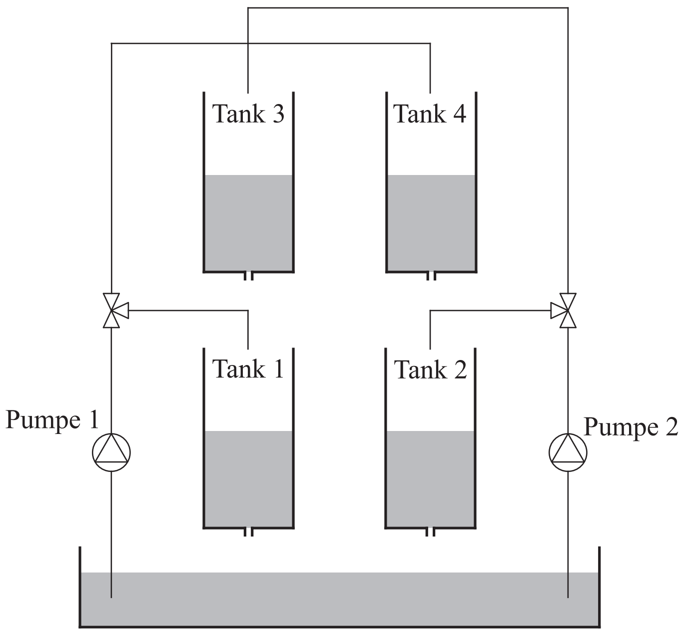
Control Systems 2: MATLAB Laboratory Course

Lab three: Quadruple Water Tank Control



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General Comment

The computer-aided laboratory course is designed in three parts as a complement to the Control Systems 2 lectures. Within each part of the laboratory, we try to make a bridge between the principles of control theory provided in the lectures, and its engineering application by studying real models in different domains. It is expected that after completing this laboratory course, the students be able to analyze the linear models' behavior of real plants, in both time and frequency domains, and also design MIMO control laws to meet some specified control objectives.

The present manuscript, is the third and last part of the computer-aided laboratory, and considers the concept of performance-oriented control design for multi-input multi-output (MIMO) systems. The tasks are designed to ensure that the students have enough programming skills to deal with variety of real engineering challenges. Sensitivity and complementary sensitivity functions will be considered as measures of performance, as well as singular values for MIMO systems. H_∞ optimal control, as the main focus of this laboratory would be investigated. An experimental MIMO model of quadruple water tank is utilized for the present laboratory, to apply the performance-oriented control design methods upon.

Chapter 1

Quadruple water-tank model

Quadruple water-tank system is a famous experimental setup for MIMO systems, which accommodates almost all the concepts of MIMO control systems. It is indeed a two-input two-output nonlinear system, and we derive its linearized model around some operational points in order to analyze it with the linear control concepts we learn through this course. Quadruple water-tank system is also simple to be constructed in reality such that the concepts of the MIMO control can be experimented and experienced.

In this section, we derive the linearized model of the quadruple water tank system, out of its nonlinear differential equations. The process is schematically depicted in Figure 1.1. We have the following differential equation for the rate of the change in water volume in every water tank in the process:

$$\frac{dV}{dt} = q_{in} - q_{out}$$

where, q_{in} and q_{out} represent the water inflow and outflow of a certain tank, respectively. Further, we have $V = Ah$, where h is the level of the water in the tank, and constant A is the tank's cross section area. Substituting the two equations, gives us the following ODE:

$$A\frac{dh}{dt} = q_{in} - q_{out}$$

According to the Bernoulli principle, the water outflow and the water level are related to each other as follows:

$$q_{out} = a\sqrt{2gh}$$

where, g represents the gravity acceleration, and a is the cross section area of the outlet hole. As a given information, consider that the pumps generate the water flow proportional to the applied voltage u,

$$q = ku$$

where u is the control input. As it can be seen in Figure 1.1, two valves are also connected to the outflow pump pipes, and divide the water coming from the pumps between the upper and lower tanks. Indicating the valve setting with the constant $\gamma \in [0, 1]$, the pump outflow is divided between the upper (q_U) and the lower tanks (q_L) as follows:

$$q_L = \gamma k u \qquad , \qquad q_U = (1 - \gamma) k u$$

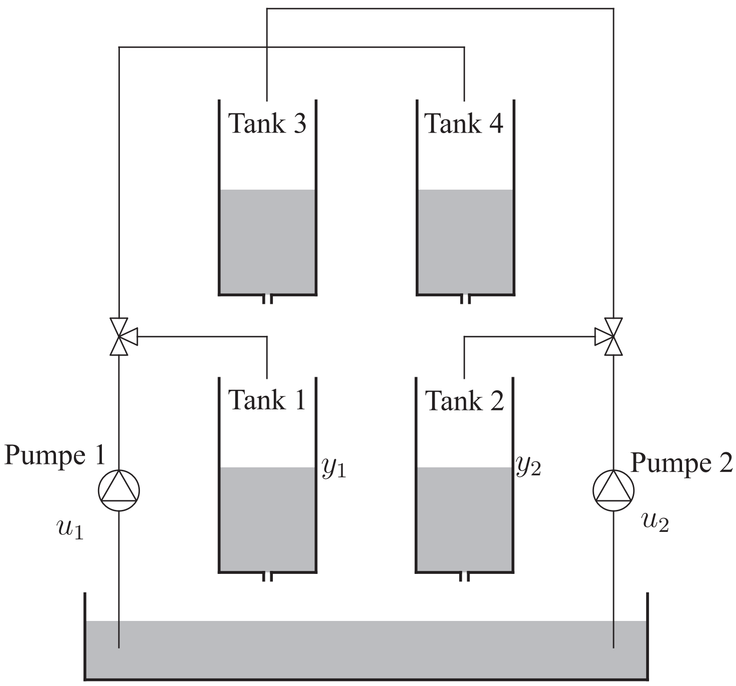

Figure 1.1: The quadruple water-tank process

Now, considering the water levels $0 < h_i \leq h_{max}$ in each tank as the states, and the pumps voltages $0 \leq u_i \leq u_{max}$ as the control inputs of our MIMO system, and also considering the couplings between the first and third tanks and the second and fourth tanks, we can derive the rate of change in the water levels as the following nonlinear differential equations:

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1} u_1 \tag{1.1}$$

$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2} u_2 \tag{1.2}$$

$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2) k_2}{A_3} u_2 \tag{1.3}$$

$$\frac{dh_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1) k_1}{A_4} u_1 \tag{1.4}$$

where, the positive k_1, and k_2 are the pumps proportional constants, and γ_1, and γ_2 determined the valves settings.
We use some sensors to measure the water heights in the lower tanks, such that the sensors' output voltages are proportional (with constant k_h) to the water heights h_i,

$$y_i = k_h h_i$$

From the equations (1.1)-(1.4), we can derive an equilibrium point h_i^0, u_i^0, and y_i^0, by solving the homogeneous system of nonlinear equations

$$\begin{aligned}
&-\frac{a_1}{A_1}\sqrt{2gh_1^0}+\frac{a_3}{A_1}\sqrt{2gh_3^0}+\frac{\gamma_1 k_1}{A_1}u_1^0=0\\
&-\frac{a_2}{A_2}\sqrt{2gh_2^0}+\frac{a_4}{A_2}\sqrt{2gh_4^0}+\frac{\gamma_2 k_2}{A_2}u_2^0=0\\
&-\frac{a_3}{A_3}\sqrt{2gh_3^0}+\frac{(1-\gamma_2)k_2}{A_3}u_2^0=0\\
&-\frac{a_4}{A_4}\sqrt{2gh_4^0}+\frac{(1-\gamma_1)k_1}{A_4}u_1^0=0
\end{aligned}$$

Now, let the following equations describe the deviations from the obtained equilibrium points:

$$\Delta h_i = h_i - h_i^0 \quad , \quad \Delta u_i = u_i - u_i^0 \quad , \quad \Delta y_i = y_i - y_i^0$$

For the overall MIMO system, introduce the state, control input, and output vectors x, u, and y

$$\mathbf{u} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} \quad , \quad \mathbf{x} = \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{pmatrix} \quad , \quad \mathbf{y} = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

where, the indices $1, 2$ for the control input u refer to the pumps, the indices $1, 2$ for the output y refer to the water levels in the lower tanks, and the indices $1, 2, 3, 4$ for the state vector refer to the water levels in each tank.
Linearizing the differential equations given in (1.1)-(1.4) around the equilibrium points $h_1^0, \ldots, h_4^0, u_1^0, u_2^0, y_1^0, y_2^0$, the state space model of the quadruple tank system is derived as follows:

$$\frac{d\mathbf{x}}{d\mathbf{t}} = \begin{pmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix} \mathbf{u} \tag{1.5}$$

$$\mathbf{y} = \begin{pmatrix} k_h & 0 & 0 & 0 \\ 0 & k_h & 0 & 0 \end{pmatrix} \mathbf{x} \tag{1.6}$$

where, the parameter T_i is defined as

$$T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$$

Transforming from the state-space model to transfer function, we are given the following two-by-two transfer matrix from the control input u, and controlled output y as follows

$$G(s) = \begin{pmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1 k_1 c_1}{1+sT_1} & \frac{(1-\gamma_2)k_2 c_1}{(1+sT_1)(1+sT_3)} \\ \frac{(1-\gamma_1)k_1 c_2}{(1+sT_2)(1+sT_4)} & \frac{\gamma_2 k_2 c_2}{1+sT_2} \end{pmatrix} \tag{1.7}$$

where,

$$c_i = \frac{T_i k_h}{A_i}$$

The zero polynomial of the transfer function in (1.7) is the greatest common deviser for the numerators of the maximal minors of $G(s)$, normalized to have the pole polynomial as denominator. We know that the zeros of a MIMO system are the zeros of the zero polynomial. The determinant of the transfer matrix $G(s)$ can be calculated as

$$\det(G(s)) = \frac{k_1 k_2 c_1 c_2 \gamma_1 \gamma_2}{\prod_{i=1}^{4}(1+T_i s)}\left[(1+T_3 s)(1+T_4 s) - \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2}\right] \tag{1.8}$$

which in this case, represents the maximal minors of $G(s)$. To obtain the zeros, the expression inside the bracket in (1.8) should be equaled to zero. The solution to the second-degree expression gives the zeros in the s-plane as follows:

$$s_{1,2} = \frac{1}{T_3 T_4}\left[-\frac{T_3+T_4}{2} \pm \sqrt{\frac{(T_3+T_4)^2}{4} - \frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2}}\right] \tag{1.9}$$

Due to the fact that $\gamma_i \in [0,1]$, it is straightforward to show if $1 < \gamma_1 + \gamma_2 \leq 2$, then we have no right-half plane zero. Otherwise, if $0 < \gamma_1 + \gamma_2 \leq 1$ the system will have right-half plane zeros.

Chapter 2

Exercises

2.1 Linearized Model of Quadruple water tank

In the previous section, the linearized model of the quadruple water-tank process was given in detail. In this chapter, we first determine the numerical model by providing the numerical values for the system, and sensor parameters, and then present the tasks. The main objective of the tasks is to make the readers familiar with how H_∞ control design is employed to achieve desired specification on sensitivity and robustness. To build up the model, we employ the following numerical values for the given parameters in (1.1)-(1.7):

1. Tanks cross section area:

$$A_1 = 15.52 \ \text{cm}^2 \quad A_2 = 90.52 \ \text{cm}^2 \quad A_3 = 15.52 \ \text{cm}^2 \quad A_4 = 15.52 \ \text{cm}^2$$

2. Outlet holes cross section area:

$$a_1 = 0.1678 \ \text{cm}^2 \quad a_2 = 0.1542 \ \text{cm}^2 \quad a_3 = 0.06743 \ \text{cm}^2 \quad a_4 = 0.06504 \ \text{cm}^2$$

3. Initial water-tank levels:

$$h_1^0 = 18 \ \text{cm} \quad h_2^0 = 18 \ \text{cm} \quad h_3^0 = 14 \ \text{cm} \quad h_4^0 = 19 \ \text{cm}$$

4. Pumps and sensors setting:

$$k_1 = 4.32 \ \frac{\text{cm}^3}{\text{Vs}} \quad k_2 = 3.74 \ \frac{\text{cm}^3}{\text{Vs}} \quad k_h = 0.2 \ \frac{\text{V}}{\text{cm}} \quad g = 981 \ \frac{\text{cm}}{\text{s}^2}$$

5. Valve setting:

$$\gamma_1 = \gamma_2 = 0.625 \tag{2.1}$$

The aggregate state vector $\mathbf{x} = (\Delta h_1 \ \ \Delta h_2 \ \ \Delta h_3 \ \ \Delta h_4)^{\mathsf{T}}$ has the following dynamics:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{2.2}$$

$$\mathbf{y} = C\mathbf{x} \tag{2.3}$$

where, $\mathbf{x} \in \mathbb{R}^{4\times 1}$ is the state vector including the levels of water in each water tank, with the given initial value $x(0) = x_0$, $\mathbf{y} \in \mathbb{R}^{2\times 1}$ are the measured levels of water in the lower tanks, and $\mathbf{u} \in \mathbb{R}^{2\times 1}$ are the pumps voltages. Using the given parameters and substituting them in the equations (1.5)-(1.6), we can derive the state space model of the quadruple water-tank:

$$A = \begin{pmatrix} -0.05644 & 0 & 0.02572 & 0 \\ 0 & -0.008892 & 0 & 0.003651 \\ 0 & 0 & -0.02572 & 0 \\ 0 & 0 & 0 & -0.02129 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.174 & 0 \\ 0 & 0.02582 \\ 0 & 0.09037 \\ 0.1044 & 0 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \end{pmatrix}$$

In the next section, the tasks are given, with the overall objective to design the H_∞ control for the quadruple water tank such that the desired performance specification are met.

2.2 Tasks

Due to the water leakage from the pumps, some unknown water leaks out the quadruple system. We model this uncertainty in the system by an unknown input n to the system. The block diagram in the Figure 2.1 depicts the uncertain model of the quadruple water tank system.

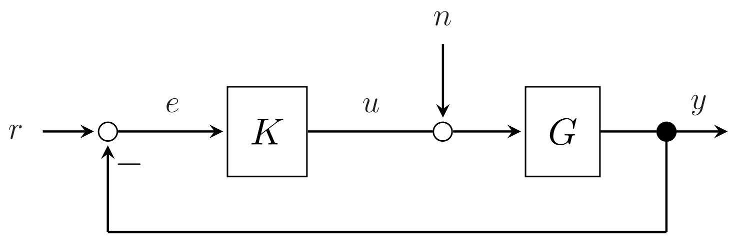

Figure 2.1: The block diagram of the quadruple water tank system with uncertainty

The objective is to design the feedback control K, such that the performance measures singular values, sensitivity and complementary sensitivity functions satisfy some specified properties.

2.2.1 Preparatory Tasks

1. Calculate the poles and zeros of the original quadruple water tank model in (2.1)-(2.2) and check the observability and controllability of the given system.

2. Derive the transfer functions from the uncertainty n to the control input (pump voltages) u, and controlled output (level of water in lower tanks) y.

3. Design a decentralized PI control for the original system in (2.1)-(2.2) by loop-shaping, according to the RGA analysis, such that, the cross-over frequency is $0.2\ \frac{\text{rad}}{\text{s}}$, and the phase margin is $\frac{\pi}{4}$.

4. Calculate the singular values of the sensitivity function S, and the complementary sensitivity function T to see if the designed PI control is satisfactory with respect to the sensitivity and robustness.

2.2.2 Main Tasks

1. Assume that we have the following weight functions in frequency domain as low-pass and high-pass filters as follows:

$$W_1 = \frac{\frac{s}{M_1} + \omega_b^1}{s + \omega_b^1} \quad , \quad W_2 = \frac{s + \beta_2 \omega_b^2}{\frac{s}{M_2} + \omega_b^2}$$

where, $M_1 = 5$, the bandwidth for the low-pass filter $\omega_b^1 = 0.5 \ \frac{\text{rad}}{\text{s}}$, $M_2 = 0.5$, $\beta_2 = 0.1$, and the high-pass filter bandwidth is $\omega_b^2 = 0.1 \ \frac{\text{rad}}{\text{s}}$. Calculate the generalized plant model P, according to the PK-structure representation, and the given MIMO system transfer function.

2. Check the followings for the obtained generalized plant model P in the previous task:

   - Existence of a stabilizing controller.
   - Sufficient condition for the controller to be proper and realizable.

3. How the generalized plant model can be adjusted if one of the items above is not satisfied? Design a perturbed generalized plant model of the quadruple water-tank system with uncertainty n, in case one of the above items are not fulfilled. If needed, consider $\epsilon = 10^{-3}$.

4. Derive the H_∞ optimal controller K, if exists.

References

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