Recap of MC

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Recap of Lecture 1.

1) Given curve $r(t) = (r_1, r_2, r_3)$. $\Longleftrightarrow$ Chapter 8

What's the length of curve between $r(a)$ and $r(b)$?

$$\text{Length} = \int_a^b |r'(t)| \cdot dt$$

Ex. $f(u) = (2u, u^2, \frac{u^3}{3})$

length between $P = (-6, 9, -9)$ and $Q = (6, 9, 9)$

$(u_P = -3)$ $(u_Q = 3)$

$f'(u) = (2, 2u, u^2)$

$$\int_{-3}^{3} |f'(u)| \cdot du = \int_{-3}^{3} \sqrt{2^2 + (2u)^2 + (u^2)^2} \cdot du$$

$$= \int_{-3}^{3} \sqrt{u^4 + 4u^2 + 4} \cdot du = \int_{-3}^{3} \sqrt{(u^2+2)^2} \cdot du$$

$$= \int_{-3}^{3} (u^2 + 2) \cdot du = \left[\frac{1}{3}u^3 + 2u\right]_{u=-3}^{3}$$

$$= \left(\frac{1}{3} \cdot 3^3 + 6\right) - \left(-\frac{1}{3} \cdot 3^3 - 6\right) = 18 + 12 = 30.$$

Quick Recap of Lecture 2:

① Limits in 2 variables.

i.e., $\lim_{(x,y) \to (x_0, y_0)} f(x,y)$

Recall: $\lim_{x \to x_0} f(x)$ $\quad \rightarrow x_0 \leftarrow$

1) Limitation is existed:
- use x_0 & y_0 directly
- use squeeze

2) Limitation not existed
- show at least 2 different result from different directions to reach $(x,y) \to (x_0, y_0)$.

Ex. $\lim_{(x,y) \to (0,0)} x \cdot \sin(xy)$

$-|x| \le x \cdot \sin(xy) \le |x|$

$\lim_{(x,y) \to (0,0)} -|x| = 0$

$\lim_{(x,y) \to (0,0)} |x| = 0$

$\therefore \lim_{(x,y) \to (0,0)} x \cdot \sin(xy) = 0$

Ex. $\lim_{(x,y) \to (0,0)} \frac{x}{y}$

when $x = y$ $\quad \lim_{(x,y) \to (0,0)} \frac{x}{y} = \lim_{(x,y) \to (0,0)} \frac{x}{x} = 1$

when $x = 0$ $\quad \lim_{(x,y) \to (0,0)} \frac{x}{y} = \lim_{y \to 0} \frac{0}{y} = 0$

$\therefore$ Not existed.

② Jacobian & Chain Rule. $Jac(F \circ G) = (Jac(F) \circ G) \cdot Jac(G)$.

Ex. $F(x,y) = (x + y^2, y)$ $\quad G(x,y) = (x - y^2, y+1)$ new x new y

$$Jac(F) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 2y \\ 0 & 1 \end{pmatrix}$$

$$Jac(G) = \begin{pmatrix} \frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} \\ \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -2y \\ 0 & 1 \end{pmatrix}$$

$Jac(F \circ G) = ?$

$$F \circ G = (\underbrace{(x - y^2)}_{x} + \underbrace{(y+1)}_{y}{}^2, \underbrace{y+1}_{y}) = (x + 2y + 1, y + 1)$$

$$Jac(F \circ G) = \begin{pmatrix} \frac{\partial(x+2y+1)}{\partial x} & \frac{\partial(x+2y+1)}{\partial y} \\ \frac{\partial(y+1)}{\partial x} & \frac{\partial(y+1)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$Jac(F) \circ G = \begin{pmatrix} 1 & 2 \cdot (y+1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2y+2 \\ 0 & 1 \end{pmatrix}$$

$$(Jac(F) \circ G) \cdot Jac(G) = \begin{pmatrix} 1 & 2y+2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

③ Given $F(x,y) = y^3 + xy + x^2 + 1$, $F(x,y) = 0$

$y'(0) = ?$

$$\begin{cases} F(x,y) = y^3 + xy + x^2 + 1 \\ F(x,y) = 0 \end{cases} \Rightarrow y^3 + xy + x^2 + 1 = 0$$

$3y^2 \cdot y' + y + x \cdot y' + 2x + 0 = 0$

$\Rightarrow y'(3y^2 + x) = -(2x + y)$

$$y' = -\frac{2x + y}{3y^2 + x}$$

$$\therefore y'(0) = -\frac{0 + y(0)}{3y^2(0) + 0} = -\frac{y(0)}{3y^2(0)} = -\frac{1}{3 \cdot y(0)}$$

what's $y(0)$?

$y^3 + xy + x^2 + 1 = 0$

when $x = 0$ $\Rightarrow y^3(0) + 0 + 0 + 1 = 0 \Rightarrow [y(0)]^3 = -1 \Rightarrow y(0) = -1$

$\therefore y'(0) = \frac{1}{3}$.

Quick Recap of L3:

① The derivative of f at P_0 in the direction u.

DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

Ex. $f(x_1, x_2) = x_1^2 + x_2^2$

$P_0 = (1, 3)$ $\quad u = (1, 2) \rightarrow u = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ $\quad \begin{cases} x_0 = 1 \\ y_0 = 3 \end{cases}$

$$\lim_{s \to 0} \frac{f(1 + \frac{1}{\sqrt{5}}s, 3 + \frac{2}{\sqrt{5}}s) - f(1,3)}{s}$$

$$= \lim_{s \to 0} \frac{(1 + \frac{1}{\sqrt{5}}s)^2 + (3 + \frac{2}{\sqrt{5}}s)^2 - 1^2 - 3^2}{s}$$

$$= \lim_{s \to 0} \frac{\cancel{1} + \frac{2}{\sqrt{5}}s + \frac{1}{5}s^2 + \cancel{9} + \frac{12}{\sqrt{5}}s + \frac{4}{5}s^2 - \cancel{1} - \cancel{9}}{s}$$

$$= \lim_{s \to 0} \frac{s^2 + \frac{14}{\sqrt{5}}s}{s} = \lim_{s \to 0} s + \frac{14}{\sqrt{5}} = \frac{14}{\sqrt{5}}$$

② Level Curve.

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Finding Level Surfaces

In Exercises 61–64, find an equation for the level surface of the function through the given point.

61. $f(x, y, z) = \sqrt{x - y} - \ln z$, $\quad (3, -1, 1)$

$f(3, -1, 1) = \sqrt{3 - (-1)} - \ln 1 = \sqrt{4} - 0 = 2$.

$\therefore \sqrt{x - y} - \ln z = 2$

③ Extreme values & Saddle points.

Discriminant / Hessian of a function $f(x,y)$:

$$D = H(f) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx} = f_{xx} \cdot f_{yy} - (f_{xy})^2.$$

$f_x(a,b) = f_y(a,b) = 0$ or $(\nabla f)(a,b) = 0$.

i) (a,b) is local max. if $f_{xx} < 0$ & $D > 0$.

ii) (a,b) is local min. if $f_{xx} > 0$ & $D > 0$.

iii) (a,b) is saddle point if $D < 0$.

Ex. $f(x,y) = x^2 + y^2$.

$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$\nabla f = 0 \Rightarrow (x,y) = (0,0)$

$f_x = 2x$, $f_y = 2y$; $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 2 \cdot 2 - 0^2 = 4 > 0$.

$f_{xx} = 2 > 0$.

$\therefore (0,0)$ is a local min.

Ex. $f(x,y) = x^3 + y^3 - 6xy$

$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x - 6y \\ 3y^2 - 6x \end{pmatrix}$ $\quad \nabla f = 0 \Rightarrow \begin{cases} 2x - 6y = 0 \\ 3y^2 - 6x = 0 \end{cases} \Rightarrow y^2 = 6y \Rightarrow y = 0$ or $y = 6$

$(0,0)$ or $(18,6)$ $\quad f_{xx} = 2 > 0$; $f_{yy} = 6y$; $f_{xy} = -6$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 2 \cdot (6y) - (-6)^2 = 12y - 36.$

$(0,0)$: $D_{(0,0)} = -36 < 0$ $\quad \therefore$ saddle point.

$(18,6)$: $D_{(18,6)} = 12 \cdot 6 - 36 = 36 > 0$ $\quad \therefore$ local min.

④ Tangent plane

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \quad (1)$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t \quad (2)$$

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \quad (3)$$

Ex. Find the tangent plane & normal line of the level surface $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1,2,4)$

$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$ $\quad \nabla f \big|_{P_0} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

tangent plane : $2(x - x_0) + 4(y - y_0) + 1 \cdot (z - z_0) = 0$

$\Rightarrow 2(x-1) + 4(y-2) + z - 4 = 0$

or $2x + 4y + z = 14$

normal line at P_0 : $x = 1 + 2t$

$y = 2 + 4t$

$z = 4 + t$

Quick Recap of L4.

① Lagrange Optimization

Q : Where on $y = x^2 + 6$ is the function $f(x,y) = x^2 + y^2$ minimal / maximal ?

$g = y - x^2 - 6 = 0$

$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x \\ 1 \end{pmatrix}$ $\quad \nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

when $\boxed{\nabla g = \lambda \cdot \nabla f}$

$\Rightarrow \begin{pmatrix} -2x \\ 1 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$\Rightarrow \begin{cases} -\cancel{2}x = \lambda \cdot \cancel{2}x \\ 1 = 2\lambda y \end{cases} \Rightarrow \begin{cases} x(\lambda + 1) = 0 \\ \lambda y = \frac{1}{2} \end{cases}$ $\quad \lambda =$

$\boxed{\text{if } x \neq 0}$: $\lambda = -1$, $y = -\frac{1}{2}$

$\because y - x^2 - 6 = 0 \Rightarrow -\frac{1}{2} - x^2 - 6 = 0$

$\Rightarrow x^2 < 0$ Impossible!

$\boxed{\text{if } x = 0}$: $y - x^2 - 6 = 0$

$\Rightarrow y = x^2 + 6 = 6$ $\quad \therefore \begin{cases} x = 0 \\ y = 6 \\ \lambda = \frac{1}{12} \end{cases}$

$\therefore (0,6)$

Q2: On $f(x,y,z) = x^2 - 4x - 4 + y^2 - 1 + z^2 = 0$, find where $g(x,y,z) = x^2 + y^2 + z^2$ is minimal.

$\nabla f = \begin{pmatrix} 2x - 4 \\ 2y \\ 2z \end{pmatrix}$ $\quad \nabla g = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

$\nabla f = \lambda \cdot \nabla g \Rightarrow \begin{pmatrix} 2x - 4 \\ 2y \\ 2z \end{pmatrix} = \lambda \cdot \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

$\begin{cases} 2x - 4 = 2\lambda x \\ 2y = 2\lambda y \\ 2z = 2\lambda z \end{cases} \Rightarrow \begin{cases} x - 2 = \lambda x \\ y = \lambda y \\ z = \lambda z \end{cases} \Rightarrow \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 0 \\ z(1-\lambda) = 0 \end{cases}$

if $y \neq 0$, $z \neq 0$

$\Rightarrow \lambda = 1$

but $x(1-\lambda) \neq 2$

Impossible.

$(\lambda \neq 1)$

if $y = 0$, $z = 0$

$\because x^2 - 4x - 4 + y^2 - 1 + z^2 = 0$

$\Rightarrow x^2 - 4x - 4 - 1 = 0$

$\Rightarrow x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$

$x = -1$ or $x = 5$.

$\therefore (-1,0,0)$, $(5,0,0)$.

$g(-1,0,0) = 1$ $\quad$ min.

$g(5,0,0) = 25$ $\quad$ max.

② $\boxed{e^{i\varphi} = \cos\varphi + \sin\varphi \cdot i}$ $\quad |e^{i\varphi}| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$e^z = e^{a+bi} = e^a \cdot e^{bi} = e^a \cdot (\cos b + i \cdot \sin b)$

$\boxed{\cos t = \frac{e^{it} + e^{-it}}{2}}$ $\quad \sin t = \frac{e^{it} - e^{-it}}{2i}$

Quick Review of L5:

① Surface Integrals.

EXAMPLE 4 Evaluate $\iint_S \sqrt{x(1+2z)}\, d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region $R: x \geq 0, y \geq 0, x + y \leq 1$ in the xy-plane (Figure 15.51).

Region: $x \geq 0$, $y \geq 0$, $x + y \leq 1$

$\iint_S \sqrt{x(1+2z)} \cdot d\sigma$

$G(x,y,z) = \sqrt{x(1+2z)} = \sqrt{x(1+2\cdot\frac{1}{2}y^2)} = \sqrt{x(1+y^2)}$

$f(x,y) = \frac{1}{2}y^2 = z$ $\quad f_x = 0$, $\quad f_y = \frac{1}{2}\cdot 2y = y$

$d\sigma = \sqrt{f_x^2 + f_y^2 + 1} \cdot dx\,dy$
$= \sqrt{0^2 + y^2 + 1} \cdot dx \cdot dy$
$= \sqrt{y^2+1} \cdot dx\,dy$

$$\iint_S \sqrt{x(1+y^2)} \cdot d\sigma$$
$$= \iint_R \sqrt{x(1+y^2)} \cdot \sqrt{1+y^2} \cdot dx\,dy$$
$$= \iint_R \sqrt{x} \cdot (1+y^2)\, dx\,dy$$
$$= \int_0^1 \int_0^{1-y} \sqrt{x} \cdot (1+y^2) \cdot dx \cdot dy$$
$$= \int_0^1 \int_0^{1-x} \sqrt{x} \cdot (1+y^2) \cdot dy \cdot dx$$
$$= \frac{284}{945} \approx 0.30$$

$\begin{cases} x \geq 0 \\ y \geq 0 \\ x+y \leq 1 \end{cases}$

consider x at first:
$0 \leq x \leq 1-y$
then consider y:
$0 \leq y \leq 1$ (constant!)

consider y at first
$0 \leq y \leq 1-x$
then x:
$0 \leq x \leq 0$

Ex. $W: x,y,z \geq 0$, $x+y+z \leq 1$
Volume = ?
$$\text{Volume} = \iiint 1 \cdot dx\,dy\,dz = \int_{0\,(x)}^1 \int_{0\,(y)}^{1-x} \int_{0\,(z)}^{1-x-y} 1 \cdot dz\,dy\,dx$$

Quick Review of L6:

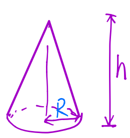
$\begin{cases} dx\,dy \to r \cdot dr \cdot d\theta \\ dx \cdot dy\,dz \to r^2 \sin\theta \cdot dr \cdot d\rho \cdot d\theta \end{cases}$

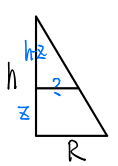
$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$

$\begin{cases} x = r\cdot\sin\theta\cdot\cos\varphi \\ y = r\cdot\sin\theta\cdot\sin\varphi \\ z = r\cdot\cos\theta \end{cases}$

Ex. Area of an unit circle D.
$$\iint_D 1 \cdot dx \cdot dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 1 \cdot r \cdot dr \cdot d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big|_0^1 \cdot d\theta$$
$$= \int_0^{2\pi} \frac{1}{2} \cdot d\theta = \frac{1}{2}\cdot\theta\Big|_0^{2\pi} = \frac{1}{2}\cdot 2\pi = \pi.$$

Ex.





$\frac{h}{R} = \frac{h-z}{?}$

$\Rightarrow ? = \frac{R(h-z)}{h}$
$= R - \frac{R}{h}z$

$$\iiint_{\text{cone}} 1 \cdot dV = \int_{z=0}^{h}\int_{\theta=0}^{2\pi} \boxed{\int_{r=0}^{R-\frac{R}{h}z} 1 \cdot r \cdot dr} \, d\theta \cdot dz$$
$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi} \frac{1}{2}r^2\Big|_0^{R-\frac{R}{h}z} \cdot d\theta \cdot dz$$
$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi} \frac{1}{2}\left(R-\frac{R}{h}z\right)^2 \cdot d\theta \cdot dz$$
$$= \int_{z=0}^{h} \cancel{2}\pi \cdot \frac{1}{\cancel{2}}\left(R-\frac{R}{h}z\right)^2 \cdot dz$$
$$= \pi R^2 \cdot \int_{z=0}^{h}\left(1-\frac{1}{h}z\right)^2 \cdot dz$$
$$= \pi R^2 \int_0^h \left(1 - \frac{2}{h}z + \frac{1}{h^2}z^2\right) dz$$
$$= \pi R^2 \left[z - \frac{1}{h}z^2 + \frac{1}{3h^2}z^3\right]_0^h$$
$$= \pi R^2 \left(h - \frac{1}{h}\cdot h^2 + \frac{1}{3h^2}\cdot h^3\right)$$
$$= \pi R^2 \left(h - h + \frac{1}{3}h\right) = \frac{1}{3}\pi h R^2.$$

Quick Review of L7: vector fields & vector integrals

① F is integrable/conservable $\Leftrightarrow$ $\frac{dF_i}{dx_j} = \frac{dF_j}{dx_i}$ $\forall\, 1 \leq i, j \leq n$

Ex: $F(x,y) = (x^2+xy, y^2)$. Is F integrable?

$\frac{dF_1}{dy} \overset{?}{=} \frac{dF_2}{dx}$

$\frac{dF_1}{dy} = x \quad \frac{dF_2}{dx} = 0 \quad \Rightarrow \frac{dF_1}{dy} \neq \frac{dF_2}{dx}$

$\therefore$ NO.

Ex: Is $F(x,y,z) = (x, 2z, y)$ conservable?

$\frac{dF_1}{dy} \overset{?}{=} \frac{dF_2}{dx} \quad 0 = 0 \quad ✓$

$\frac{dF_1}{dz} \overset{?}{=} \frac{dF_3}{dx} \quad 0 = 0 \quad ✓$

$\frac{dF_2}{dz} \overset{?}{=} \frac{dF_3}{dy} \quad 2 \neq 1 \quad$ No!

② Divergence of a vector field.

$$\text{div}(F) = \frac{dF_1}{dx_1} + \frac{dF_2}{dx_2} + \cdots + \frac{dF_n}{dx_n}$$

Motivation on divergence:

Div > 0 : more outflow than inflow.

Div < 0 : more inflow than outflow.

Ex. $F(x,y) = (x, y)$
$\text{div}(F) = \frac{dx}{dx} + \frac{dy}{dy} = 1 + 1 = 2$

Ex. $F(x,y) = (x^2, y^2)$
$\text{div}(F) = 2x + 2y = 2(x+y)$

③ Curl of a vector field.

in $\mathbb{R}^2$: $F=(F_1,F_2)$

$$\operatorname{curl}(F) = \frac{dF_2}{dx} - \frac{dF_1}{dy}$$

- curl of F at (a,b)
  - >0 : counter-clockwise rotation.
  - <0 : clockwise rotation.
  - $=0$: no rotation.

in $\mathbb{R}^3$: $F=(F_1,F_2,F_3)$

$$\operatorname{curl}(F) = \left(\frac{dF_3}{dy} - \frac{dF_2}{dz},\ \frac{dF_1}{dz} - \frac{dF_3}{dx},\ \frac{dF_2}{dx} - \frac{dF_1}{dy}\right)$$

Ex. $F(x,y) = (y^3-9y,\ x^3-9x)$

$\operatorname{curl}(F) = \frac{dF_2}{dx} - \frac{dF_1}{dy} = (3x^2-9) - (3y^2-9) = 3x^2-3y^2$

at $(1,0)$: $3(x^2-y^2) = 3 > 0$ → counter-clockwise rotation.

at $(3,0)$: $3(x^2-y^2) = 3\cdot 9 = 27 > 0$ → counter-clockwise rotation.

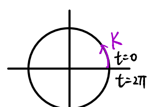
Quick Review of L8: Line integrals.

① $g: \mathbb{R}^n \to \mathbb{R}$ scalar field : $\int_a^b g(f(t))\cdot\|f'(t)\|\cdot dt$ = integrating g along K.

Ex. $f(t) = (\cos t, \sin t)$. $0 \le t \le 2\pi$.

$g(x,y) = x$

- Integrate g along K:



$g(f(t)) = g(\cos t, \sin t) = \cos t$

$$\int_K g\cdot ds = \int_{t=0}^{2\pi} \cos t\cdot dt = \sin t\Big|_{t=0}^{2\pi} = \sin(2\pi) - \sin(0) = 0$$

② Line integral of F over K:

$$\int_K F \bullet ds = \int_{t=a}^{b} F(f(t)) \bullet f'(t)\cdot dt.$$

Ex. $F(x,y) = (1,2)$ $\quad F(\sin t, \cos t) =$

Integrate along K

K = unit circle = $f(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

Calculate the line integrate of F over K:

$$\int_K F\bullet ds = \int_a^b F(f(t))\bullet f'(t)\cdot dt$$

$F(f(t)) = F(\cos t, \sin t) = (1,2)$

$f'(t) = (-\sin t, \cos t)$

$= \int_0^{2\pi} (1,2)\bullet(-\sin t, \cos t)\cdot dt$

$= \int_0^{2\pi} (-\sin t + 2\cos t)\cdot dt$

$= \left[\cos t + 3\sin t\right]_{t=0}^{2\pi}$

$= (\cos 2\pi + 3\sin 2\pi) - (\cos 0 + 3\sin 0)$

$= (1+0) - (1+0) = 0$

Ex. $F(x,y) = (-y,x)$; $f(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

$$\int_K F\bullet ds = \int_0^{2\pi} F(f(t))\cdot f'(t)\cdot dt$$

$F(f(t)) = F(\cos t, \sin t) = (-\sin t, \cos t)$

$f'(t) = (-\sin t, \cos t)$

$= \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\cdot dt$

$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\cdot dt = \int_0^{2\pi} 1\cdot dt = 2\pi.$

③ Line integral of a gradient field.

$$\int_K \nabla g \bullet ds = g(f(b)) - g(f(a))$$

where K: $f: [a,b] \to \mathbb{R}^n$

$g: \mathbb{R}^n \to \mathbb{R}$.

Quick Review of L9:

Ex. $f'' + 8f' + 20f = 0$.

1) Find all complex functions $f: \mathbb{C} \to \mathbb{C}$ satisfying $f'' + 8f' + 20f = 0$.

$x^2 + 8x + 20 = 0$

$\Rightarrow x = \frac{-8 \pm \sqrt{8^2 - 4\cdot 20}}{2} = -4 \pm \frac{1}{2}\sqrt{-16} = -4 \pm \frac{1}{2}\cdot 4i = -4 \pm 2i$

$\therefore$ all solutions $A\cdot e^{(-4-2i)t} + B\cdot e^{(-4+2i)t}$ where $A, B \in \mathbb{C}$

2) Find all real functions $f: \mathbb{R} \to \mathbb{R}$ satisfying $f'' + 8f' + 20f = 0$.

$A = A_0 + A_1 i \qquad A_0, A_1 \in \mathbb{R}$

$B = B_0 + B_1 i \qquad B_0, B_1 \in \mathbb{R}$

$(A_0 + A_1 i)\cdot e^{(-4-2i)t} + (B_0 + B_1 i)\cdot e^{(-4+2i)t}$

$= (A_0 + A_1 i)\cdot e^{-4t}\cdot e^{-2it} + (B_0 + B_1 i)\cdot e^{-4t}\cdot e^{2it}$

$= e^{-4t}\Big[(A_0 + A_1 i)\cdot(\cos 2t - \sin(2t)\cdot i)$

$\quad + (B_0 + B_1 i)\cdot(\cos 2t + \sin(2t)\cdot i)\Big]$

Recall: $e^{i\varphi} = \cos\varphi + \sin\varphi\cdot i$

$\to e^{i\cdot(-2t)} = \cos(-2t) + \sin(-2t)\cdot i$

$\to e^{i\cdot 2t} = \cos(2t) + \sin(2t)\cdot i$

$= e^{-4t}\Big[A_0\cdot\cos 2t - A_0\sin(2t) i + A_1\cdot\cos 2t\cdot i + A_1\cdot\sin(2t)\cdot\cancel{i}$

$\quad + B_0\cdot\cos 2t + B_0\cdot\sin(2t)\cdot i + B_1\cdot\cos 2t\cdot i - B_1\cdot\sin(2t)\cdot\cancel{i}$

$= e^{-4t}\left(A_0\cdot\cos 2t + A_1\cdot\sin(2t) + B_0\cdot\cos 2t - B_1\cdot\sin(2t)\right)$

$\quad + \underbrace{i\cdot e^{-4t}\left(-A_0\cdot\sin(2t) + A_1\cdot\cos 2t + B_0\cdot\sin(2t) + B_1\cdot\cos 2t\right)}_{=0}$

$\Rightarrow \begin{cases} -A_0 + B_0 = 0 \Rightarrow \boxed{B_0 = A_0} \\ A_1 + B_1 = 0 \Rightarrow \boxed{B_1 = -A_1} \end{cases}$

$\therefore$ The real parts would be:

$e^{-4t}(A_0\cdot\cos 2t + A_1\cdot\sin 2t + A_0\cdot\cos 2t + A_1\sin 2t)$

$= A_0\,(2e^{-4t}\cdot\cos 2t) + A_1\cdot(2\cdot e^{-4t}\cdot\sin 2t)$

Ex. $y' = K$

$$\frac{dy}{dx} = K \Rightarrow dy = K \cdot dx$$

$$\Rightarrow \int dy = \int K \cdot dx \Rightarrow y = Kx + C$$

Ex. $y' = Ky$

$$\frac{dy}{dx} = Ky \Rightarrow \frac{1}{y} \cdot dy = K \cdot dx$$

$$\Rightarrow \int \frac{1}{y} \cdot dy = \int K \cdot dx \Rightarrow \ln|y| = Kx + C$$

Quick Review of L10:

Ex. $f' - \frac{1}{t} f = t \cdot \cos t$

Try $f = A(t) \cdot t$

$f' = A'(t) \cdot t + A(t)$

$\Rightarrow A'(t) \cdot t + A(t) - A(t) = t \cdot \cos t$

$\Rightarrow A'(t) \cdot t = t \cdot \cos t$

$A'(t) = \cos t$

$\therefore A(t) = \sin t + C$

$\therefore f = (\sin t + c) \cdot t$

$\frac{df}{dt} - \frac{1}{t} f = 0$

$\frac{df}{dt} = \frac{1}{t} \cdot f$

$\frac{1}{f} df = \frac{1}{t} dt$

$\int \frac{1}{f} \cdot df = \int \frac{1}{t} \cdot dt$

$\ln|f| = \ln|t| + C$

$\ln|f| = \ln|t| + C$

$e^{\ln|f|} = e^{\ln|t| + c}$

$\Rightarrow |f| = e^{c} \cdot e^{\ln|t|}$

$\Rightarrow |f| = A \cdot |t|$

$\rightarrow$ Try $f = A(t) \cdot t$

Ex. $y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$

let $u = \frac{y}{x} \Rightarrow y = xu$. (only use x & u)

$y' = u + x \cdot u'$

$\Rightarrow u + x \cdot u' = u^2 + u$

$\Rightarrow x \cdot u' = u^2$

$\Rightarrow \frac{1}{u^2} \cdot u' = \frac{1}{x}$

$\Rightarrow \frac{1}{u^2} \cdot \frac{du}{dx} = \frac{1}{x}$

$\Rightarrow \frac{1}{u^2} \cdot du = \frac{1}{x} \cdot dx$

$\Rightarrow \int \frac{1}{u^2} \cdot du = \int \frac{1}{x} \cdot dx$

$\Rightarrow -\frac{1}{u} = \ln|x| + C$

$\Rightarrow u = -\frac{1}{\ln|x| + C}$

$\Rightarrow \frac{y}{x} = -\frac{1}{\ln|x| + C}$

$\Rightarrow y = -\frac{x}{\ln|x| + C}$

Ex. $y' = \frac{-(y + \cos x)}{3y^2 + x}$, $y(\pi) = 0$, $y(0) = ?$

S1: $\frac{-(y + \cos x)}{3y^2 + x} = \frac{-M}{N}$ $\quad M = y + \cos x$, $N = 3y^2 + x$

S2: $N_x = 1$, $M_y = 1$

$\varphi_y = N$, $\varphi_x = M$

S3: $\varphi = \int N \cdot dy = \int (3y^2 + x) \cdot dy = y^3 + xy + C(x)$

S4: $\begin{cases} M = \varphi_x = y + C'(x) \\ M = y + \cos x \end{cases} \Rightarrow C'(x) = \cos x$

$\Rightarrow C(x) = \sin x + C$

S5: $\therefore \varphi = y^3 + xy + \sin x + C = 0$

$\because y(\pi) = 0$

$\therefore 0 + \pi \cdot 0 + \sin \pi + C = 0$

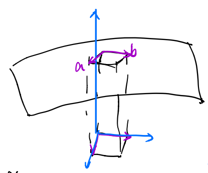
$\Rightarrow C = 0$

$\therefore \varphi = y^3 + xy + \sin x$

$y(0) = ?$

$\varphi = 0, x = 0 \Rightarrow 0 = y^3(0) \Rightarrow y(0) = 0.$

Surface Area:



$\vec{a} = (dx, 0, f_x \cdot dx) = dx(1, 0, f_x)$

$\vec{b} = (0, dy, f_y \cdot dy) = dy(0, 1, f_y)$

$\underbrace{\vec{a} \times \vec{b}}_{= ds} = \langle -f_x, -f_y, 1 \rangle \, dx \, dy$ cross product.

$\iint ds =$

$$\begin{matrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{matrix}$$