《Attack Resilient H_2, H_∞ and L_1 state estimator》

$$x(t+1) = Ax(t) \qquad x \in \mathbb{R}^n \qquad \|e\|_0 \leq k < m$$
$$y(t) = Cx(t) + e(t) \qquad y \in \mathbb{R}^m$$

Definition (projection map)

$P_I = [e_{i_1}, \cdots, e_{i_p}] \in \mathbb{R}^{e \times m}$ is a projection map

e_i is the ith canonical basis vector of the space $\mathbb{R}^m$

$|I| = p = m-k$

Redefine the system model for Local estimator:

$$y_I(t) = P_I y(t) \in \mathbb{R}^k$$
$$C_I = P_I C$$
$$\boxed{\begin{aligned} x(t+1) &= Ax(t) \\ y_I(t) &= C_I x(t) + e(t) \end{aligned}}$$

Estimator Design

Local estimator + Global Fusion.

① Local estimators:

1° $V \triangleq \{I \subset \{1,2,\cdots,m\} : |I| = m-k\}$

(Group of Local estimators)

Define $|V| = \binom{m}{p}$

2° Find estimator Gain (K^I):

If (A, C_I) is detectable, then

$$K^I = PC_I^T (C_I P C_I^T)^{-1}$$
$$P = A[P - PC_I^T (C_I P C_I^T)^{-1} - C_I P]A^T \quad \text{[Matlab: idare]}$$

(Riccatic equation)

3° estimate: $\hat{x}^I(t+1) = A\hat{x}^I(t) - K^I(y_I(t) - C_I \hat{x}^I(t))$

error: $e^I(t) \triangleq x(t) - \hat{x}^I(t)$

residual: $r^I(t) \triangleq y_I(t) - C_I \hat{x}^I(t)$

4° check if the local estimation is valid:

$$F^I(k^I) = \begin{bmatrix} A + K^I C_I \\ C_I \end{bmatrix}$$

① $\|r^I\|_2 \leq \|F^I(k^I)\|_{2\to 2} \Rightarrow$ valid

$$V(t) \triangleq \{ I \subset \{1,2,\cdots,m\} : \|r^I(0:t)\|_2 \leq \|F^I(k^I)\|_{2\to 2} \}$$

② $\|r^I\|_2 \leq \|F^I(k^I)\|_{2\to\infty} \Rightarrow$ valid

$$V(t) \triangleq \{ I \in \{1,2,\cdots,m\} : \|r^I(0:t)\|_2 \leq \|F^I(k^I)\|_{2\to\infty} \}$$

③ $$V(t) \triangleq \{ I \in \{1,2,\cdots,m\} : \|r^I(0:t)\|_\infty \leq \|F^I(k^I)\|_{\infty\to\infty} \}$$

5° Global Fusion:

$$\hat{x}(t) = \begin{cases} \frac{1}{|V(t)|} \sum_{I(t) \in V(t)} \hat{x}_i^I(t) & ① \\ \frac{1}{2}\left(\min_{I \in V(t)} \hat{x}_i^I(t) + \max_{J \in V(t)} \hat{x}_i^J(t)\right) & ② ③ \end{cases}$$

$\boxed{\text{Algorithm}}$

1. Initialization: k (number of attacks), (A, C) (system), $\hat{x}_0 = \text{zeros}(n,1)$

2. $|I| = m - k$, $|\mathcal{V}| = \lfloor m/k \rfloor \triangleq a$, $\mathcal{V} = \{I_1, I_2, \cdots, I_a\}$

   $P_I = [e_{i_1}, \cdots, e_{i_{|I|}}]^T \in \mathbb{R}^{|I| \times m}$ (e_i: canonical basis of $\mathbb{R}^m$)

   $C_I = P_I C$, $y_I(t) = P_I y(t)$

3. For $I \subset \mathcal{V}$:

$$\begin{cases} P = \text{idare}(A^T, C^T, \text{zeros}(\text{size}(A^T)), 0, [\,], [\,]) \\ k^I = P C_I^T (C_I P C_I^T)^{-1} \end{cases}$$

   if isnan(P):

   break for.

else:

$$\hat{x}^{I}(t+1) = A\hat{x}(t) - K^{I}(y_{I}(t+1) - C_{I}\hat{x}(t))$$

$$r^{I}(t+1) = y_{I}(t+1) - C_{I}\hat{x}(t)$$

End For.

4. $\vartheta' = \{ I : \|r^{I}(t+1)\|_2 \leq \|F(K^{I})\|_{2\to 2} \}$

where, $F(K^{I}) = \begin{bmatrix} A + K^{I}C_{I} \\ C_{I} \end{bmatrix}$

If ϑ' is ϕ:

Luenburger estimator

else:

$$\hat{x}(t+1) = \frac{1}{|\vartheta'|} \sum_{I \in \vartheta'} \hat{x}^{I}(t+1)$$