C.1.5 33, 36

33. Doubling your money Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

Assume the investment is X, and the needed years is t.

$X(1+6.25\%)^t = 2X$

$\Rightarrow 1.0625^t = 2$

$\Rightarrow t = \log_{1.0625} 2 \approx 11.43$

"log of 2 to base 1.0625"

36. Eliminating a disease Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

a. to reduce the number of cases to 1000?

b. to eliminate the disease; that is, to reduce the number of cases to less than 1?

a. Assume a years are needed.

$10{,}000\,(1-20\%)^a = 1000$

$\Rightarrow 10 \times 0.8^a = 1$

$\Rightarrow 0.8^a = 0.1$

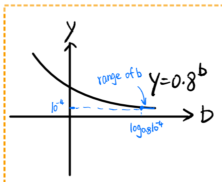
$\Rightarrow a = \log_{0.8} 0.1 \approx 10.32$

b. Assume b years are needed.

$10{,}000\,(1-20\%)^b < 1$

$\Rightarrow 0.8^b < 10^{-4}$

$\Rightarrow b > \log_{0.8} 10^{-4} \quad (b > 41.28)$



C 1.6

In Exercises 7–10, determine from its graph whether the function is one-to-one.

7. $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \ge 0 \end{cases}$

8. $f(x) = \begin{cases} 2x + 6, & x \le -3 \\ x + 4, & x > -3 \end{cases}$

9. $f(x) = \begin{cases} 1 - \dfrac{x}{2}, & x \le 0 \\ \dfrac{x}{x+2}, & x > 0 \end{cases}$

9. When $x \le 0$. $\to$ It's a line $\Rightarrow$ Two nodes can decide it.

$(0,1)$ $(-2,2)$ $\leftarrow$ choose any two points.

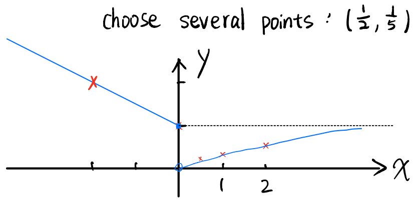
When $x > 0$.

$\frac{x}{x+2} = \frac{1}{1+\frac{2}{x}}$ $\quad x\uparrow\ \frac{2}{x}\downarrow\ (1+\frac{2}{x})\downarrow\ (\frac{1}{1+\frac{2}{x}})\uparrow \Rightarrow$ Increasing fun.

But, does it have limitation? $\frac{2}{x}\ (x>0) \quad \therefore \frac{2}{x} > 0$

$\Rightarrow 1+\frac{2}{x} > 1 \Rightarrow \frac{1}{1+\frac{2}{x}} \in (0,1)$

choose several points: $(\frac{1}{2},\frac{1}{5})$ $(1,\frac{1}{3})$ $(2,\frac{1}{2})$



Yes.

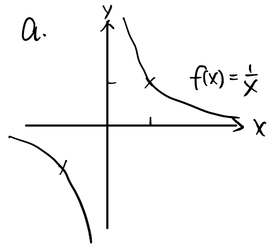
It's one-to-one.

Recall: A function f(x) is one-to-one on a domain D if $f(x_1) \ne f(x_2)$ $\forall x_1 \ne x_2$ in D.

18. a. Graph the function $f(x) = 1/x$. What symmetry does the graph have?

b. Show that f is its own inverse.

a.



symmetric about $y=x$

b.

Recall:

Δ Definition of inverse function: prerequisite.

Given a one-to-one function f on a domain D with range R, then the inverse function f^{-1} is defined by:

$f^{-1}(b) = a$ if $f(a) = b$ $\langle$ Domain of f^{-1} is R, Range of f^{-1} is D $\rangle$

$y = \frac{1}{x} \quad (x \ne 0, y \ne 0)$

$\Rightarrow x = \frac{1}{y}$

Interchange x and y: $y = \frac{1}{x} \quad \therefore f^{-1}(x) = \frac{1}{x}$

$\therefore f(x) = f^{-1}(x)$

Theory and Examples

75. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.

$\because f(x)$ is a one-to-one function

$\therefore \forall x_1, x_2 \in D,\ f(x_1) \ne f(x_2)$

$\therefore \forall x_1, x_2 \in D,\ -f(x_1) \ne -f(x_2)$

$\therefore \forall x_1, x_2 \in D,\ g(x_1) \ne g(x_2)$

$\therefore g(x)$ is also a one-to-one function.

76. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.

$\because f(x)$ is never zero

$\therefore \forall x_3 \in D,\ f(x_3) \ne 0$

$\because f(x)$ is one-to-one

$\therefore \forall x_1, x_2 \in D,\ f(x_1) \ne f(x_2)$

$\because g(x) = \frac{1}{x}$ is one-to-one and $f(x)$ is never zero.

$\therefore \forall x_1, x_2 \in D,\ \frac{1}{f(x_1)} \ne \frac{1}{f(x_2)}$

$\therefore \forall x_1, x_2 \in D,\ h(x_1) \ne h(x_2)$

$\therefore h(x)$ is one-to-one.

77. Suppose that the range of g lies in the domain of f so that the composition $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

Tip: $f \circ g = f[g(x)]$ $\langle R_g$ lies in $D_f \rangle$

$\because g$ is one-to-one

$\therefore \forall x_1, x_2 \in D_g,\ g(x_1) \ne g(x_2)$

$\because f$ is one-to-one

$\therefore \forall g(x_1), g(x_2) \in R_g,\ f[g(x_1)] \ne f[g(x_2)]$

$\therefore f[g(x)]$ (i.e. $f \circ g$) is one-to-one.

(78) If a composition $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

$\because f\circ g = f[g(x)]$ is one-to-one

$\therefore R_g$ lies in D_f ;

$\forall x_1, x_2 \in D_g,\ f[g(x_1)] \neq f[g(x_2)]$

$\therefore \forall x_1, x_2 \in D_g,\ g(x_1) \neq g(x_2)$

(Because if $\exists x_1, x_2 \in D_g, g(x_1) = g(x_2)$,

then $\exists x_1, x_2 \in D_g,\ f[g(x_1)] = f[g(x_2)]$ conflic !)

$\therefore g$ is one-to-one.

OR write in another way:

If g is not one-to-one,

then $\exists x_1, x_2 \in D_g,\ g(x_1) = g(x_2)$

thus, $\exists x_1, x_2 \in D_g,\ f[g(x_1)] = f[g(x_2)]$

Under this assumption, $f\circ g$ can NOT be one-to-one.

Therefore, this assumption is wrong.

$\therefore g$ must be one-to-one.

(81) Start with the graph of $y = \ln x$. Find an equation of the graph that results from

a. shifting down 3 units. **b.** shifting right 1 unit.

c. shifting left 1, up 3 units. **d.** shifting down 4, right 2 units.

e. reflecting about the y-axis.

f. reflecting about the line $y = x$.

Recall: down $-$ left $+$

up $+$ right $-$

a. $y = \ln x - 3$ b. $y = \ln(x-1)$

c. $y = \ln(x+1) + 3$ d. $y = \ln(x-2) - 4$

e. $y = \ln(-x)$ f: $y = e^x$

For (f)

Way 1: [direct memory] $y = e^x$ and $y = \ln x$ are symmetrical to $y = x$

Way 2: [By definition: $f^{-1}(x)$ can be obtained by reflecting the graph of $f(x)$ about the line $y = x$.]

$y = \ln x = \log_e x$

$\Rightarrow e^y = x$

Interchange x and y: $y = e^x$

(79.) Find a formula for the inverse function f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

a. $f(x) = \dfrac{100}{1 + 2^{-x}}$ **b.** $f(x) = \dfrac{50}{1 + 1.1^{-x}}$

c. $f(x) = \dfrac{e^x - 1}{e^x + 1}$ **d.** $f(x) = \dfrac{\ln x}{2 - \ln x}$

a. $f(x) = \dfrac{100}{1+2^{-x}}$

$y = \dfrac{100}{1+2^{-x}}$

$1 + 2^{-x} = \dfrac{100}{y}$

$2^{-x} = \dfrac{100}{y} - 1$

$-x = \log_2\left(\dfrac{100}{y} - 1\right)$

$\Rightarrow x = -\log_2\left(\dfrac{100}{y} - 1\right)$

Interchange x and y:

$f^{-1}(x) = -\log_2\left(\dfrac{100}{x} - 1\right)$

Verification:

$(f\circ f^{-1})(x) = f[f^{-1}(x)]$

$= f\left[-\log_2\left(\dfrac{100}{x} - 1\right)\right]$

$= \dfrac{100}{1 + 2^{\log_2\left(\frac{100}{x} - 1\right)}}$

$= \dfrac{100}{1 + \left(\frac{100}{x} - 1\right)}$

$= \dfrac{100}{\frac{100}{x}}$

$= x$

$(f^{-1}\circ f)(x) = f^{-1}[f(x)]$

$= f^{-1}\left[\dfrac{100}{1+2^{-x}}\right]$

$= -\log_2\left(\dfrac{100}{\frac{100}{1+2^{-x}}} - 1\right)$

b. $f(x) = \dfrac{50}{1 + 1.1^{-x}}$

$y = \dfrac{50}{1 + 1.1^{-x}}$

$1 + 1.1^{-x} = \dfrac{50}{y}$

$1.1^{-x} = \dfrac{50}{y} - 1$

$-x = \log_{1.1}\left(\dfrac{50}{y} - 1\right)$

$x = -\log_{1.1}\left(\dfrac{50}{y} - 1\right)$

Interchang x and y:

$f^{-1}(x) = -\log_{1.1}\left(\dfrac{50}{x} - 1\right)$

Verification:

$(f\circ f^{-1})(x) = f[f^{-1}(x)]$

$= f\left[-\log_{1.1}\left(\dfrac{50}{x} - 1\right)\right]$

$= \dfrac{50}{1 + 1.1^{\log_{1.1}\left(\frac{50}{x} - 1\right)}}$

$= \dfrac{50}{1 + \left(\frac{50}{x} - 1\right)}$

$= \dfrac{50}{\frac{50}{x}}$

$= x$

$(f^{-1}\circ f)(x) = f^{-1}[f(x)]$

$= f^{-1}\left[\dfrac{50}{1 + 1.1^{-x}}\right]$

c. $f(x) = \dfrac{e^x - 1}{e^x + 1}$

$y = \dfrac{e^x - 1}{e^x + 1}$

$= \dfrac{(e^x + 1) - 2}{e^x + 1}$

$= 1 - \dfrac{2}{e^x + 1}$

$\Rightarrow \dfrac{2}{e^x + 1} = 1 - y$

$e^x + 1 = \dfrac{2}{1-y}$

$e^x = \dfrac{2}{1-y} - 1 = \dfrac{2 - (1-y)}{1-y}$

$e^x = \dfrac{1+y}{1-y}$

$\Rightarrow x = \ln\left(\dfrac{1+y}{1-y}\right)$

Interchange x and y:

$f^{-1}(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

Verification:

$(f\circ f^{-1})(x) = f[f^{-1}(x)]$

$= f\left[\ln\left(\dfrac{1+x}{1-x}\right)\right]$

$= \dfrac{e^{\ln\left(\frac{1+x}{1-x}\right)} - 1}{e^{\ln\left(\frac{1+x}{1-x}\right)} + 1}$

$= \dfrac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1}$

$= \dfrac{(1+x) - (1-x)}{(1+x) + (1-x)}$

$= \dfrac{1 + x - 1 + x}{1 + x + 1 - x}$

$= \dfrac{2x}{2}$

$= x$

d. $f(x) = \dfrac{\ln x}{2 - \ln x}$

$y = \dfrac{\ln x}{2 - \ln x} = \dfrac{\ln x - 2 + 2}{2 - \ln x}$

$y = -1 + \dfrac{2}{2 - \ln x}$

$\Rightarrow \dfrac{2}{2 - \ln x} = y + 1$

$2 - \ln x = \dfrac{2}{y+1}$

$\Rightarrow \ln x = 2 - \dfrac{2}{y+1} = \dfrac{2y}{y+1}$

$\Rightarrow e^{\left(\frac{2y}{y+1}\right)} = x$

Interchange x and y: $y = e^{\left(\frac{2x}{x+1}\right)}$

$f^{-1}(x) = e^{\left(\frac{2x}{x+1}\right)}$

Verification:

$(f\circ f^{-1})(x) = f[f^{-1}(x)]$

$= f\left[e^{\left(\frac{2x}{x+1}\right)}\right] = \dfrac{\ln e^{\left(\frac{2x}{x+1}\right)}}{2 - \ln e^{\left(\frac{2x}{x+1}\right)}}$

$= \dfrac{\frac{2x}{x+1}}{2 - \frac{2x}{x+1}}$ $\times(x+1)$ $= \dfrac{2x}{2(x+1) - 2x}$

$= \dfrac{2x}{2} = x$

$$= -\log_2\left(1+2^{-x}-1\right)$$
$$= -\log_2 2^{-x}$$
$$= \log_2 (2^{-x})^{-1}$$
$$= \log_2 2^{x}$$
$$= x$$

$$= -\log_{1.1}\left(\frac{50}{\frac{50}{1+1.1^{-x}}}-1\right)$$
$$= -\log_{1.1}\left(1+1.1^{-x}-1\right)$$
$$= -\log_{1.1} 1.1^{-x}$$
$$= -(-x)$$
$$= x$$

$$(f^{-1}\circ f)(x) = f^{-1}[f(x)]$$
$$= f^{-1}\left[\frac{e^x-1}{e^x+1}\right]$$
$$= f^{-1}\left[\frac{e^x+1}{e^x+1}-\frac{2}{e^x+1}\right]$$
$$= f^{-1}\left[1-\frac{2}{e^x+1}\right]$$
$$= \ln\left[\frac{1+\left(1-\frac{2}{e^x+1}\right)}{1-\left(1-\frac{2}{e^x+1}\right)}\right]$$
$$= \ln\left[\frac{2-\frac{2}{e^x+1}}{\frac{2}{e^x+1}}\right]$$
$$= \ln\left[\frac{\frac{2e^x+2-2}{e^x+1}}{\frac{2}{e^x+1}}\right]$$
$$= \ln\left[\frac{\frac{2e^x}{e^x+1}}{\frac{2}{e^x+1}}\right]$$
$$= \ln\left(\frac{2e^x}{2}\right)$$
$$= \ln(e^x)$$
$$= x$$

$$(f^{-1}\circ f)(x) = f^{-1}[f(x)]$$
$$= f^{-1}\left[\frac{\ln x}{2-\ln x}\right]$$
$$= f^{-1}\left[\frac{\ln x-2}{2-\ln x}+\frac{2}{2-\ln x}\right]$$
$$= f^{-1}\left[-1+\frac{2}{2-\ln x}\right]$$
$$= e^{\left[\frac{2\left(-1+\frac{2}{2-\ln x}\right)}{-1+\frac{2}{2-\ln x}+1}\right]}$$
$$= e^{\frac{2\left(-1+\frac{2}{2-\ln x}\right)}{\frac{2}{2-\ln x}}}$$
$$= e^{\frac{\cancel{2}\left(-1+\frac{2}{2-\ln x}\right)\times(2-\ln x)}{\cancel{2}}}$$
$$= e^{(-2+\ln x+2)}$$
$$= e^{\ln x} = e^{\log_e x}$$
$$= x$$