11

Mobile Robots

The previous chapters deal mainly with articulated manipulators that represent the large majority of robots used in industrial settings. However, *mobile robots* are becoming increasingly important in advanced applications, in view of their potential for autonomous intervention. This chapter presents techniques for modelling, planning and control of *wheeled* mobile robots. The structure of the kinematic constraints arising from the pure rolling of the wheels is first analyzed; it is shown that such constraints are in general *nonholonomic* and consequently reduce the local mobility of the robot. The *kinematic model* associated with the constraints is introduced to describe the instantaneous admissible motions, and conditions are given under which it can be put in *chained form*. The *dynamic model*, that relates the admissible motions to the generalized forces acting on the robot DOFs, is then derived. The peculiar nature of the kinematic model, and in particular the existence of *flat outputs*, is exploited to devise *trajectory planning* methods that guarantee that the nonholonomic constraints are satisfied. The structure of *minimum-time trajectories* is also analyzed. The *motion control* problem for mobile robots is then discussed, with reference to two basic motion tasks, i.e., *trajectory tracking* and *posture regulation*. The chapter concludes by surveying some techniques for *odometric localization* that is necessary to implement feedback control schemes.

11.1 Nonholonomic Constraints

Wheels are by far the most common mechanism to achieve locomotion in mobile robots. Any wheeled vehicle is subject to kinematic constraints that reduce in general its local mobility, while leaving intact the possibility of reaching arbitrary configurations by appropriate manoeuvres. For example, any driver knows by experience that, while it is impossible to move instantaneously a car in the direction orthogonal to its heading, it is still possible to park it arbitrarily, at least in the absence of obstacles. It is therefore important to analyze in detail the structure of these constraints.

In accordance with the terminology introduced in Sect. B.4, consider a mechanical system whose *configuration* $\boldsymbol{q} \in \mathcal{C}$ is described by a vector of *generalized coordinates*, and assume that the *configuration space* $\mathcal{C}$ (i.e., the space of all possible robot configurations) coincides[1] with $\mathbb{R}^n$. The motion of the system that is represented by the evolution of $\boldsymbol{q}$ over time may be subject to constraints that can be classified under various criteria. For example, they may be expressed as equalities or inequalities (respectively, *bilateral* or *unilateral* constraints), and they may depend explicitly on time or not (*rheonomic* or *scleronomic* constraints). In this chapter, only bilateral scleronomic constraints will be considered.

Constraints that can be put in the form

$$h_i(\boldsymbol{q}) = 0 \qquad i = 1, \ldots, k < n \tag{11.1}$$

are called *holonomic* (or *integrable*). In the following, it is assumed that the functions $h_i : \mathcal{C} \mapsto \mathbb{R}$ are of class C^∞ (*smooth*) and independent. The effect of holonomic constraints is to reduce the space of accessible configurations to a subset of $\mathcal{C}$ with dimension $n - k$. A mechanical system for which all the constraints can be expressed in the form (11.1) is called *holonomic*.

In the presence of holonomic constraints, the implicit function theorem can be used in principle to solve the equations in (11.1) by expressing k generalized coordinates as a function of the remaining $n-k$, so as to eliminate them from the formulation of the problem. However, in general this procedure is only valid locally, and may introduce singularities. A convenient alternative is to replace the original generalized coordinates with a reduced set of $n - k$ new coordinates that are directly defined on the accessible subspace, in such a way that the available DOFs are effectively characterized. The mobility of the reduced system thus obtained is completely equivalent to that of the original mechanism.

Holonomic constraints are generally the result of mechanical interconnections between the various bodies of the system. For example, prismatic and revolute joints used in robot manipulators are a typical source of such constraints, and joint variables are an example of reduced sets of coordinates in the above sense. Constraints of the form (11.1) may also arise in particular operating conditions; for example, one may mention the case of a kinematically redundant manipulator that moves while keeping the end-effector fixed at a certain pose (*self-motion*).

Constraints that involve generalized coordinates and velocities

$$a_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \qquad i = 1, \ldots, k < n$$

[1] This assumption is taken for simplicity. In the general case, the configuration space $\mathcal{C}$ may be identified with a Euclidean space only on a local basis, because its global geometric structure is more complex; this will be further discussed in Chap. 12. The material presented in this chapter is, however, still valid.

are called *kinematic*. They constrain the instantaneous admissible motion of the mechanical system by reducing the set of generalized velocities that can be attained at each configuration. Kinematic constraints are generally expressed in *Pfaffian form*, i.e., they are linear in the generalized velocities:

$$\boldsymbol{a}_i^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0 \qquad i = 1, \ldots, k < n, \tag{11.2}$$

or, in matrix form

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \mathbf{0}. \tag{11.3}$$

Vectors $\boldsymbol{a}_i : \mathcal{C} \mapsto \mathrm{I\!R}^n$ are assumed to be smooth as well as linearly independent.

Clearly, the existence of k holonomic constraints (11.1) implies that of an equal number of kinematic constraints:

$$\frac{dh_i(\boldsymbol{q})}{dt} = \frac{\partial h_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = 0 \qquad i = 1, \ldots, k.$$

However, the converse is not true in general. A system of kinematic constraints in the form (11.3) may or may not be integrable to the form (11.1). In the negative case, the kinematic constraints are said to be *nonholonomic* (or *non-integrable*). A mechanical system that is subject to at least one such constraint is called *nonholonomic*.

Nonholonomic constraints reduce the mobility of the mechanical system in a completely different way with respect to holonomic constraints. To appreciate this fact, consider a single Pfaffian constraint

$$\boldsymbol{a}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0. \tag{11.4}$$

If the constraint is holonomic, it can be integrated and written as

$$h(\boldsymbol{q}) = c, \tag{11.5}$$

where $\partial h/\partial \boldsymbol{q} = \gamma(\boldsymbol{q})\, \boldsymbol{a}^T(\boldsymbol{q})$, with $\gamma(\boldsymbol{q}) \neq 0$ an *integrating factor* and c an integration constant. Therefore, there is a loss of *accessibility* in the configuration space, because the motion of the mechanical system in $\mathcal{C}$ is confined to a particular *level surface* of the scalar function h. This surface, which depends on the initial configuration $\boldsymbol{q}_0$ through the value of $h(\boldsymbol{q}_0) = c$, has dimension $n - 1$.

Assume instead that the constraint (11.4) is nonholonomic. In this case, generalized velocities are indeed constrained to belong to a subspace of dimension $n-1$, i.e., the null space of matrix $\boldsymbol{a}^T(\boldsymbol{q})$. Nevertheless, the fact that the constraint is non-integrable means that there is no loss of accessibility in $\mathcal{C}$ for the system. In other words, while the number of DOFs decreases to $n - 1$ due to the constraint, the number of generalized coordinates cannot be reduced, not even locally.

The conclusion just drawn for the case of a single constraint is general. An n-dimensional mechanical system subject to k nonholonomic constraints

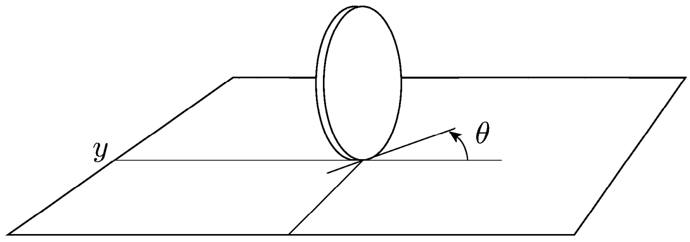

Fig. 11.1. Generalized coordinates for a disk rolling on a plane

can access its whole configuration space $\mathcal{C}$, although at any configuration its generalized velocities must belong to an $(n-k)$-dimensional subspace.

The following is a classical example of nonholonomic mechanical system, that is particularly relevant in the study of mobile robots.

Example 11.1

Consider a disk that rolls without slipping on the horizontal plane, while keeping its *sagittal plane* (i.e., the plane that contains the disk) in the vertical direction (Fig. 11.1). Its configuration is described by three[2] generalized coordinates: the Cartesian coordinates (x, y) of the contact point with the ground, measured in a fixed reference frame, and the angle θ characterizing the orientation of the disk with respect to the x axis. The configuration vector is therefore $\boldsymbol{q} = [\,x \quad y \quad \theta\,]^T$.

The *pure rolling* constraint for the disk is expressed in the Pfaffian form as

$$\dot{x}\sin\theta - \dot{y}\cos\theta = [\,\sin\theta \quad -\cos\theta \quad 0\,]\,\dot{\boldsymbol{q}} = 0, \tag{11.6}$$

and entails that, in the absence of slipping, the velocity of the contact point has zero component in the direction orthogonal to the sagittal plane. The angular velocity of the disk around the vertical axis instead is unconstrained.

Constraint (11.6) is nonholonomic, because it implies no loss of accessibility in the configuration space of the disk. To substantiate this claim, consider that the disk can be driven from any initial configuration $\boldsymbol{q}_i = [\,x_i \quad y_i \quad \theta_i\,]^T$ to any final configuration $\boldsymbol{q}_f = [\,x_f \quad y_f \quad \theta_f\,]^T$ through the following sequence of movements that do not violate constraint (11.6):

1. rotate the disk around its vertical axis so as to reach the orientation θ_v for which the *sagittal axis* (i.e., the intersection of the sagittal plane and the horizontal plane) goes through the final contact point (x_f, y_f);
2. roll the disk on the plane at a constant orientation θ_v until the contact point reaches its final position (x_f, y_f);
3. rotate again the disk around its vertical axis to change the orientation from θ_v to θ_f.

[2] One could add to this description an angle ϕ measuring the rotation of the disk around the horizontal axis passing through its centre. Such a coordinate is however irrelevant for the analysis presented in this chapter, and is therefore ignored in the following.

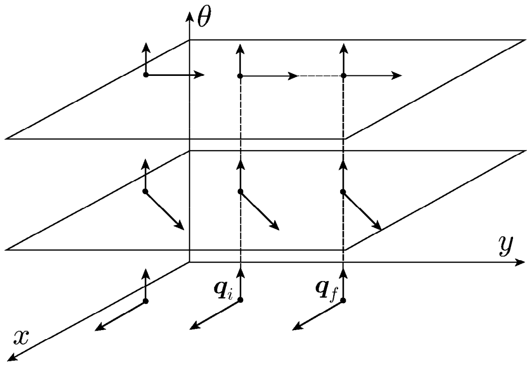

Fig. 11.2. A local representation of the configuration space for the rolling disk with an example manoeuvre that transfers the configuration from $\boldsymbol{q}_i$ to $\boldsymbol{q}_f$ (*dashed line*)

An example of this manoeuvre is shown in Fig. 11.2. Two possible directions of instantaneous motion are shown at each configuration: the first, that is aligned with the sagittal axis, moves the contact point while keeping the orientation constant (rolling); the second varies the orientation while keeping the contact point fixed (rotation around the vertical axis).

It is interesting to note that, in addition to wheeled vehicles, there exist other robotic systems that are nonholonomic in nature. For example, the pure rolling constraint also arises in manipulation problems with round-fingered robot hands. Another kind of nonholonomic behaviour is found in multibody systems that 'float' freely (i.e., without a fixed base), such as manipulators used in space operations. In fact, in the absence of external generalized forces, the conservation of the angular momentum represents a non-integrable Pfaffian constraint for the system.

11.1.1 Integrability Conditions

In the presence of Pfaffian kinematic constraints, integrability conditions can be used to decide whether the system is holonomic or nonholonomic.

Consider first the case of a *single* Pfaffian constraint:

$$\boldsymbol{a}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \sum_{j=1}^{n} a_j(\boldsymbol{q})\dot{q}_j = 0. \tag{11.7}$$

For this constraint to be integrable, there must exist a scalar function $h(\boldsymbol{q})$ and an integrating factor $\gamma(\boldsymbol{q}) \neq 0$ such that the following condition holds:

$$\gamma(\boldsymbol{q})a_j(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial q_j} \qquad j = 1,\dots,n. \tag{11.8}$$

The converse is also true: if there exists an integrating factor $\gamma(\boldsymbol{q}) \neq 0$ such that $\gamma(\boldsymbol{q})\boldsymbol{a}(\boldsymbol{q})$ is the gradient of a scalar function $h(\boldsymbol{q})$, constraint (11.7) is integrable. By using Schwarz theorem on the symmetry of second derivatives, the integrability condition (11.8) may be replaced by the following system of partial differential equations:

$$\frac{\partial(\gamma a_k)}{\partial q_j} = \frac{\partial(\gamma a_j)}{\partial q_k} \qquad j, k = 1, \ldots, n, \quad j \neq k, \tag{11.9}$$

that does not contain the unknown function $h(\boldsymbol{q})$. Note that condition (11.9) implies that a Pfaffian constraint with constant coefficients a_j is always holonomic.

Example 11.2

Consider the following kinematic constraint in $\mathcal{C} = \mathrm{I\!R}^3$:

$$\dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 = 0.$$

The holonomy condition (11.9) gives

$$\begin{aligned}
\frac{\partial\gamma}{\partial q_2} &= \gamma + q_1\frac{\partial\gamma}{\partial q_1}\\
\frac{\partial\gamma}{\partial q_3} &= \frac{\partial\gamma}{\partial q_1}\\
q_1\frac{\partial\gamma}{\partial q_3} &= \frac{\partial\gamma}{\partial q_2}.
\end{aligned}$$

By substituting the second and third equations into the first, it is easy to conclude that the only solution is $\gamma = 0$. Therefore, the constraint is nonholonomic.

Example 11.3

Consider the pure rolling constraint (11.6). In this case, the holonomy condition (11.9) gives

$$\begin{aligned}
\sin\theta\frac{\partial\gamma}{\partial y} &= -\cos\theta\frac{\partial\gamma}{\partial x}\\
\cos\theta\frac{\partial\gamma}{\partial\theta} &= \gamma\sin\theta\\
\sin\theta\frac{\partial\gamma}{\partial\theta} &= -\gamma\cos\theta.
\end{aligned}$$

Squaring and adding the last two equations gives $\partial\gamma/\partial\theta = \pm\gamma$. Assume for example $\partial\gamma/\partial\theta = \gamma$. Using this in the above equations leads to

$$\begin{aligned}
\gamma\cos\theta &= \gamma\sin\theta\\
\gamma\sin\theta &= -\gamma\cos\theta
\end{aligned}$$

whose only solution is $\gamma = 0$. The same conclusion is reached by letting $\partial\gamma/\partial\theta = -\gamma$. This confirms that constraint (11.6) is nonholonomic.

The situation becomes more complicated when dealing with a system of $k > 1$ kinematic constraints of the form (11.3). In fact, in this case it may happen that the single constraints are not integrable if taken separately, but the whole system is integrable. In particular, if $p \leq k$ independent linear combinations of the constraints

$$\sum_{i=1}^{k} \gamma_{ji}(\boldsymbol{q})\boldsymbol{a}_i^T(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad j = 1, \dots, p$$

are integrable, there exist p independent scalar functions $h_j(\boldsymbol{q})$ such that

$$\text{span}\left\{\frac{\partial h_1(\boldsymbol{q})}{\partial \boldsymbol{q}}, \dots, \frac{\partial h_p(\boldsymbol{q})}{\partial \boldsymbol{q}}\right\} \subset \text{span}\left\{\boldsymbol{a}_1^T(\boldsymbol{q}), \dots, \boldsymbol{a}_k^T(\boldsymbol{q})\right\} \quad \forall \boldsymbol{q} \in \mathcal{C}.$$

Therefore, the configurations that are accessible for the mechanical system belong to the $(n-p)$-dimensional subspace consisting of the particular level surfaces of the functions h_j:

$$\{\boldsymbol{q} \in \mathcal{C}\colon h_1(\boldsymbol{q}) = c_1, \dots, h_p(\boldsymbol{q}) = c_p\}$$

on which the motion is started (see Problem 11.2). In the case $p = k$, the system of kinematic constraints (11.3) is completely integrable, and hence holonomic.

Example 11.4

Consider the system of Pfaffian constraints

$$\begin{aligned} \dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 &= 0 \\ \dot{q}_1 + \dot{q}_2 + q_1\dot{q}_3 &= 0. \end{aligned}$$

Taken separately, these constraints are found to be non-integrable (in particular, the first is the nonholonomic constraint of Example 11.2). However, subtracting the second from the first gives

$$(q_1 - 1)(\dot{q}_2 - \dot{q}_3) = 0$$

so that $\dot{q}_2 = \dot{q}_3$, because the constraints must be satisfied for any value of $\boldsymbol{q}$. The assigned system of constraints is then equivalent to

$$\begin{aligned} \dot{q}_2 &= \dot{q}_3 \\ \dot{q}_1 + (1 + q_1)\dot{q}_2 &= 0, \end{aligned}$$

which can be integrated as

$$\begin{aligned} q_2 - q_3 &= c_1 \\ \log(q_1 + 1) + q_2 &= c_2 \end{aligned}$$

with integration constants c_1, c_2.

The integrability conditions of a system of Pfaffian kinematic constraints are quite complex and derive from a fundamental result of differential geometry known as *Frobenius theorem*. However, as shown later in the chapter, it is possible to derive such conditions more directly from a different perspective.

11.2 Kinematic Model

The system of k Pfaffian constraints (11.3) entails that the admissible generalized velocities at each configuration $\boldsymbol{q}$ belong to the $(n-k)$-dimensional null space of matrix $\boldsymbol{A}^T(\boldsymbol{q})$. Denoting by $\{\boldsymbol{g}_1(\boldsymbol{q}), \ldots, \boldsymbol{g}_{n-k}(\boldsymbol{q})\}$ a basis of $\mathcal{N}(\boldsymbol{A}^T(\boldsymbol{q}))$, the admissible trajectories for the mechanical system can then be characterized as the solutions of the nonlinear dynamic system

$$\dot{\boldsymbol{q}} = \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{q}) u_j = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u} \qquad m = n - k, \tag{11.10}$$

where $\boldsymbol{q} \in \mathbb{R}^n$ is the state vector and $\boldsymbol{u} = [\, u_1 \quad \ldots \quad u_m \,]^T \in \mathbb{R}^m$ is the input vector. System (11.10) is said to be *driftless* because one has $\dot{\boldsymbol{q}} = \boldsymbol{0}$ if the input is zero.

The choice of the *input vector fields* $\boldsymbol{g}_1(\boldsymbol{q}), \ldots, \boldsymbol{g}_m(\boldsymbol{q})$ (and thus of matrix $\boldsymbol{G}(\boldsymbol{q})$) in (11.10) is not unique. Correspondingly, the components of $\boldsymbol{u}$ may have different meanings. In general, it is possible to choose the basis of $\mathcal{N}(\boldsymbol{A}^T(\boldsymbol{q}))$ in such a way that the u_js have a physical interpretation, as will be shown later for some examples of mobile robots. In any case, vector $\boldsymbol{u}$ may not be directly related to the actual control inputs, that are in general forces and/or torques. For this reason, Eq. (11.10) is referred to as the *kinematic model* of the constrained mechanical system.

The holonomy or nonholonomy of constraints (11.3) can be established by analyzing the controllability[3] properties of the associated kinematic model (11.10). In fact, two cases are possible:

1. If system (11.10) is controllable, given two arbitrary configurations $\boldsymbol{q}_i$ and $\boldsymbol{q}_f$ in $\mathcal{C}$, there exists a choice of $\boldsymbol{u}(t)$ that steers the system from $\boldsymbol{q}_i$ to $\boldsymbol{q}_f$, i.e., there exists a trajectory that joins the two configurations and satisfies the kinematic constraints (11.3). Therefore, these do not affect in any way the accessibility of $\mathcal{C}$, and they are (completely) nonholonomic.
2. If system (11.10) is not controllable, the kinematic constraints (11.3) reduce the set of accessible configurations in $\mathcal{C}$. Hence, the constraints are partially or completely integrable depending on the dimension $\nu < n$ of the accessible configuration space. In particular:

[3] Refer to Appendix D for a short survey of nonlinear controllability theory, including the necessary tools from differential geometry.

2a. If $m < \nu < n$, the loss of accessibility is not maximal, and thus constraints (11.3) are only partially integrable. The mechanical system is still nonholonomic.
2b. If $\nu = m$, the loss of accessibility is maximal, and constraints (11.3) are completely integrable. Therefore, the mechanical system is holonomic.

Note how this particular viewpoint, i.e., the equivalence between controllability and nonholonomy, was already implicitly adopted in Example 11.1, where the controllability of the kinematic system was proven *constructively*, i.e., by exhibiting a reconfiguration manoeuvre. A more systematic approach is to take advantage of the controllability conditions for nonlinear driftless systems. In particular, controllability may be verified using the *accessibility rank condition*

$$\dim \Delta_{\mathcal{A}}(\boldsymbol{q}) = n, \tag{11.11}$$

where $\Delta_{\mathcal{A}}$ is the *accessibility distribution* associated with system (11.10), i.e., the involutive closure of distribution $\Delta = \mathrm{span}\{\boldsymbol{g}_1, \dots, \boldsymbol{g}_m\}$. The following cases may occur:

1. If (11.11) holds, system (11.10) is controllable and the kinematic constraints (11.3) are (completely) nonholonomic.
2. If (11.11) does not hold, system (11.10) is not controllable and the kinematic constraints (11.3) are at least partially integrable. In particular, let

$$\dim \Delta_{\mathcal{A}}(\boldsymbol{q}) = \nu < n.$$

Then

2a. If $m < \nu < n$, constraints (11.3) are only partially integrable.
2b. If $\nu = m$, constraints (11.3) are completely integrable, and hence holonomic. This happens when $\Delta_{\mathcal{A}}$ coincides with $\Delta = \mathrm{span}\{\boldsymbol{g}_1, \dots, \boldsymbol{g}_m\}$, i.e., when the latter distribution is involutive.

It is easy to verify that, in the case of a single kinematic constraint (11.7), the integrability condition given by (11.9) is equivalent to the involutivity of $\Delta = \mathrm{span}\{\boldsymbol{g}_1, \dots, \boldsymbol{g}_{n-1}\}$. Another remarkable situation is met when the number of Pfaffian constraints is $k = n - 1$; in this case, the associated kinematic model (11.10) consists of a single vector field $\boldsymbol{g}$ ($m = 1$). Hence, $n - 1$ Pfaffian constraints are always integrable, because the distribution associated with a single vector field is always involutive. For example, a mechanical system with two generalized coordinates that is subject to a scalar Pfaffian constraint is always holonomic.

In the following, the kinematic models of two wheeled vehicles of particular interest will be analyzed in detail. A large part of the existing mobile robots have a kinematic model that is equivalent to one of these two.

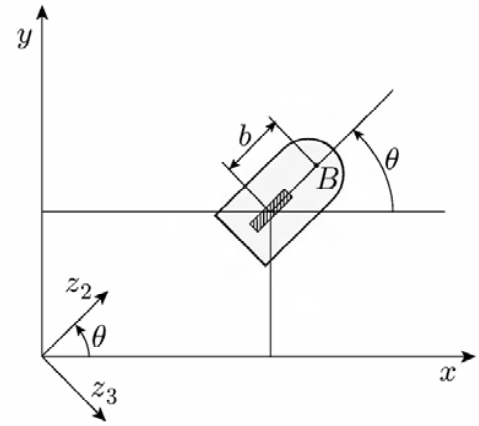

Fig. 11.3. Generalized coordinates for a unicycle

11.2.1 Unicycle

A *unicycle* is a vehicle with a single orientable wheel. Its configuration is completely described by $\boldsymbol{q} = [\,x \quad y \quad \theta\,]^T$, where (x, y) are the Cartesian coordinates of the contact point of the wheel with the ground (or equivalently, of the wheel centre) and θ is the orientation of the wheel with respect to the x axis (see Fig. 11.3).

As already seen in Example 11.1, the pure rolling constraint for the wheel is expressed as

$$\dot{x}\sin\theta - \dot{y}\cos\theta = [\,\sin\theta \quad -\cos\theta \quad 0\,]\,\dot{\boldsymbol{q}} = 0, \tag{11.12}$$

entailing that the velocity of the contact point is zero in the direction orthogonal to the sagittal axis of the vehicle. The line passing through the contact point and having such direction is therefore called *zero motion line*. Consider the matrix

$$\boldsymbol{G}(\boldsymbol{q}) = [\,\boldsymbol{g}_1(\boldsymbol{q}) \quad \boldsymbol{g}_2(\boldsymbol{q})\,] = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix},$$

whose columns $\boldsymbol{g}_1(\boldsymbol{q})$ and $\boldsymbol{g}_2(\boldsymbol{q})$ are, for each $\boldsymbol{q}$, a basis of the null space of the matrix associated with the Pfaffian constraint. All the admissible generalized velocities at $\boldsymbol{q}$ are therefore obtained as a linear combination of $\boldsymbol{g}_1(\boldsymbol{q})$ and $\boldsymbol{g}_2(\boldsymbol{q})$. The kinematic model of the unicycle is then

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega, \tag{11.13}$$

where the inputs v and ω have a clear physical interpretation. In particular, v is the *driving velocity*, i.e., the modulus[4] (with sign) of the contact point

[4] Note that v is given by the angular speed of the wheel around its horizontal axis multiplied by the wheel radius.

velocity vector, whereas the *steering velocity* ω is the wheel angular speed around the vertical axis.

The Lie bracket of the two input vector fields is

$$[\boldsymbol{g}_1, \boldsymbol{g}_2](\boldsymbol{q}) = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix},$$

that is always linearly independent from $\boldsymbol{g}_1(\boldsymbol{q}), \boldsymbol{g}_2(\boldsymbol{q})$. Therefore, the iterative procedure (see Sect. D.2) for building the accessibility distribution $\Delta_{\mathcal{A}}$ ends with

$$\dim \Delta_{\mathcal{A}} = \dim \Delta_2 = \dim \mathrm{span}\{\boldsymbol{g}_1, \boldsymbol{g}_2, [\boldsymbol{g}_1, \boldsymbol{g}_2]\} = 3.$$

This indicates that the unicycle is controllable with degree of nonholonomy $\kappa = 2$, and that constraint (11.12) is nonholonomic — the same conclusion reached in Example 11.3 by applying the integrability condition.

A unicycle in the strict sense (i.e., a vehicle equipped with a single wheel) is a robot with a serious problem of balance in static conditions. However, there exist vehicles that are kinematically equivalent to a unicycle but more stable from a mechanical viewpoint. Among these, the most important are the *differential drive* and the *synchro drive* vehicles, already introduced in Sect. 1.2.2.

For the differential drive mobile robot of Fig. 1.13, denote by (x, y) the Cartesian coordinates of the midpoint of the segment joining the two wheel centres, and by θ the common orientation of the fixed wheels (hence, of the vehicle body). Then, the kinematic model (11.13) of the unicycle also applies to the differential drive vehicle, provided that the driving and steering velocities v and ω are expressed as a function of the actual velocity inputs, i.e., the angular speeds ω_R and ω_L of the right and left wheel, respectively. Simple arguments (see Problem 11.6) can be used to show that there is a one-to-one correspondence between the two sets of inputs:

$$v = \frac{r\,(\omega_R + \omega_L)}{2} \qquad \omega = \frac{r\,(\omega_R - \omega_L)}{d}, \tag{11.14}$$

where r is the radius of the wheels and d is the distance between their centres.

The equivalence with the kinematic model (11.13) is even more straightforward for the *synchro drive* mobile robot of Fig. 1.14, whose control inputs are indeed the driving velocity v and the steering velocity ω, that are common to the three orientable wheels. The Cartesian coordinates (x, y) may represent in this case any point of the robot (for example, its centroid), while θ is the common orientation of the wheels. Note that, unlike a differential drive vehicle, the orientation of the body of a synchro drive vehicle never changes, unless a third actuator is added for this specific purpose.

11.2.2 Bicycle

Consider now a *bicycle*, i.e., a vehicle having an orientable wheel and a fixed wheel arranged as in Fig. 11.4. A possible choice for the generalized coordi-

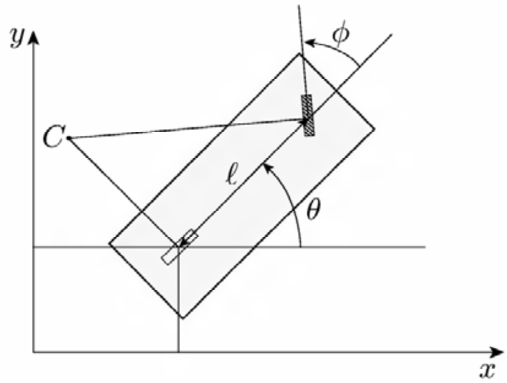

Fig. 11.4. Generalized coordinates and instantaneous centre of rotation for a bicycle

nates is $\boldsymbol{q} = [\,x \quad y \quad \theta \quad \phi\,]^T$, where (x, y) are the Cartesian coordinates of the contact point between the rear wheel and the ground (i.e., of the rear wheel centre), θ is the orientation of the vehicle with respect to the x axis, and ϕ is the steering angle of the front wheel with respect to the vehicle.

The motion of the vehicle is subject to two pure rolling constraints, one for each wheel:

$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0 \tag{11.15}$$

$$\dot{x} \sin\theta - \dot{y}\cos\theta = 0, \tag{11.16}$$

where (x_f, y_f) is the Cartesian position of the centre of the front wheel. The geometric meaning of these constraints is obvious: the velocity of the centre of the front wheel is zero in the direction orthogonal to the wheel itself, while the velocity of the centre of the rear wheel is zero in the direction orthogonal to the sagittal axis of the vehicle. The zero motion lines of the two wheels meet at a point C called *instantaneous centre of rotation* (Fig. 11.4), whose position depends only on (and changes with) the configuration $\boldsymbol{q}$ of the bicycle. Each point of the vehicle body then moves instantaneously along an arc of circle with centre in C (see also Problem 11.7).

Using the rigid body constraint

$$\begin{aligned} x_f &= x + \ell\cos\theta \\ y_f &= y + \ell\sin\theta, \end{aligned}$$

where ℓ is the distance between the wheels, constraint (11.15) can be rewritten as

$$\dot{x}\sin(\theta + \phi) - \dot{y}\cos(\theta + \phi) - \ell\,\dot{\theta}\cos\phi = 0. \tag{11.17}$$

The matrix associated with the Pfaffian constraints (11.16), (11.17) is then

$$\boldsymbol{A}^T(\boldsymbol{q}) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta + \phi) & -\cos(\theta + \phi) & -\ell\cos\phi & 0 \end{bmatrix},$$

with constant rank $k = 2$. The dimension of its null space is $n - k = 2$, and all the admissible velocities at $\boldsymbol{q}$ may be written as a linear combination of a basis of $\mathcal{N}(\boldsymbol{A}^T(\boldsymbol{q}))$, for example

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi/\ell \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2.$$

Since the front wheel is orientable, it is immediate to set $u_2 = \omega$, where ω is the *steering* velocity. The expression of u_1 depends instead on how the vehicle is driven.

If the bicycle has *front-wheel drive*, one has directly $u_1 = v$, where v is the *driving* velocity of the front wheel. The corresponding kinematic model is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega. \tag{11.18}$$

Denoting by $\boldsymbol{g}_1(\boldsymbol{q})$ and $\boldsymbol{g}_2(\boldsymbol{q})$ the two input vector fields, simple computations give

$$\boldsymbol{g}_3(\boldsymbol{q}) = [\boldsymbol{g}_1, \boldsymbol{g}_2](\boldsymbol{q}) = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ -\cos\phi/\ell \\ 0 \end{bmatrix} \qquad \boldsymbol{g}_4(\boldsymbol{q}) = [\boldsymbol{g}_1, \boldsymbol{g}_3](\boldsymbol{q}) = \begin{bmatrix} -\sin\theta/\ell \\ \cos\theta/\ell \\ 0 \\ 0 \end{bmatrix},$$

both linearly independent from $\boldsymbol{g}_1(\boldsymbol{q})$ and $\boldsymbol{g}_2(\boldsymbol{q})$. Hence, the iterative procedure for building the accessibility distribution $\Delta_{\mathcal{A}}$ ends with

$$\dim \Delta_{\mathcal{A}} = \dim \Delta_3 = \dim \operatorname{span}\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\} = 4.$$

This means that the front-wheel drive bicycle is controllable with degree of nonholonomy $\kappa = 3$, and constraints (11.15), (11.16) are (completely) nonholonomic.

The kinematic model of a bicycle with *rear-wheel drive* can be derived by noting that in this case the first two equations must coincide with those of the unicycle model (11.13). It is then sufficient to set $u_1 = v/\cos\phi$ to obtain

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega, \tag{11.19}$$

where v is the *driving* velocity of the rear wheel.[5] In this case, one has

$$\boldsymbol{g}_3(\boldsymbol{q})=[\boldsymbol{g}_1,\boldsymbol{g}_2](\boldsymbol{q}) = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{1}{\ell\cos^2\phi} \\ 0 \end{bmatrix} \qquad \boldsymbol{g}_4(\boldsymbol{q})=[\boldsymbol{g}_1,\boldsymbol{g}_3](\boldsymbol{q}) = \begin{bmatrix} -\dfrac{\sin\theta}{\ell\cos^2\phi} \\ \dfrac{\cos\theta}{\ell\cos^2\phi} \\ 0 \\ 0 \end{bmatrix},$$

again linearly independent from $\boldsymbol{g}_1(\boldsymbol{q})$ and $\boldsymbol{g}_2(\boldsymbol{q})$. Hence, the rear-wheel drive bicycle is also controllable with degree of nonholonomy $\kappa = 3$.

Like the unicycle, the bicycle is also unstable in static conditions. Kinematically equivalent vehicles that are mechanically balanced are the *tricycle* and the *car-like* robot, introduced in Sect. 1.2.2 and shown respectively in Fig. 1.15 and 1.16. In both cases, the kinematic model is given by (11.18) or by (11.19) depending on the wheel drive being on the front or the rear wheels. In particular, (x, y) are the Cartesian coordinates of the midpoint of the rear wheel axle, θ is the orientation of the vehicle, and ϕ is the steering angle.

11.3 Chained Form

The possibility of transforming the kinematic model (11.10) of a mobile robot in a canonical form is of great interest for solving planning and control problems with efficient, systematic procedures. Here, the analysis is limited to systems with two inputs, like the unicycle and bicycle models.

A $(2, n)$ *chained form* is a two-input driftless system

$$\dot{\boldsymbol{z}} = \boldsymbol{\gamma}_1(\boldsymbol{z})v_1 + \boldsymbol{\gamma}_2(\boldsymbol{z})v_2,$$

whose equations are expressed as

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ &\;\vdots \\ \dot{z}_n &= z_{n-1} v_1. \end{aligned} \tag{11.20}$$

Using the following notation for a 'repeated' Lie bracket:

$$\mathrm{ad}_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2 = [\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2] \qquad \mathrm{ad}^k_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2 = [\boldsymbol{\gamma}_1, \mathrm{ad}^{k-1}_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2],$$

[5] Note that the kinematic model (11.19) is no longer valid for $\phi = \pm\pi/2$, where the first vector field is not defined. This corresponds to the mechanical jam in which the front wheel is orthogonal to the sagittal axis of the vehicle. This singularity does not arise in the front-wheel drive bicycle (11.18), that in principle can still pivot around the rear wheel contact point in such a situation.

one has for system (11.20)

$$\gamma_1 = \begin{bmatrix} 1 \\ 0 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \end{bmatrix} \quad \gamma_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathrm{ad}^k_{\gamma_1}\gamma_2 = \begin{bmatrix} 0 \\ \vdots \\ (-1)^k \\ \vdots \\ 0 \end{bmatrix},$$

where $(-1)^k$ is the $(k+2)$-th component. This implies that the system is controllable, because the accessibility distribution

$$\Delta_{\mathcal{A}} = \mathrm{span}\,\{\gamma_1, \gamma_2, \mathrm{ad}_{\gamma_1}\gamma_2, \ldots, \mathrm{ad}^{n-2}_{\gamma_1}\gamma_2\}$$

has dimension n. In particular, the degree of nonholonomy is $\kappa = n - 1$.

There exist necessary and sufficient conditions for transforming a generic two-input driftless system

$$\dot{\boldsymbol{q}} = \boldsymbol{g}_1(\boldsymbol{q})u_1 + \boldsymbol{g}_2(\boldsymbol{q})u_2 \tag{11.21}$$

in the chained form (11.20) via coordinate and input transformations

$$\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{q}) \qquad \boldsymbol{v} = \boldsymbol{\beta}(\boldsymbol{q})\boldsymbol{u}. \tag{11.22}$$

In particular, it can be shown that systems like (11.21) with dimension n not larger than 4 can *always* be put in chained form. This applies, for example, to the kinematic models of the unicycle and the bicycle.

There also exist sufficient conditions for transformability in chained form that are relevant because they are constructive. Define the distributions

$$\begin{aligned} \Delta_0 &= \mathrm{span}\,\{\boldsymbol{g}_1, \boldsymbol{g}_2, \mathrm{ad}_{\boldsymbol{g}_1}\boldsymbol{g}_2, \ldots, \mathrm{ad}^{n-2}_{\boldsymbol{g}_1}\boldsymbol{g}_2\} \\ \Delta_1 &= \mathrm{span}\,\{\boldsymbol{g}_2, \mathrm{ad}_{\boldsymbol{g}_1}\boldsymbol{g}_2, \ldots, \mathrm{ad}^{n-2}_{\boldsymbol{g}_1}\boldsymbol{g}_2\} \\ \Delta_2 &= \mathrm{span}\,\{\boldsymbol{g}_2, \mathrm{ad}_{\boldsymbol{g}_1}\boldsymbol{g}_2, \ldots, \mathrm{ad}^{n-3}_{\boldsymbol{g}_1}\boldsymbol{g}_2\}. \end{aligned}$$

Assume that, in a certain set, it is $\dim \Delta_0 = n$, Δ_1 and Δ_2 are involutive, and there exists a scalar function $h_1(\boldsymbol{q})$ whose differential $d\boldsymbol{h}_1$ satisfies

$$d\boldsymbol{h}_1 \cdot \Delta_1 = 0 \qquad d\boldsymbol{h}_1 \cdot \boldsymbol{g}_1 = 1,$$

where the symbol $\cdot$ denotes the inner product between a row vector and a column vector — in particular, $\cdot\Delta_1$ is the inner product with any vector generated by distribution Δ_1. In this case, system (11.21) can be put in the form (11.20) through the coordinate transformation[6]

$$z_1 = h_1$$

[6] This transformation makes use of the Lie derivative (see Appendix D).

$$
\begin{aligned}
z_2 &= L_{\boldsymbol{g}_1}^{n-2} h_2 \\
&\vdots \\
z_{n-1} &= L_{\boldsymbol{g}_1} h_2 \\
z_n &= h_2,
\end{aligned}
$$

where h_2 must be chosen independent of h_1 and such that $dh_2 \cdot \Delta_2 = 0$. The input transformation is given by

$$
\begin{aligned}
v_1 &= u_1 \\
v_2 &= \left(L_{\boldsymbol{g}_1}^{n-1} h_2 \right) u_1 + \left(L_{\boldsymbol{g}_2} L_{\boldsymbol{g}_1}^{n-2} h_2 \right) u_2.
\end{aligned}
$$

In general, the coordinate and input transformations are not unique.

Consider the kinematic model (11.13) of the unicycle. With the change of coordinates

$$
\begin{aligned}
z_1 &= \theta \\
z_2 &= x \cos\theta + y \sin\theta \\
z_3 &= x \sin\theta - y \cos\theta
\end{aligned}
\tag{11.23}
$$

and the input transformation

$$
\begin{aligned}
v &= v_2 + z_3 v_1 \\
\omega &= v_1,
\end{aligned}
\tag{11.24}
$$

one obtains the (2,3) chained form

$$
\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 v_1.
\end{aligned}
\tag{11.25}
$$

Note that, while z_1 is simply the orientation θ, coordinates z_2 and z_3 represent the position of the unicycle in a moving reference frame whose z_2 axis is aligned with the sagittal axis of the vehicle (see Fig. 11.3).

As for mobile robots with bicycle-like kinematics, consider for example the model (11.19) corresponding to the rear-wheel drive case. Using the change of coordinates

$$
\begin{aligned}
z_1 &= x \\
z_2 &= \frac{1}{\ell} \sec^3\theta \tan\phi \\
z_3 &= \tan\theta \\
z_4 &= y
\end{aligned}
$$

and the input transformation

$$\begin{aligned} v &= \frac{v_1}{\cos\theta} \\ \omega &= -\frac{3}{\ell} v_1 \sec\theta \sin^2\phi + \frac{1}{\ell} v_2 \cos^3\theta \cos^2\phi, \end{aligned}$$

the (2,4) chained form is obtained:

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ \dot{z}_4 &= z_3 v_1. \end{aligned}$$

This transformation is defined everywhere in the configuration space, with the exception of points where $\cos\theta = 0$. The equivalence between the two models is then subject to the condition $\theta \neq \pm k\pi/2$, with $k = 1, 2, \ldots$.

11.4 Dynamic Model

The derivation of the dynamic model of a mobile robot is similar to the manipulator case, the main difference being the presence of nonholonomic constraints on the generalized coordinates. An important consequence of nonholonomy is that exact linearization of the dynamic model via feedback is no longer possible. In the following, the Lagrange formulation is used to obtain the dynamic model of an n-dimensional mechanical system subject to $k < n$ kinematic constraints in the form (11.3), and it is shown how this model can be partially linearized via feedback.

As usual, define the Lagrangian $\mathcal{L}$ of the mechanical system as the difference between its kinetic and potential energy:

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - \mathcal{U}(\boldsymbol{q}), \tag{11.26}$$

where $\boldsymbol{B}(\boldsymbol{q})$ is the (symmetric and positive definite) inertia matrix of the mechanical system. The Lagrange equations are in this case

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau} + \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda}, \tag{11.27}$$

where $\boldsymbol{S}(\boldsymbol{q})$ is an $(n \times m)$ matrix mapping the $m = n - k$ external inputs $\boldsymbol{\tau}$ to generalized forces performing work on $\boldsymbol{q}$, $\boldsymbol{A}(\boldsymbol{q})$ is the transpose of the $(k \times n)$ matrix characterizing the kinematic constraints (11.3), and $\boldsymbol{\lambda} \in \mathbb{R}^m$ is the vector of *Lagrange multipliers*. The term $\boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda}$ represents the vector of reaction forces at the generalized coordinate level. It has been assumed that the number of available inputs matches the number of DOFs (*full actuation*), that is, in turn, equal to the number n of generalized coordinates minus the number k of constraints.

Using (11.26), (11.27), the *dynamic model* of the constrained mechanical system is expressed as

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau} + \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda} \tag{11.28}$$

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}, \tag{11.29}$$

where

$$\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\right)^T + \left(\frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T.$$

Consider now a matrix $\boldsymbol{G}(\boldsymbol{q})$ whose columns are a basis for the null space of $\boldsymbol{A}^T(\boldsymbol{q})$, so that $\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{0}$. One can replace the constraint given by (11.29) with the kinematic model

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} = \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{q})\, v_i, \tag{11.30}$$

where $\boldsymbol{v} \in \mathbb{R}^m$ is the vector of *pseudo-velocities*;[7] for example, in the case of a unicycle the components of this vector are the driving velocity v and the steering velocity ω. Moreover, the Lagrange multipliers in (11.28) can be eliminated premultiplying both sides of the equation by $\boldsymbol{G}^T(\boldsymbol{q})$. This leads to the *reduced dynamic model*

$$\boldsymbol{G}^T(\boldsymbol{q})\left(\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) = \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau}, \tag{11.31}$$

a system of m differential equations.

Differentiation of (11.30) with respect to time gives

$$\ddot{\boldsymbol{q}} = \dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{v} + \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{v}}.$$

Premultiplying this by $\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})$ and using the reduced dynamic model (11.31), one obtains

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{v}} + \boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v}) = \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau}, \tag{11.32}$$

where

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{q}) &= \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\\ \boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v}) &= \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{v} + \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{n}(\boldsymbol{q}, \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v}),\end{aligned}$$

[7] In the dynamic modeling context, the use of this term emphasizes the difference between $\boldsymbol{v}$ and $\dot{\boldsymbol{q}}$, that are the actual (generalized) velocities of the mechanical system.

with $\boldsymbol{M}(\boldsymbol{q})$ positive definite and

$$\dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{v} = \sum_{i=1}^{m} \left(v_i \frac{\partial \boldsymbol{g}_i}{\partial \boldsymbol{q}}(\boldsymbol{q}) \right) \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v}.$$

This finally leads to the *state-space reduced model*

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} \tag{11.33}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{m}(\boldsymbol{q},\boldsymbol{v}) + \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau}, \tag{11.34}$$

that represents in a compact form the kinematic and dynamic models of the constrained system as a set of $n+m$ differential equations.

Suppose now that

$$\det\left(\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\right) \neq 0,$$

an assumption on the 'control availability' that is satisfied in many cases of interest. It is then possible to perform a *partial linearization via feedback* of (11.33), (11.34) by letting

$$\boldsymbol{\tau} = \left(\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\right)^{-1} \left(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{m}(\boldsymbol{q},\boldsymbol{v})\right), \tag{11.35}$$

where $\boldsymbol{a} \in \mathbb{R}^m$ is the *pseudo-acceleration* vector. The resulting system is

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} \tag{11.36}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{a}. \tag{11.37}$$

Note the structure of this system: the first n equations are the kinematic model, of which the last m — that represent the inclusion of m integrators on the input channels — are a *dynamic extension*. If the system is unconstrained and fully actuated, it is $\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{S}(\boldsymbol{q}) = \boldsymbol{I}_n$; then, the feedback law (11.35) simply reduces to an inverse dynamics control analogous to (8.57), and correspondingly the closed-loop system is equivalent to n decoupled double integrators.

The implementation of the feedback control (11.35) in principle requires the measurement of $\boldsymbol{v}$, and this may not be available. However, pseudo-velocities can be computed via the kinematic model as

$$\boldsymbol{v} = \boldsymbol{G}^\dagger(\boldsymbol{q})\dot{\boldsymbol{q}} = \left(\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\right)^{-1} \boldsymbol{G}^T(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{11.38}$$

provided that $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are measured. Note that the left pseudo-inverse of $\boldsymbol{G}(\boldsymbol{q})$ has been used here.

By defining the state $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{v}) \in \mathbb{R}^{n+m}$ and the input $\boldsymbol{u} = \boldsymbol{a} \in \mathbb{R}^m$, system (11.36), (11.37) can be expressed as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} = \begin{bmatrix} \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{bmatrix} \boldsymbol{u}, \tag{11.39}$$

i.e., a nonlinear system with drift also known as the *second-order kinematic model* of the constrained mechanical system. Accordingly, Eq. (11.36) is sometimes called *first-order kinematic model*. In view of the results recalled in Appendix D, the controllability of the latter guarantees the controllability of system (11.39).

Summarizing, in nonholonomic mechanical systems — such as wheeled mobile robots — it is possible to 'cancel' the dynamic effects via nonlinear state feedback, provided that the dynamic parameters are exactly known and the complete state of the system (generalized coordinates and velocities $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$) is measured.

Under these assumptions, the control problem can be directly addressed at the (pseudo-)velocity level, i.e., by choosing $\boldsymbol{v}$ in such a way that the kinematic model

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v}$$

behaves as desired. From $\boldsymbol{v}$, it is possible to derive the actual control inputs at the generalized force level through (11.35). Since $\boldsymbol{a} = \dot{\boldsymbol{v}}$ appears in this equation, the pseudo-velocities $\boldsymbol{v}$ must be differentiable with respect to time.

Example 11.5

For illustration, the above procedure for deriving, reducing and partially linearizing the dynamic model is now applied to the unicycle. Let m be the mass of the unicycle, I its moment of inertia around the vertical axis through its centre, $\boldsymbol{\tau}_1$ the driving force and $\boldsymbol{\tau}_2$ the steering torque. With the kinematic constraint expressed as (11.12), the dynamic model (11.28), (11.29) takes on the form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} + \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \lambda$$

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0.$$

In this case one has

$$\begin{aligned} \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \boldsymbol{0} \\ \boldsymbol{G}(\boldsymbol{q}) &= \boldsymbol{S}(\boldsymbol{q}) \\ \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q}) &= \boldsymbol{I} \\ \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{B}\dot{\boldsymbol{G}}(\boldsymbol{q}) &= \boldsymbol{0}, \end{aligned}$$

and thus the reduced model in state-space is obtained as

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} \\ \dot{\boldsymbol{v}} &= \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau} \end{aligned}$$

where

$$\boldsymbol{M}^{-1}(\boldsymbol{q}) = \begin{bmatrix} 1/m & 0 \\ 0 & 1/I \end{bmatrix}.$$

By using the input transformation

$$\boldsymbol{\tau} = \boldsymbol{M}\boldsymbol{u} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \boldsymbol{u},$$

the second-order kinematic model is obtained as

$$\dot{\xi} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

with the state vector $\xi = [\,x \quad y \quad \theta \quad v \quad \omega\,]^T \in \mathbb{R}^5$.

11.5 Planning

As with manipulators, the problem of planning a trajectory for a mobile robot can be broken down in finding a *path* and defining a *timing law* on the path. However, if the mobile robot is subject to nonholonomic constraints, the first of these two subproblems becomes more difficult than in the case of manipulators. In fact, in addition to meeting the boundary conditions (interpolation of the assigned points and continuity of the desired degree) the path must also satisfy the nonholonomic constraints at all points.

11.5.1 Path and Timing Law

Assume that one wants to plan a trajectory $\boldsymbol{q}(t)$, for $t \in [t_i, t_f]$, that leads a mobile robot from an initial configuration $\boldsymbol{q}(t_i) = \boldsymbol{q}_i$ to a final configuration $\boldsymbol{q}(t_f) = \boldsymbol{q}_f$ in the absence of obstacles. The trajectory $\boldsymbol{q}(t)$ can be broken down into a geometric path $\boldsymbol{q}(s)$, with $d\boldsymbol{q}(s)/ds \neq 0$ for any value of s, and a timing law $s = s(t)$, with the parameter s varying between $s(t_i) = s_i$ and $s(t_f) = s_f$ in a monotonic fashion, i.e., with $\dot{s}(t) \geq 0$, for $t \in [t_i, t_f]$. A possible choice for s is the arc length along the path; in this case, it would be $s_i = 0$ and $s_f = L$, where L is the length of the path.

The above space-time separation implies that

$$\dot{\boldsymbol{q}} = \frac{d\boldsymbol{q}}{dt} = \frac{d\boldsymbol{q}}{ds}\,\dot{s} = \boldsymbol{q}'\dot{s},$$

where the prime symbol denotes differentiation with respect to s. The generalized velocity vector is then obtained as the product of the vector $\boldsymbol{q}'$, which is directed as the tangent to the path in configuration space, by the scalar $\dot{s}$, that varies its modulus. Note that the vector $[\,x' \quad y'\,]^T \in \mathbb{R}^2$ is directed as