) it is

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$

  and then

$$\boldsymbol{\jmath}_{Oi} = \boldsymbol{z}_{i-1}.$$

In summary, the Jacobian in (3.5) can be partitioned into the (3×1) column vectors $\boldsymbol{\jmath}_{Pi}$ and $\boldsymbol{\jmath}_{Oi}$ as

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{\jmath}_{P1} & & \boldsymbol{\jmath}_{Pn} \\ & \dots & \\ \boldsymbol{\jmath}_{O1} & & \boldsymbol{\jmath}_{On} \end{bmatrix}, \tag{3.29}$$

where

$$\begin{bmatrix} \boldsymbol{\jmath}_{Pi} \\ \boldsymbol{\jmath}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \boldsymbol{z}_{i-1} \\ \mathbf{0} \end{bmatrix} & \text{for a \textit{prismatic} joint} \\ \begin{bmatrix} \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1}) \\ \boldsymbol{z}_{i-1} \end{bmatrix} & \text{for a \textit{revolute} joint.} \end{cases} \tag{3.30}$$

The expressions in (3.30) allow Jacobian computation in a simple, systematic way on the basis of direct kinematics relations. In fact, the vectors $\boldsymbol{z}_{i-1}$, $\boldsymbol{p}_e$ and $\boldsymbol{p}_{i-1}$ are all functions of the joint variables. In particular:

- $\boldsymbol{z}_{i-1}$ is given by the third column of the rotation matrix $\boldsymbol{R}_{i-1}^0$, i.e.,

$$\boldsymbol{z}_{i-1} = \boldsymbol{R}_1^0(q_1)\dots\boldsymbol{R}_{i-1}^{i-2}(q_{i-1})\boldsymbol{z}_0 \tag{3.31}$$

  where $\boldsymbol{z}_0 = [0 \quad 0 \quad 1]^T$ allows the selection of the third column.
- $\boldsymbol{p}_e$ is given by the first three elements of the fourth column of the transformation matrix $\boldsymbol{T}_e^0$, i.e., by expressing $\widetilde{\boldsymbol{p}}_e$ in the (4×1) homogeneous form

$$\widetilde{\boldsymbol{p}}_e = \boldsymbol{A}_1^0(q_1)\dots\boldsymbol{A}_n^{n-1}(q_n)\widetilde{\boldsymbol{p}}_0 \tag{3.32}$$

  where $\widetilde{\boldsymbol{p}}_0 = [0 \quad 0 \quad 0 \quad 1]^T$ allows the selection of the fourth column.

- $\boldsymbol{p}_{i-1}$ is given by the first three elements of the fourth column of the transformation matrix $\boldsymbol{T}_{i-1}^0$, i.e., it can be extracted from

$$\widetilde{\boldsymbol{p}}_{i-1} = \boldsymbol{A}_1^0(q_1)\dots\boldsymbol{A}_{i-1}^{i-2}(q_{i-1})\widetilde{\boldsymbol{p}}_0. \tag{3.33}$$

The above equations can be conveniently used to compute the translational and rotational velocities of any point along the manipulator structure, as long as the direct kinematics functions relative to that point are known.

Finally, notice that the Jacobian matrix depends on the frame in which the end-effector velocity is expressed. The above equations allow computation of the geometric Jacobian with respect to the base frame. If it is desired to represent the Jacobian in a different Frame u, it is sufficient to know the relative rotation matrix $\boldsymbol{R}^u$. The relationship between velocities in the two frames is

$$\begin{bmatrix} \dot{\boldsymbol{p}}_e^u \\ \boldsymbol{\omega}_e^u \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^u & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^u \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \boldsymbol{\omega}_e \end{bmatrix},$$

which, substituted in (3.4), gives

$$\begin{bmatrix} \dot{\boldsymbol{p}}_e^u \\ \boldsymbol{\omega}_e^u \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^u & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^u \end{bmatrix} \boldsymbol{J}\dot{\boldsymbol{q}}$$

and then

$$\boldsymbol{J}^u = \begin{bmatrix} \boldsymbol{R}^u & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^u \end{bmatrix} \boldsymbol{J}, \tag{3.34}$$

where $\boldsymbol{J}^u$ denotes the geometric Jacobian in Frame u, which has been assumed to be time-invariant.

3.2 Jacobian of Typical Manipulator Structures

In the following, the Jacobian is computed for some of the typical manipulator structures presented in the previous chapter.

3.2.1 Three-link Planar Arm

In this case, from (3.30) the Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_3 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_3 - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p}_3 - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}.$$

Computation of the position vectors of the various links gives

$$\boldsymbol{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_1 = \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \end{bmatrix}$$

$$\boldsymbol{p}_3 = \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix}$$

while computation of the unit vectors of revolute joint axes gives

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

since they are all parallel to axis z_0. From (3.29) it is

$$\boldsymbol{J} = \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} & -a_2s_{12} - a_3s_{123} & -a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} & a_2c_{12} + a_3c_{123} & a_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \tag{3.35}$$

In the Jacobian (3.35), only the three non-null rows are relevant (the rank of the matrix is at most 3); these refer to the two components of linear velocity along axes x_0, y_0 and the component of angular velocity about axis z_0. This result can be derived by observing that three DOFs allow specification of at most three end-effector variables; v_z, ω_x, ω_y are always null for this kinematic structure. If orientation is of no concern, the (2×3) Jacobian for the positional part can be derived by considering just the first two rows, i.e.,

$$\boldsymbol{J}_P = \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} & -a_2s_{12} - a_3s_{123} & -a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} & a_2c_{12} + a_3c_{123} & a_3c_{123} \end{bmatrix}. \tag{3.36}$$

3.2.2 Anthropomorphic Arm

In this case, from (3.30) the Jacobian is

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_3 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_3 - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p}_3 - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}.$$

Computation of the position vectors of the various links gives

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{p}_2 = \begin{bmatrix} a_2c_1c_2 \\ a_2s_1c_2 \\ a_2s_2 \end{bmatrix}$$

$$\boldsymbol{p}_3 = \begin{bmatrix} c_1(a_2c_2 + a_3c_{23}) \\ s_1(a_2c_2 + a_3c_{23}) \\ a_2s_2 + a_3s_{23} \end{bmatrix}$$

while computation of the unit vectors of revolute joint axes gives

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}.$$

From (3.29) it is

$$\boldsymbol{J} = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{3.37}$$

Only three of the six rows of the Jacobian (3.37) are linearly independent. Having 3 DOFs only, it is worth considering the upper (3×3) block of the Jacobian

$$\boldsymbol{J}_P = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix} \tag{3.38}$$

that describes the relationship between the joint velocities and the end-effector linear velocity. This structure does not allow an arbitrary angular velocity $\boldsymbol{\omega}$ to be obtained; in fact, the two components ω_x and ω_y are not independent $(s_1\omega_y = -c_1\omega_x)$.

3.2.3 Stanford Manipulator

In this case, from (3.30) it is

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_6 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_6 - \boldsymbol{p}_1) & \boldsymbol{z}_2 \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} \end{bmatrix.$$
$$\begin{matrix} \boldsymbol{z}_3 \times (\boldsymbol{p}_6 - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p}_6 - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p}_6 - \boldsymbol{p}_5) \\ \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}.$$

Computation of the position vectors of the various links gives

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{p}_3 = \boldsymbol{p}_4 = \boldsymbol{p}_5 = \begin{bmatrix} c_1s_2d_3 - s_1d_2 \\ s_1s_2d_3 + c_1d_2 \\ c_2d_3 \end{bmatrix}$$

$$\boldsymbol{p}_6 = \begin{bmatrix} c_1s_2d_3 - s_1d_2 + \big(c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5\big)d_6 \\ s_1s_2d_3 + c_1d_2 + \big(s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5\big)d_6 \\ c_2d_3 + (-s_2c_4s_5 + c_2c_5)d_6 \end{bmatrix},$$

while computation of the unit vectors of joint axes gives

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad z_2 = z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \quad z_5 = \begin{bmatrix} c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5 \\ s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}.$$

The sought Jacobian can be obtained by developing the computations as in (3.29), leading to expressing end-effector linear and angular velocity as a function of joint velocities.

3.3 Kinematic Singularities

The Jacobian in the differential kinematics equation of a manipulator defines a linear mapping

$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3.39}$$

between the vector $\dot{\boldsymbol{q}}$ of joint velocities and the vector $\boldsymbol{v}_e = [\dot{\boldsymbol{p}}_e^T \quad \boldsymbol{\omega}_e^T]^T$ of end-effector velocity. The Jacobian is, in general, a function of the configuration $\boldsymbol{q}$; those configurations at which $\boldsymbol{J}$ is rank-deficient are termed *kinematic singularities*. To find the singularities of a manipulator is of great interest for the following reasons:

a) Singularities represent configurations at which mobility of the structure is reduced, i.e., it is not possible to impose an arbitrary motion to the end-effector.

b) When the structure is at a singularity, infinite solutions to the inverse kinematics problem may exist.

c) In the neighbourhood of a singularity, small velocities in the operational space may cause large velocities in the joint space.

Singularities can be classified into:

- *Boundary* singularities that occur when the manipulator is either outstretched or retracted. It may be understood that these singularities do not represent a true drawback, since they can be avoided on condition that the manipulator is not driven to the boundaries of its reachable workspace.
- *Internal* singularities that occur inside the reachable workspace and are generally caused by the alignment of two or more axes of motion, or else by the attainment of particular end-effector configurations. Unlike the above, these singularities constitute a serious problem, as they can be encountered anywhere in the reachable workspace for a planned path in the operational space.

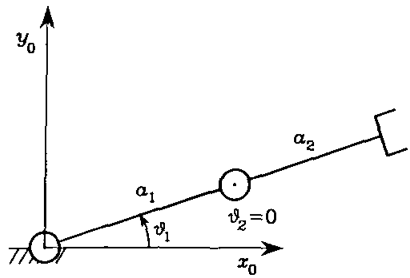

Fig. 3.3. Two-link planar arm at a boundary singularity

Example 3.2

To illustrate the behaviour of a manipulator at a singularity, consider a two-link planar arm. In this case, it is worth considering only the components $\dot{p}_x$ and $\dot{p}_y$ of the linear velocity in the plane. Thus, the Jacobian is the (2×2) matrix

$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}. \tag{3.40}$$

To analyze matrix rank, consider its determinant given by

$$\det(\boldsymbol{J}) = a_1 a_2 s_2. \tag{3.41}$$

For $a_1, a_2 \neq 0$, it is easy to find that the determinant in (3.41) vanishes whenever

$$\vartheta_2 = 0 \qquad \vartheta_2 = \pi,$$

ϑ_1 being irrelevant for the determination of singular configurations. These occur when the arm tip is located either on the outer ($\vartheta_2 = 0$) or on the inner ($\vartheta_2 = \pi$) boundary of the reachable workspace. Figure 3.3 illustrates the arm posture for $\vartheta_2 = 0$.

By analyzing the differential motion of the structure in such configuration, it can be observed that the two column vectors $[\,-(a_1 + a_2)s_1 \quad (a_1 + a_2)c_1\,]^T$ and $[\,-a_2 s_1 \quad a_2 c_1\,]^T$ of the Jacobian become parallel, and thus the Jacobian rank becomes one; this means that the tip velocity components are not independent (see point **a)** above).

3.3.1 Singularity Decoupling

Computation of internal singularities via the Jacobian determinant may be tedious and of no easy solution for complex structures. For manipulators having a spherical wrist, by analogy with what has already been seen for inverse kinematics, it is possible to split the problem of singularity computation into two separate problems:

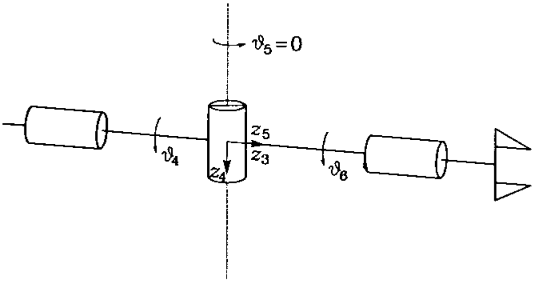

Fig. 3.4. Spherical wrist at a singularity

- computation of *arm singularities* resulting from the motion of the first 3 or more links,
- computation of *wrist singularities* resulting from the motion of the wrist joints.

For the sake of simplicity, consider the case $n = 6$; the Jacobian can be partitioned into (3×3) blocks as follows:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} \end{bmatrix} \tag{3.42}$$

where, since the outer 3 joints are all revolute, the expressions of the two right blocks are respectively

$$\boldsymbol{J}_{12} = \begin{bmatrix} \boldsymbol{z}_3 \times (\boldsymbol{p}_e - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p}_e - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p}_e - \boldsymbol{p}_5) \end{bmatrix}$$

$$\boldsymbol{J}_{22} = \begin{bmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}. \tag{3.43}$$

As singularities are typical of the mechanical structure and do not depend on the frames chosen to describe kinematics, it is convenient to choose the origin of the end-effector frame at the intersection of the wrist axes (see Fig. 2.32). The choice $\boldsymbol{p} = \boldsymbol{p}_W$ leads to

$$\boldsymbol{J}_{12} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$

since all vectors $\boldsymbol{p}_W - \boldsymbol{p}_i$ are parallel to the unit vectors $\boldsymbol{z}_i$, for $i = 3, 4, 5$, no matter how Frames $3, 4, 5$ are chosen according to DH convention. In view of this choice, the overall Jacobian becomes a block lower-triangular matrix. In this case, computation of the determinant is greatly simplified, as this is given by the product of the determinants of the two blocks on the diagonal, i.e.,

$$\det(\boldsymbol{J}) = \det(\boldsymbol{J}_{11})\det(\boldsymbol{J}_{22}). \tag{3.44}$$

In turn, a true *singularity decoupling* has been achieved; the condition

$$\det(\boldsymbol{J}_{11}) = 0$$

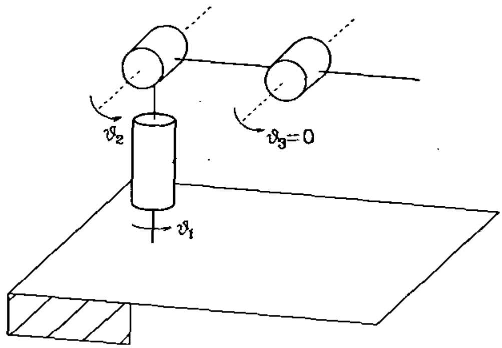

Fig. 3.5. *Anthropomorphic arm at an elbow singularity*

leads to determining the *arm singularities*, while the condition

$$\det(\boldsymbol{J}_{22}) = 0$$

leads to determining the *wrist singularities*.

Notice, however, that this form of Jacobian does not provide the relationship between the joint velocities and the end-effector velocity, but it leads to simplifying singularity computation. Below the two types of singularities are analyzed in detail.

3.3.2 Wrist Singularities

On the basis of the above singularity decoupling, wrist singularities can be determined by inspecting the block $\boldsymbol{J}_{22}$ in (3.43). It can be recognized that the wrist is at a singular configuration whenever the unit vectors $\boldsymbol{z}_3$, $\boldsymbol{z}_4$, $\boldsymbol{z}_5$ are linearly dependent. The wrist kinematic structure reveals that a singularity occurs when $\boldsymbol{z}_3$ and $\boldsymbol{z}_5$ are aligned, i.e., whenever

$$\vartheta_5 = 0 \qquad \vartheta_5 = \pi.$$

Taking into consideration only the first configuration (Fig. 3.4), the loss of mobility is caused by the fact that rotations of equal magnitude about opposite directions on ϑ_4 and ϑ_6 do not produce any end-effector rotation. Further, the wrist is not allowed to rotate about the axis orthogonal to $\boldsymbol{z}_4$ and $\boldsymbol{z}_3$, (see point **a**) above). This singularity is naturally described in the joint space and can be encountered anywhere inside the manipulator reachable workspace; as a consequence, special care is to be taken in programming an end-effector motion.

3.3.3 Arm Singularities

Arm singularities are characteristic of a specific manipulator structure; to illustrate their determination, consider the anthropomorphic arm (Fig. 2.23),

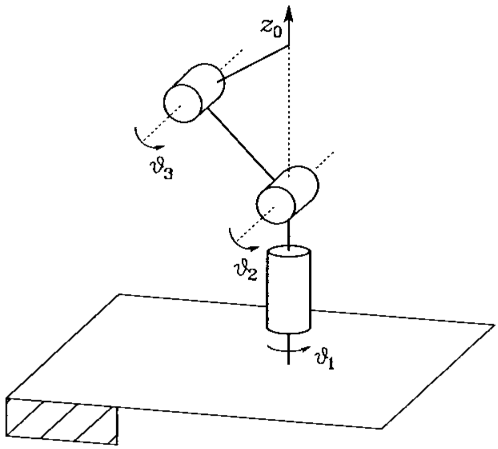

Fig. 3.6. Anthropomorphic arm at a shoulder singularity

whose Jacobian for the linear velocity part is given by (3.38). Its determinant is

$$\det(\boldsymbol{J}_P) = -a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23}).$$

Like in the case of the planar arm of Example 3.2, the determinant does not depend on the first joint variable.

For $a_2, a_3 \neq 0$, the determinant vanishes if $s_3 = 0$ and/or $(a_2 c_2 + a_3 c_{23}) = 0$. The first situation occurs whenever

$$\vartheta_3 = 0 \qquad \vartheta_3 = \pi$$

meaning that the elbow is outstretched (Fig. 3.5) or retracted, and is termed *elbow singularity*. Notice that this type of singularity is conceptually equivalent to the singularity found for the two-link planar arm.

By recalling the direct kinematics equation in (2.66), it can be observed that the second situation occurs when the wrist point lies on axis z_0 (Fig. 3.6); it is thus characterized by

$$p_x = p_y = 0$$

and is termed *shoulder singularity*.

Notice that the whole axis z_0 describes a continuum of singular configurations; a rotation of ϑ_1 does not cause any translation of the wrist position (the first column of $\boldsymbol{J}_P$ is always null at a shoulder singularity), and then the kinematics equation admits infinite solutions; moreover, motions starting from the singular configuration that take the wrist along the z_1 direction are not allowed (see point **b)** above).

If a spherical wrist is connected to an anthropomorphic arm (Fig. 2.26), the arm direct kinematics is different. In this case the Jacobian to consider represents the block $\boldsymbol{J}_{11}$ of the Jacobian in (3.42) with $\boldsymbol{p} = \boldsymbol{p}_W$. Analyzing its determinant leads to finding the same singular configurations, which are relative to different values of the third joint variables, though — compare (2.66) and (2.70).

Finally, it is important to remark that, unlike the wrist singularities, the arm singularities are well identified in the operational space, and thus they can be suitably avoided in the end-effector trajectory planning stage.

3.4 Analysis of Redundancy

The concept of *kinematic redundancy* has been introduced in Sect. 2.10.2; redundancy is related to the number n of DOFs of the structure, the number m of operational space variables, and the number r of operational space variables necessary to specify a given task.

In order to perform a systematic analysis of redundancy, it is worth considering differential kinematics in lieu of direct kinematics (2.82). To this end, (3.39) is to be interpreted as the differential kinematics mapping relating the n components of the joint velocity vector to the $r \leq m$ components of the velocity vector $\boldsymbol{v}_e$ of concern for the specific task. To clarify this point, consider the case of a 3-link planar arm; that is not intrinsically redundant ($n = m = 3$) and its Jacobian (3.35) has 3 null rows accordingly. If the task does not specify ω_z ($r = 2$), the arm becomes functionally redundant and the Jacobian to consider for redundancy analysis is the one in (3.36).

A different case is that of the anthropomorphic arm for which only position variables are of concern ($n = m = 3$). The relevant Jacobian is the one in (3.38). The arm is neither intrinsically redundant nor can become functionally redundant if it is assigned a planar task; in that case, indeed, the task would set constraints on the 3 components of end-effector linear velocity.

Therefore, the differential kinematics equation to consider can be formally written as in (3.39), i.e.,

$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{3.45}$$

where now $\boldsymbol{v}_e$ is meant to be the $(r \times 1)$ vector of end-effector velocity of concern for the specific task and $\boldsymbol{J}$ is the corresponding $(r \times n)$ Jacobian matrix that can be extracted from the geometric Jacobian; $\dot{\boldsymbol{q}}$ is the $(n \times 1)$ vector of joint velocities. If $r < n$, the manipulator is kinematically redundant and there exist $(n - r)$ *redundant DOFs*.

The Jacobian describes the linear mapping from the joint velocity space to the end-effector velocity space. In general, it is a function of the configuration. In the context of differential kinematics, however, the Jacobian has to be regarded as a constant matrix, since the instantaneous velocity mapping is of interest for a given posture. The mapping is schematically illustrated in Fig. 3.7 with a typical notation from set theory.

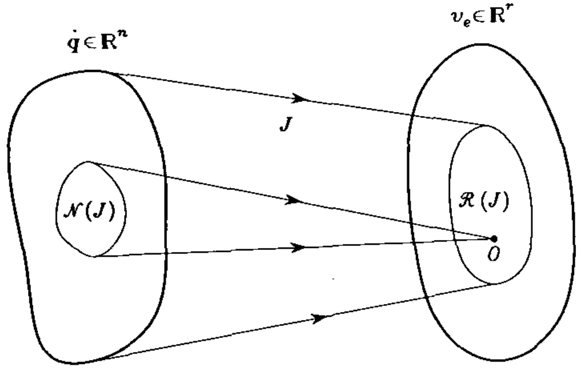

Fig. 3.7. Mapping between the joint velocity space and the end-effector velocity space

The differential kinematics equation in (3.45) can be characterized in terms of the *range* and *null* spaces of the mapping;[2] specifically, one has that:

- The *range* space of $\boldsymbol{J}$ is the subspace $\mathcal{R}(\boldsymbol{J})$ in $\mathbb{R}^r$ of the end-effector velocities that can be generated by the joint velocities, in the given manipulator posture.
- The *null* space of $\boldsymbol{J}$ is the subspace $\mathcal{N}(\boldsymbol{J})$ in $\mathbb{R}^n$ of joint velocities that do not produce any end-effector velocity, in the given manipulator posture.

If the Jacobian has *full rank*, one has

$$\dim\big(\mathcal{R}(\boldsymbol{J})\big) = r \qquad \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n - r$$

and the range of $\boldsymbol{J}$ spans the entire space $\mathbb{R}^r$. Instead, if the Jacobian degenerates at a *singularity*, the dimension of the range space decreases while the dimension of the null space increases, since the following relation holds:

$$\dim\big(\mathcal{R}(\boldsymbol{J})\big) + \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n$$

independently of the rank of the matrix $\boldsymbol{J}$.

The existence of a subspace $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$ for a redundant manipulator allows determination of systematic techniques for handling redundant DOFs. To this end, if $\dot{\boldsymbol{q}}^*$ denotes a solution to (3.45) and $\boldsymbol{P}$ is an $(n \times n)$ matrix so that

$$\mathcal{R}(\boldsymbol{P}) \equiv \mathcal{N}(\boldsymbol{J}),$$

the joint velocity vector

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^* + \boldsymbol{P}\dot{\boldsymbol{q}}_0, \tag{3.46}$$

with arbitrary $\dot{\boldsymbol{q}}_0$, is also a solution to (3.45). In fact, premultiplying both sides of (3.46) by $\boldsymbol{J}$ yields

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}^* + \boldsymbol{J}\boldsymbol{P}\dot{\boldsymbol{q}}_0 = \boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v}_e$$

[2] See Sect. A.4 for the linear mappings.

since $\boldsymbol{J}\boldsymbol{P}\dot{\boldsymbol{q}}_0 = \boldsymbol{0}$ for any $\dot{\boldsymbol{q}}_0$. This result is of fundamental importance for redundancy resolution; a solution of the kind (3.46) points out the possibility of choosing the vector of arbitrary joint velocities $\dot{\boldsymbol{q}}_0$ so as to exploit advantageously the redundant DOFs. In fact, the effect of $\dot{\boldsymbol{q}}_0$ is to generate *internal motions* of the structure that do not change the end-effector position and orientation but may allow, for instance, manipulator reconfiguration into more dexterous postures for execution of a given task.

3.5 Inverse Differential Kinematics

In Sect. 2.12 it was shown how the inverse kinematics problem admits closed-form solutions only for manipulators having a simple kinematic structure. Problems arise whenever the end-effector attains a particular position and/or orientation in the operational space, or the structure is complex and it is not possible to relate the end-effector pose to different sets of joint variables, or else the manipulator is redundant. These limitations are caused by the highly nonlinear relationship between joint space variables and operational space variables.

On the other hand, the differential kinematics equation represents a linear mapping between the joint velocity space and the operational velocity space, although it varies with the current configuration. This fact suggests the possibility to utilize the differential kinematics equation to tackle the inverse kinematics problem.

Suppose that a motion trajectory is assigned to the end-effector in terms of $\boldsymbol{v}_e$ and the initial conditions on position and orientation. The aim is to determine a feasible joint trajectory $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ that reproduces the given trajectory.

By considering (3.45) with $n = r$, the joint velocities can be obtained via simple inversion of the Jacobian matrix

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}_e. \tag{3.47}$$

If the initial manipulator posture $\boldsymbol{q}(0)$ is known, joint positions can be computed by integrating velocities over time, i.e.,

$$\boldsymbol{q}(t) = \int_0^t \dot{\boldsymbol{q}}(\varsigma)d\varsigma + \boldsymbol{q}(0).$$

The integration can be performed in discrete time by resorting to numerical techniques. The simplest technique is based on the Euler integration method; given an integration interval Δt, if the joint positions and velocities at time t_k are known, the joint positions at time $t_{k+1} = t_k + \Delta t$ can be computed as

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \dot{\boldsymbol{q}}(t_k)\Delta t. \tag{3.48}$$

This technique for inverting kinematics is independent of the solvability of the kinematic structure. Nonetheless, it is necessary that the *Jacobian* be *square* and of *full rank*; this demands further insight into the cases of *redundant* manipulators and kinematic *singularity* occurrence.

3.5.1 Redundant Manipulators

When the manipulator is *redundant* ($r < n$), the Jacobian matrix has more columns than rows and infinite solutions exist to (3.45). A viable solution method is to formulate the problem as a constrained linear optimization problem.

In detail, once the end-effector velocity $\boldsymbol{v}_e$ and Jacobian $\boldsymbol{J}$ are given (for a given configuration $\boldsymbol{q}$), it is desired to find the solutions $\dot{\boldsymbol{q}}$ that satisfy the linear equation in (3.45) and *minimize* the quadratic cost functional of joint velocities[3]

$$g(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}}$$

where $\boldsymbol{W}$ is a suitable $(n \times n)$ symmetric positive definite weighting matrix.

This problem can be solved with the *method of Lagrange multipliers*. Consider the modified cost functional

$$g(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}} + \boldsymbol{\lambda}^T(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}}),$$

where $\boldsymbol{\lambda}$ is an $(r \times 1)$ vector of unknown multipliers that allows the incorporation of the constraint (3.45) in the functional to minimize. The requested solution has to satisfy the necessary conditions:

$$\left(\frac{\partial g}{\partial \dot{\boldsymbol{q}}}\right)^T = \boldsymbol{0} \qquad \left(\frac{\partial g}{\partial \boldsymbol{\lambda}}\right)^T = \boldsymbol{0}.$$

From the first one, it is $\boldsymbol{W}\dot{\boldsymbol{q}} - \boldsymbol{J}^T\boldsymbol{\lambda} = \boldsymbol{0}$ and thus

$$\dot{\boldsymbol{q}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T\boldsymbol{\lambda} \tag{3.49}$$

where the inverse of $\boldsymbol{W}$ exists. Notice that the solution (3.49) is a minimum, since $\partial^2 g/\partial \dot{\boldsymbol{q}}^2 = \boldsymbol{W}$ is positive definite. From the second condition above, the constraint

$$\boldsymbol{v}_e = \boldsymbol{J}\dot{\boldsymbol{q}}$$

is recovered. Combining the two conditions gives

$$\boldsymbol{v}_e = \boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\boldsymbol{\lambda};$$

under the assumption that $\boldsymbol{J}$ has full rank, $\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T$ is an $(r \times r)$ square matrix of rank r and thus can be inverted. Solving for $\boldsymbol{\lambda}$ yields

$$\boldsymbol{\lambda} = (\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{v}_e$$

[3] Quadratic forms and the relative operations are recalled in Sect. A.6.

which, substituted into (3.49), gives the sought optimal solution

$$\dot{\boldsymbol{q}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{v}_e. \tag{3.50}$$

Premultiplying both sides of (3.50) by $\boldsymbol{J}$, it is easy to verify that this solution satisfies the differential kinematics equation in (3.45).

A particular case occurs when the weighting matrix $\boldsymbol{W}$ is the identity matrix $\boldsymbol{I}$ and the solution simplifies into

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}\boldsymbol{v}_e; \tag{3.51}$$

the matrix

$$\boldsymbol{J}^{\dagger} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1} \tag{3.52}$$

is the *right pseudo-inverse* of $\boldsymbol{J}$.[4] The obtained solution locally minimizes the norm of joint velocities.

It was pointed out above that if $\dot{\boldsymbol{q}}^*$ is a solution to (3.45), $\dot{\boldsymbol{q}}^* + \boldsymbol{P}\dot{\boldsymbol{q}}_0$ is also a solution, where $\dot{\boldsymbol{q}}_0$ is a vector of arbitrary joint velocities and $\boldsymbol{P}$ is a projector in the null space of $\boldsymbol{J}$. Therefore, in view of the presence of redundant DOFs, the solution (3.51) can be modified by the introduction of another term of the kind $\boldsymbol{P}\dot{\boldsymbol{q}}_0$. In particular, $\dot{\boldsymbol{q}}_0$ can be specified so as to satisfy an additional constraint to the problem.

In that case, it is necessary to consider a new cost functional in the form

$$g'(\dot{\boldsymbol{q}}) = \frac{1}{2}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0)^T(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0);$$

this choice is aimed at minimizing the norm of vector $\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0$; in other words, solutions are sought which satisfy the constraint (3.45) and are as close as possible to $\dot{\boldsymbol{q}}_0$. In this way, the objective specified through $\dot{\boldsymbol{q}}_0$ becomes unavoidably a secondary objective to satisfy with respect to the primary objective specified by the constraint (3.45).

Proceeding in a way similar to the above yields

$$g'(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0)^T(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0) + \boldsymbol{\lambda}^T(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}});$$

from the first necessary condition it is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^T\boldsymbol{\lambda} + \dot{\boldsymbol{q}}_0 \tag{3.53}$$

which, substituted into (3.45), gives

$$\boldsymbol{\lambda} = (\boldsymbol{J}\boldsymbol{J}^T)^{-1}(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}}_0).$$

Finally, substituting $\boldsymbol{\lambda}$ back in (3.53) gives

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}\boldsymbol{v}_e + (\boldsymbol{I}_n - \boldsymbol{J}^{\dagger}\boldsymbol{J})\dot{\boldsymbol{q}}_0. \tag{3.54}$$

---

[4] See Sect. A.7 for the definition of the pseudo-inverse of a matrix.

As can be easily recognized, the obtained solution is composed of two terms. The first is relative to minimum norm joint velocities. The second, termed *homogeneous solution*, attempts to satisfy the additional constraint to specify via $\dot{\boldsymbol{q}}_0$;[5] the matrix $(\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})$ is one of those matrices $\boldsymbol{P}$ introduced in (3.46) which allows the projection of the vector $\dot{\boldsymbol{q}}_0$ in the null space of $\boldsymbol{J}$, so as not to violate the constraint (3.45). A direct consequence is that, in the case $\boldsymbol{v}_e = \boldsymbol{0}$, is is possible to generate *internal motions* described by $(\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})\dot{\boldsymbol{q}}_0$ that reconfigure the manipulator structure without changing the end-effector position and orientation.

Finally, it is worth discussing the way to specify the vector $\dot{\boldsymbol{q}}_0$ for a convenient utilization of redundant DOFs. A typical choice is

$$\dot{\boldsymbol{q}}_0 = k_0 \left(\frac{\partial w(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T \tag{3.55}$$

where $k_0 > 0$ and $w(\boldsymbol{q})$ is a (secondary) objective function of the joint variables. Since the solution moves along the direction of the gradient of the objective function, it attempts to *maximize* it *locally* compatible to the primary objective (kinematic constraint). Typical objective functions are:

- The *manipulability measure*, defined as

$$w(\boldsymbol{q}) = \sqrt{\det\big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)} \tag{3.56}$$

  which vanishes at a singular configuration; thus, by maximizing this measure, redundancy is exploited to move away from singularities.[6]
- The *distance from mechanical joint limits*, defined as

$$w(\boldsymbol{q}) = -\frac{1}{2n} \sum_{i=1}^{n} \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right)^2 \tag{3.57}$$

  where q_{iM} (q_{im}) denotes the maximum (minimum) joint limit and $\bar{q}_i$ the middle value of the joint range; thus, by maximizing this distance, redundancy is exploited to keep the joint variables as close as possible to the centre of their ranges.
- The *distance from an obstacle*, defined as

$$w(\boldsymbol{q}) = \min_{\boldsymbol{p},\boldsymbol{o}} \|\boldsymbol{p}(\boldsymbol{q}) - \boldsymbol{o}\| \tag{3.58}$$

  where $\boldsymbol{o}$ is the position vector of a suitable point on the obstacle (its centre, for instance, if the obstacle is modelled as a sphere) and $\boldsymbol{p}$ is the

[5] It should be recalled that the additional constraint has secondary priority with respect to the primary kinematic constraint.

[6] The manipulability measure is given by the product of the singular values of the Jacobian (see Problem 3.8).

position vector of a generic point along the structure; thus, by maximizing this distance, redundancy is exploited to avoid collision of the manipulator with an obstacle (see also Problem 3.9).[7]

3.5.2 Kinematic Singularities

Both solutions (3.47) and (3.51) can be computed only when the Jacobian has full rank. Hence, they become meaningless when the manipulator is at a singular configuration; in such a case, the system $\boldsymbol{v}_e = \boldsymbol{J}\dot{\boldsymbol{q}}$ contains linearly dependent equations.

It is possible to find a solution $\dot{\boldsymbol{q}}$ by extracting all the linearly independent equations only if $\boldsymbol{v}_e \in \mathcal{R}(\boldsymbol{J})$. The occurrence of this situation means that the assigned path is physically executable by the manipulator, even though it is at a singular configuration. If instead $\boldsymbol{v}_e \notin \mathcal{R}(\boldsymbol{J})$, the system of equations has no solution; this means that the operational space path cannot be executed by the manipulator at the given posture.

It is important to underline that the inversion of the Jacobian can represent a serious inconvenience not only at a singularity but also in the neighbourhood of a singularity. For instance, for the Jacobian inverse it is well known that its computation requires the computation of the determinant; in the neighbourhood of a singularity, the determinant takes on a relatively small value which can cause large joint velocities (see point **c)** in Sect. 3.3). Consider again the above example of the shoulder singularity for the anthropomorphic arm. If a path is assigned to the end-effector which passes nearby the base rotation axis (geometric locus of singular configurations), the base joint is forced to make a rotation of about π in a relatively short time to allow the end-effector to keep tracking the imposed trajectory.

A more rigorous analysis of the solution features in the neighbourhood of singular configurations can be developed by resorting to the singular value decomposition (SVD) of matrix $\boldsymbol{J}$.[8]

An alternative solution overcoming the problem of inverting differential kinematics in the neighbourhood of a singularity is provided by the so-called *damped least-squares (DLS) inverse*

$$\boldsymbol{J}^\star = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T + k^2\boldsymbol{I})^{-1} \tag{3.59}$$

where k is a damping factor that renders the inversion better conditioned from a numerical viewpoint. It can be shown that such a solution can be

[7] If an obstacle occurs along the end-effector path, it is opportune to invert the order of priority between the kinematic constraint and the additional constraint; in this way the obstacle may be avoided, but one gives up tracking the desired path.

[8] See Sect. A.8.

obtained by reformulating the problem in terms of the minimization of the cost functional

$$g''(\dot{\boldsymbol{q}}) = \frac{1}{2}(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}})^T(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}}) + \frac{1}{2}k^2\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}},$$

where the introduction of the first term allows a finite inversion error to be tolerated, with the advantage of norm-bounded velocities. The factor k establishes the relative weight between the two objectives, and there exist techniques for selecting optimal values for the damping factor (see Problem 3.10).

3.6 Analytical Jacobian

The above sections have shown the way to compute the end-effector velocity in terms of the velocity of the end-effector frame. The Jacobian is computed according to a *geometric technique* in which the contributions of each joint velocity to the components of end-effector linear and angular velocity are determined.

If the end-effector pose is specified in terms of a minimal number of parameters in the operational space as in (2.80), it is natural to ask whether it is possible to compute the Jacobian via differentiation of the direct kinematics function with respect to the joint variables. To this end, an *analytical technique* is presented below to compute the Jacobian, and the existing relationship between the two Jacobians is found.

The translational velocity of the end-effector frame can be expressed as the time derivative of vector $\boldsymbol{p}_e$, representing the origin of the end-effector frame with respect to the base frame, i.e.,

$$\dot{\boldsymbol{p}}_e = \frac{\partial \boldsymbol{p}_e}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3.60}$$

For what concerns the rotational velocity of the end-effector frame, the minimal representation of orientation in terms of three variables $\boldsymbol{\phi}_e$ can be considered. Its time derivative $\dot{\boldsymbol{\phi}}_e$ in general differs from the angular velocity vector defined above. In any case, once the function $\boldsymbol{\phi}_e(\boldsymbol{q})$ is known, it is formally correct to consider the Jacobian obtained as

$$\dot{\boldsymbol{\phi}}_e = \frac{\partial \boldsymbol{\phi}_e}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_\phi(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3.61}$$

Computing the Jacobian $\boldsymbol{J}_\phi(\boldsymbol{q})$ as $\partial\boldsymbol{\phi}_e/\partial\boldsymbol{q}$ is not straightforward, since the function $\boldsymbol{\phi}_e(\boldsymbol{q})$ is not usually available in direct form, but requires computation of the elements of the relative rotation matrix.

Upon these premises, the differential kinematics equation can be obtained as the time derivative of the direct kinematics equation in (2.82), i.e.,

$$\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \dot{\boldsymbol{\phi}}_e \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_\phi(\boldsymbol{q}) \end{bmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3.62}$$

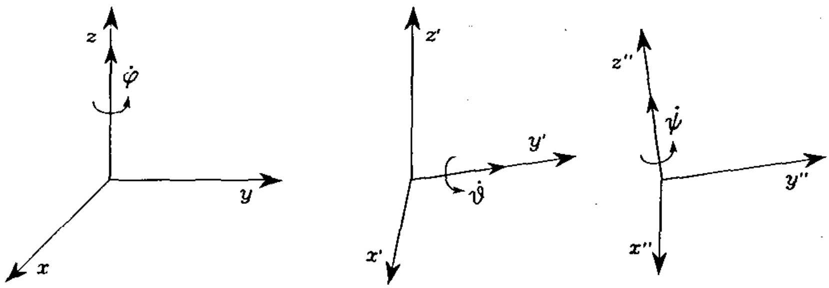

Fig. 3.8. Rotational velocities of Euler angles ZYZ in current frame

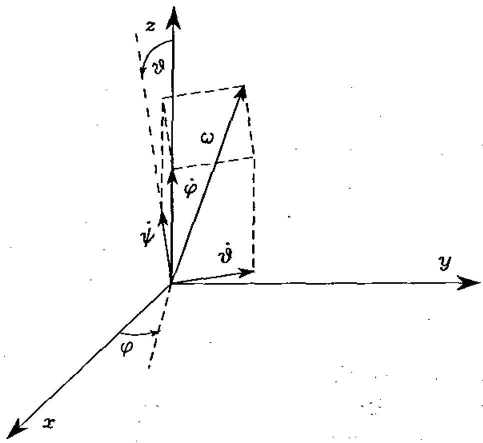

Fig. 3.9. Composition of elementary rotational velocities for computing angular velocity

where the *analytical Jacobian*

$$\boldsymbol{J}_A(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}(\boldsymbol{q})}{\partial \boldsymbol{q}} \tag{3.63}$$

is different from the geometric Jacobian $\boldsymbol{J}$, since the end-effector angular velocity $\boldsymbol{\omega}_e$ with respect to the base frame is not given by $\dot{\boldsymbol{\phi}}_e$.

It is possible to find the relationship between the angular velocity $\boldsymbol{\omega}_e$ and the rotational velocity $\dot{\boldsymbol{\phi}}_e$ for a given set of orientation angles. For instance, consider the Euler angles ZYZ defined in Sect. 2.4.1; in Fig. 3.8, the vectors corresponding to the rotational velocities $\dot{\varphi}$, $\dot{\vartheta}$, $\dot{\psi}$ have been represented with reference to the current frame. Figure 3.9 illustrates how to compute the contributions of each rotational velocity to the components of angular velocity about the axes of the reference frame:

- as a result of $\dot{\varphi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\varphi}\,[0 \quad 0 \quad 1]^T$
- as a result of $\dot{\vartheta}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\vartheta}\,[-s_\varphi \quad c_\varphi \quad 0]^T$

- as a result of $\dot{\psi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\psi}\,[c_\varphi s_\vartheta \quad s_\varphi s_\vartheta \quad c_\vartheta]^T$,

and then the equation relating the angular velocity $\boldsymbol{\omega}_e$ to the time derivative of the Euler angles $\dot{\boldsymbol{\phi}}_e$ is[9]

$$\boldsymbol{\omega}_e = \boldsymbol{T}(\boldsymbol{\phi}_e)\dot{\boldsymbol{\phi}}_e, \tag{3.64}$$

where, in this case,

$$\boldsymbol{T} = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\vartheta \\ 0 & c_\varphi & s_\varphi s_\vartheta \\ 1 & 0 & c_\vartheta \end{bmatrix}.$$

The determinant of matrix $\boldsymbol{T}$ is $-s_\vartheta$, which implies that the relationship cannot be inverted for $\vartheta = 0, \pi$. This means that, even though all rotational velocities of the end-effector frame can be expressed by means of a suitable angular velocity vector $\boldsymbol{\omega}_e$, there exist angular velocities which cannot be expressed by means of $\dot{\boldsymbol{\phi}}_e$ when the orientation of the end-effector frame causes $s_\vartheta = 0$.[10] In fact, in this situation, the angular velocities that can be described by $\dot{\boldsymbol{\phi}}_e$ should have linearly dependent components in the directions orthogonal to axis z ($\omega_x^2 + \omega_y^2 = \dot{\vartheta}^2$). An orientation for which the determinant of the transformation matrix vanishes is termed *representation singularity* of $\boldsymbol{\phi}_e$.

From a physical viewpoint, the meaning of $\boldsymbol{\omega}_e$ is more intuitive than that of $\dot{\boldsymbol{\phi}}_e$. The three components of $\boldsymbol{\omega}_e$ represent the components of angular velocity with respect to the base frame. Instead, the three elements of $\dot{\boldsymbol{\phi}}_e$ represent nonorthogonal components of angular velocity defined with respect to the axes of a frame that varies as the end-effector orientation varies. On the other hand, while the integral of $\dot{\boldsymbol{\phi}}_e$ over time gives $\boldsymbol{\phi}_e$, the integral of $\boldsymbol{\omega}_e$ does not admit a clear physical interpretation, as can be seen in the following example.

Example 3.3

Consider an object whose orientation with respect to a reference frame is known at time $t = 0$. Assign the following time profiles to $\boldsymbol{\omega}$:

- $\boldsymbol{\omega} = [\pi/2 \quad 0 \quad 0]^T \quad 0 \le t \le 1 \qquad \boldsymbol{\omega} = [0 \quad \pi/2 \quad 0]^T \quad 1 < t \le 2$,
- $\boldsymbol{\omega} = [0 \quad \pi/2 \quad 0]^T \quad 0 \le t \le 1 \qquad \boldsymbol{\omega} = [\pi/2 \quad 0 \quad 0]^T \quad 1 < t \le 2$.

The integral of $\boldsymbol{\omega}$ gives the same result in the two cases

$$\int_0^2 \boldsymbol{\omega}\,dt = [\pi/2 \quad \pi/2 \quad 0]^T$$

but the final object orientation corresponding to the second timing law is clearly different from the one obtained with the first timing law (Fig. 3.10).

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[9] This relation can also be obtained from the rotation matrix associated with the three angles (see Problem 3.11).

[10] In Sect. 2.4.1, it was shown that for this orientation the inverse solution of the Euler angles degenerates.

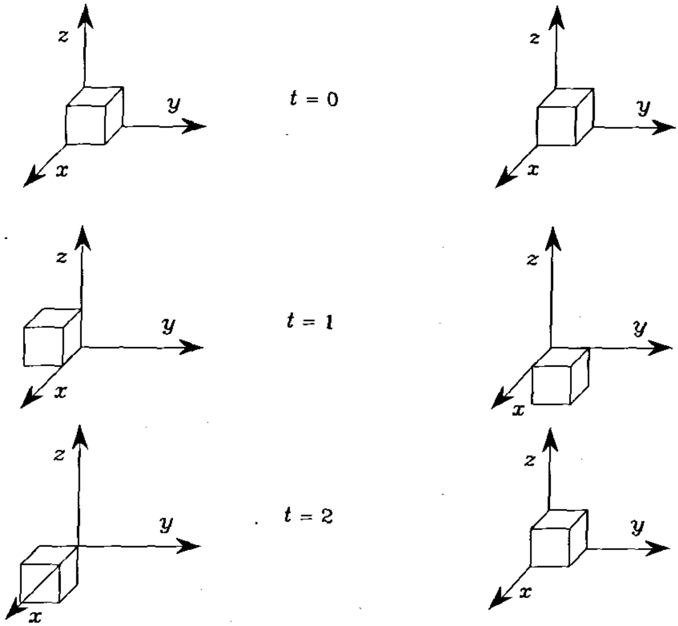

Fig. 3.10. Nonuniqueness of orientation computed as the integral of angular velocity

Once the transformation $\boldsymbol{T}$ between $\boldsymbol{\omega}_e$ and $\dot{\boldsymbol{\phi}}_e$ is given, the analytical Jacobian can be related to the geometric Jacobian as

$$\boldsymbol{v}_e = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{T}(\boldsymbol{\phi}_e) \end{bmatrix} \dot{\boldsymbol{x}}_e = \boldsymbol{T}_A(\boldsymbol{\phi}_e)\dot{\boldsymbol{x}}_e \tag{3.65}$$

which, in view of (3.4), (3.62), yields

$$\boldsymbol{J} = \boldsymbol{T}_A(\boldsymbol{\phi})\boldsymbol{J}_A. \tag{3.66}$$

This relationship shows that $\boldsymbol{J}$ and $\boldsymbol{J}_A$, in general, differ. Regarding the use of either one or the other in all those problems where the influence of the Jacobian matters, it is anticipated that the geometric Jacobian will be adopted whenever it is necessary to refer to quantities of clear physical meaning, while the analytical Jacobian will be adopted whenever it is necessary to refer to differential quantities of variables defined in the operational space.

For certain manipulator geometries, it is possible to establish a substantial equivalence between $\boldsymbol{J}$ and $\boldsymbol{J}_A$. In fact, when the DOFs cause rotations of the end-effector all about the same fixed axis in space, the two Jacobians are essentially the same. This is the case of the above three-link planar arm. Its geometric Jacobian (3.35) reveals that only rotations about axis z_0 are permitted. The (3×3) analytical Jacobian that can be derived by considering the end-effector position components in the plane of the structure and defining the end-effector orientation as $\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$ coincides with the matrix that is obtained by eliminating the three null rows of the geometric Jacobian.

3.7 Inverse Kinematics Algorithms

In Sect. 3.5 it was shown how to invert kinematics by using the differential kinematics equation. In the numerical implementation of (3.48), computation of joint velocities is obtained by using the inverse of the Jacobian evaluated with the joint variables at the previous instant of time

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \boldsymbol{J}^{-1}(\boldsymbol{q}(t_k))\boldsymbol{v}_e(t_k)\Delta t.$$

It follows that the computed joint velocities $\dot{\boldsymbol{q}}$ do not coincide with those satisfying (3.47) in the continuous time. Therefore, reconstruction of joint variables $\boldsymbol{q}$ is entrusted to a numerical integration which involves *drift* phenomena of the solution; as a consequence, the end-effector pose corresponding to the computed joint variables differs from the desired one.

This inconvenience can be overcome by resorting to a solution scheme that accounts for the *operational space error* between the desired and the actual end-effector position and orientation. Let

$$\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{x}_e \tag{3.67}$$

be the expression of such error.

Consider the time derivative of (3.67), i.e.,

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}}_e \tag{3.68}$$

which, according to differential kinematics (3.62), can be written as

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}}_d - \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3.69}$$

Notice in (3.69) that the use of operational space quantities has naturally lead to using the analytical Jacobian in lieu of the geometric Jacobian. For this equation to lead to an *inverse kinematics algorithm*, it is worth relating the computed joint velocity vector $\dot{\boldsymbol{q}}$ to the error $\boldsymbol{e}$ so that (3.69) gives a differential equation describing error evolution over time. Nonetheless, it is necessary to choose a relationship between $\dot{\boldsymbol{q}}$ and $\boldsymbol{e}$ that ensures convergence of the error to zero.

Having formulated inverse kinematics in algorithmic terms implies that the joint variables $\boldsymbol{q}$ corresponding to a given end-effector pose $\boldsymbol{x}_d$ are accurately computed only when the error $\boldsymbol{x}_d - \boldsymbol{k}(\boldsymbol{q})$ is reduced within a given threshold; such settling time depends on the dynamic characteristics of the error differential equation. The choice of $\dot{\boldsymbol{q}}$ as a function of $\boldsymbol{e}$ permits finding inverse kinematics algorithms with different features.