C4.4: 12, 40, 73, 107, 116
C4.5: 5, 38, 79
C4.6: 59

C4.4.

Graphing Functions

In Exercises 9–70, graph the function using appropriate methods from the graphing procedures presented just before Example 9, identifying the coordinates of any local extreme points and inflection points. Then find coordinates of absolute extreme points, if any.

9. $y = x^2 - 4x + 3$ **10.** $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$ **12.** $y = x(6 - 2x)^2$

Recall:

EXAMPLE 9 Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2} \qquad x\text{-intercept at } x = -1,\ y\text{-intercept at } y = 1$$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2} \qquad \text{Critical points: } x = -1, x = 1$$

$$f''(x) = \frac{(1+x^2)^2\cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4} = \frac{4x(x^2-3)}{(1+x^2)^3} \qquad \text{After some algebra, including canceling the common factor } (1+x^2)$$

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.
4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.
6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$
$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. \qquad \text{Dividing by } x^2$$

We see that $f(x) \to 1$ as $x \to \infty$ and that $f(x) \to 1$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote. Since the function is continuous on $(-\infty, \infty)$, there are no vertical asymptotes.

7. The graph of f is sketched in Figure 4.32. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

1. Domain of f: $(-\infty, +\infty)$

2. $y' = (6-2x)^2 + x\cdot 2(6-2x)(-2)$
$= (6-2x)(6-2x-4x)$
$= 2(3-x)\cdot 6(1-x)$
$= 12(x-1)(x-3)$

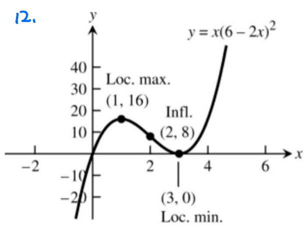
$y'' = 12[1\times(x-3) + (x-1)\times 1]$
$= 12(2x-4)$
$= 24(x-2)$

$\therefore$ y is concave up on $(2, +\infty)$ and concave down on $(-\infty, 2)$.
There is a inflection ponit at $x = 2$
$(2, 8)$

$\therefore$ y rises on $(-\infty, 1) \cup (3, +\infty)$ and falls on $(1, 3)$.

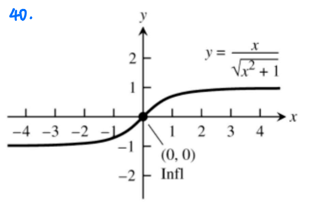
$\therefore$ Local maximum is at $x = 1$: $(1, 16)$
Local minimum is at $x = 3$: $(3, 0)$

12.



40. $y = \frac{x}{\sqrt{x^2+1}}$

40.



1. Domain: $x \in (-\infty, +\infty)$

2. $y' = \frac{1\cdot(x^2+1)^{\frac{1}{2}} - x\cdot\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x}{x^2+1}$
$= \frac{(x^2+1)^{\frac{1}{2}} - x^2(x^2+1)^{-\frac{1}{2}}}{x^2+1} = \frac{\sqrt{x^2+1} - x^2\cdot\frac{1}{\sqrt{x^2+1}}}{x^2+1} \quad \times\sqrt{x^2+1} \quad \times\sqrt{x^2+1}$
$= \frac{x^2+1-x^2}{(x^2+1)^{\frac{3}{2}}} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$ or $(x^2+1)^{-\frac{3}{2}} > 0$

$\therefore$ y increases on $(-\infty, +\infty)$

$y'' = -\frac{3}{2}\cdot(x^2+1)^{-\frac{5}{2}}\cdot 2x = -\frac{3x}{(x^2+1)^{5/2}}$

$\therefore$ y is concave up on $(-\infty, 0)$ and concave down on $(0, +\infty)$

There is a inflation point at $x = 0$. $(0, 0)$

$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{\frac{x^2+1}{x^2}}} = \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$ ← boundary line

$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{\frac{x^2+1}{x^2}}} = \lim_{x\to-\infty} -\frac{1}{\sqrt{1+\frac{1}{x^2}}} = -1$

Sketching the General Shape, Knowing y'

Each of Exercises 71–92 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure described in this section to sketch the general shape of the graph of f.

71. $y' = 2 + x - x^2$ **72.** $y' = x^2 - x - 6$

73. $y' = x(x-3)^2$ **74.** $y' = x^2(2-x)$

1) Find y''
2) Sketch the general shape of the graph of f.

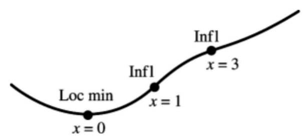
1) $y' = x(x-3)^2$

$y'' = (x-3)^2 + x\cdot 2(x-3)\cdot 1 = (x-3)(x-3+2x) = 3(x-3)(x-1)$

$y' > 0$
$\Rightarrow x > 0$
$\therefore$ y rises on $(0, +\infty)$ and falls on $(-\infty, 0)$.

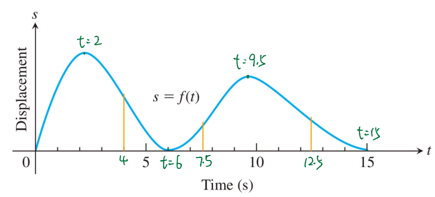
$y'' > 0$
$\Rightarrow 3(x-3)(x-1) > 0$
$\Rightarrow x \in (-\infty, 1) \cup (3, +\infty)$
$\therefore$ y is concave up on $(-\infty, 1)$ and $(3, +\infty)$ and concave down on $(1, 3)$.

$\therefore$ Inflection Point at $x = 1$ and $x = 3$.



Motion Along a Line The graphs in Exercises 107 and 108 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? Toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** Acceleration equal to zero? **(d)** When is the acceleration positive? Negative?

107.



(a) Tip: The object is moving away when s is increasing as t increases.
The object is moving away the origin when $t \in (0, 2) \cup (6, 9.5)$.
The object is moving toward the origin when $t \in (2, 6) \cup (9.5, 15)$

(b) Tip: Velocity equals to zero when the slope of the tangent line for $y = s(t)$ is horizontal.
When t is approximately at $t = 2, 6, 9.5$.

(c) Tip: The acceleration will be zero when the curve $y = s(t)$ reaches its inflection.
When t is approximately 4, 7.5, 12.5

(d) Tip: The acceleration is positive when it's concave up; the acceleration is negative when it's concave down.
Acceleration is positive: $t \in (4, 7.5) \cup (12.5, 15)$
Acceleration is negative: $t \in (0, 4) \cup (7.5, 12.5)$.

116. Parabolas

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c,\ a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

(a) y will reach vertex at $x = -\frac{b}{2a}$

$\therefore\ y = a\times\left(-\frac{b}{2a}\right)^2 + b\cdot\left(-\frac{b}{2a}\right) + c$

$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$

$= \frac{-b^2}{4a} + c$ or $\frac{4ac-b^2}{4a}$ when $x = -\frac{b}{2a}$

$\therefore$ coordinates of the vertex : $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

(b) $y' = 2ax + b \quad y'' = 2a$

The parabola is concave up when $y'' > 0 \Rightarrow a > 0$

The parabola is concave down when $y'' < 0 \Rightarrow a < 0$

C4.5

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

5. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ 6. $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1}$

Recall:

THEOREM 6—L'Hôpital's Rule
Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

$f(x) = 1-\cos x \qquad f(a) = 1-\cos 0 = 0$

$g(x) = x^2 \qquad g(a) = 0^2 = 0$

$a = 0 \qquad f$ and g are differentiable on ...

The requirements are met.

$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}\cdot\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{2}\cdot\lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{2}\cdot\cos 0 = \frac{1}{2}$

Evaluate :

$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2\cdot(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2\cdot(1+\cos x)}$

$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{1+\cos x} = 1\times 1\times\frac{1}{2} = \frac{1}{2}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

38. $\lim_{x\to 0^+}(\ln x - \ln\sin x)$

$\lim_{x\to 0^+}(\ln x - \ln\sin x) = \lim_{x\to 0^+}\ln\frac{x}{\sin x} = \ln\boxed{\lim_{x\to 0^+}\frac{x}{\sin x}}$

$= \ln\lim_{x\to 0^+}\frac{1}{\cos x} = \ln 1 = 0.$

79. Continuous extension Find a value of c that makes the function

$$f(x) = \begin{cases} \frac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

$\lim_{x\to 0}\frac{9x-3\cdot\sin 3x}{5x^3} \overset{\text{l'Hôpital's Rule}}{=\!=} \lim_{x\to 0}\frac{9-3\cdot\cos 3x\cdot 3}{15x^2}$

$= \lim_{x\to 0}\frac{9}{15}\frac{1-\cos 3x}{x^2} = \frac{3}{5}\lim_{x\to 0}\frac{3\sin 3x}{2x} = \frac{9}{10}\lim_{x\to 0}\frac{\sin 3x}{x}$

$= \frac{9}{10}\cdot\lim_{x\to 0}\frac{3\cdot\cos x}{1} = \frac{27}{10}\lim_{x\to 0}\cos x = \frac{27}{10}$

C4.6

59. Wilson lot size formula One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

a. Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula.*)

b. Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

(a) $A(q) = \frac{km}{q} + \frac{h}{2}q + cm$

$A'(q) = -km\cdot\frac{1}{q^2} + \frac{h}{2}$

$A''(q) = -km\cdot(-2)\cdot q^{-3} = 2km\cdot\frac{1}{q^3}$

$A'(q) = 0$

$\Rightarrow \frac{h}{2} - \frac{km}{q^2} = 0$

$hq^2 - 2km = 0$

$q^2 = \frac{2km}{h}$

$\because q > 0 \quad \therefore q = \sqrt{\frac{2km}{h}}$

$A''\left(\sqrt{\frac{2km}{h}}\right) = 2km\cdot\frac{1}{\frac{2km}{h}\cdot\sqrt{\frac{2km}{h}}} = \frac{h}{\sqrt{\frac{2km}{h}}} > 0$

$\therefore$ when $q = \sqrt{\frac{2km}{h}}$, there is a minimum average weekly cost.

Another solution : Recall : $x + \frac{a}{x}$ will reach min. at $\sqrt{a}$

$A(q) = \frac{h}{2}\left(q + \frac{\frac{2km}{h}}{q}\right) + cm$

$\therefore A(q)$ will reach its minimum when $q = \sqrt{\frac{2km}{h}}$

(b) $A(q) = \frac{(k+bq)m}{q} + \frac{h}{2}q + cm = \frac{km}{q} + \frac{h}{2}q + m(c+b)$

$A'(q) = -\frac{km}{q^2} + \frac{h}{2} \qquad A''(q) = -km\cdot(-2)\cdot q^{-3} = \frac{km}{q^3}$

$A'(q) = 0$

$\Rightarrow \frac{h}{2} - \frac{km}{q^2} = 0 \Rightarrow q^2 = \frac{2km}{h}$

$\because q > 0$

$\therefore q = \sqrt{\frac{2km}{h}}$

$\because A''\left(\sqrt{\frac{2km}{h}}\right) > 0$

$\therefore$ There is a minimum value at $q = \sqrt{\frac{2km}{h}}$.