C7.2 Confusion of d( ) : Ex: $d(\sin^2 x) = 2\cdot\sin x\cdot d(\sin x) = 2\sin x\cdot\cos x\cdot dx$

$\Leftrightarrow (\sin^2 x)' = 2\sin x\cdot(\sin x)' = 2\sin x\cdot\cos x$

Separable Differential Equations

Solve the differential equation in Exercises 9–22.

17. $\dfrac{dy}{dx} = 2x\sqrt{1-y^2}, \quad -1 < y < 1$

17. $\dfrac{dy}{dx} = 2x\cdot\sqrt{1-y^2}$

$\dfrac{1}{\sqrt{1-y^2}}\cdot dy = 2x\cdot dx$ Hint: $(\sin^{-1}t)' = \dfrac{1}{\sqrt{1-t^2}}$

$\Rightarrow \displaystyle\int \frac{1}{\sqrt{1-y^2}}\cdot dy = \int 2x\cdot dx$

$\sin^{-1}y = x^2 + C \quad (-1<y<1)$

$\therefore y = \sin(x^2+c)$

Summary:

Step1: split all the terms related to x to one side, and all the terms related to y to another side.

Step2: Put integral to both sides.

22. $\dfrac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

$\dfrac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

$= (e^x+1)(e^{-y}+1)$

$\Rightarrow \dfrac{1}{e^{-y}+1}\cdot dy = (e^x+1)\,dx$

$\displaystyle\int \frac{1}{e^{-y}+1}\cdot dy = \int (e^x+1)\,dx$

$\displaystyle\int \frac{e^y}{1+e^y}\cdot dy = \int (e^x+1)\,dx$

$\displaystyle\int \frac{1}{1+e^y}\cdot d(e^y) = \int (e^x+1)\,dx$

$\ln(1+e^y) = e^x + x + C$

C8.4

Initial Value Problems

Solve the initial value problems in Exercises 67–70 for x as a function of t.

69. $(t^2+2t)\dfrac{dx}{dt} = 2x+2 \quad (t, x > 0), \quad x(1) = 1$

$(t^2+2t)\cdot\dfrac{dx}{dt} = 2x+2 \Rightarrow (t^2+2t)\cdot dx = (2x+2)\cdot dt$

$\Rightarrow \dfrac{1}{2x+2}\,dx = \dfrac{1}{t^2+2t}\cdot dt$

$\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{1}{x+1}\,dx = \int \frac{1}{t^2+2t}\cdot dt = \int \frac{1}{t(t+2)}\cdot dt = \frac{1}{2}\int\left(\frac{1}{t} - \frac{1}{t+2}\right)\cdot dt$

$\Rightarrow \frac{1}{2}\cdot\ln|x+1| = \frac{1}{2}\left(\ln|t| - \ln|t+2|\right) + C$

$\Rightarrow \ln|x+1| = \ln|t| - \ln|t+2| + C = \ln\left|\dfrac{t}{t+2}\right| + C$

$\because x, t > 0 \quad \therefore \ln(x+1) = \ln\left(\dfrac{t}{t+2}\right) + C$

$\because x(1) = 1, \therefore \ln(2) = \ln\left(\frac{1}{3}\right) + c \Rightarrow c = \ln 2 - \ln\frac{1}{3} = \ln 6$

$\therefore \ln(x+1) = \ln\left(\dfrac{t}{t+2}\right) + \ln 6 = \ln\left(\dfrac{6t}{t+2}\right) \Rightarrow x+1 = \dfrac{6t}{t+2} \Rightarrow x = \dfrac{6t}{t+2} - 1$

Recall:

Solving Linear Equations

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by multiplying both sides by a *positive* function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x)\cdot y$. We will show how to find v in a moment, but first we

C16.2

First-Order Linear Equations

Solve the differential equations in Exercises 1–14.

5. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

$x\cdot\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

$\dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{1}{x} - \dfrac{1}{x^2}$

$P(x) = \dfrac{2}{x} \quad Q(x) = \dfrac{1}{x} - \dfrac{1}{x^2}$

$v(x) = e^{\int P(x)\cdot dx} = e^{\int \frac{2}{x}\cdot dx} = e^{2\cdot\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$

Times $v(x)$ to both sides.

$x^2\cdot\dfrac{dy}{dx} + 2x\cdot y = x - 1$

$d(x^2y) = (x^2)'\cdot y + x^2\cdot y'$
$= 2x\cdot dx\cdot y + x^2\cdot dy$
$= 2xy\cdot dx + x^2\cdot dy$
$\dfrac{d(x^2y)}{dx} = 2xy + x^2\cdot\dfrac{dy}{dx}$

$\dfrac{d(x^2y)}{dx} = x - 1$

$\displaystyle\int d(x^2\cdot y) = \int (x-1)\cdot dx$

$x^2y = \frac{1}{2}x^2 - x + C$

$\Rightarrow y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{C}{x^2}, \quad x > 0.$

13. $\sin\theta\dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

$\sin\theta\cdot\dfrac{dr}{d\theta} + (\cos\theta)\cdot r = \tan\theta, \quad 0 < \theta < \dfrac{\pi}{2}$

$\dfrac{dr}{d\theta} + \dfrac{\cos\theta}{\sin\theta}\cdot r = \dfrac{\tan\theta}{\sin\theta}$

$\dfrac{dr}{d\theta} + \dfrac{1}{\tan\theta}\cdot r = \dfrac{1}{\cos\theta}$

$P(\theta) = \dfrac{1}{\tan\theta}, \quad Q(\theta) = \dfrac{1}{\cos\theta}$

$v(\theta) = e^{\int P(\theta)\,d\theta} = e^{\int \frac{1}{\tan\theta}\cdot d\theta = \int \frac{\cos\theta}{\sin\theta}\cdot d\theta = \int \frac{1}{\sin\theta}\cdot d(\sin\theta) = \ln|\sin\theta|} = |\sin\theta|$

$\because 0 < \theta < \frac{\pi}{2} \quad \therefore v(\theta) = \sin\theta$

$\sin\theta\cdot\dfrac{dr}{d\theta} + \dfrac{\sin\theta}{\tan\theta}\cdot r = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \dfrac{d(\sin\theta\cdot r)}{d\theta} = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \displaystyle\int d(\sin\theta\cdot r) = \int \frac{\sin\theta}{\cos\theta}\cdot d\theta = -\int \frac{1}{\cos\theta}\cdot d(\cos\theta)$

$= -\ln|\cos\theta| + C = -\ln\cos\theta = \ln\dfrac{1}{\cos\theta} + C$

$\Rightarrow \sin\theta\cdot r = \ln\dfrac{1}{\cos\theta} + C$

$\Rightarrow r = \dfrac{1}{\sin\theta}\left(\ln\dfrac{1}{\cos\theta} + C\right)$

Solving Initial Value Problems

Solve the initial value problems in Exercises 15–20.

15. $\frac{dy}{dt} + 2y = 3, \quad y(0) = 1$

16. $t\frac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

$$t \cdot \frac{dy}{dt} + 2y = t^3, \quad t > 0$$

$$\frac{dy}{dt} + \frac{2}{t} \cdot y = t^2$$

$$P(t) = \frac{2}{t}, \quad Q(t) = t^2$$

$$v(t) = e^{\int P(t)dt} = e^{\int \frac{2}{t} \cdot dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2$$

$$t^2 \cdot \left(\frac{dy}{dt} + \frac{2}{t} \cdot y\right) = t^2 \cdot t^2$$

$$\Rightarrow t^2 \cdot \frac{dy}{dt} + 2t \cdot y = t^4$$

$$\frac{d(t^2 y)}{dt} = t^4$$

$$\int d(t^2 y) = \int t^4 \cdot dt$$

$$t^2 y = \frac{1}{5}t^5 + C$$

$$\because y(2) = 1$$

$$\therefore 2^2 \cdot 1 = \frac{1}{5} \times 2^5 + C$$

$$\Rightarrow C = -\frac{12}{5}$$

$$\therefore t^2 y = \frac{1}{5}t^5 - \frac{12}{5}$$

$$\therefore y = \frac{1}{5} \cdot t^3 - \frac{12}{5} \cdot \frac{1}{t^2}$$

32.

HISTORICAL BIOGRAPHY
Jacob Bernoulli
(1654–1705)
www.bit.ly/2NUSHeH

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

For example, in the equation

$$\frac{dy}{dx} - y = e^{-x}y^2,$$

we have $n = 2$, so that $u = y^{1-2} = y^{-1}$ and $du/dx = -y^{-2}\,dy/dx$. Then $dy/dx = -y^2\,du/dx = -u^{-2}\,du/dx$. Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x}u^{-2},$$

or, equivalently,

$$\frac{du}{dx} + u = -e^{-x}.$$

This last equation is linear in the (unknown) dependent variable u.

Solve the Bernoulli equations in Exercises 29–32.

29. $y' - y = -y^2$ **30.** $y' - y = xy^2$

31. $xy' + y = y^{-2}$ **32.** $x^2y' + 2xy = y^3$

Understanding of Bernoulli Differential Equation:

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n$$

$$\text{let } u = y^{1-n}$$

$$\Rightarrow du = (1-n) \cdot y^{-n} \cdot dy$$

$$\Rightarrow dy = \frac{1}{(1-n)} \cdot y^n \cdot du$$

$$\therefore \frac{1}{dx} \cdot \frac{1}{(1-n)} \cdot y^n \cdot du + P(x) \cdot y = Q(x) \cdot y^n$$

$$\Rightarrow \frac{du}{dx} \cdot y^n + P(x)(1-n)y = Q(x) \cdot (1-n) \cdot y^n$$

$$\Rightarrow \frac{du}{dx} + (1-n) \cdot P(x) \cdot y^{1-n} = Q(x) \cdot (1-n)$$

$$\Rightarrow \frac{du}{dx} + (1-n) \cdot P(x)\, u = Q(x) \cdot (1-n)$$

$$x^2 y' + 2xy = y^3$$

$$x^2 \cdot \frac{dy}{dx} + 2xy = y^3$$

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = \frac{1}{x^2} \cdot y^3$$

$$P(x) = \frac{2}{x}, \quad Q(x) = \frac{1}{x^2}, \quad n = 3$$

$$\text{let } u = y^{1-n} = y^{1-3} = y^{-2}$$

$$\text{thus, } \frac{du}{dx} + (1-3) \cdot P(x) \cdot u = (1-3) \cdot Q(x)$$

$$v(x) = e^{\int -2 \cdot P(x) \cdot dx} = e^{-2\int \frac{2}{x} \cdot dx} = e^{-4 \cdot \ln|x|} = |x|^{-4} = x^{-4}$$

$$x^{-4} \cdot \left(\frac{du}{dx} + (1-3) \cdot \frac{2}{x} \cdot u\right) = x^{-4} \cdot (1-3) \cdot \frac{1}{x^2}$$

$$\frac{d(u \cdot x^{-4})}{dx} = -2 \cdot x^{-6}$$

$$\Rightarrow \int d(u \cdot x^{-4}) = \int (-2) \cdot x^{-6} dx$$

$$\Rightarrow u \cdot x^{-4} = (-2) \cdot \frac{1}{-5} \cdot x^{-5} + C$$

$$u \cdot x^{-4} = \frac{2}{5}x^{-5} + C$$

$$u = \frac{2}{5} \cdot x^{-1} + Cx^4$$

$$y^{-2} = \frac{2}{5}x^{-1} + Cx^4 \quad \Rightarrow y = \left(\frac{2}{5x} + Cx^4\right)^{-\frac{1}{2}}$$

C16.4

Phase Lines and Solution Curves

In Exercises 1–8,

a. Identify the equilibrium values. Which are stable and which are unstable?

b. Construct a phase line. Identify the signs of y' and y''.

c. Sketch several solution curves.

6. $y' = y - \sqrt{y}, \quad y > 0$

Recall:

DEFINITION If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

$y' = 0 \Rightarrow y - \sqrt{y} = 0 \Rightarrow y = \sqrt{y} \Rightarrow y^2 = y \Rightarrow y^2 - y = 0$

$\Rightarrow y(y-1) = 0 \quad \because y > 0 \quad \therefore y = 1$

(a) $y=1$ is an unstable equilibrium.

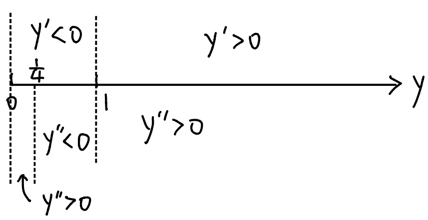
$\because \lim_{y \to +\infty} y - \sqrt{y} = +\infty$ (move away)

(b) $y' = y - \sqrt{y} = y - y^{\frac{1}{2}}$

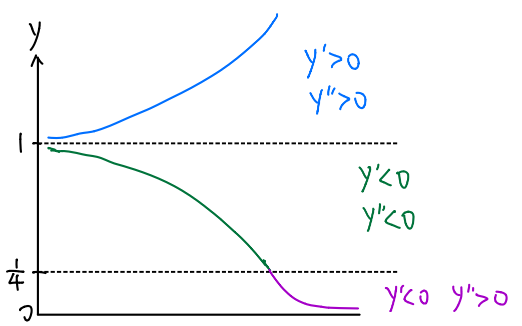
$y'' = y' - \frac{1}{2} \cdot y^{-\frac{1}{2}} \cdot y' = y'\left(1 - \frac{1}{2\sqrt{y}}\right) = (y - \sqrt{y})\left(1 - \frac{1}{2\sqrt{y}}\right)$

$= \sqrt{y}(\sqrt{y} - 1)\left(1 - \frac{1}{2\sqrt{y}}\right) = (\sqrt{y} - 1)\left(\sqrt{y} - \frac{1}{2}\right)$

$y'' = 0 \Rightarrow \sqrt{y} = 1$ or $\sqrt{y} = \frac{1}{2} \Rightarrow y = 1$ or $\frac{1}{4}$



(c)



14. Controlling a population The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

a. Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time:

$$\frac{dP}{dt} = rP(M - P)(P - m),$$

where P is the population of the deer and r is a positive constant of proportionality. Include a phase line.

b. Explain how this model differs from the logistic model $dP/dt = rP(M - P)$. Is it "more or less reasonable than the logistic model?

c. Show that if $P > M$ for all t, then $\lim_{t\to\infty} P(t) = M$.

d. What happens if $P < m$ for all t?

e. Discuss the solutions to the differential equation. What are the equilibrium points of the model? Explain the dependence of the steady-state value of P on the initial values of P. About how many permits should be issued?

$\frac{dP}{dt} = r \cdot P(M-P)(P-m)$

$P' = r \cdot P(M-P)(P-m) \Rightarrow \begin{cases} P>M, & P'<0 \\ m<P<M, & P'>0 \\ 0<P<m, & P'<0 \end{cases}$

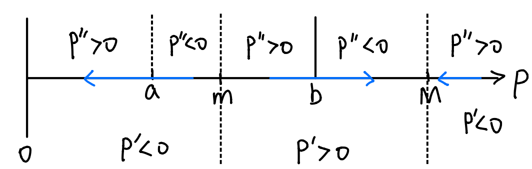
$P' = 0 \Rightarrow P = 0$ or m or M

$P' = r(MP - P^2)(P-m) = r(MP^2 - mMP - P^3 + mP^2)$

$P'' = r(2MP - mM - 3P^2 + 2mP) = r\left[-3P^2 + 2(M+m)P - mM\right]$

When $P'' = 0$, $P = \frac{-2(m+M) \pm \sqrt{4(m+M)^2 - 12Mm}}{-6}$

$\therefore P = \underbrace{\frac{(m+M) + \sqrt{m^2 + M^2 - Mm}}{3}}_{b''}$ or $\underbrace{\frac{(m+M) - \sqrt{m^2 + M^2 - Mm}}{3}}_{a''}$

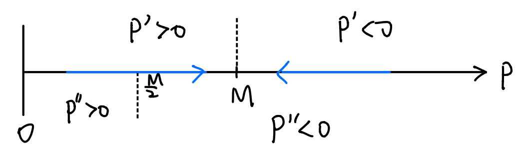


(a) It's reasonable. Because when $P>M$, then $P \to M$ as $t \to \infty$; when $P<m$, $P \to 0$ as $t \to \infty$; when $m<P<M$, then $P \to M$ as $t \to \infty$.

(b) $\frac{dP}{dt} = rP(M-P) \quad P' = 0 \Rightarrow P = 0$ or $P = M$

$P'' = r\left[(M-P) + P(-1)\right] = r(M - 2P)$

$P'' = 0 \Rightarrow P = \frac{M}{2}$



$\frac{dP}{dt} = r \cdot P(M-P)(P-m)$ is more realistic

(c) when $P>M$, $P'<0$, $P''>0$

$\because P = M$ is a solution to $P' = 0$

$\therefore$ it won't cross solution trajectory

$\therefore P \to M$ as $t \to \infty$

(d) $P<m$,

$P \to 0$ as $t \to +\infty$