C3.6 : 5, 14, 63, 82

C3.8 : 2, 10, 27, 53, 102, 104

C 3.11 : 6, 15, 41

C3.6.

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = dy/dx = f'(g(x))g'(x)$.

5. $y = \sqrt{u}, \quad u = \sin x$

6. $y = \sin u, \quad u = x - \cos x$

$y = \sqrt{u}$, $u = \sin x$ $\Rightarrow$ $y = \sqrt{\sin x} = (\sin x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}\cdot(\sin x)^{-\frac{1}{2}}\cdot\cos x = \frac{1}{2}\cdot\sqrt{\frac{1}{\sin x}}\cdot\cos x = \frac{\cos x}{2\cdot\sqrt{\sin x}}$$

OR

$$\frac{dy}{dx} = f'(g(x))\cdot g'(x) = f'(u)\cdot(\sin x)' = \frac{1}{2\sqrt{u}}\cdot\cos x = \frac{\cos x}{2\cdot\sqrt{\sin x}}$$

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

14. $u = 3x^2 - 4x + 6$

$y = f(u) = \sqrt{u}$, $u = g(x) = 3x^2 - 4x + 6$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot(6x-4) = \frac{3x-2}{\sqrt{3x^2-4x+6}}$$

In Exercises 51–70, find dy/dt.

63. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

$u = g(t) = 1 + \tan^4\left(\frac{t}{12}\right)$

Note: $(\tan x)' = \sec^2 x$

$y = f(u) = u^3$

$$\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dt} = 3u^2\cdot\left[0 + 4\cdot\tan^3\left(\frac{t}{12}\right)\cdot\sec^2\left(\frac{t}{12}\right)\cdot\frac{1}{12}\right]$$

$$= 3\cdot\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\cdot\left[\frac{1}{3}\cdot\tan^3\left(\frac{t}{12}\right)\cdot\sec^2\left(\frac{t}{12}\right)\right]$$

$$= \left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\cdot\tan^3\left(\frac{t}{12}\right)\cdot\sec^2\left(\frac{t}{12}\right)$$

Finding Derivative Values

In Exercises 81–86, find the value of $(f \circ g)'$ at the given value of x.

81. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

82. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1-x}, \quad x = -1$

Solution 1 :

$$f\circ g = f[g(x)] = f\left[\frac{1}{1-x}\right] = 1 - \frac{1}{\frac{1}{1-x}} = 1-(1-x) = x$$

$(f\circ g)' = 1$ $\therefore$ when $x=-1$, $(f\circ g)'(-1) = 1$

Solution 2:

$g'(x) = (-1)\cdot(1-x)^{-2}\cdot(-1) = (1-x)^{-2}$ $\quad g'(-1) = 2^{-2} = \frac{1}{4}$ $\quad g(-1) = \frac{1}{2}$

$f'(u) = 0 + \frac{1}{u^2} = \frac{1}{u^2}$

$(f\circ g)'(-1) = f'[g(-1)]\cdot g'(-1) = f'\left(\frac{1}{2}\right)\cdot g'(-1) = 4\times\frac{1}{4} = 1$

C3.8

Derivatives of Inverse Functions

In Exercises 1–4:

a. Find $f^{-1}(x)$.

b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points, $df^{-1}/dx = 1/(df/dx)$.

1. $f(x) = 2x + 3, \quad a = -1$

2. $f(x) = \frac{x+2}{1-x}, \quad a = \frac{1}{2}$

(a) $y = \frac{x+2}{1-x}$

$y(1-x) = x+2$

$y - xy = x+2$

$xy + x = y-2$

$x(y+1) = y-2$

$x = \frac{y-2}{y+1}$

Interchang x and y:

$y = \frac{x-2}{x+1}$

$\therefore f^{-1}(x) = \frac{x-2}{x+1}$

(b) $f(x) = \frac{x+2}{1-x} = \frac{x-1+3}{1-x} = -1 + \frac{3}{1-x} = -\left[1+\frac{3}{x-1}\right]$

$f^{-1}(x) = \frac{x-2}{x+1} = \frac{x+1-3}{x+1} = 1 - \frac{3}{x+1} = -\left[\frac{3}{x+1} - 1\right]$

Solution 1:

$$\frac{3}{x} \xrightarrow{\text{right 1 unit}} \frac{3}{x-1} \xrightarrow{\text{up 1}} 1 + \frac{3}{x-1} \xrightarrow[y]{\text{reflect}} -\left[1+\frac{3}{x-1}\right]$$

$$\xrightarrow{\text{left 1 unit}} \frac{3}{x+1} \xrightarrow{\text{down 1}} \frac{3}{x+1} - 1 \xrightarrow[y]{\text{reflect}} -\left[\frac{3}{x+1} - 1\right]$$

Solution 2: Find Asympotic Line

For $f(x)$: $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{x+2}{1-x} = \lim_{x\to+\infty}\frac{-(1-x)+3}{1-x} = \lim_{x\to+\infty} -1 + \frac{3}{1-x} = -1$

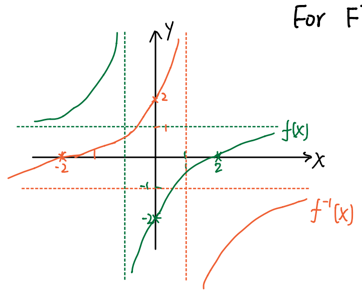
$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{x+2}{1-x} = \lim_{x\to-\infty} -1 + \frac{3}{1-x} = -1$ $\Rightarrow y = -1$

Another Line : $x = 1$

For $F^{-1}(x)$: $\lim_{x\to-\infty} f^{-1}(x) = \lim_{x\to-\infty}\frac{x-2}{x+1} = \lim_{x\to-\infty}\frac{x+1-3}{x+1} = \lim_{x\to-\infty} 1 - \frac{3}{x+1} = 1$

$\lim_{x\to+\infty} f^{-1}(x) = \lim_{x\to+\infty}\frac{x-2}{x+1} = \lim_{x\to-\infty} 1 - \frac{3}{x+1} = 1$

$\Rightarrow y = 1$ Another Line : $x = -1$



(c) $f(x) = \frac{x+2}{1-x} = \frac{x-1+3}{1-x} = -1 + \frac{3}{1-x} = -1 + 3\cdot(1-x)^{-1}$

$$\frac{df}{dx} = 0 + 3\cdot(-1)\cdot(1-x)^{-2}\cdot(-1) = \frac{3}{(1-x)^2}$$

$$f'\left(x=\tfrac{1}{2}\right) = \frac{3}{\left(1-\frac{1}{2}\right)^2} = \frac{3}{\frac{1}{4}} = 12$$

$$f^{-1}(x) = \frac{x-2}{x+1} = \frac{x+1-3}{x+1} = 1 - \frac{3}{x+1} = 1 - 3(x+1)^{-1}$$

$$\frac{df^{-1}}{dx} = 0 - 3\cdot(-1)\cdot(x+1)^{-2}\cdot 1 = \frac{3}{(x+1)^2}$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(\frac{1}{2})} = \left.\frac{df^{-1}}{dx}\right|_{x=5} = \frac{3}{(5+1)^2} = \frac{3}{36} = \frac{1}{12}$$

$$f\left(\tfrac{1}{2}\right) = \frac{\frac{1}{2}+2}{1-\frac{1}{2}} = 1+4 = 5$$

10. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

Goal: $g^{-1\prime}(x=0) = ?$

Recall:

THEOREM 3—The Derivative Rule for Inverses
If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \qquad (1)$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

$\because$ $g(x)$ passes through $(0,0)$ with slope 2.

$\therefore g(0)=0,\ g'(0)=2$

$\therefore g^{-1}(0)=0$

$$\left.\frac{dg^{-1}}{dx}\right|_{x=0} = \frac{1}{g'[g^{-1}(0)]} = \frac{1}{g'(0)} = \frac{1}{2}$$

Derivatives of Logarithms

In Exercises 11–40, find the derivative of y with respect to x, t, or θ, as appropriate.

27. $y = \dfrac{\ln x}{1 + \ln x}$

$$y = \frac{\ln x}{1+\ln x} = \frac{1+\ln x - 1}{1+\ln x} = 1 - \frac{1}{1+\ln x} = 1-(1+\ln x)^{-1}$$

$$y' = 0-(-1)\cdot(1+\ln x)^{-2}\cdot\frac{1}{x} = \frac{1}{x\cdot(1+\ln x)^2}$$

Logarithmic Differentiation

In Exercises 41–54, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

53. $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$

54. $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

$$y = \left(\frac{x(x-2)}{x^2+1}\right)^{1/3}$$

$$\ln y = \frac{1}{3}\ln\left(\frac{x(x-2)}{x^2+1}\right) = \frac{1}{3}\cdot\left[\ln x(x-2) - \ln(x^2+1)\right]$$

$$= \frac{1}{3}\cdot\left[\ln x + \ln(x-2) - \ln(x^2+1)\right]$$

$$\frac{d}{dx}\ln y = \frac{1}{y}\cdot y' = \frac{1}{3}\left[\frac{1}{x}+\frac{1}{x-2}-\frac{1}{x^2+1}\cdot 2x\right]$$

$$\Rightarrow \frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x}+\frac{1}{x-2}-\frac{2x}{x^2+1}\right)$$

$$\Rightarrow y' = \frac{1}{3}\cdot\sqrt[3]{\frac{x(x-2)}{x^2+1}}\cdot\left(\frac{1}{x}+\frac{1}{x-2}-\frac{2x}{x^2+1}\right)$$

102. Show that $\lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = e^x$ for any $x > 0$.

Recall:

THEOREM 4—The Number e as a Limit The number e can be calculated as the limit

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$= \lim_{n\to\infty}\left[\left[1+\frac{1}{\frac{n}{x}}\right]^{\frac{n}{x}}\right]^x = \left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right]^x = e^x \text{ for any } x>0.$$

104. Using mathematical induction, show that for $n > 1$,

$$\frac{d^n}{dx^n}\ln x = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

When $n=1$

$$\frac{d}{dx}\ln x = \frac{1}{x} = (-1)^{1-1}\cdot\frac{(1-1)!}{x^1} = \frac{1}{x}$$

If the statement holds for $n=k$,

then we would have: $\dfrac{d^k}{dx^k}\cdot\ln x = (-1)^{k-1}\cdot\dfrac{(k-1)!}{x^k}$

For $n=k+1$, we would have:

$$\frac{d^{k+1}}{dx^{k+1}}\cdot\ln x = \frac{d}{dx}\left[(-1)^{k-1}\cdot\frac{(k-1)!}{x^k}\right]$$

$$= (-1)^{k-1}\cdot(k-1)!\cdot(-k)\cdot x^{-k-1}$$

$$= (-1)^k\cdot k!\cdot x^{-(k+1)} = (-1)^k\cdot\frac{k!}{x^{k+1}}$$

Thus by mathematical induction the result is established for all $n \geq 1$

Idea of mathematical induction:

Base case: the equation holds. (For example, $n=1$)

Assume $n=k$ still holds, then calculate for $n=k+1$ case based on $n=k$ case. If it still holds for $n=k+1$ case, it means that the equation hold for all case according to the chain.

$n=1$ holds $\rightarrow$ $n=1+1$ holds $\rightarrow$ $n=2+1$ holds $\rightarrow \cdots$ always holds.

C3.11

6. Common linear approximations at $x = 0$ Find the linearizations of the following functions at $x = 0$.

a. $\sin x$ **b.** $\cos x$ **c.** $\tan x$ **d.** e^x **e.** $\ln(1 + x)$

Recall :

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

(a) $f(x) = \sin x \quad f'(x) = \cos x$

At $a = 0$, $L(x) = f(0) + f'(0)\cdot(x-0)$

$= \sin 0 + \cos 0 \cdot x$

$= x$

(b) $f(x) = \cos x \quad f'(x) = -\sin x$

At $a = 0$, $L(x) = f(0) + f'(0)\cdot(x-0)$

$= \cos 0 - \sin 0 \cdot x$

$= 1$

(c) $f(x) = \tan x \quad f'(x) = \sec^2 x$

At $a = 0$, $L(x) = f(0) + f'(0)\cdot(x-0)$

$= \tan 0 + \sec^2 0 \cdot x$

$= 0 + \frac{1}{1^2} \cdot x$

$= x$

(d) $f(x) = e^x \quad f'(x) = e^x$

At $a = 0$, $L(x) = f(0) + f'(0)\cdot(x-0)$

$= e^0 + e^0 \cdot x$

$= 1 + x$

(e) $f(x) = \ln(1+x) \quad f'(x) = \frac{1}{1+x}$

At $a = 0$, $L(x) = f(0) + f'(0)\cdot(x-0)$

$= \ln 1 + \frac{1}{1+0} \cdot x$

$= 0 + x = x$

15. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

$f(x) = (1+x)^k \quad f'(x) = k\cdot(1+x)^{k-1}\cdot 1 = k(1+x)^{k-1}$

At $x = 0$, $L(x) = f(0) + f'(0)\cdot(x-0)$

$= (1+0)^k + k\cdot(1+0)^{k-1} \cdot x$

$= 1 + kx$

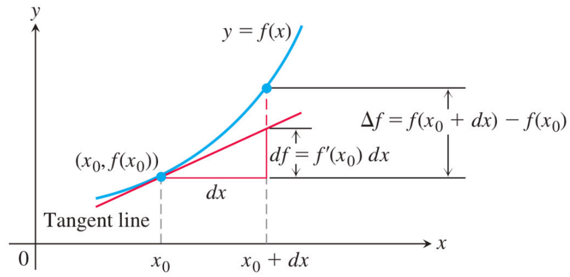
Approximation Error

In Exercises 39–44, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a. the change $\Delta f = f(x_0 + dx) - f(x_0)$;

b. the value of the estimate $df = f'(x_0)\,dx$; and

c. the approximation error $|\Delta f - df|$.



39. $f(x) = x^2 + 2x, \quad x_0 = 1, \quad dx = 0.1$

40. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$

41. $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.1$

$f(x) = x^3 - x \quad f'(x) = 3x^2 - 1$

(a) $\Delta f = f(x_0 + dx) - f(x_0)$

$= f(1+0.1) - f(1)$

$= (1.1^3 - 1.1) - (1 - 1)$

$= 0.231$

(b) $df = f'(x_0)\cdot dx$

$= f'(1)\cdot 0.1 = (3\times 1^2 - 1)\times 0.1 = 0.2$

(c) $|\Delta f - df| = |0.231 - 0.2| = 0.031$