C5.4

11, 15, 34.

Evaluating Integrals

Evaluate the integrals in Exercises 1–34.

11. $\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta \, d\theta$

31. $\int_0^{1/2} \frac{4}{\sqrt{1-x^2}} dx$

13. $\int_{\pi/2}^{0} \frac{1+\cos 2t}{2} dt$

15. $\int_0^{\pi/4} \tan^2 x \, dx$

11. $\csc\theta = \frac{1}{\sin\theta}$ (cosecant)

$\cot\theta = \frac{1}{\tan\theta}$

$$\int_{\pi/4}^{3\pi/4} \csc\theta \cdot \cot\theta \cdot d\theta = \int_{\pi/4}^{3\pi/4} \frac{1}{\sin\theta} \cdot \frac{1}{\tan\theta} \cdot d\theta = \int_{\pi/4}^{3\pi/4} \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} d\theta$$

$$= \int_{\sin\frac{\pi}{4}}^{\sin\frac{3\pi}{4}} \frac{\cancel{\cos\theta}}{\sin^2\theta} \cdot \frac{1}{\cancel{\cos\theta}} d(\sin\theta) = \int_{\frac{1}{\sqrt2}}^{\frac{1}{\sqrt2}} \frac{1}{\sin^2\theta} \cdot d(\sin\theta) = \int_{\frac{1}{\sqrt2}}^{\frac{1}{\sqrt2}} \frac{1}{t^2} \cdot dt$$

$$= -\frac{1}{t}\Big]_{\frac{1}{\sqrt2}}^{\frac{1}{\sqrt2}} = 0$$

OR

$$\int_{\pi/4}^{3\pi/4} \csc\theta \cdot \cot\theta \cdot d\theta = \Big[-\csc\theta\Big]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} = -\csc\left(\tfrac{3}{4}\pi\right) - \left(-\csc\left(\tfrac{\pi}{4}\right)\right)$$

$$= -\sqrt2 - (-\sqrt2) = 0$$

15. $$\int_0^{\frac{\pi}{4}} \tan^2 x \cdot dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \cdot dx = \tan x - x\Big]_0^{\frac{\pi}{4}}$$

$$= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0) = 1 - \frac{\pi}{4} - (0 - 0) = 1 - \frac{\pi}{4}$$

Hint:

$\sec x = \frac{1}{\cos x}$, $(\tan x)' = \sec^2 x$

$\tan^2 x = \sec^2 x - 1 = \frac{1}{\cos^2\theta} - 1 = \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta}$

31. Recall: $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$

$$\int_0^{\frac12} \frac{4}{\sqrt{1-x^2}} \cdot dx = 4\int_0^{\frac12} \frac{1}{\sqrt{1-x^2}} \cdot dx = 4 \cdot \left[\sin^{-1} x\right]_0^{\frac12} = 4\left[\sin^{-1}\left(\tfrac12\right) - \sin^{-1}(0)\right]$$

$$= 4\left[\frac{\pi}{6} - 0\right] = \frac{2}{3}\pi.$$

75. Cost from marginal cost The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

$$c(100) - c(1) = \int_1^{100} \frac{1}{2\sqrt{x}} \cdot dx = \left[\sqrt{x}\right]_1^{100} = \sqrt{100} - \sqrt{1} = 10 - 1 = 9$$

Hint: $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

OR

$c(x) = \int_0^x \frac{1}{2\sqrt{x}} \cdot dx = \left[\sqrt{x}\right]_0^x = \sqrt{x} - 0 = \sqrt{x}$

$c(100) - c(1) = \sqrt{100} - \sqrt{1} = 10 - 1 = 9.$

In Exercises 76–78, guess an antiderivative and validate your guess by differentiation. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

76. Revenue from marginal revenue Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - 2/(x+1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

$$\int_0^3 2 - \frac{2}{(x+1)^2} \cdot dx$$

$$= \int_0^3 2 - \frac{2}{(x+1)^2} \cdot d(x+1) \qquad \text{let } t = x+1$$

$$= \int_1^4 2 - \frac{2}{t^2} \cdot dt \qquad \left(t + \frac{1}{t}\right)' = 1 - \frac{1}{t^2}$$

$$= 2\int_1^4 1 - \frac{1}{t^2} \cdot dt$$

$$= 2\left[t + \frac{1}{t}\right]_1^4 = 2\left[\left(4 + \tfrac14\right) - (1+1)\right] = 2\left(2 + \tfrac14\right) = 4.5$$

C5.5

Evaluating Indefinite Integrals

In Exercises 1–16, make the given substitutions to evaluate the indefinite integrals.

6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx, \quad u = 1 + \sqrt{x}$

7. $\int \sin 3x \, dx, \quad u = 3x$ **8.** $\int x \sin(2x^2)$

9. $\int \sec 2t \tan 2t \, dt, \quad u = 2t$

10. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2} dt, \quad u = 1 - \cos\frac{t}{2}$

11. $\int \frac{9r^2 \, dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

6. $\int \frac{(1+\sqrt{x})^{\frac13}}{\sqrt{x}} \cdot dx \qquad u = 1 + \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} \cdot dx \Rightarrow 2 \cdot du = \frac{1}{\sqrt{x}} \cdot dx$

$$\int \frac{(1+\sqrt{x})^{\frac13}}{\sqrt{x}} \cdot dx = \int (1+\sqrt{x})^{\frac13} \cdot \frac{1}{\sqrt{x}} \cdot dx$$

$$= \int u^{\frac13} \cdot 2 \cdot du = 2\int u^{\frac13} \cdot du = 2 \cdot \frac34 u^{\frac43} + C = \frac32 \cdot u^{\frac43} + C$$

$$= \frac32 \cdot (1+\sqrt{x})^{\frac43} \cdot dx$$

9. $\int \sec 2t \cdot \tan 2t \cdot dt$, $u = 2t$ $\qquad du = 2 \cdot dt \Rightarrow dt = \frac12 \cdot du$

$$= \int \sec u \cdot \tan u \cdot \frac12 \cdot dt$$

$$= \frac12 \int \sec u \cdot \tan u \cdot dt$$

secant

$(\sec x)' = \sec x \cdot \tan x.$

$$= \frac12 \cdot \sec u + C$$

11. $\int \frac{9r^2 \cdot dr}{\sqrt{1-r^3}}$, $u = 1 - r^3$ $\qquad du = -3r^2 dr \Rightarrow dr = \frac{1}{-3r^2} \cdot du$

$$= \int \overset{3}{\cancel{9}} \cdot \cancel{r^2} \cdot \frac{1}{\sqrt{u}} \cdot \frac{1}{-\cancel{3r^2}} \cdot du$$

$$= -3 \cdot \int u^{-\frac12} \cdot du = -3 \cdot u^{\frac12} \cdot 2 + C = -6u^{\frac12} + C$$

$$= -6(1-r^3)^{\frac12} + C$$

29. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

29. $\int x^{\frac{1}{2}}\cdot\sin(x^{\frac{3}{2}}+1)\cdot dx$

Let $u=x^{\frac{3}{2}}+1$ $\quad du=\frac{3}{2}\cdot x^{\frac{1}{2}}\cdot dx=\frac{3\sqrt{x}}{2}\cdot dx$

$\Rightarrow dx=\frac{2}{3\sqrt{x}}\cdot du$

$\int \cancel{\sqrt{x}}\cdot\sin u\cdot\frac{2}{3\cancel{\sqrt{x}}}\cdot du$

$=\frac{2}{3}\int\sin u\cdot du=-\frac{2}{3}\cos u+C$

$=-\frac{2}{3}\cos(x^{\frac{3}{2}}+1)+C$

39. $\int\frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

39. Let $u=2-\frac{1}{x}$ $\quad du=\frac{1}{x^2}\cdot dx\Rightarrow dx=x^2\cdot du$

$\int\frac{1}{x^2}\cdot\sqrt{2-\frac{1}{x}}\cdot dx=\int x^{-2}\cdot\sqrt{u}\cdot x^2\cdot du=\int u^{\frac{1}{2}}\cdot du=\frac{2}{3}\cdot u^{\frac{3}{2}}+C$

$=\frac{2}{3}\cdot(2-\frac{1}{x})^{\frac{3}{2}}+C$

C5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–48.

15. $\int_0^1\sqrt{t^5+2t}\,(5t^4+2)\,dt$

Recall:

THEOREM 7—Substitution in Definite Integrals
If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x)=u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx=\int_{g(a)}^{g(b)}f(u)\,du.$$

15. $\int_0^1\sqrt{t^5+2t}\,(5t^4+2)\cdot dt$ $\quad$ let $u=t^5+2t$

$du=(5t^4+2)\cdot dt$

$=\int_0^3\sqrt{u}\cdot du$

$=\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^3=\frac{2}{3}\cdot3^{\frac{3}{2}}-\frac{2}{3}\cdot0^{\frac{3}{2}}=\frac{2}{3}\cdot\sqrt{3^3}=2\sqrt{3}$

31. $\int_2^4\frac{dx}{x(\ln x)^2}$

31. $\int_2^4\frac{1}{x\cdot(\ln x)^2}\cdot dx$ $\quad u=\ln x$ $\quad du=\frac{1}{x}\cdot dx$

$=\int_2^4\frac{1}{(\ln x)^2}\cdot\frac{1}{x}\cdot dx=\int_{\ln2}^{\ln4}\frac{1}{u^2}\cdot du=\left[-\frac{1}{u}\right]_{\ln2}^{\ln4}$

$=-\frac{1}{\ln4}-\left(-\frac{1}{\ln2}\right)=-\frac{1}{\ln2^2}+\frac{1}{\ln2}=-\frac{1}{2\cdot\ln2}+\frac{1}{\ln2}=\frac{1}{2\cdot\ln2}=\frac{1}{\ln4}$

34. $\int_{\pi/4}^{\pi/2}\cot t\,dt$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot t\cdot dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos t}{\sin t}\cdot dt$ $\quad u=\sin t$ $\quad du=\cos t\cdot dt$

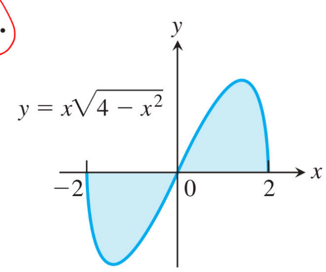
$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{\sin t}\cdot\cos t\cdot dt=\int_{\frac{1}{\sqrt2}}^{1}\frac{1}{u}\cdot du=\left[\ln u\right]_{\frac{1}{\sqrt2}}^{1}=\ln1-\ln\sqrt{2}^{-1}$

$=0+\ln\sqrt{2}=\ln\sqrt{2}$

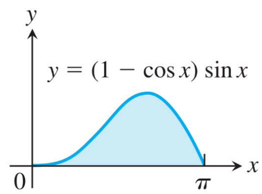
Area

Find the total areas of the shaded regions in Exercises 49–64.

49.



50.



49. Area $=-\int_{-2}^{0}x\cdot\sqrt{4-x^2}\cdot dx+\int_0^2 x\cdot\sqrt{4-x^2}\cdot dx$

$f(x)=x\cdot\sqrt{4-x^2}$

$f(-x)=-x\cdot\sqrt{4-x^2}$ $\quad\therefore f(x)=-f(-x)$

$\therefore f(x)$ is symetric to $(0,0)$

$\therefore -\int_{-2}^{0}x\cdot\sqrt{4-x^2}\cdot dx=\int_0^2 x\cdot\sqrt{4-x^2}\cdot dx$

$\therefore$ Area $=2\int_0^2 x\cdot\sqrt{4-x^2}\cdot dx$ $\quad$ let $u=4-x^2$

$du=-2x\cdot dx$

$=-\int_0^2\sqrt{4-x^2}\cdot(-2x)\cdot dx$

$=-\int_4^0\sqrt{u}\cdot du$

$=-\left[\frac{2}{3}\cdot u^{\frac{3}{2}}\right]_4^0$

$=-\frac{2}{3}\left(0^{\frac{3}{2}}-4^{\frac{3}{2}}\right)=-\frac{2}{3}\left(0-\sqrt{4\times4\times4}\right)=-\frac{2}{3}\cdot(-8)=\frac{16}{3}$

C8.1

Recall:

Integration by Parts Formula—Differential Version

$$\int u\,dv=uv-\int v\,du \qquad (2)$$

Integration by Parts Formula

$$\int u(x)v'(x)\,dx=u(x)v(x)-\int v(x)u'(x)\,dx \qquad (1)$$

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

8. $\int xe^{3x}\,dx$

$d(e^{3x})=e^{3x}\cdot3$

$\int x\cdot e^{3x}\cdot dx=\frac{1}{3}\int x\cdot3e^{3x}\cdot dx=\frac{1}{3}\int x\cdot d(e^{3x})=\frac{1}{3}\left(x\cdot e^{3x}-\int e^{3x}\cdot dx\right)$

$=\frac{1}{3}\left(x e^{3x}-\frac{1}{3}\cdot e^{3x}\right)+C=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+C$

17. $\displaystyle\int (x^2 - 5x)e^x\,dx$

$$\int (x^2-5x)\cdot e^x\cdot dx \qquad d(e^x) = e^x\cdot dx$$

$$= \int (x^2 e^x - 5x\cdot e^x)\cdot dx$$

$$= \int x^2 e^x\cdot dx - \int 5x\cdot e^x\cdot dx$$

$$\boxed{\begin{aligned}\int x^2 e^x\cdot dx &= \int x^2\cdot d(e^x) = x^2\cdot e^x - \int e^x\cdot d(x^2)\\ &= x^2\cdot e^x - \int e^x\cdot 2x\cdot dx\end{aligned}}$$

$$= x^2\cdot e^x - \int 2x\cdot e^x\cdot dx - \int 5x\cdot e^x\cdot dx$$

$$= x^2\cdot e^x - \int 7x\cdot e^x\cdot dx$$

$$= x^2\cdot e^x - 7\int x\cdot e^x\cdot dx \quad = x^2\cdot e^x - 7\cdot\int x\cdot d(e^x)$$

$$= x^2 e^x - 7\left(x\cdot e^x - \int e^x\cdot dx\right)$$

$$= x^2\cdot e^x - 7\left(x\cdot e^x - e^x\right) + C$$

$$= e^x\left(x^2 - 7x + 7\right) + C$$

OR $\displaystyle\int (x^2-5x)\cdot e^x\cdot dx = \int (x^2-5x)\cdot d(e^x)$

$$= (x^2-5x)\cdot e^x - \int e^x\cdot d(x^2-5x)$$

$$= (x^2-5x)\cdot e^x - \int e^x(2x-5)\cdot dx$$

$$= e^x(x^2-5x) - \int (2x-5)\cdot d(e^x)$$

$$= e^x(x^2-5x) - \left[e^x(2x-5) - \int e^x\cdot d(2x-5)\right]$$

$$= e^x(x^2-5x) - \left[e^x(2x-5) - \int e^x\cdot 2\cdot dx\right]$$

$$= e^x(x^2-5x) - e^x(2x-5) + 2\int e^x\cdot dx$$

$$= e^x(x^2-5x-2x+5) + 2\cdot e^x + C$$

$$= e^x(x^2-7x+7) + C$$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\displaystyle\int e^{\sqrt{3s+9}}\,ds$ **26.** $\displaystyle\int_0^1 x\sqrt{1-x}\,dx$

$$\int e^{\sqrt{3s+9}}\cdot ds \qquad \text{let } u = 3s+9 \qquad du = 3\cdot ds$$

$$= \int e^{\sqrt{u}}\cdot\tfrac{1}{3}\,du$$

let $t = \sqrt{u}$

$$= \tfrac{1}{3}\int e^{\sqrt{u}}\cdot du = \tfrac{1}{3}\int e^t\cdot d(t^2) = \tfrac{1}{3}\int e^t\cdot 2t\cdot dt$$

$$= \tfrac{2}{3}\int e^t\cdot t\cdot dt = \tfrac{2}{3}\int t\cdot d(e^t)$$

$$= \tfrac{2}{3}\left(t\cdot e^t - \int e^t\cdot dt\right) = \tfrac{2}{3}\left(t\cdot e^t - e^t\right) + C$$

$$= \tfrac{2}{3}\left(\sqrt{u}\cdot e^{\sqrt{u}} - e^{\sqrt{u}}\right) + C$$

$$= \tfrac{2}{3}\left(\sqrt{3s+9}\cdot e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$$

Evaluating Integrals

Evaluate the integrals in Exercises 31–56. Some integrals do not require integration by parts.

39. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

$$\int x^3\cdot\sqrt{x^2+1}\cdot dx \qquad \text{let } u = x^2+1 \qquad du = 2x\cdot dx$$

$$= \int \sqrt{x^2+1}\cdot x^2\cdot x\cdot dx \qquad \Rightarrow x^2 = u-1$$

$$= \int \sqrt{u}\cdot(u-1)\cdot\tfrac{1}{2}\cdot du$$

$$= \tfrac{1}{2}\int u^{\frac{1}{2}}(u-1)\cdot du$$

$$= \tfrac{1}{2}\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\cdot du$$

$$= \tfrac{1}{2}\left[\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right] + C$$

$$= \tfrac{1}{5}\cdot u^{\frac{5}{2}} - \tfrac{1}{3}u^{\frac{3}{2}} + C$$

$$= \tfrac{1}{5}\cdot(x^2+1)^{\frac{5}{2}} - \tfrac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$