# Robust Consensus Control of Leader-Follower Networked System

Yu Zheng, Olugbenga Anubi, *Member, IEEE*

***Abstract*—This paper designs robust consensus control strategies for multi-agent systems with leader-following communication topology. The dynamics of agents are considered as linear time-invariant with external disturbance at leader and heterogeneous matching uncertainties. By system decomposition, the consensus problem can be seen as stability problem, and one-time control design can work for even infinite agents by employing the algebraic connectivity of graph. Robust consensus design is to constraint the $\ell_2$ gain between disturbance and consensus error in the presence of bounded model mismatching uncertainties. Finally, a simulation example on satellite formation control is given to show the extension of consensus control on formation control.**

## I. Introduction

Multi-agent systems have attract a significant amount of research interests in the past decade due not only to its piratical potential in many application areas, such as formation control of mobile robots [1], cruise control of the string of vehicles [2], task assignment in intelligent building [3] and more, but also to the arising theoretical challenges in cooperative control of multiple agents. The main challenge is distributed control strategy design based on partial and relative information without an intervention of a central controller [4].

One of the fundamental problems is consensus control: by employing communication capability of network, the consensus on common control goal, relative information, or global information is achieved by using distributed control strategies. For example, in formation control, all agents would be aware of the state information of agents in the sensing range by using relative sensors, and the information would be shared through wireless communication network [5]. To maintain the desired formation, the group of agents might response unanticipated situations, which requires them response distributively in order to achieve asymptotically convergence of agreement on what changes took place and the control goal. The asymptotically convergence to a common cooperative value is seen as consensus in literature [6].

Until recently, many topics on consensus control have still been studied in a large amount of researches, readers would see [7, 8] for cluster consensus, [9, 10] for iterative learning-based consensus, [11, 12, 13] for robust consensus, [14, 15] for resilient consensus, [16, 17] for leader-follower consensus and a bunch of extension on various application cases, like formation control [4], consensus tracking [18], cruise control [19] and more. Robust consensus topics, in literature, are focusing on external disturbance [20], model uncertainty of agents [13], nonlinearity [21], heterogeneous matching uncertainties [12], communication time delay [11], asymmetric interconnection perturbations [22] and uncertain network graphs [23]. In this paper, we mainly discuss robust consensus with dynamics mismatching uncertainties and unknown external disturbance. And a direct extension on formation control is also proposed.

The reminder of this paper is organized as follows: in section II, for self-consistence, the notations and preliminary introduction of graph theory are presented. In section III, under unknown disturbance introduced through leader, a $\ell_2$ gain based robust consensus controller is given, and corresponding LMI is given to constraint the feasibility of control design. In section IV, mismatching uncertainties among agents' dynamics are also considered in robust consensus control. In section V, a direct extension on formation control is clarified and in section VI two robust consensus control designs are tested on satellite formation control example. All conclusion remarks follow in section VII.

The authors are with Department of Electrical and Computer Engineering, FAMU-FSU College of Engineering.yz19b@fsu.edu, oanubi@fsu.edu

Olugbenga Moses Anubi is the corresponding author.

## II. Notation and Graph Theory

The following notions and conventions are employed throughout the paper: $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times m}, \mathbb{S}^n$ denote the space of real numbers, real vectors of length $n$, real matrices of $n$ rows and $m$ columns, and symmetric matrix of $n$ rows and columns respectively. $\mathbb{R}_+$ denotes positive real numbers. Normal-face lower-case letters ($x \in \mathbb{R}$) are used to represent real scalars, bold-face lower-case letter ($\mathbf{x} \in \mathbb{R}^n$) represents vectors, while normal-face upper case ($X \in \mathbb{R}^{n\times m}$) represents matrices. $X^\top$ denotes the transpose of the matrix $X$. By $Q \succeq 0$, it is meant that $Q$ is a positive semi-definite symmetric matrix, i.e $\mathbf{x}^\top Q\mathbf{x} \geq 0\ \forall \mathbf{x} \neq 0$ and $Q \succ 0$ denotes positive definiteness which is defined with strict $>$ instead. $\otimes$ denotes Kronecker product admitting below property:

*Lemma 2.1:* The product of two Kronecker products yields another Kronecker product, such that

$$(A\otimes B)(C\otimes D) = AC\otimes BD$$

An important lemma, in performance-based robust control, is introduced:

*Lemma 2.2 (Kalman-Yakubovich-Popov (KYP)[24]):* Given $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ with $(A,B)$ controllable and $det(j\omega I - A) \neq 0\ \forall\ \omega \in \mathbb{R}$, and a Hermitian matrix

$M \in \mathbb{S}^{n+m}$, the FDI

$$\begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix}^{\star} M \begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix} \preceq 0$$

holds for all $\omega \in \mathbb{R} \cap \{\infty\}$ if and only if the LMI

$$M + \begin{bmatrix}A^\top P + PA & PB \\ B^\top P & 0\end{bmatrix} \preceq 0$$

admits a Hermitian solution $P$. The corresponding equivalence for strict inequalities holds even if $(A,B)$ is not controllable.

It is natural to model communication activities among agents by a graph. Thus, we will go though some necessary definitions and lemmas in graph theory here. A directed graph $\mathscr{G}(\mathscr{V},\mathscr{A})$ contains vertices, denoted by $\mathscr{V} = \{v_1, v_2, \ldots, v_n\}$, and edges, denoted by $\mathscr{A} \in \mathscr{V} \times \mathscr{V}$. $a = (\alpha, \beta) \in \mathscr{A}$ represents an ordered edge in $\mathscr{A}$ with tail $\alpha$ and head $\beta$. The in-degree of vertex $\alpha$, denoted as $d_i(\alpha)$, is the number of edges with head as $\alpha$, and all in-degrees is collected in the diagonal of the in-degree matrix $D$. By assuming the vertices $\mathscr{G}$ are enumerated, the adjacency matrix, denoted by $G(\mathscr{G})$, is a square matrix of size $|\mathscr{V}|$, defined as [25]

$$G_{ij} = \begin{cases} 1 & \text{if } (\alpha_i, \alpha_j) \in \mathscr{G} \\ 0 & \text{otherwise} \end{cases}.$$

Then, the Laplacian matrix of the graph $\mathscr{G}$ is given as

$$L = D - G$$

Notice that the rows of $L$ necessarily sum to zero. And $L$ has zero eigenvalue associated with eigenvector $\mathbf{1}$ such that $L\mathbf{1} = 0$ [26].

Spanning tree is a directed tree that contains all the nodes and some edges. A digraph is said to have a spanning tree if there exists at least one node that has a directed path to all other nodes. [25].

*Lemma 2.3:* If the directed graph $\mathscr{G}$ contains a spanning tree, then the 0 eigenvalue of Laplacian matrix $L$ is simple, and all other eigenvalues of $L$ have positive real parts.

In literature, people usually describe leader-follower graph by describing followers' graph using a Laplacian matrix $L$ and the connectivity between leader and followers using a pinning matrix $P$ for [27]. Then the whole graph is described by a new matrix $\hat{L} = L + P$. In this paper, we describe the leader-follower graph without isolating the leader, such that the Laplacian matrix is shown in the form

$$L = \begin{bmatrix}0 & 0 \\ l_1 & L_2\end{bmatrix}.$$

A leader-follower graph is said to have spanning tree if there is at least one follower connect to the leader and the follower network graph contains a spanning tree.

## III. Robust Consensus against External Disturbance

In this section, we initially talk about robust consensus control against external disturbance. Formal definition of robust consensus is given, and a system decomposition method is summarized. Based on the decomposed systems, we only need to design controller once for all agents by utilizing the algebraic connectivity of graph (second largest eigenvalue of Laplacian matrix). A LMI-based control design will be given, consensus is also achieved by the robust consensus design.

Consider a general linear agent dynamics for $N$ agents in a leader-follower system:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= A\mathbf{x}_1 + \mathbf{w} && \text{(leader)} \\ \dot{\mathbf{x}}_i &= A\mathbf{x}_i + B\mathbf{u}_i && \text{(followers)} \end{aligned} \tag{1}$$

where, $\mathbf{x}_i, \mathbf{w} \in \mathbb{R}^n$ are the states of $i$-th agent and the external disturbance introduced by the leader respectively, $\mathbf{u}_i \in \mathbb{R}^p$ is the control inputs of $i$-th agent, and the set of followers' indexes is denoted by $\Pi_f = \{2, 3, \cdots, N\}$.

The below assumption is used thoughout the paper:

*Assumption 1:* The directed graph of the leader-follower system in (1) is assumed to contain a spanning tree.

The notions of consensus and robust consensus are defined below

*Definition 1 (Consensus):* The system in (1) is said to reach consensus if and only if

$$\lim_{t\to\infty} \|\mathbf{x}_i - \mathbf{x}_j\| = 0, \ \ \forall i, j = 1, 2, \cdots, N$$

*Definition 2 (Robust Consensus):* The system in (1) is said to reach robust consensus if

$$\|\mathbf{x}_i - \mathbf{x}_j\|_{\mathscr{L}_2} \le \gamma \|\mathbf{w}\|_{\mathscr{L}_2}, \ \ \forall i, j = 1, 2, \cdots, N, i \ne j \tag{2}$$

*Remark 1:* A sufficient condition of (2) is given as

$$\|\mathbf{x}_i - \mathbf{x}_1\|_{\mathscr{L}_2} \le \frac{\gamma}{2} \|\mathbf{w}\|_{\mathscr{L}_2}, \ \ \forall i, j = 1, 2, \cdots, N, i \ne j \tag{3}$$

Define consensus error as $\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_1$ ($\mathbf{e}_1 \equiv 0$), and consider the below consensus control law:

$$\mathbf{u}_i = -cK \sum_{j \in \mathscr{N}_i} (\mathbf{x}_i - \mathbf{x}_j) \tag{4}$$

where, $c \in \mathbb{R}$ is a coupling control parameter designed later, $\mathscr{N}_i$ denotes the communication set of $i$-th agent. Then the collective closed-loop error dynamics is given as

$$\dot{\mathbf{e}} = (I_N \otimes A - cL \otimes BK)\mathbf{e} - \bar{\mathbf{w}} \tag{5}$$

where, $\bar{\mathbf{w}} = [0 \ \ \mathbf{w}^\top \ \ \cdots \ \ \mathbf{w}^\top]^\top$, $\mathbf{e} = [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N]^\top$, and $\mathbf{e}_1 \equiv 0$.

*Lemma 3.1 (System Decomposition):* To achieve robust consensus on (1) under consensus control law in (4), it is equivalent to achieve the robust performance

$$\|\varepsilon_i\|_{\mathscr{L}_2} \le \frac{\gamma}{2} \|\tilde{\mathbf{w}}_i\|_{\mathscr{L}_2} \tag{6}$$

for all $i \in \Pi_f$, where $\varepsilon_i$ is given by

$$\dot{\varepsilon}_i = (A - c\lambda_i BK)\varepsilon_i - \tilde{\mathbf{w}}_i, \ \ i \in \Pi_f \tag{7}$$

where $\lambda_i$ is the non-negative eigenvalue of $L$.

*Proof:* By finding the schur transformation for Laplacian matrix $L$ such that

$$Y^{-1}LY = \begin{bmatrix}0 & 0 \\ 0 & U\end{bmatrix} \triangleq \Sigma,$$

where $Y$ is unitary matrix, $U$ is a upper triangular matrix with $L$'s eigenvalues on its diagonal, we can define an auxiliary error $\tilde{\mathbf{e}} = (Y^{-1} \otimes I_n)\mathbf{e}$, such that $\tilde{\mathbf{e}} = 0$ if and only if $\mathbf{e} = 0$. By using *Lemma 2.1*, an auxiliary error dynamics of (5) is given by

$$\dot{\tilde{\mathbf{e}}} = (I_N \otimes A - c\Sigma \otimes BK)\tilde{\mathbf{e}} - \left(Y^{-1} \otimes I_n\right)\bar{\mathbf{w}} \tag{8}$$

Notice that $\tilde{\mathbf{e}}_1 \equiv 0$, let $\varepsilon = [\tilde{\mathbf{e}}_2^\top, \tilde{\mathbf{e}}_3^\top, \cdots, \tilde{\mathbf{e}}_N^\top]^\top$, then

$$\dot{\varepsilon} = (I_{N-1} \otimes A - cU \otimes BK)\varepsilon - \tilde{\mathbf{w}} \tag{9}$$

where $\tilde{\mathbf{w}} = (Y^{-1} \otimes I_n)[\mathbf{w}^\top \ \mathbf{w}^\top \ \cdots \ \mathbf{w}^\top]^\top$. Then, due to diagonalization, the auxiliary error dynamics in (9) can be decomposed to (7). Since $Y$ is unitary matrix, we have $\|\varepsilon_i\|_{\mathscr{L}_2} = \|\mathbf{e}_i\|_{\mathscr{L}_2}$ and $\|\tilde{\mathbf{w}}_i\|_2 = \|\mathbf{w}\|_2$. Thus, if (6) is achieved for all $i \in \Pi_f$, then (2) is achieved. ■

*Remark 2:* Based on the decomposed error system (6), consensus on (1) is achieved if $(A - c\lambda_i BK)$ for all $i \in \Pi_f$.

Next, the following theorem states the robust consensus on (1) is achieved.

*Theorem 3.2:* The leader-follower system in (1) achieves robust consensus under the control law in (4) with

$$c = \frac{1}{\min\limits_{i\in\Pi_f} \mathrm{Re}(\lambda_i)}, \quad K = QB^\top P, \tag{10}$$

where $P = X^{-1}, Q$ are given by solving below LMI

$$\begin{bmatrix} XA^\top + AX - 2BQB^\top & -I_n & X \\ -I_n & -\frac{\alpha}{4}I_n & 0 \\ X & 0 & -I_n \end{bmatrix} \prec 0 \tag{11}$$
$$X \succeq 0, \quad Q \succeq 0, \quad \alpha \geq 0$$

Furthermore, the $\mathscr{L}_2$ gain for robust consensus is given by $\gamma = \sqrt{\alpha}$.

*Proof:* As stated in *Lemma 3.1*, the robust consensus is achieved if

$$\|\varepsilon_i\|_{\mathscr{L}_2} \leq \frac{\gamma}{2}\|\tilde{\mathbf{w}}_i\|_{\mathscr{L}_2}, \quad \forall i \in \Pi_f$$

which is equivalent to require

$$\left\|\frac{E_i(s)}{W_i(s)}\right\|_\infty \leq \frac{\gamma}{2}, \quad \forall i \in \Pi_f$$

For convenience, the subsequent statement are all with $\forall i \in \Pi_f$. By using KYP lemma, it is equivalent to

$$\begin{bmatrix} (A - c\lambda_i BK)^\top P + P(A - c\lambda_i BK) & -P \\ -P & -\frac{\gamma^2}{4}I_n \end{bmatrix} + \begin{bmatrix} I_n \\ 0 \end{bmatrix}\begin{bmatrix} I_n & 0 \end{bmatrix} \prec 0$$

By applying schur complement on the left hand side, an equivalent condition is given by

$$\begin{bmatrix} (A - c\lambda_i BK)^\top P + P(A - c\lambda_i BK) & -P & I_n \\ -P & -\frac{\gamma^2}{4}I_n & 0 \\ I_n & 0 & -I_n \end{bmatrix} \prec 0$$

Multiplying $\begin{bmatrix} P^{-1} & & \\ & I & \\ & & I \end{bmatrix}$ on both sides, let $X \triangleq P^{-1}$, then

$$\begin{bmatrix} XA^\top + AX - c\lambda_i(BKX)^\top - c\lambda_i BKX & -I_n & X \\ -I_n & -\frac{\gamma^2}{4}I_n & 0 \\ X & 0 & -I_n \end{bmatrix} \prec 0$$

Let $K$ and $c$ are defined by (10), a sufficient condition is given by

$$\begin{bmatrix} XA^\top + AX - 2BQB^\top & -I_n & X \\ -I_n & -\frac{\gamma^2}{4}I_n & 0 \\ X & 0 & -I_n \end{bmatrix} \prec 0$$

Let $\alpha = \gamma^2 \geq 0$, then we finish the proof. ■

*Corollary 2.1:* If the robust consensus is reached as stated in *Theorem 3.2*, then consensus on (1) is also achieved by controller in (4) with the control setting in (10).

*Proof:* The consensus on (1) is achieved if $A - c\lambda_i BK$ is Hurwitz for all $i \in \Pi_f$. By employing Lyapunov function, the equivalent condition is given as

$$(A - c\lambda_i BK)^\top P + P(A - c\lambda_i BK) \prec 0, \quad \forall i \in \Pi_f$$

By multiplying $X \triangleq P^{-1}$ on the both sides, and substituting the control factorization (10), it is easy to obtain below LMI:

$$XA^\top + AX - 2BQB^\top \prec 0$$

which can be guaranteed by (11). ■

## IV. Robust Consensus against Mismatching Uncertainty

in this section, except from external disturbance, we also consider model uncertainty in robust design. We assume followers have mismatching uncertainties around the a model agreement. The agreed model is assumed as the leader's dynamics.

Consider the below agent dynamics with model parameter uncertainty

$$\begin{aligned} \dot{\mathbf{x}}_1 &= A_0\mathbf{x}_1 + \mathbf{w} && \text{(leader)} \\ \dot{\mathbf{x}}_i &= A_i\mathbf{x}_i + B\mathbf{u}_i && \text{(followers)} \end{aligned} \tag{12}$$

where $A_0$ is the dynamics leader has, and it is also the agreement on dynamics of the heterogeneous following agents, $A_i = (I + \Delta_i)A_0$ is the dynamics of $i$-th follower considering model mismatching uncertainty $\Delta_i A_0$. The assumption of spanning tree in the leader-follower graph still exists. And the below uncertainty consideration is assumed:

$$\bar{\sigma}(\Delta_i) \leq \delta, \quad \forall i \in \Pi_f \tag{13}$$

By defining consensus error as $\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_1$, it follows

$$\dot{\mathbf{e}}_i = A_0\mathbf{e}_i + B\mathbf{u}_i + \Delta_i\mathbf{e}_i + (\Delta_i A_0\mathbf{x}_1 - \mathbf{w})$$

The performance criterion is to keep the worst-case energy of the consensus error $\mathbf{e}$ as small as possible over all disturbance $\mathbf{w}$ of unit energy, in the presence of model mismatching uncertainties.

Consider the same consensus control law in (4), the collective closed-loop error dynamics is given as

$$\dot{\mathbf{e}} = (I_N \otimes A_0 - cL \otimes BK)\mathbf{e} + \Delta\mathbf{e} + (\Delta\bar{\mathbf{x}}_1 - \bar{\mathbf{w}}) \tag{14}$$

where,

$$\Delta = \begin{bmatrix} 0 & & & \\ & \Delta_2 & & \\ & & \ddots & \\ & & & \Delta_N \end{bmatrix}, \bar{\mathbf{x}}_1 = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_1 \end{bmatrix} \in \mathbb{R}^{Nn}, \bar{\mathbf{w}} = \begin{bmatrix} 0 \\ \mathbf{w} \\ \vdots \\ \mathbf{w} \end{bmatrix} \in \mathbb{R}^{Nn}$$

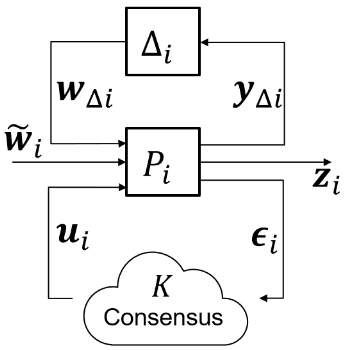


Fig. 1. Robust control block with model uncertainty ($P_i$ denotes the input-output transfer function for (16))

By following the proof of *Lemma 3.1*, $\tilde{\mathbf{e}} = (Y^{-1} \otimes I_n)\mathbf{e}$, then the collective auxiliary error dynamics is given by

$$\dot{\tilde{\mathbf{e}}} = (I_N \otimes A_0 - c\Sigma \otimes BK)\tilde{\mathbf{e}} + \Delta\tilde{\mathbf{e}} + (Y^{-1} \otimes I_n)(\Delta\bar{\mathbf{x}}_1 - \bar{\mathbf{w}})$$

Let $\tilde{\mathbf{w}} \triangleq (Y^{-1} \otimes I_n)(\Delta\bar{\mathbf{x}}_1 - \bar{\mathbf{w}})$, then the decomposed systems are

$$\dot{\varepsilon}_i = (A_0 - c\lambda_i BK)\varepsilon_i + \Delta_i\varepsilon_i + \tilde{\mathbf{w}}_i, \quad \forall i \in \Pi_f \tag{15}$$

## A. S-procedure-based Approach

Based on (15), let $\mathbf{y}_{\Delta i} = A_0\varepsilon_i, \mathbf{z}_i = \varepsilon_i$, $\mathbf{w}_{\Delta i} = \delta_i \mathbf{y}_{\Delta i}$, then the below augmented systems are considered for all $i \in \Pi_f$. The corresponding block diagram of $i$-th system shown in *Fig. 1*.

$$\begin{bmatrix} \dot{\varepsilon}_i \\ \mathbf{y}_{\Delta i} \\ \mathbf{z}_i \end{bmatrix} = \begin{bmatrix} A_0 - c\lambda_i BK & I & I \\ A_0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_i \\ \mathbf{w}_{\Delta i} \\ \tilde{\mathbf{w}}_i \end{bmatrix} \tag{16}$$
$$\mathbf{w}_{\Delta i} = \delta_i \mathbf{y}_{\Delta i}, \quad \forall i \in \Pi_f$$

The control goal is to minimize the $\mathscr{L}_2$ gain between $\tilde{\mathbf{w}}_i$ and $\mathbf{z}_i$ for all $i \in \Pi_f$ in the presence of the model mismatching uncertainty, such that

$$\|\mathbf{w}_{\Delta i}\|_2^2 \leq \mathbf{y}_{\Delta i}^\top(\delta^2 I_n)\mathbf{y}_{\Delta i} \to \|\mathbf{z}_i\|_{\mathscr{L}_2} \leq \tilde{\gamma}\|\tilde{\mathbf{w}}_i\|_{\mathscr{L}_2} \tag{17}$$

*Theorem 4.1:* Consider the uncertainty assumption in (13), the leader-follower system in (12) reaches robust consensus under the consensus control law in (4) if

$$c = \frac{1}{\min\limits_{i \in \Pi_f} \mathrm{Re}(\lambda_i)}, \quad K = QB^\top X^{-1}, \tag{18}$$

where, $X, Q$ are given by solving the following LMI with $\tau > 0$

$$\begin{bmatrix} A_0X + XA_0^\top - 2BQB^\top & I_n & I_n & XA_0^\top & X \\ I_n & -\tau I_n & 0 & 0 & 0 \\ I_n & 0 & -\alpha I_n & 0 & 0 \\ A_0X & 0 & 0 & -\frac{1}{\tau}\Sigma_\Delta^{-1} & 0 \\ X & 0 & 0 & 0 & -I_n \end{bmatrix} \preceq 0,$$
$$X \succeq 0, \quad Q \succeq 0, \quad \alpha \geq 0 \tag{19}$$

with $\Sigma_\Delta = \delta^2 I_n$. Furthermore, $\mathscr{L}_2$ gain is given as $\tilde{\gamma} = \sqrt{\alpha}$.

*Proof:* Consider the control goal in (17), let $P_i$ represent the transfer function between $[\mathbf{y}_{\Delta i}^\top \ \mathbf{z}_i^\top]^\top$ and $[\mathbf{w}_{\Delta i}^\top \ \tilde{\mathbf{w}}_i^\top]^\top$, then the left side is equivalent to

$$\begin{bmatrix} \mathbf{y}_{\Delta i} \\ \mathbf{z}_i \\ \mathbf{w}_{\Delta i} \\ \tilde{\mathbf{w}}_i \end{bmatrix}^\star \begin{bmatrix} -\Sigma_\Delta & & & \\ & 0 & & \\ & & I_n & \\ & & & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\Delta i} \\ \mathbf{z}_i \\ \mathbf{w}_{\Delta i} \\ \tilde{\mathbf{w}}_i \end{bmatrix} \preceq 0,$$

$$\begin{bmatrix} P_i \\ I \end{bmatrix}^\star \begin{bmatrix} -\Sigma_\Delta & & & \\ & 0 & & \\ & & I_n & \\ & & & 0 \end{bmatrix} \begin{bmatrix} P_i \\ I \end{bmatrix} \preceq 0$$

the right side is equivalent to

$$\begin{bmatrix} P_i \\ I \end{bmatrix}^\star \begin{bmatrix} 0 & & & \\ & I_n & & \\ & & 0 & \\ & & & -\tilde{\gamma}^2 I_n \end{bmatrix} \begin{bmatrix} P_i \\ I \end{bmatrix} \preceq 0$$

By s-procedure, there exists $\tau$ such that (17) is equivalent to

$$\begin{bmatrix} P_i \\ I \end{bmatrix}^\star \begin{bmatrix} \tau\Sigma_\Delta & & & \\ & I_n & & \\ & & -\tau I_n & \\ & & & -\tilde{\gamma}^2 I_n \end{bmatrix} \begin{bmatrix} P_i \\ I \end{bmatrix} \preceq 0$$

By bounded real lemma, it follows

$$\begin{bmatrix} (A_0 - c\lambda_i BK)^\top P + P(A_0 - c\lambda_i BK) & P & P \\ P & 0 & 0 \\ P & 0 & 0 \end{bmatrix} +$$
$$\begin{bmatrix} A_0 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}^\top \begin{bmatrix} \tau\Sigma_\Delta & & & \\ & I_n & & \\ & & -\tau I_n & \\ & & & -\tilde{\gamma}^2 I_n \end{bmatrix} \begin{bmatrix} A_0 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix} \preceq 0$$

$$\begin{bmatrix} (A_0 - c\lambda_i BK)^\top P + P(A_0 - c\lambda_i BK) & P & P \\ P & -\tau I_n & 0 \\ P & 0 & -\tilde{\gamma}^2 I_n \end{bmatrix} +$$
$$\begin{bmatrix} A_0 & I_n \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau\Sigma_\Delta & \\ & I_n \end{bmatrix} \begin{bmatrix} A_0 & 0 & 0 \\ I_n & 0 & 0 \end{bmatrix} \preceq 0$$

Multiplying $\begin{bmatrix} P^{-1} & & \\ & I & \\ & & I \end{bmatrix}$ on both sides, using the control factorization in (18) yields

$$\begin{bmatrix} P^{-1}A_0^\top + A_0P^{-1} - 2c\lambda_i BQB^\top & I_n & I_n \\ I_n & -\tau I_n & 0 \\ I_n & 0 & -\tilde{\gamma}^2 I_n \end{bmatrix}$$
$$+ \begin{bmatrix} P^{-1}A_0 & P^{-1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau\Sigma_\Delta & \\ & I_n \end{bmatrix} \begin{bmatrix} A_0^\top P^{-1} & 0 & 0 \\ P^{-1} & 0 & 0 \end{bmatrix} \preceq 0$$

By choosing $c = \frac{1}{\min_{i \in \Pi_f} \mathrm{Re}(\lambda_i)}$, we have $-2c\lambda_i BQB^\top \preceq -2BQB^\top$ for all $i \in \Pi_f$. Finally, by using schur complement, the sufficient LMI in (19) is obtained. ■

### B. Passivity-based Approach

*Theorem 4.2:* Consider the uncertainty assumption in (13), the leader-follower system (12) reaches the robust consensus under consensus control law in (4) if

$$c = \frac{1}{\min_{i\in\Pi_f} \mathrm{Re}(\lambda_i)}, \quad K = QB^\top X^{-1},$$

where $X, Q$ are solved by solving the following LMI with any $\tau > 0$

$$\begin{bmatrix} A_0X + XA_0^\top - 2BQB^\top & I_n & I_n & \sqrt{1+\tau\delta^2}X \\ I_n & -\alpha I_n & 0 & 0 \\ I_n & 0 & -\tau I_n & 0 \\ \sqrt{1+\tau\delta^2}X & 0 & 0 & -I_n \end{bmatrix} \prec 0,$$
$$X \succeq 0, \quad Q \succeq 0, \quad \alpha \geq 0 \tag{20}$$

Furthermore, the $\mathscr{L}_2$ gain for robust consensus is given as $\gamma = 2\sqrt{\alpha}$

*Proof:* By assumption in (13),

$$\|\mathbf{v}_i\|^2 - \delta^2\|\varepsilon_i\|^2 \leq 0$$

Consider a positive definite function $V = \varepsilon_i^\top P \varepsilon_i$ with $P \succeq 0$, then

$$\dot{V} = \varepsilon_i^\top \left( P(A_0 - c\lambda_i BK) + (A_0 - c\lambda_i BK)^\top P \right) \varepsilon_i + \varepsilon_i^\top P\bar{\mathbf{w}} + \bar{\mathbf{w}}^\top P\varepsilon_i + \varepsilon_i^\top P\mathbf{v}_i + \mathbf{v}_i^\top P\varepsilon_i$$

Let $c = \frac{1}{\min_{i\in\Pi_f} \mathrm{Re}(\lambda_i)}$, and $K = QB^\top P$, then

$$\begin{aligned} \dot{V} &= \varepsilon_i^\top (PA_0 + A_0^\top P)\varepsilon_i - 2c\lambda_i\varepsilon_i^\top PBQB^\top P\varepsilon_i + \varepsilon_i^\top P\bar{\mathbf{w}} + \\ &\quad \bar{\mathbf{w}}^\top P\varepsilon_i + \varepsilon_i^\top P\mathbf{v}_i + \mathbf{v}_i^\top P\varepsilon_i \\ &\leq \varepsilon_i^\top (PA_0 + A_0^\top P)\varepsilon_i - 2\varepsilon_i^\top PBQB^\top P\varepsilon_i + \varepsilon_i^\top P\bar{\mathbf{w}} + \\ &\quad \bar{\mathbf{w}}^\top P\varepsilon_i + \varepsilon_i^\top P\mathbf{v}_i + \mathbf{v}_i^\top P\varepsilon_i \end{aligned}$$

Next, adding and subtracting $(\frac{\gamma^2}{4}\|\mathbf{w}\|^2 - \|\varepsilon_i\|^2) + \tau(\|\mathbf{v}_i\|^2 - \delta^2\|\mathbf{x}_i\|^2)$ yields

$$\dot{V} \leq \begin{bmatrix} \varepsilon_i \\ \bar{\mathbf{w}} \\ \mathbf{v}_i \end{bmatrix}^\top M \begin{bmatrix} \varepsilon_i \\ \bar{\mathbf{w}} \\ \mathbf{v}_i \end{bmatrix} + (\frac{\gamma^2}{4}\|\mathbf{w}\|^2 - \|\varepsilon_i\|^2)$$

where,

$$M = \begin{bmatrix} PA_0 + A_0^\top P - 2PBQB^\top P + (1+\tau\delta^2)I_n & P & P \\ P & -\frac{\gamma^2}{4}I_n & 0 \\ P & 0 & -\tau I_n \end{bmatrix}$$

Thus, a sufficient condition of robust consensus is $M \prec 0$. Let $X = P^{-1}$, multiplying $\begin{bmatrix} P^{-1} & & \\ & I & \\ & & I \end{bmatrix}$ before and after $M$ yields

$$\begin{bmatrix} A_0X + XA_0^\top - 2BQB^\top & I_n & I_n \\ I_n & -\frac{\gamma^2}{4}I_n & 0 \\ I_n & 0 & -\tau I_n \end{bmatrix} + \begin{bmatrix} \sqrt{1+\tau\delta^2}X \\ 0 \\ 0 \end{bmatrix} I_n \begin{bmatrix} \sqrt{1+\tau\delta^2}X & 0 & 0 \end{bmatrix} \prec 0$$

Then, by using schur complement, the LMI in (20) is obtained. ■

## V. Extension on Formation control

In this section, we briefly talk about the extension of consensus control on displacement-based formation control. Given reference configuration of formation as a set of constant states $\{\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_N\}, \mathbf{q}_i \in \mathbb{R}^n, i \in \{1\} \cup \Pi_f$, the displacement formation control is to maintain the desired formation by achieving consensus on the displacement error that is defined as

$$\xi_i = \mathbf{x}_i - \mathbf{q}_i \tag{21}$$

Without loss of generality, let's consider a general linear dynamics for all agents without external disturbance and model uncertainties, then the dynamics of displacement error can be given by

$$\begin{aligned} \dot{\xi}_1 &= A\xi_1 + A\mathbf{q}_1 \\ \dot{\xi}_i &= A\xi_i + B\mathbf{u}_i + A\mathbf{q}_i \end{aligned}$$

Thus, the displacement error becomes the state $\mathbf{x}$ we talked previously. In order to use the previous control control designs, a compensate part should be added in consensus control law, such that

$$B\mathbf{u}_i = -A\mathbf{q}_i - cBK \sum_{j\in\mathcal{N}_i} (\xi_i - \xi_j) \tag{22}$$

## VI. Simulation

In this section, we implement the control designs on a satellite formation control example [28]. It contains 5 following satellites and 1 leading satellite. We use same system parameters as in [28] so that readers can compare easily. Assume that a virtual satellite is moving in a circular orbit of radius R with an angular rate $\omega_0 = 0.0015$, the dynamical parameter used is given by

$$A = \begin{bmatrix} 0_3 & I_3 \\ A_1 & A_2 \end{bmatrix}, B = \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega_0^2 & 0 \\ 0 & 0 & -\omega_0^2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 2\omega_0 & 0 \\ -2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In order to force the six satellites to from a regular hexagon shape with a separation of 250m in the plane tangent to the orbit of the virtual satellite by an angle 60°, the formation reference configuration is given by $\mathbf{q}_1 = [125, 125, 0, 0, 0, 0]^\top$, $\mathbf{q}_2 = [250, 62.5, 62.5\sqrt{3}, 0, 0, 0]^\top$, $\mathbf{q}_3 = [125, 0, 125\sqrt{3}, 0, 0, 0]^\top$, $\mathbf{q}_4 = [-125, 0, 125\sqrt{3}, 0, 0, 0]^\top$, $\mathbf{q}_5 = [-250, 62.5, 62.5\sqrt{3}, 0, 0, 0]^\top$, $\mathbf{q}_6 = [-125, 125, 0, 0, 0, 0]^\top$. The communication graph used for our simulation is shown below, and communication weights are all assumed as 1.

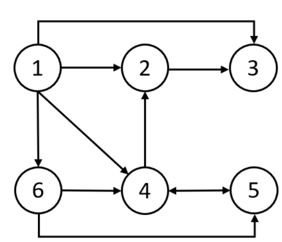


Fig. 2. Communication Topology (contains a spanning tree)

### A. Robust Control Against Disturbance

The external disturbance is introduced from leader which is simulated as an exogenous input $Asin(\omega t)$ with $A = 1, \omega = 10$.

The solved control gains by *Theorem 3.2* are $c = \frac{1}{\min\limits_{i \in \Pi_f} \mathrm{Re}(\lambda_i)} + 0.1 = 1.1$ and

$$K = \begin{bmatrix} 5.4716 & -0.0056 & 0.0002 & 4.5299 & -0.0010 & 0.0003 \\ 0.0027 & 5.4789 & 0.0009 & -0.0010 & 4.5343 & 0.0005 \\ 0.0002 & 0.0006 & 5.4811 & 0.0003 & 0.0005 & 4.5357 \end{bmatrix}$$

The corresponding $\ell_2$ gain between the external disturbance and the consensus error is $\gamma = 0.5726$. The formation control result is shown in *figure 3* and the consensus on displacement errors of 6 states are shown in *figure 4*.

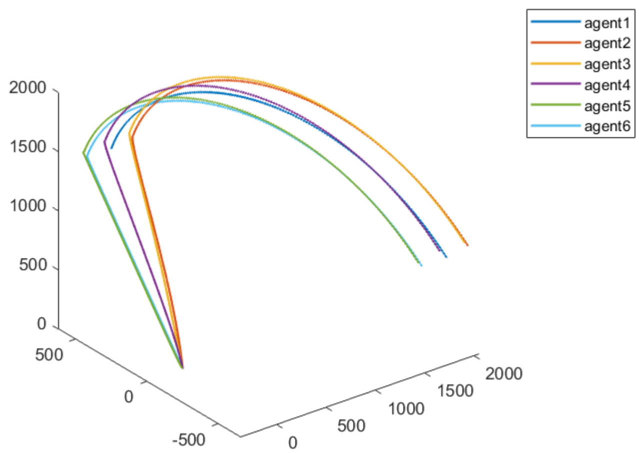


Fig. 3. Formation trajectory (Agent 1 is leader that starts from $(0, 500, 866\sqrt{3})$ with initial velocities $(1.5, 0, 0)$, all followers start from $(0, 0, 0)$ with initial zero velocities. The controller is given by *Theorem 3.2*)

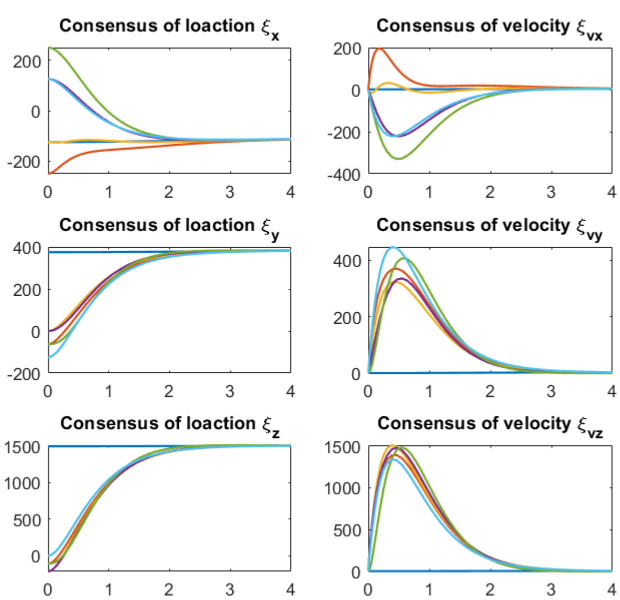


Fig. 4. Consensus on displacement error of locations and velocities under external disturbance (controller is given by *Theorem 3.2*)

### B. Robust Control against Mismatching Uncertainty

The model uncertainty considered in simulation is given by $\Delta_2 = 0.5I_n, \Delta_3 = 0.1I_n, \Delta_4 = 0.07I_n, \Delta_5 = 0.03I_n, \Delta_6 = 0.09I_n$, the bound of uncertainty is set as $\delta = 0.5$. External disturbance setting is same as the one in the last subsection.

By line search, a smallest $\ell_2$ gain $\gamma = 2.7608$ is found as $\tau = 4.6301$. Then, by solving the LMI in (19), the control gain in (18) is given by $c = \frac{1}{\min\limits_{i \in \Pi_f} \mathrm{Re}(\lambda_i)} + 0.1 = 1.1$ and

$$K = \begin{bmatrix} 13.5781 & -0.0068 & 0.0000 & 7.6598 & 0.0000 & 0.0000 \\ 0.0068 & 13.5782 & 0.0000 & 0.0000 & 7.6598 & 0.0000 \\ 0.0000 & 0.0000 & 13.5779 & 0.0000 & 0.0000 & 7.6597 \end{bmatrix}$$

The *Fig. 5* and *Fig. 6* shows the robust controller designed in *Theorem 4.1* can force satellites to maintain desired formation .

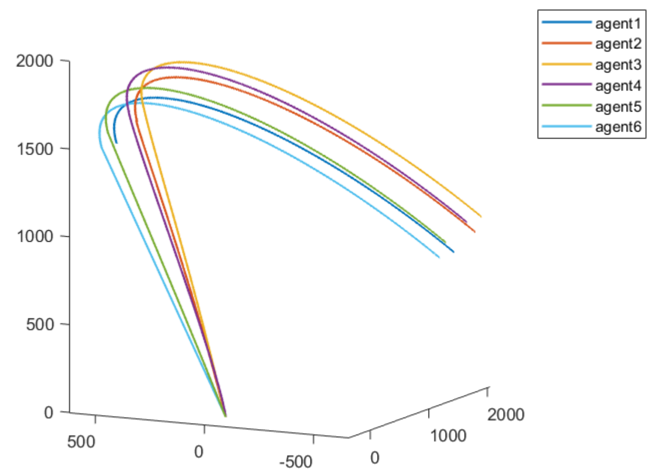


Fig. 5. Formation trajectory under model uncertainties and external disturbance (controller is given by *Theorem 4.1*)

## VII. Conclusion

This paper designs robust consensus controllers against external disturbance and model mismatching uncertainty. $H_\infty$ control design approach and s-procedure are used, and simpler consensus control factorization is proposed compared to literature. Furthermore, the system decomposition on the consensus error system of multi-agent systems provides a way to design control strategy even for infinite number of agents.

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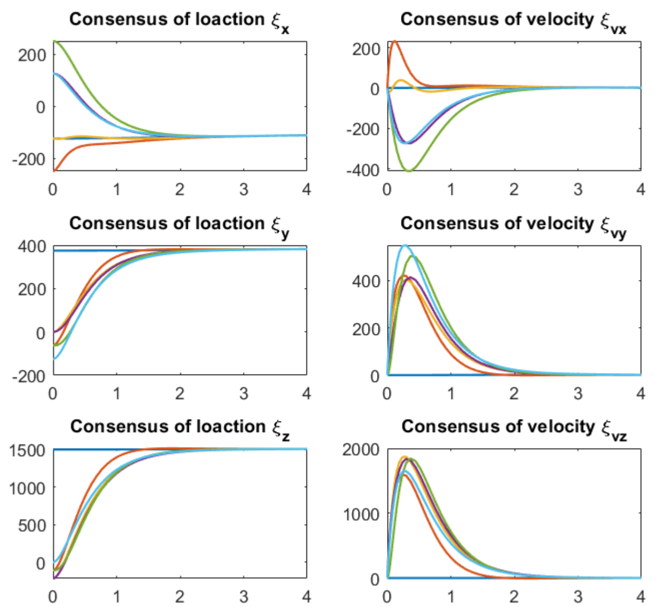


Fig. 6. Consensus on displacement error of locations and velocities under model uncertainties and external disturbance (controller is given by *Theorem 4.1*)

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