C3.1 : 18, 22, 29, 36
C3.2 : 6, 17*, 56
C3.3 : 15, 16, 21, 56
C3.5 : 9, 10, 64.

C3.1

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

17. $f(x) = \sqrt{x}$, $(4, 2)$ **18.** $f(x) = \sqrt{x+1}$, $(8, 3)$

$f(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

Recall :

> **DEFINITION** The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is
> $$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h},$$
> provided this limit exists.

$$f'(8) = \lim_{h\to0} \frac{f(8+h) - f(8)}{h} = \lim_{h\to0} \frac{\sqrt{8+h+1} - \sqrt{8+1}}{h}$$

$$= \lim_{h\to0} \frac{\sqrt{9+h} - 3}{h} = \lim_{h\to0} \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h\cdot(\sqrt{9+h}+3)} = \lim_{h\to0} \frac{h}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h\to0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

$\therefore$ Slope is $\frac{1}{6}$

Equation : $y = \frac{1}{6}(x-8)+3$

$\Rightarrow y = \frac{1}{6}x + \frac{5}{3}$

In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x - 3x^2$, $x = 1$ **20.** $y = x^3 - 2x + 7$, $x = -2$

21. $y = \frac{1}{x-1}$, $x = 3$ **22.** $y = \frac{x-1}{x+1}$, $x = 0$

$$y = \frac{x-1}{x+1} = \frac{x+1-2}{x+1} = 1 - \frac{2}{x+1}$$

$$f'(0) = \lim_{h\to0} \frac{\left(1-\frac{2}{0+h+1}\right) - \left(1 - \frac{2}{0+1}\right)}{h}$$

$$= \lim_{h\to0} \frac{2 - \frac{2}{h+1}}{h}$$

$$= \lim_{h\to0} \frac{2(h+1)-2}{h(h+1)} = \lim_{h\to0} \frac{2h}{h(h+1)} = \lim_{h\to0} \frac{2}{h+1} = \frac{2}{0+1} = 2$$

OR

$$f'(0) = \lim_{h\to0} \frac{\frac{h-1}{h+1} - \frac{-1}{1}}{h} = \lim_{h\to0} \frac{\frac{h-1}{h+1}+1}{h}$$

$$= \lim_{h\to0} \frac{(h-1)+(h+1)}{h(h+1)} = \lim_{h\to0} \frac{2h}{h(h+1)} = \lim_{h\to0} \frac{2}{h+1} = \frac{2}{0+1} = 2$$

Rates of Change

29. Object dropped from a tower An object is dropped from the top of a 100-m-high tower. Its height above ground after t s is $100 - 4.9t^2$ m. How fast is it falling 2 s after it is dropped?

$f(t) = 100 - 4.9t^2$ $f(t)$: height above ground at t s.

$$f'(2) = \lim_{h\to0} \frac{[100-4.9(2+h)^2] - [100 - 4.9\times2^2]}{h} = \lim_{h\to0} \frac{4.9(2-2-h)(2+2+h)}{h}$$

$$= \lim_{h\to0} \frac{-4.9\cdot h(h+4)}{h} = \lim_{h\to0} -4.9(h+4) = -19.6$$

$\therefore$ the speed is 19.6 m/sec.

36. Does the graph of

$$g(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent line at the origin? Give reasons for your answer.

Analysis of the question :

whether a function has a tangent line at x_0 depends on whether $\lim_{h\to0}\frac{f(x_0+h)-f(x_0)}{h}$ exists.

$$\lim_{h\to0} \frac{(0+h)\cdot\sin\frac{1}{0+h} - 0}{h} = \lim_{h\to0} \sin\frac{1}{h}$$

$\because \lim_{h\to0} \sin\frac{1}{h}$ does Not exist

$\therefore f(x)$ has no tangent at the origin.

C3.2

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

6. $r(s) = \sqrt{2s+1}$; $r'(0)$, $r'(1)$, $r'(1/2)$

$$r'(0) = \lim_{h\to0} \frac{\sqrt{2(0+h)+1} - \sqrt{2\times0+1}}{h} = \lim_{h\to0} \frac{\sqrt{2h+1}-1}{h}$$

$$= \lim_{h\to0} \frac{(\sqrt{2h+1}-1)(\sqrt{2h+1}+1)}{h\cdot(\sqrt{2h+1}+1)} = \lim_{h\to0} \frac{2\cancel{h}}{\cancel{h}\cdot(\sqrt{2h+1}+1)}$$

$$= \frac{2}{\sqrt{2\times0+1}+1} = \frac{2}{1+1} = 1$$

Similar to $r'(1)$ & $r'(\frac{1}{2})$

OR

$$r'(s) = \lim_{h\to0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h} = \lim_{h\to0} \frac{(\sqrt{2(s+h)+1} - \sqrt{2s+1})(\sqrt{2(s+h)+1}+\sqrt{2s+1})}{h(\sqrt{2(s+h)+1}+\sqrt{2s+1})}$$

$$= \lim_{h\to0} \frac{2(s+h)+1-(2s+1)}{h(\sqrt{2(s+h)+1}+\sqrt{2s+1})} = \lim_{h\to0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{2(s+h)+1}+\sqrt{2s+1})}$$

$$= \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}} = \frac{1}{\sqrt{2s+1}}$$

$\therefore r'(0) = \frac{1}{\sqrt{0+1}} = 1$, $r'(1) = \frac{1}{\sqrt{2+1}} = \frac{1}{\sqrt{3}}$, $r'(\frac{1}{2}) = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \dfrac{8}{\sqrt{x-2}}, \quad (x, y) = (6, 4)$

Step 1: Differentiate the function.

Step 2: Find the tangent line at (6,4)

S1: $\lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \dfrac{\frac{8}{\sqrt{x+h-2}} - \frac{8}{\sqrt{x-2}}}{h} = 8\cdot\lim\limits_{h\to0} \dfrac{\sqrt{x-2}-\sqrt{x+h-2}}{h\sqrt{x+h-2}\cdot\sqrt{x-2}}$

$= 8\cdot\lim\limits_{h\to0} \dfrac{(x-2)-(x+h-2)}{h\sqrt{x+h-2}\cdot\sqrt{x-2}\,(\sqrt{x-2}+\sqrt{x+h-2})} = 8\cdot\lim\limits_{h\to0}\dfrac{-\cancel{h}}{\cancel{h}\sqrt{x+h-2}\cdot\sqrt{x-2}\,(\sqrt{x-2}+\sqrt{x+h-2})}$

$= -8\cdot \times \dfrac{1}{\sqrt{x-2}\cdot\sqrt{x-2}\cdot(\sqrt{x-2}+\sqrt{x-2})} = \dfrac{-8}{(x-2)\cdot2\sqrt{x-2}} = -\dfrac{4}{(x-2)\cdot\sqrt{x-2}}$

$\therefore f'(x) = -\dfrac{4}{(x-2)\cdot\sqrt{x-2}}$

S2: $f'(6) = -\dfrac{4}{(6-2)\cdot\sqrt{6-2}} = -\dfrac{1}{2}$ $\quad\therefore$ the slope is $-\frac{1}{2}$

Equation of the tangent line at (6,4):

$y = -\frac{1}{2}(x-6)+4 = -\frac{1}{2}x+7$

56. Tangent line to $y=\sqrt{x}$ Does any tangent line to the curve $y=\sqrt{x}$ cross the x-axis at $x=-1$? If so, find an equation for the line and the point of tangency. If not, why not?

If there is a tangent line to the curve $y=\sqrt{x}$ cross the x-axis at $x=-1$, then we could assume the line function:

$g(x) = k(x+1)+0 = k(x+1)$ | or $g(x)=kx+b$ put $(-1,0)$ inside $k\cdot(-1)+b=0 \Rightarrow b=k$

$f(x)=\sqrt{x}$

$\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \lim\limits_{h\to0}\dfrac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to0}\dfrac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \lim\limits_{h\to0}\dfrac{1}{\sqrt{x+h}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

Assume tangent point is $(x_0, \sqrt{x_0})$, then $k=\dfrac{1}{2\sqrt{x_0}}$

$\therefore \dfrac{1}{2\sqrt{x_0}}(x_0+1) = \sqrt{x_0}$

$x_0+1 = 2x_0$

$x_0 = 1$

$\therefore$ the slope is $\frac{1}{2}$, the tangent point is $(1,1)$,

the tangent line is $y=\frac{1}{2}x+\frac{1}{2}$.

C3.3.

In Exercises 13–16, find y' **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y=(3-x^2)(x^3-x+1)$ **14.** $y=(2x+3)(5x^2-4x)$

15. $y=(x^2+1)\left(x+5+\dfrac{1}{x}\right)$ **16.** $y=(1+x^2)(x^{3/4}-x^{-3})$

(a)

Recall: Product Rule

Derivative Product Rule
If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

15. $y=(x^2+1)(x+5+\frac{1}{x})$

$y' = \frac{d}{dx}(x^2+1)(x+5+\frac{1}{x}) + (x^2+1)\cdot\frac{d}{dx}(x+5+\frac{1}{x})$

$= 2x(x+5+\frac{1}{x}) + (x^2+1)(1+0-\frac{1}{x^2})$

$= 2x^2+10x+2+x^2+1-1-\frac{1}{x^2}$

$= 3x^2+10x-\frac{1}{x^2}+2$

16. $y=(1+x^2)(x^{3/4}-x^{-3})$

$y' = \frac{d}{dx}(1+x^2)(x^{3/4}-x^{-3}) + (1+x^2)\frac{d}{dx}(x^{3/4}-x^{-3})$

$= (0+2x)(x^{3/4}-x^{-3}) + (1+x^2)\left[\frac{3}{4}x^{-\frac{1}{4}}-(-3)\cdot x^{-4}\right]$

$= 2x^{7/4}-2x^{-2}+\frac{3}{4}x^{-1/4}+3\cdot x^{-4}+\frac{3}{4}\cdot x^{7/4}+3\cdot x^{-2}$

$= \frac{11}{4}x^{7/4}+\frac{3}{4}x^{-1/4}+3x^{-4}+x^{-2}$

(b)

15. $y=(x^2+1)(x+5+\frac{1}{x}) = x^3+5x^2+x+x+5+\frac{1}{x} = x^3+5x^2+2x+\frac{1}{x}+5$

$y'=3x^2+10x+2-\frac{1}{x^2}$

16. $y=(1+x^2)(x^{3/4}-x^{-3}) = x^{3/4}-x^{-3}+x^{11/4}-x^{-1}$

$y' = \frac{3}{4}\cdot x^{-1/4}-(-3)\cdot x^{-4}+\frac{11}{4}\cdot x^{7/4}-(-1)\cdot x^{-2}$

$= \frac{11}{4}\cdot x^{7/4}+\frac{3}{4}\cdot x^{-1/4}+x^{-2}+3x^{-4}$

Find the derivatives of the functions in Exercises 17–40.

21. $v=(1-t)(1+t^2)^{-1}$ **22.** $w=(2x-7)^{-1}(x+5)$

Recall:

Derivative Quotient Rule
If u and v are differentiable at x and if $v(x)\neq0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

$v=(1-t)(1+t^2)^{-1} = \dfrac{1-t}{1+t^2}$

$\dfrac{d}{dt}\left(\dfrac{1-t}{1+t^2}\right) = \dfrac{(1+t^2)\cdot(-1)-(1-t)\cdot2t}{(1+t^2)^2} = \dfrac{-1-t^2-2t+2t^2}{(1+t^2)^2}$

$= \dfrac{t^2-2t-1}{(1+t^2)^2}$

56. a. Horizontal tangent lines Find equations for the horizontal tangent lines to the curve $y=x^3-3x-2$. Also find equations for the lines that are perpendicular to these tangent lines at the points of tangency.

b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent line at this point.

(a) $y=x^3-3x-2$

$y'=3x^2-3$

$\because$ horizontal tangent lines

$\therefore$ slope $=0$

$\therefore 3x^2-3=0 \Rightarrow x=\pm1$

when $x=1$, the tangent point is $(1,-4)$,

the tangent line is : $y=0\cdot x-4 \Rightarrow y=-4$

the line perpendicular to $y=-4$ is $x=1$;

when $x=-1$, the tangent point is $(-1,0)$,

the tangent line is : $y=0\cdot x+0 \Rightarrow y=0$

the line perpendicular to $y=0$ is $x=-1$.

(b) $y'=3x^2-3$

$x^2\geq0$

$\Rightarrow 3x^2-3\geq-3$

$\therefore$ the smallest slope is -3.

when $3x^2-3=-3$

$3x^2=0$

$x=0$

$\therefore$ the point is $(0,-2)$

The slope of the line that perpendicular to the tangent line is $\frac{1}{3}$. $(\frac{1}{3}\times(-3)=-1)$

$\therefore$ Equation of the perpendicular line is:

$y=\frac{1}{3}(x-0)-2$

$\Rightarrow y=\frac{1}{3}x-2$

C3.5.

Derivatives

In Exercises 1–18, find dy/dx.

9. $y = xe^{-x} \sec x$

10. $y = (\sin x + \cos x) \sec x$

$\sec x = \frac{1}{\cos x}$

9. $y = \frac{x \cdot e^{-x}}{\cos x}$

$$\frac{dy}{dx} = \frac{-\sin x \cdot x \cdot e^{-x} + \cos x \cdot [e^{-x} + x \cdot e^{-x} \cdot (-1)]}{\cos^2 x}$$

$$= \frac{e^{-x}[\cos x - x \cdot \sin x - x \cdot \cos x]}{\cos^2 x}$$

$$= e^{-x}(\sec x - x \cdot \tan x \cdot \sec x - x \cdot \sec x)$$

$$= e^{-x} \cdot \sec x\,(1 - x \cdot \tan x - x)$$

10. $y = (\sin x + \cos x) \cdot \sec x = \tan x + 1$

$\frac{dy}{dx} = \sec^2 x$

Note: $\frac{d}{dx} \tan x = \sec^2 x$.

64. Assume that a particle's position on the x-axis is given by

$$x = 3\cos t + 4\sin t,$$

where x is measured in meters and t is measured in seconds.

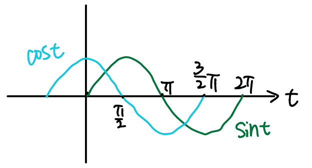
a. Find the particle's position when $t = 0$, $t = \pi/2$, and $t = \pi$.

b. Find the particle's velocity when $t = 0$, $t = \pi/2$, and $t = \pi$.

(a) when $t=0$, $x = 3 \cdot \cos 0 + 4 \cdot \sin 0 = 3$

when $t = \frac{\pi}{2}$, $x = 3 \cdot \cos\frac{\pi}{2} + 4 \sin\frac{\pi}{2} = 3 \times 0 + 4 \times 1 = 4$

when $t = \pi$, $x = 3 \cdot \cos\pi + 4 \cdot \sin\pi = 3 \times (-1) + 4 \times 0 = -3$



(b) $x(t) = 3 \cdot \cos t + 4 \cdot \sin t$

$x' = \frac{dx}{dt} = 3 \cdot (-\sin t) + 4\cos t = 4\cos t - 3\sin t$

when $t = 0$, $x' = 4 \cdot \cos 0 - 3 \cdot \sin 0 = 4$

when $t = \frac{\pi}{2}$, $x' = 4 \cdot \cos\frac{\pi}{2} - 3 \cdot \sin\frac{\pi}{2} = 0 - 3 \times 1 = -3$

when $t = \pi$, $x' = 4 \cdot \cos\pi - 3 \cdot \sin\pi = 4 \times (-1) - 3 \times 0 = -4$