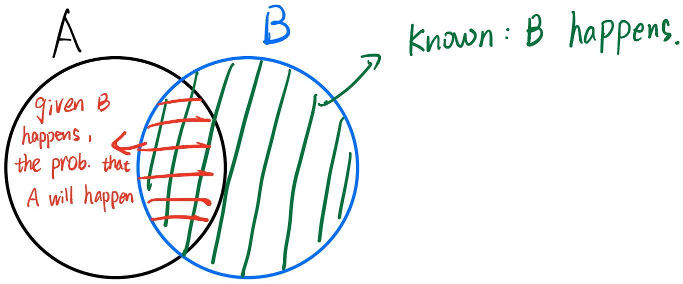
For Tutorial 2:

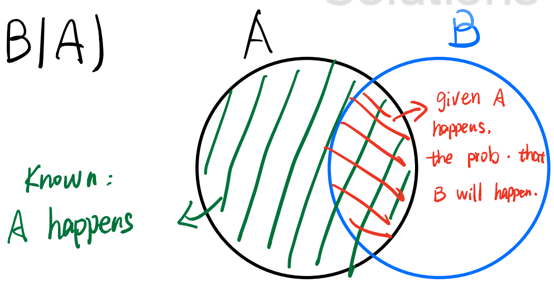
- $P(A|B)$ vs. $P(B|A)$

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

$P(A|B)$:



$P(B|A)$



Assume: Total: 100 $\quad |A| = 20 \quad |B| = 30 \quad |A \cap B| = 10$

$$P(A) = \frac{|A|}{\text{Total}} = \frac{20}{100}$$

$$P(B) = \frac{|B|}{\text{Total}} = \frac{30}{100}$$

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{10}{30}$$

$$P(B|A) = \frac{|A \cap B|}{|A|} = \frac{10}{20}$$

$$P(A \cap B) = \frac{|A \cap B|}{\text{Total}} = \frac{10}{100}$$

$$= P(A|B) \cdot P(B) = \frac{|A \cap B|}{\cancel{|B|}} \cdot \frac{\cancel{|B|}}{\text{Total}}$$

$$= P(B|A) \cdot P(A) = \frac{|A \cap B|}{\cancel{|A|}} \cdot \frac{\cancel{|A|}}{\text{Total}}$$

Overview of Totorial 4 :

- Expectation of $g(x)$:
  1

$$E(X) = \int_{-\infty}^{+\infty} x \cdot \underset{\uparrow \, (p.d.f)}{f(x)} \cdot dx$$

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) \cdot f(x) \cdot dx$$

- $f(x,y) \longrightarrow E(X)$:
  2, 3, 4, 5, 6, 7, 9

Marginal Density Function of X :

$$\int_{\text{all } y} f(x,y) \cdot dy = \sum_{\text{all } y} f(x,y)$$

- $E(XY) = \sum_{(x,y)} x \cdot y \cdot p(x,y)$ / $E(aX - bY) = aE(X) - bE(Y)$
  7, 8, 9

- $V(aX - bY + c) = a^2 \cdot V(X) + b^2 \cdot V(Y) - 2ab\,Cov(X,Y)$
  8, 9

- $Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$
  9, 10, 11

$$\begin{aligned}
&E[(X-\mu_X)(Y-\mu_Y)] \\
&= E[XY - X\mu_Y - \mu_X Y + \mu_X \cdot \mu_Y] \\
&= E(XY) - \mu_Y \cdot E(X) - \mu_X E(Y) + \mu_X \cdot \mu_Y \\
&= E(XY) - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y \\
&= E(XY) - E(X) \cdot E(Y)
\end{aligned}$$

- $\rho_{XY} = \dfrac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$
  12.

$Cov(X,Y) = \begin{cases} 0, & \text{No linear Correlation between } x \text{ and } Y. \\ >0, & \text{Overall, } X \uparrow \longrightarrow Y \uparrow \\ <0, & \text{Overall, } X \uparrow \longrightarrow Y \searrow \end{cases}$

$\rho_{XY} = \begin{cases} 0, & \text{No linear Correlation between } x \text{ and } Y. \\ & \text{If } |\rho_{XY}| \text{ is closer to 1, then } x \text{ and } Y \text{ are closer to a linear relationship.} \end{cases}$

1. A continuous random variable X has the density function

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere}. \end{cases}$$

Find the expected value of $g(X) = e^{2X/3}$.

Key : $E(X) = \int_{-\infty}^{+\infty} x \cdot f(x) \cdot dx$ Expectation of the r.v. x.

$E[g(x)] = \int_{-\infty}^{+\infty} g(x) \cdot f(x) \cdot dx$ Expectation of the function g of r.v. x.

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) \cdot f(x) \cdot dx = \int_0^{+\infty} e^{\frac{2}{3}x} \cdot e^{-x} \cdot dx = \int_0^{+\infty} e^{-\frac{1}{3}x} \cdot dx$$

$$= \left[-3e^{-\frac{1}{3}x}\right]_0^{+\infty} = 0 - \left(-3e^0\right) = 3$$

$$\boxed{\begin{array}{l} x \to +\infty \\ -\frac{1}{3}x \to -\infty \\ e^{-\frac{1}{3}x} \to 0 \end{array}}$$

$$\boxed{\begin{array}{l} \left[e^{-\frac{1}{3}x}\right]' \\ = e^{-\frac{1}{3}x} \cdot \left(-\frac{1}{3}\right) \\ \Rightarrow \left[-3e^{-\frac{1}{3}x}\right]' = e^{-\frac{1}{3}x} \end{array}}$$

2. A fast food restaurant operates both a drive-through facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-through and walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(X)$.

Marginal Density Function g of x :

$$g(x) = \int_0^1 \frac{2}{3}(x+2y) \cdot dy = \frac{2}{3}x(1-0) + \int_0^1 \frac{4}{3}y \cdot dy$$

$$= \frac{2}{3}x + \left[\frac{2}{3}y^2\right]_0^1 = \frac{2}{3}x + \left(\frac{2}{3} - 0\right) = \frac{2}{3}(x+1), \quad 0 \le x \le 1$$

$$E(X) = \int_{-\infty}^{+\infty} x \cdot g(x) \cdot dx = \int_0^1 \frac{2}{3}x(x+1) \cdot dx = \frac{2}{3}\int_0^1 (x^2 + x) \cdot dx$$

$$= \frac{2}{3}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1 = \frac{2}{3} \times \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{9}$$

3. If the joint probability distribution of X and Y is given by

$$f(x,y) = \frac{x+y}{30}, \qquad \text{for x} = 0,\ 1,\ 2,\ 3;\ \text{y} = 0,\ 1,\ 2,$$

find $E(X)$.

Marginal Density Function g of x:

$$g(0) = f(0,0) + f(0,1) + f(0,2) = \frac{0}{30} + \frac{1}{30} + \frac{2}{30} = \frac{3}{30} = \frac{1}{10}$$

Similarly,

$$g(1) = f(1,0) + f(1,1) + f(1,2) = \frac{1}{30} + \frac{2}{30} + \frac{3}{30} = \frac{6}{30} = \frac{1}{5}$$

$$g(2) = f(2,0) + f(2,1) + f(2,2) = \frac{2}{30} + \frac{3}{30} + \frac{4}{30} = \frac{9}{30} = \frac{3}{10}$$

$$g(3) = f(3,0) + f(3,1) + f(3,2) = \frac{3}{30} + \frac{4}{30} + \frac{5}{30} = \frac{12}{30} = \frac{2}{5}$$

$$E(X) = \sum_{x=0}^{3} x \cdot g(x) = 0 \times \frac{1}{10} + 1 \times \frac{1}{5} + 2 \times \frac{3}{10} + 3 \times \frac{2}{5} = \frac{1}{5} + \frac{3}{5} + \frac{6}{5} = 2$$

4. Each rear tire on an experimental airplane is supposed to be filled to a pressure of 40 pounds per square inch (psi). Let X denote the actual air pressure of the right tire and Y denote the actual air pressure of the left tire. Suppose that X and Y are random variables with the joint density function

$$f(x,y) = \begin{cases} \frac{3}{3920000}(x^2+y^2) & 30 \le x \le 50, 30 \le y \le 50 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(X)$.

Marginal Density Function g of x:

$$g(x) = \int_{30}^{50} f(x,y) \cdot dy = \frac{3}{392 \times 10^4} x^2 \times (50-30) + \frac{3}{392 \times 10^4} \int_{30}^{50} y^2 \cdot dy$$

$$= \frac{3}{196 \times 10^3} x^2 + \frac{3}{392 \times 10^4} \times \left[\frac{1}{3} y^3\right]_{30}^{50}$$

$$= \frac{3}{196 \times 10^3} x^2 + \frac{\cancel{3}}{\underset{4}{\cancel{392}} \times \underset{10}{\cancel{10^4}}} \times \cancel{\frac{1}{3}} \times \overset{=\cancel{98000}}{(50^3 - 30^3)} = \frac{3}{196 \times 10^3} x^2 + \frac{1}{40}$$

$$E(X) = \int_{30}^{50} x \cdot g(x) \cdot dx = \int_{30}^{50} \frac{3}{196 \times 10^3} x^3 + \frac{1}{40} x \, dx = \left[\frac{3}{196 \times 10^3} \times \frac{1}{4} x^4 + \frac{1}{80} x^2\right]_{30}^{50}$$

$$= 40.816$$

5. Let X, Y and Z have the joint probability density function

$$f(x,y,z) = \begin{cases} 3xy^2z & 0<x,y<1, 0<z<2, \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(X)$, $E(Y)$ and $E(Z)$.

Marginal Density Function g of x:

$$g(x) = \begin{cases} 0, \text{ elsewhere} \\ \int_0^2 \int_0^1 f(x,y,z) \cdot dy \cdot dz \end{cases}$$

$$\boxed{g(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}}$$

$$= x \cdot \int_0^2 z \cdot \underbrace{\int_0^1 3y^2 \cdot dy}_{[y^3]_0^1 = 1} \cdot dz = x \cdot \int_0^2 z \, dz = x \cdot \left[\tfrac{1}{2} z^2\right]_0^2 = 2x, \; 0<x<1$$

$$E(X) = \int_0^1 x \cdot g(x) \cdot dx = \int_0^1 x \cdot 2x \cdot dx = \left[\tfrac{2}{3} x^3\right]_0^1 = \tfrac{2}{3}$$

Similarly,

$$g(y) = \begin{cases} 0, \text{ elsewhere} \\ \int_0^2 \int_0^1 3xy^2 z \cdot dx \cdot dz = 3y^2 \int_0^2 z \underbrace{\int_0^1 x \cdot dx}_{\left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}} \cdot dz \end{cases}$$

$$\boxed{g(y) = \begin{cases} 3y^2, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}}$$

$$= 3y^2 \int_0^2 \tfrac{1}{2} z \cdot dz = 3y^2 \left[\tfrac{1}{4} z^2\right]_0^2 = 3y^2 \times 1 = 3y^2, \; 0<y<1$$

$$E(Y) = \int_0^1 y \cdot g(y) \cdot dy = \int_0^1 \underbrace{y \cdot 3y^2}_{3y^3} \cdot dy = \left[\tfrac{3}{4} y^4\right]_0^1 = \tfrac{3}{4}$$

$$\boxed{\left[\tfrac{3}{4} y^4\right]' = 3y^3}$$

$$g(z) = \begin{cases} 0, \text{ elsewhere} \\ \int_0^1 \int_0^1 3xy^2 z \cdot dx \cdot dy = z \int_0^1 3y^2 \cdot \underbrace{\int_0^1 x \cdot dx}_{\left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}} \cdot dy = \frac{1}{2} z \int_0^1 3y^2 \cdot dy \end{cases}$$

$$= \tfrac{1}{2} z \cdot [y^3]_0^1 = \tfrac{1}{2} z, \; 0<z<2$$

$$\boxed{g(z) = \begin{cases} \frac{1}{2} z, & 0<z<2 \\ 0, & \text{elsewhere} \end{cases}}$$

$$E(Z) = \int_0^2 z \cdot g(z) \cdot dz = \int_0^2 z \cdot \tfrac{1}{2} z \cdot dz = \tfrac{1}{2} \left[\tfrac{1}{3} z^3\right]_0^2 = \tfrac{1}{6} \times 8 = \tfrac{4}{3}$$

6. From a sack of fruit containing 3 oranges, 2 apples, and 3 bananas, a random sample of 4 pieces of fruit is selected. If X is the number of oranges and Y is the number of apples in the sample, find $E(X)$, $E(Y)$, $E(XY)$ and $Cov(X,Y)$.

$\#O: 3 \quad \#A: 2 \quad \#B: 3 \longrightarrow$ Choose 4 $\begin{cases} \#O: x \leq 3 \\ \#A: Y \leq 2 \\ \#B: 4-(x+y) \leq 3 \end{cases}$

X	Y	P(X,Y)
3	1	$\binom{3}{3}\binom{2}{1}\binom{3}{0} / \binom{8}{4} = \frac{2}{70}$
3	0	$\frac{3}{70}$
2	2	$\frac{3}{70}$
2	1	$\frac{18}{70}$
2	0	$9/70$
1	2	$9/70$
1	1	$18/70$
1	0	$3/70$
0	2	$3/70$
0	1	$2/70$

O, A, B

$\longrightarrow \binom{3}{x}\binom{2}{y}\binom{3}{4-x-y} / \binom{8}{4}$

$$P(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{70}$$

X	3	2	1	0
P(X)	5/70	30/70	30/70	5/70

Y	2	1	0
P(Y)	15/70	40/70	15/70

$$E(X) = \sum_{x=0}^{3} x \cdot P(X) = 0 \times \frac{5}{70} + 1 \times \frac{30}{70} + 2 \times \frac{30}{70} + 3 \times \frac{5}{70}$$
$$= 105/70 = \frac{3}{2}$$

$$E(Y) = \sum_{y=0}^{2} y \cdot P(y) = 0 \times \frac{15}{70} + 1 \times \frac{40}{70} + 2 \times \frac{15}{70} = 1$$

$$E(XY) = \sum_{y=0}^{2} \sum_{x=0}^{3} x \cdot y \cdot P(X,Y) = \frac{1}{70}\left(3 \times 2 + 4 \times 3 + 2 \times 18 + 2 \times 9 + 1 \times 18\right) = \frac{9}{7}$$

$$Cov(X,Y) = E\{(X-\mu_x)(Y-\mu_y)\} = E\left[\left(X-\tfrac{3}{2}\right)(Y-1)\right]$$
$$= E\left(XY - X - \tfrac{3}{2}Y + \tfrac{3}{2}\right) = E(XY) - E(X) - \tfrac{3}{2}E(Y) + \tfrac{3}{2}$$
$$= \frac{9}{7} - \frac{3}{2} - \frac{3}{2} \times 1 + \frac{3}{2} = -\frac{3}{14}$$

$\longrightarrow Cov(X,Y) \begin{cases} = 0 & \text{No correlation between X and Y.} \\ > 0 \\ < 0 \end{cases}$

7. Suppose that X and Y have the following joint probability function:

			y	
	$f(x,y)$	1	3	5
x	2	0.10	0.20	0.10
	4	0.15	0.30	0.15

Find

(a) $E(2X-3Y)$

(b) $E(XY)$

(a) $E(2X-3Y) = 2E(X) - 3E(Y)$

$$P(X) = \begin{cases} 0.1+0.2+0.1 = 0.4, & x=2 \\ 0.15+0.30+0.15 = 0.6, & X=4 \end{cases}$$

$$P(Y) = \begin{cases} 0.1+0.15 = 0.25, & y=1 \\ 0.2+0.3 = 0.5, & y=3 \\ 0.1+0.15 = 0.25, & y=5 \end{cases}$$

$E(X) = \sum X \cdot P(X) = 2\times 0.4 + 4\times 0.6 = 0.8+2.4 = 3.2$

$E(Y) = \sum y \cdot P(Y) = 1\times 0.25 + 3\times 0.5 + 5\times 0.25 = 3$

$E(2X-3Y) = 2E(X) - 3E(Y) = 2\times 3.2 - 3\times 3 = -2.6$

(b) $E(XY) = \sum_{(x,y)} X \cdot y \cdot P(X,y)$

$= 2\times 0.1 + 6\times 0.2 + 10\times 0.1 + 4\times 0.15 + 12\times 0.3 + 20\times 0.15$

$= 9.6$

$E(X)\cdot E(Y) = E(XY) \longleftrightarrow X$ and Y are independent

8. A green and a red die are rolled. Let X be the outcome of the green die and let Y be the sum of the outcomes. Find

(a) $E(X+Y)$
(b) $E(X-Y)$
(c) $E(XY)$
(d) $V(2X-Y)$
(e) $V(X+3Y-5)$

X : outcome of green die Y : sum of the outcomes.

(a) $E(X+Y) = E(X)+E(Y)$

$E(X) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$

Z : outcome of red die ($Y = X+Z$)

$E(Y) = E(X+Z) = E(X)+E(Z) = 2\times E(X) = 7$

$E(X+Y) = E(X)+E(Y) = 10.5$

(b) $E(X-Y) = E(X)-E(Y) = E(X)-2E(X) = -E(X) = -\frac{7}{2}$

(c) The outcomes of green and red die are independent →

$E(X\cdot Z) = E(X)\cdot E(Z) = [E(X)]^2$

$E(XY) = E[X(X+Z)] = E(X^2+XZ) = E(X^2)+E(XZ)$

$= E(X^2)+[E(X)]^2 = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)+\left(\frac{7}{2}\right)^2$

$= \frac{91}{6}+\frac{49}{4} = 15\frac{1}{6}+12\frac{1}{4} = 27\frac{5}{12}$

(d) $V(2X-Y) = V(2X-X-Z) = V(X-Z) \xlongequal[\text{independent}]{X \text{ and } Z \text{ are}} V(X)+V(Z)$

$= 2\cdot V(X) = 2\times 2\frac{11}{12} = 5\frac{5}{6}$

$V(X) = E(X^2)-\mu_X^2 = \frac{91}{6}-\frac{49}{4} = 2\frac{11}{12}$

Way2: $V(2X-Y) = 4V(X)+V(Y)-4Cov(X,Y)$

$V(Y) = V(X+Z) = V(X)+V(Z) = 2V(X)$

$$Cov(X,Y) = E(X,Y) - E(X)\cdot E(Y) = 27\frac{5}{12} - \frac{7}{2}\times 7 = 2\frac{11}{12}$$

$$\Rightarrow V(2X-Y) = 6V(X) - 4Cov(X,Y)$$

$$= 6\times 2\frac{11}{12} - 4\times 2\frac{11}{12} = 2\times 2\frac{11}{12} = 5\frac{5}{6}$$

(e) $V(X+3Y-5) = V(X+3Y) = V(X+3X+3Z) = V(4X+3Z)$

$$=16V(X) + 9V(Z) = 25\cdot V(X) = 25\times 2\frac{11}{12} = 72\frac{11}{12}$$

Way 2: $V(X+3Y-5) = V(X+3Y)$

$$= V(X) + 9V(Y) + 6Cov(X,Y)$$

$$= 19\cdot V(X) + 6\times 2\frac{11}{12}$$

$$= 19\times 2\frac{11}{12} + 6\times 2\frac{11}{12} = 25\times 2\frac{11}{12} = 72\frac{11}{12}$$

9. The joint probability density function of random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0\le x\le 1, \quad 0\le y\le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $E(X+Y)$ and $E(XY)$

(b) Find $V(X)$, $V(Y)$, $\text{Cov}(X,Y)$

(c) Find $V(X+Y)$

Marginal Density Function g of x:

$$g(x) = \begin{cases} 0, \text{ elsewhere} \\ \int_0^1 \frac{3}{2}(x^2+y^2)\cdot dy = \frac{3}{2}x^2(1-0) + \int_0^1 \frac{3}{2}y^2\cdot dy = \frac{3}{2}x^2 + \left[\frac{1}{2}y^3\right]_0^1 \\ = \frac{3}{2}x^2 + \frac{1}{2}, \quad 0\le x\le 1 \end{cases}$$

$$E(X) = \int_0^1 x\cdot g(x)\cdot dx = \int_0^1 x\cdot\left(\frac{3}{2}x^2 + \frac{1}{2}\right)\cdot dx = \left[\frac{3}{8}x^4 + \frac{1}{4}x^2\right]_0^1 = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$$

Similarly, $g(y) = \begin{cases} 0, \text{ elsewhere} \\ \int_0^1 \frac{3}{2}(x^2+y^2)\cdot dx = \frac{3}{2}y^2 + \frac{1}{2}, \quad 0\le y\le 1 \end{cases}$

$$E(Y) = \frac{5}{8}$$

(a) $E(X+Y) = E(X) + E(Y) = \frac{10}{8} = \frac{5}{4}$

$$E(XY) = \int_0^1 \int_0^1 xy \cdot f(x,y) \cdot dx \cdot dy = \int_0^1 \int_0^1 xy \cdot \frac{3}{2}(x^2+y^2) \cdot dx \cdot dy$$

$$= \frac{3}{2} \int_0^1 \int_0^1 x^3 y + x y^3 \cdot dx \cdot dy = \frac{3}{2} \int_0^1 \left[\frac{y}{4} x^4 + \frac{y^3}{2} x^2 \right]_{x=0}^1 \cdot dy$$

$$= \frac{3}{2} \int_0^1 \left(\frac{y}{4} + \frac{y^3}{2} \right) \cdot dy = \frac{3}{2} \cdot \left[\frac{1}{8} y^2 + \frac{1}{8} y^4 \right]_0^1 = \frac{3}{2} \left(\frac{1}{8} + \frac{1}{8} \right) = \frac{3}{8}$$

(b) $V(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_0^1 x^2 \cdot g(x) \cdot dx = \int_0^1 x^2 \left(\frac{3}{2} x^2 + \frac{1}{2} \right) \cdot dx = \left[\frac{3}{10} x^5 + \frac{1}{6} x^3 \right]_{x=0}^1$$

$$= \frac{3}{10} + \frac{1}{6} = \frac{14}{30} = \frac{7}{15}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{7}{15} - \left(\frac{5}{8} \right)^2 = \frac{7}{15} - \frac{25}{64} = \frac{73}{960}$$

also, $V(Y) = E(Y^2) - [E(Y)]^2 = \frac{73}{960}$

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y) = \frac{3}{8} - \frac{5}{8} \times \frac{5}{8} = -\frac{1}{64}$$

(c) $V(X+Y) = V(X) + V(Y) + 2 \cdot Cov(X,Y)$

$$= \frac{73}{960} \times 2 + 2 \times \left(-\frac{1}{64} \right) = \frac{73}{480} - \frac{1}{32} = \frac{73-15}{480}$$

$$= \frac{58}{480} = \frac{29}{240}$$

10. Find the covariance of random variables X and Y, if their joint probability distribution is given as

	$f(x,y)$	x = 1	2	3	P(Y)
y	1	0.05	0.05	0.1	0.2
	2	0.05	0.1	0.35	0.5
	3	0	0.2	0.1	0.3
	P(X)	0.1	0.35	0.55	1

Cov(X, Y) ?

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$

x	1	2	3
P(X)	0.1	0.35	0.55
y	1	2	3
P(Y)	0.2	0.5	0.3

$$E(X) = \sum_{x=1}^{3} x\cdot P(x) = 1\times 0.1 + 2\times 0.35 + 3\times 0.55 = 2.45$$

$$E(Y) = \sum_{y=1}^{3} y\cdot P(Y) = 1\times 0.2 + 2\times 0.5 + 3\times 0.3 = 2.1$$

$$E(XY) = \sum_{(x,y)} x\cdot y\cdot P(x,y) = 1\times 0.05 + 2\times 0.05 + 3\times 0.1 + 2\times 0.05$$
$$+ 4\times 0.1 + 6\times 0.35 + 3\times 0 + 6\times 0.2 + 9\times 0.1 = 5.15$$

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$
$$= 5.15 - 2.45\times 2.1 = 0.005$$

11. Let X denote the diameter of an armored electric cable and Y denote the diameter of the ceramic mold that makes the cable. Both X and Y are scaled so that they range between 0 and 1. Suppose that X and Y have the joint density

$$f(x,y) = \begin{cases} \frac{1}{y} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find Cov(X,Y)

(b) Find $P(X+Y > \frac{1}{2})$

(a) $$g(x) = \begin{cases} 0, & \text{elsewhere} \\ \int_x^1 \frac{1}{y}\cdot dy = [\ln y]_x^1 = -\ln x, & 0<X<1 \end{cases}$$

$$E(X) = \int_0^1 x\cdot g(x)\cdot dx = \int_0^1 x\cdot(-\ln x)\cdot dx = \int_0^1(-x\cdot\ln x - \tfrac{1}{2}x)\cdot dx + \int_0^1 \tfrac{1}{2}x\cdot dx$$
$$= \left[-\tfrac{1}{2}x^2\ln x\right]_{x=0}^1 + \left[\tfrac{1}{4}x^2\right]_0^1 = \frac{1}{4}$$

$$\boxed{\left[\tfrac{1}{2}x^2\cdot\ln x\right]' = x\cdot\ln x + \tfrac{1}{2}x}$$

$$g(y) = \begin{cases} 0, & \text{elsewhere} \\ \int_0^y \frac{1}{y} \cdot dx = \frac{1}{y} \int_0^y dx = \frac{1}{y}(y-0) = 1, & 0<y<1 \end{cases}$$

$$E(Y) = \int_0^1 y \cdot g(y) \cdot dy = \int_0^1 y \cdot 1 \cdot dy = \left[\tfrac{1}{2}y^2\right]_{y=0}^{1} = \tfrac{1}{2}$$

$$E(XY) = \int_0^1 \int_0^y xy \cdot f(x,y) \cdot dx \cdot dy$$

$$= \int_0^1 \underbrace{\int_0^y x \cdot dx}_{\left[\frac{1}{2}x^2\right]_{x=0}^{y} = \frac{1}{2}y^2} \cdot dy = \int_0^1 \tfrac{1}{2}y^2 \cdot dy = \left[\tfrac{1}{6}y^3\right]_{y=0}^{1} = \tfrac{1}{6}$$

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$$= \tfrac{1}{6} - \tfrac{1}{4} \times \tfrac{1}{2} = \tfrac{1}{24}$$

(b) $P(X+Y > \frac{1}{2})$

$$= P(X \geq \tfrac{1}{4}) + P(X<\tfrac{1}{4}, Y > \tfrac{1}{2}-X)$$

$$= \int_{\frac{1}{4}}^{1} g(x) \cdot dx + \int_0^{\frac{1}{4}} \int_{\frac{1}{2}-x}^{1} f(x,y) \cdot dy \cdot dx$$

$$\boxed{\begin{aligned} &[x \cdot \ln x - x]' \\ &= \ln x + x \cdot \tfrac{1}{x} - 1 \\ &= \ln x \end{aligned}}$$

$$= \int_{\frac{1}{4}}^{1} -\ln x \, dx + \int_0^{\frac{1}{4}} \underbrace{\left[\ln y\right]_{\frac{1}{2}-x}^{1}}_{0 - \ln(\frac{1}{2}-x)} \cdot dx$$

$$\boxed{\begin{aligned} &\left[(\tfrac{1}{2}-x) - (\tfrac{1}{2}-x) \cdot \ln(\tfrac{1}{2}-x)\right]' \\ &= -1 - \left[-1 \times \ln(\tfrac{1}{2}-x) + (\tfrac{1}{2}-x) \times \tfrac{1}{\frac{1}{2}-x} \cdot (-1)\right] \\ &= -1 + \ln(\tfrac{1}{2}-x) + 1 = \ln(\tfrac{1}{2}-x) \end{aligned}}$$

$$= \int_{\frac{1}{4}}^{1} -\ln x \cdot dx - \int_0^{\frac{1}{4}} \ln(\tfrac{1}{2}-x) \cdot dx$$

$$= \left[x - x \cdot \ln x\right]_{\frac{1}{4}}^{1} - \left[(\tfrac{1}{2}-x) - (\tfrac{1}{2}-x) \cdot \ln(\tfrac{1}{2}-x)\right]_0^{\frac{1}{4}}$$

$$= (1-0) - (\tfrac{1}{4} - \tfrac{1}{4}\ln\tfrac{1}{4}) - \left[(\tfrac{1}{4} - \tfrac{1}{4}\ln\tfrac{1}{4}) - (\tfrac{1}{2} - \tfrac{1}{2}\ln(\tfrac{1}{2}))\right]$$

$$= 1 - (\cancel{\tfrac{1}{2}} - \tfrac{1}{2}\ln\tfrac{1}{4}) + (\cancel{\tfrac{1}{2}} - \tfrac{1}{2}\ln\tfrac{1}{2})$$

$$= 1 + \ln\tfrac{1}{2} - \tfrac{1}{2}\ln\tfrac{1}{2} = 1 + \tfrac{1}{2}\ln\tfrac{1}{2} \approx 0.6534$$

$$OR = P\left(X \geq \tfrac{1}{4}\right) - \int_{\frac{1}{2}-0}^{\frac{1}{2}-\frac{1}{4}} -\ln t \cdot dt \leftarrow \int_0^{\frac{1}{4}} \ln\left(\tfrac{1}{2}-x\right) \cdot dx$$

Let $t = \frac{1}{2} - x$ $\Rightarrow x = \frac{1}{2} - t$ $\Rightarrow dx = -dt$

$$= 1 - P\left(X < \tfrac{1}{4}\right) + \int_{\frac{1}{2}}^{\frac{1}{4}} \ln x \cdot dx$$

$$= 1 - \int_0^{\frac{1}{4}} -\ln x \cdot dx - \int_{\frac{1}{4}}^{\frac{1}{2}} \ln x \cdot dx$$

$$= 1 + \int_0^{\frac{1}{4}} \ln x \cdot dx - \int_{\frac{1}{4}}^{\frac{1}{2}} \ln x \cdot dx$$

$$= 1 + \left[x \cdot \ln x - x\right]_0^{\frac{1}{4}} - \left[x \cdot \ln x - x\right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= 1 + \left(\tfrac{1}{4} \cdot \ln \tfrac{1}{4} - \tfrac{1}{4}\right) - \left[\left(\tfrac{1}{2} \cdot \ln \tfrac{1}{2} - \tfrac{1}{2}\right) - \left(\tfrac{1}{4} \cdot \ln \tfrac{1}{4} - \tfrac{1}{4}\right)\right]$$

$$= 1 + \tfrac{1}{2} \ln \tfrac{1}{4} - \cancel{\tfrac{1}{2}} - \tfrac{1}{2} \ln \tfrac{1}{2} + \cancel{\tfrac{1}{2}}$$

$$= 1 + \tfrac{1}{2}\left(2 \cdot \ln \tfrac{1}{2} - \ln \tfrac{1}{2}\right) = 1 + \tfrac{1}{2} \ln \tfrac{1}{2} \approx 0.6534$$

12. Given a random variable X, with standard deviation σ_X, and a random variable $Y = a + bX$, show that if $b < 0$, the correlation coefficient $\rho_{XY} = -1$, and if $b > 0$, $\rho_{XY} = 1$.

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$Cov(X,Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right]$$

$$= \sigma_{XY} = E(XY) - E(X) \cdot E(Y)$$

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$$= E\left[X(a+bX)\right] - E(X) \cdot E(a+bX)$$

$$= E(aX + bX^2) - E(X) \cdot \left[a + b\,E(X)\right]$$

$$= a\,E(X) + b\,E(X^2) - a\,E(X) - b\left[E(X)\right]^2$$

$$= b\left(E(X^2) - \mu_X^2\right) = b\,\sigma_X^2$$

$$V(Y) = V(a+bX)$$

$$= V(bX)$$

$$= b^2\,V(X)$$

$$\Rightarrow \sigma_Y = |b|\,\sigma_X$$

If $b < 0$, then $\rho_{XY} = \dfrac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \dfrac{b\,\sigma_X^2}{\sigma_X \cdot (-b)\,\sigma_X} = -1$

If $b > 0$, then $\rho_{XY} = \dfrac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \dfrac{b \cdot \sigma_X^2}{\sigma_X \cdot b\,\sigma_X} = 1$