C2.5 : 15, 45, 61, 62

C2.6 : 21, 23, 35, 95, 105*

C2.5

At what points are the functions in Exercises 13–32 continuous?

13. $y = \frac{1}{x-2} - 3x$ **14.** $y = \frac{1}{(x+2)^2} + 4$

15. $y = \frac{x+1}{x^2 - 4x + 3}$ **16.** $y = \frac{x+3}{x^2 - 3x - 10}$

15. $y = \frac{x+1}{x^2-4x+3} = \frac{x+1}{(x-1)(x-3)}$

When $x \neq 1$ and $x \neq 3$, the function is continous.

OR

When $x \in (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$

45. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

If $f(x)$ is continuous at every x, then $f(x)$ needs to be continuous at $x = 3$.

$3^2 - 1 = 2 \times a \times 3$

$9 - 1 = 6a$

$6a = 8$

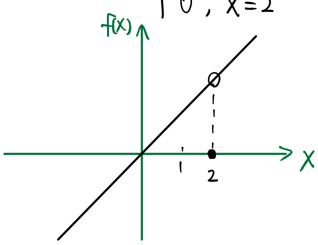
$a = \frac{4}{3}$

61. Removable discontinuity Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

Recall: What's removable discontinuity.

at $x = 2$ is called a "removable discontinuity" since by changing the function definition at that one point, we can create a new function that is continuous at $x = 2$. Similarly, $x = 4$

Ex. $f(x) = \begin{cases} x, & x \neq 2 \\ 0, & x = 2 \end{cases}$



$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x = 2$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x = 2$

$\therefore \lim_{x \to 2} f(x) = 2$

$f(2) = 0$

$\therefore \lim_{x \to 2} f(x) \neq f(2)$

$\therefore f(x)$ is not continuous at $x = 2$

By changing the definition at $x = 2$ (i.e., $f(2) = 2$). Then, the new function could be continuous at every x.

62. Nonremovable discontinuity Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

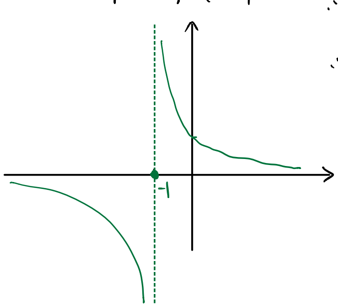
$g(x) = \begin{cases} \frac{1}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases}$

$\lim_{x \to -1^+} g(x) = +\infty$ $\lim_{x \to -1^-} g(x) = -\infty$

$\therefore \lim_{x \to -1^+} g(x) \neq \lim_{x \to -1^-} g(x)$

$\therefore \lim_{x \to -1} g(x)$ does NOT exist.

$\therefore g(x)$ is discontinuous at $x = -1$



No matter how we change the definition at $x = -1$, the new function won't be continous at $x = -1$.

C2.6

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. Write ∞ or $-\infty$ where appropriate.

21. $f(x) = \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$ **22.** $h(x) = \frac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$

(a) $f(x) = \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3} = \frac{3 + 5 \cdot (\frac{1}{x})^5 - (\frac{1}{x})^7}{6 \cdot (\frac{1}{x})^4 - 7 \cdot (\frac{1}{x})^6 + 3 \cdot (\frac{1}{x})^7}$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3} \underset{\div x^3}{=} \lim_{x \to +\infty} \frac{3x^4 + 5(\frac{1}{x}) - (\frac{1}{x})^3}{6 - 7 \cdot (\frac{1}{x})^2 + 3(\frac{1}{x})^3} = +\infty$

(b) $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3} \underset{\div x^3}{=} \lim_{x \to -\infty} \frac{3x^4 + 5(\frac{1}{x}) - (\frac{1}{x})^3}{6 - 7 \cdot (\frac{1}{x})^2 + 3(\frac{1}{x})^3} = +\infty$

Limits as $x \to \infty$ or $x \to -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x.

Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36. Write ∞ or $-\infty$ where appropriate.

23. $\lim_{x \to \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$ **24.** $\lim_{x \to -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

$\lim_{x \to \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}} = \lim_{x \to \infty} \sqrt{\frac{8 - 3(\frac{1}{x})^2}{2 + (\frac{1}{x})}} = \lim_{x \to \infty} \sqrt{\frac{8+0}{2+0}} = \sqrt{4} = 2$

35. $\lim_{x \to \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$

$\lim_{x \to \infty} \frac{x-3}{\sqrt{4x^2+25}} = \lim_{x \to \infty} \sqrt{\frac{x^2 - 6x + 9}{4x^2 + 25}} = \lim_{x \to \infty} \sqrt{\frac{1 - 6 \cdot (\frac{1}{x}) + 9 \cdot (\frac{1}{x})^2}{4 + 25 \cdot (\frac{1}{x})^2}}$

$= \lim_{x \to \infty} \sqrt{\frac{1 - 0 + 0}{4 + 0}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 93 and 94.

93. If f has the constant value $f(x) = k$, then $\lim_{x \to \infty} f(x) = k$.

94. If f has the constant value $f(x) = k$, then $\lim_{x \to -\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 95–98.

95. $\lim_{x \to 0} \frac{-1}{x^2} = -\infty$ **96.** $\lim_{x \to 0} \frac{1}{|x|} = \infty$

According to the definition of limitation:

For every real number $-B < 0$, we always can find a $\delta > 0$ such that for all x that statifies $0 < |x - 0| < \delta$, we have

$-\frac{1}{x^2} < -B$ $\quad 0 < |x - 0| < \delta$

$\Rightarrow \frac{1}{x^2} > B$ $\quad 0 < |x| < \delta$

$\Rightarrow x^2 < \frac{1}{B}$

$\Rightarrow |x| < \frac{1}{\sqrt{B}}$

$\therefore \delta_{max} = \frac{1}{\sqrt{B}}$ $\quad (\delta \in (0, \frac{1}{\sqrt{B}}])$

$\therefore$ For every number $-B < 0$, there exists a corresponding number $\delta = \frac{1}{\sqrt{B}}$ (or smaller positive number) such that $-\frac{1}{x^2} < -B$ whenever $0 < |x - 0| < \delta$ $\therefore \lim_{x \to 0} -\frac{1}{x^2} = -\infty$

95. For every real number $-B<0$, we must find a $\delta>0$ such that for all x, $0<|x-0|<\delta \Rightarrow \frac{-1}{x^2}<-B$. Now, $-\frac{1}{x^2}<-B<0 \Leftrightarrow \frac{1}{x^2}>B>0 \Leftrightarrow x^2<\frac{1}{B} \Leftrightarrow |x|<\frac{1}{\sqrt{B}}$. Choose $\delta=\frac{1}{\sqrt{B}}$, or any smaller positive value, then $0<|x|<\delta \Rightarrow |x|<\frac{1}{\sqrt{B}} \Rightarrow \frac{-1}{x^2}<-B$ so that $\lim\limits_{x\to 0} -\frac{1}{x^2}=-\infty$.

Oblique Asymptotes

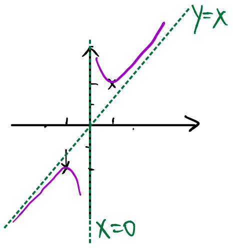
Graph the rational functions in Exercises 105–110. Include the graphs and equations of the asymptotes.

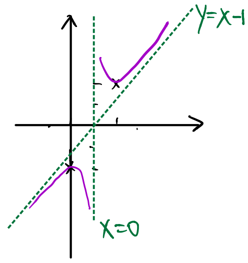
105. $y = \dfrac{x^2}{x-1}$ **106.** $y = \dfrac{x^2+1}{x-1}$

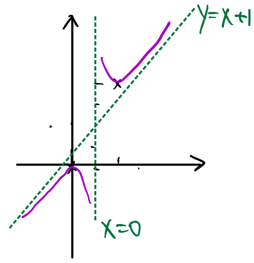
$$y=\frac{x^2}{x-1}$$

$$y=\frac{x^2-1+1}{x-1}=\frac{x^2-1}{x-1}+\frac{1}{x-1}=x+1+\frac{1}{x-1}=(x-1)+\frac{1}{x-1}+2$$

$$f_0(x)=x+\frac{1}{x}\xrightarrow{\text{right 1 unit}} f_1(x)=x-1+\frac{1}{x-1}\xrightarrow{\text{up 2 units}} f_2(x)=(x-1)+\frac{1}{x-1}+2$$







OR

Find asymptotic line for $y=\frac{x^2}{x-1}$

one is : $x-1=0 \Rightarrow x=1$

another one : $y=ax+b$

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x}{x-1}=\lim_{x\to\infty}\frac{1}{1-\frac{1}{x}}=1$$

$$b=\lim_{x\to\infty} f(x)-ax=\lim_{x\to\infty}\frac{x^2}{x-1}-x=\lim_{x\to\infty}\frac{x^2-x^2+x}{x-1}=\lim_{x\to\infty}\frac{x}{x-1}=\lim_{x\to\infty}\frac{1}{1-\frac{1}{x}}=1$$

$\therefore$ $y=x+b$,

Then, you can draw the graph based on finded asymptotic lines.