EXERCISES 4.3

Evaluate the limits in Exercises 1–32.

1. $\lim_{x\to0}\frac{3x}{\tan 4x}$
2. $\lim_{x\to2}\frac{\ln(2x-3)}{x^2-4}$
3. $\lim_{x\to0}\frac{\sin ax}{\sin bx}$
4. $\lim_{x\to0}\frac{1-\cos ax}{1-\cos bx}$
5. $\lim_{x\to0}\frac{\sin^{-1}x}{\tan^{-1}x}$
6. $\lim_{x\to1}\frac{x^{1/3}-1}{x^{2/3}-1}$

2. $$\lim_{x\to2}\frac{\ln(2x-3)\to0}{x^2-4\to0}$$
$$=\lim_{x\to2}\frac{\frac{1}{2x-3}}{2x}=\frac{1}{2x(2x-3)}=\frac{1}{4\times1}=\frac14$$

6. $$\lim_{x\to1}\frac{x^{\frac13}-1\to0}{x^{\frac23}-1\to0}$$
$$=\lim_{x\to1}\frac{\frac13x^{-\frac23}}{\frac23x^{-\frac13}}=\lim_{x\to1}\frac12x^{-\frac23-(-\frac13)}=\frac12\lim_{x\to1}x^{-\frac13}=\frac12$$

Way 2: $$\lim_{x\to1}\frac{x^{\frac13}-1}{x^{\frac23}-1}=\lim_{x\to1}\frac{\cancel{x^{\frac13}-1}}{(\cancel{x^{\frac13}-1})(x^{\frac13}+1)}$$
$$=\lim_{x\to1}\frac{1}{x^{\frac13}+1}=\frac12$$

15. $\lim_{x\to0}\frac{x-\sin x}{x-\tan x}$

17. $\lim_{x\to0+}\frac{\sin^2 x}{\tan x-x}$

19. $\lim_{t\to\pi/2}\frac{\sin t}{t}$

15. $$\lim_{x\to0}\frac{x-\sin x}{x-\tan x}\qquad (\tan x)'=\sec^2x=\frac{1}{\cos^2x}$$
$$=\lim_{x\to0}\frac{1-\cos x}{1-\frac{1}{\cos^2x}}$$
$$=\lim_{x\to0}\frac{\cos^2x(1-\cos x)}{\cos^2x-1}$$
$$=\lim_{x\to0}\frac{-\cos^2x\,(\cancel{\cos x-1})\to-1}{(\cancel{\cos x-1})(\cos x+1)\to2}=-\frac12$$

17. $$\lim_{x\to0^+}\frac{\sin^2x}{\tan x-x}$$
$$\lim_{x\to0^+}\frac{2\sin x\cos x}{\frac{1}{\cos^2x}-1}$$
$$=\lim_{x\to0^+}\frac{2\sin x\cdot\cos^3x}{1-\cos^2x}$$
$$=\lim_{x\to0^+}\frac{2\cancel{\sin x}\cdot\cos^3x\to2^+}{\sin^{\cancel{2}}x\to0^+}=+\infty$$

19. $$\lim_{t\to\frac{\pi}{2}}\frac{\sin t\to1}{t\to\frac{\pi}{2}}=\frac{2}{\pi}$$

6 Steps:

1) What's domain? continuous / discontinuous.
2) $f'(x)$ ↗ ↘
3) $f''(x)$ ∩ concave (<0) convex ∪ (>0)
4) Any Asymptotes?
5) Even / odd / neither
6) A sketch of f.

① $f(x)=\sqrt{1+x^2}=(1+x^2)^{\frac12}$

1) $1+x^2\geq0 \quad \therefore x\in\mathbb{R}$

2) $f'(x)=\frac12(1+x^2)^{-\frac12}\cdot2x=x(1+x^2)^{-\frac12}$

$f'(x)>0$ when $x>0$ ↗

$f'(x)<0$ when $x<0$ ↘

3) $$f''(x)=\left(\frac{x}{\sqrt{1+x^2}}\right)'=\frac{\sqrt{1+x^2}-x\cdot x\cdot(1+x^2)^{-\frac12}}{1+x^2}$$
$$=(1+x^2)^{-\frac12}-x^2\cdot(1+x^2)^{-\frac32}$$

when $f''(x)>0 \Rightarrow (1+x^2)^{-\frac12}>x^2(1+x^2)^{-\frac32} \Rightarrow 1+x^2>x^2$ always true.

$f''(x)>0$ for all $x\in\mathbb{R}$. convex ∪

4) $f(x)=(1+x^2)^{\frac12}$

The graph of $y=f(x)$ has a **horizontal asymptote** $y=L$ if

either $\lim_{x\to\infty}f(x)=L$ or $\lim_{x\to-\infty}f(x)=L$, or both.

The graph of $y=f(x)$ has a **vertical asymptote** at $x=a$ if

either $\lim_{x\to a-}f(x)=\pm\infty$ or $\lim_{x\to a+}f(x)=\pm\infty$, or both.

The straight line $y=ax+b$ (where $a\neq0$) is an **oblique asymptote** of the graph of $y=f(x)$ if

either $\lim_{x\to-\infty}(f(x)-(ax+b))=0$ or $\lim_{x\to\infty}(f(x)-(ax+b))=0$,

or both.

$$\lim_{x\to+\infty}f(x)-(ax+b)=\lim_{x\to+\infty}\frac{[\sqrt{1+x^2}-(ax+b)][\sqrt{1+x^2}+(ax+b)]}{[\sqrt{1+x^2}+(ax+b)]}$$
$$=\lim_{x\to+\infty}\frac{(1+x^2)-(ax+b)^2}{[\sqrt{1+x^2}+(ax+b)]}=\lim_{x\to+\infty}\frac{x^2(1-a^2)-2abx+(1-b^2)}{\sqrt{1+x^2}+(ax+b)}=0.$$

$1-a^2=0\Rightarrow a^2=1\Rightarrow a=\pm1$

if $a=1$, then: $\lim_{x\to+\infty}\frac{-2bx+(1-b^2)\to-2bx}{\sqrt{1+x^2}+x+b\to2x+b\to2x}$ $\quad \approx\sqrt{x^2}=x$ $\quad\to-b=0\Rightarrow b=0$

if $a=-1$, then: $\lim_{x\to+\infty}\frac{2bx+(1-b^2)\to2bx}{\sqrt{1+x^2}-x+b\to x-x+b\to b}\to2x\to+\infty$ ✗

$\therefore y=x$

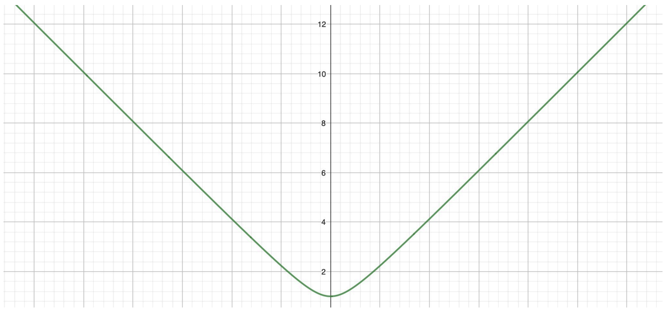
Similarly: $y=-x$

5) $f(x) = (1+x^2)^{\frac{1}{2}}$

$f(-x) = (1+x^2)^{\frac{1}{2}}$

$\Rightarrow f(x) = f(-x)$ Even

6)



② $f(x) = \dfrac{x^2-5x+6}{x-1}$ $\quad \begin{matrix} 1 & -2 \\ 1 & -3 \end{matrix}$

1) $f(x) = \dfrac{x^2-5x+6}{x-1} = \dfrac{(x-2)(x-3)}{x-1}$

$x-1 \neq 0 \Rightarrow x \neq 1$

2) $f'(x) = \dfrac{(2x-5)(x-1) - (x^2-5x+6)\cdot 1}{(x-1)^2}$

$= \dfrac{2x^2 - \cancel{2x} - \cancel{5x} + 5 - x^2 + \cancel{5x} - 6}{(x-1)^2}$

$= \dfrac{x^2-2x-1}{(x-1)^2}$

when $f'(x) > 0$

$x^2-2x-1 > 0$

$x > 1+\sqrt{2}$ or $x < 1-\sqrt{2}$

$\therefore f'(x) > 0$ $(\nearrow)$ when $x \in (1+\sqrt{2}, +\infty) \cup (-\infty, 1-\sqrt{2})$

$f'(x) < 0$ $(\searrow)$ when $x \in (1-\sqrt{2}, 1+\sqrt{2})$

3) $f''(x) = \dfrac{(2x-2)(x-1)^2 - (x^2-2x-1)\cdot 2(x-1)}{(x-1)^4}$

$= \dfrac{2(x-1)^2 - 2(x^2-2x-1)}{(x-1)^3}$

$= 2\,\dfrac{\cancel{x^2} - \cancel{2x} + 1 - \cancel{x^2} + \cancel{2x} + 1}{(x-1)^3}$

$= \dfrac{4}{(x-1)^3}$

when $f''(x) < 0$

$\dfrac{4}{(x-1)^3} < 0 \Rightarrow (x-1)^3 < 0 \Rightarrow x-1 < 0 \Rightarrow x < 1$

$\therefore f''(x) < 0$ $\cap$ concave when $x \in (-\infty, 1)$

$f''(x) > 0$ $\cup$ convex when $x \in (1, +\infty)$

4) $\lim\limits_{x\to+\infty} f(x) - (ax+b)$

$= \lim\limits_{x\to+\infty} \dfrac{(x-2)(x-3)}{x-1} - (ax+b)$

$= \lim\limits_{x\to+\infty} \dfrac{(x-2)(x-3) - (x-1)(ax+b)}{x-1}$

$= \lim\limits_{x\to+\infty} \dfrac{(x^2-5x+6) - [ax^2 + x(b-a) - b]}{x-1}$

$= \lim\limits_{\substack{x\to+\infty \\ (x\to-\infty)}} \dfrac{x^2(1-a) - x(5+b-a) + (6+b)}{x-1} = 0.$

$1-a = 0 \Rightarrow a = 1$

$\lim\limits_{\substack{x\to+\infty \\ (x\to-\infty)}} \dfrac{-x(b+4) + (b+6)}{x-1}$

$= -(b+4) = 0$

$\Rightarrow b = -4.$

$\therefore \begin{cases} a = 1 \\ b = -4 \end{cases}$

$\Rightarrow y = x-4$

$\lim\limits_{x\to1^+} \dfrac{\overset{-}{(x-2)}\,\overset{-}{(x-3)}}{\underset{+}{x-1}} \to +$

$= +\infty$

$\lim\limits_{x\to1^-} \dfrac{\overset{-}{(x-2)}\,\overset{-}{(x-3)}}{\underset{-}{x-1}} \to -$

$= -\infty$

$\Rightarrow x = 1$

5) $f(x) = \dfrac{x^2-5x+6}{x-1}$

$f(-x) = \dfrac{x^2+5x+6}{-x-1} = \dfrac{(x+2)(x+3)}{-(x+1)} \neq f(x)$

$f(x) + f(-x) = \dfrac{(x-2)(x-3)}{x-1} - \dfrac{(x+2)(x+3)}{x+1}$

$= \dfrac{(x-2)(x-3)(x+1) - (x+2)(x+3)(x-1)}{(x-1)(x+1)}$

$= \dfrac{[x^3 + x^2(-2-3+1) + x(6-2-3) + 6] - [x^3 + x^2(2+3-1) + x(6-2-3) - 6]}{(x-1)(x+1)}$

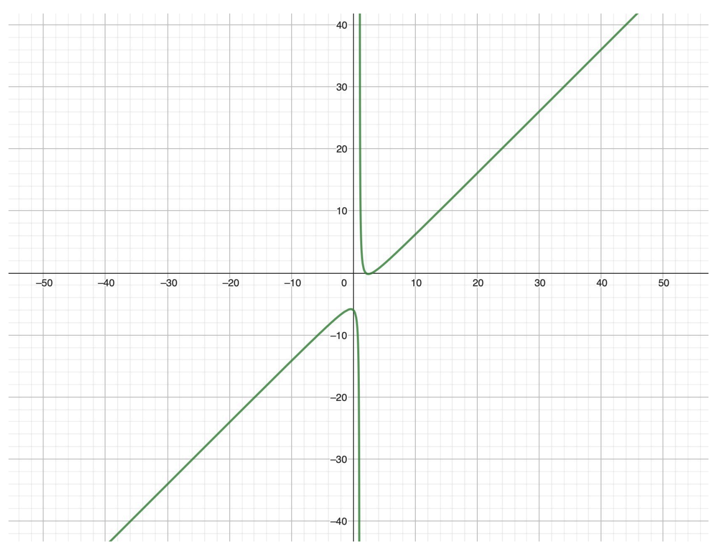
$= \dfrac{(x^3-4x^2+x+6) - (x^3+4x^2+x-6)}{(x-1)(x+1)} \neq 0.$

$(x+a)(x+b)(x+c)$

$= x^3 + x^2(a+b+c) + x(ab+ac+bc) + abc$

Neither.

6)



20. **(Maximizing rental profit)** All 80 rooms in a motel will be rented each night if the manager charges \$40 or less per room. If he charges \$$(40+x)$ per room, then $2x$ rooms will remain vacant. If each rented room costs the manager \$10 per day and each unrented room \$2 per day in overhead, how much should the manager charge per room to maximize his daily profit?

Assuming he charges \$$(40+x)$ per room

profit = In − Out

$= (40+x)(80-2x) - [\#\text{rent} \times 10 + \#\text{remain} \times 2]$ $\quad -10(80-2x)-4x$

$= (40+x)(80-2x) - [10(80-2x) + 2\cdot 2x]$

$= 3200 - \cancel{80x} + \cancel{80x} - 2x^2 - 800 + 20x - 4x$

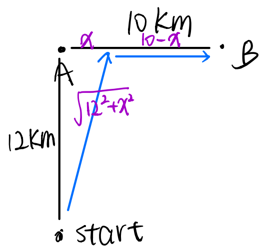
$f(x) = -2x^2 + 16x + 2400$

$f'(x) = -4x + 16 \qquad f''(x) = -4 < 0$

when $f'(x) = 0$, $x = 4$, $f(x)$ will reach max.

$\therefore$ should charge 44.

21. **(Minimizing travel time)** You are in a dune buggy in the desert 12 km due south of the nearest point A on a straight east-west road. You wish to get to point B on the road 10 km east of A. If your dune buggy can average 15 km/h travelling over the desert and 39 km/h travelling on the road, toward what point on the road should you head in order to minimize your travel time to B?



Total travel time =

$$f(x) = \frac{(12^2+x^2)^{\frac{1}{2}}}{15} + \frac{10-x}{39}$$

$$f'(x) = \frac{\cancel{\tfrac{1}{2}}\cdot(12^2+x^2)^{-\frac{1}{2}}\cdot \cancel{2}x}{15} + \frac{0-1}{39}$$

$$= \frac{x}{15\sqrt{12^2+x^2}} - \frac{1}{39}$$

when $f'(x) = 0$, $x = 5$.

$f(5) = \frac{13}{15} + \frac{5}{39} = 0.9949$

$$f''(x) = \frac{1}{15}\cdot\frac{\sqrt{12^2+x^2} - \frac{x^2}{\sqrt{x^2+12^2}}}{12^2+x^2} = \frac{12^2}{15(12^2+x^2)^{\frac{3}{2}}} > 0$$

$f(0) \approx 1.0564 \qquad f(10) = \frac{\sqrt{244}}{15} \approx 1.0414$

$\because f(5) < f(0)$, $f(5) < f(10)$

$\therefore$ head for point 5 km east of A

③ $f(x) = x^2 \cdot e^{-x^2}$

1) $x \in \mathbb{R}$

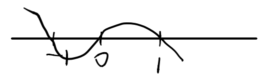
2) $f'(x) = 2x\cdot e^{-x^2} + x^2\cdot e^{-x^2}\cdot(-2x)$

$= 2xe^{-x^2}(1-x^2)$

when $f'(x) > 0$

$2xe^{-x^2}(1-x^2) > 0$

$x(1-x)(1+x) > 0$



$\Rightarrow x \in (-\infty,-1)\cup(0,1)$

$\therefore f'(x) > 0$ (↗) when $x \in (-\infty,-1)\cup(0,1)$

$f'(x) < 0$ (↘) when $x \in (-1,0)\cup(1,+\infty)$

3) $f'(x) = e^{-x^2}\cdot 2x(1-x^2) = e^{-x^2}(2x-2x^3)$

$f''(x) = e^{-x^2}\cdot(-2x)\cdot 2x(1-x^2) + e^{-x^2}(2-6x^2)$

$= e^{-x^2}\cdot -4x^2(1-x^2) + 2e^{-x^2}(1-3x^2)$

$= 2e^{-x^2}[-2x^2(1-x^2) + (1-3x^2)] = 2e^{-x^2}[-2x^2+2x^4+1-3x^2]$

$= 2e^{-x^2}\cdot(2x^4-5x^2+1)$

When $f''(x) > 0$

$2e^{-x^2}(2x^4-5x^2+1) > 0$

$\Rightarrow 2x^4-5x^2+1 > 0$.

let $x^2 = t$ $(t \geq 0)$

$2t^2-5t+1 > 0$

$t^* = \frac{5\pm\sqrt{25-4\cdot 2}}{4} = \frac{5\pm\sqrt{17}}{4}$

$\Rightarrow t > \frac{5+\sqrt{17}}{4}$ or $t < \frac{5-\sqrt{17}}{4}$
$(x^2) \qquad (x^2)$

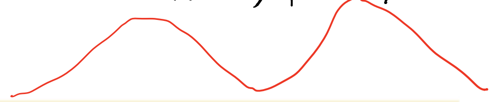
$\therefore$ ↗ : $x \in \left(\left(\frac{5+\sqrt{17}}{4}\right)^{\frac{1}{2}}, +\infty\right) \cup \left(-\infty, -\left(\frac{5+\sqrt{17}}{4}\right)^{\frac{1}{2}}\right)$
$\cup \left(-\left(\frac{5-\sqrt{17}}{4}\right)^{\frac{1}{2}}, \left(\frac{5-\sqrt{17}}{4}\right)^{\frac{1}{2}}\right)$

$\therefore f''(x) > 0$ ∪ convex
when $x \in (-\infty,-1.51)\cup(-0.47,0.47)\cup(1.51,+\infty)$

$f''(x) < 0$ ∩ concave
when $x \in (-1.51,-0.47)\cup(0.47,1.51)$

4) $\lim_{x\to+\infty} x^2\cdot e^{-x^2} = 0$ $\quad\therefore$ Horizontal asymptote : $y=0$.

$\lim_{x\to-\infty} x^2\cdot e^{-x^2} = 0$



The graph of $y = f(x)$ has a **horizontal asymptote** $y = L$ if

either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, or both.

5) $f(x) = x^2\cdot e^{-x^2}$

$f(-x) = (-x)^2\cdot e^{-(-x)^2} = x^2\cdot e^{-x^2} = f(x)$

$\therefore$ Even.

6)