This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TPWRS.2019.2941906, IEEE Transactions on Power Systems

Approximating Economic Dispatch by Linearizing Transmission Losses

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***Abstract*—We study a non-convex *Transmission-Constrained Economic Dispatch (TCED) problem* that uses the traditional linear DC model of the transmission system together with a non-linear representation of losses. This problem is typically approximated by the convex problem obtained by linearizing the constraints around some base-case state. Electricity prices and dispatch decisions are then chosen based on the resulting linearly-constrained economic dispatch (LCED) problem. Different LCED problems have been suggested in the literature and they are all derived using one of two linearization techniques, which we call *direct* and *indirect* linearization, respectively. An LCED problem often used in practice that uses Loss Distribution Factors (LDFs) is derived using indirect linearization and is termed the *common LCED problem*. This paper studies the assumptions required to recover the optimal dispatch of the non-convex TCED problem from the solution of the common LCED problem. We show that the common LCED problem may have multiple minimizers, in which case small perturbations of the base-case state may result in large dispatch approximation error. Furthermore, even if the base-case state matches a minimizer of the non-convex TCED problem, it is proven that there does not always exist a choice of LDFs such that the optimal dispatch of the TCED problem is also optimal for the common LCED problem. On the other hand, such LDFs do exist and are identified for the special case where no line limits are binding.**

Nomenclature

General Notation for a Column Vector v

- $\dot{v}$ Vector v with element ρ removed
- $\mathring{v}$ Vector v with element σ removed
- v^0 Base-case value for vector v
- $v^\star$ Locally optimal value for vector v
- v_i Element i of vector v
- $\|v\|_\infty$ Infinity norm of vector v
- $\|v\|_1$ One norm of vector v
- v^T Transpose of vector v

General Notation for a Matrix M

- $\dot{M}$ Matrix M with column ρ removed
- $\mathring{M}$ Matrix M with column σ removed
- M^T Transpose of matrix M
- M^{-1} Inverse of matrix M
- $M^{-\mathsf{T}}$ Inverse transpose of matrix M
- M_{ij} Element in row i and column j of matrix M
- M_i Column i of matrix M
- $|M|$ Element-wise absolute value of matrix M

General Notation for a Vector Valued Function f

- $f_i(\cdot)$ Element i of the function $f(\cdot)$
- $\nabla f(\cdot)$ Jacobian of $f(x)$ with respect to its arguement
- $\nabla \mathring{f}(\cdot)$ Jacobian $\nabla f(\cdot)$ with column σ removed
- $\nabla f_i(\cdot)$ Column i of the Jacobian $\nabla f(\cdot)$
- $\nabla_x f(\cdot, y)$ Partial Jacobian of function $f(x, y)$ with respect to x

Transmission System Graph

- $\mathcal{G} := (\mathbb{N}, \mathbb{E})$ Undirected transmission system graph
- $\mathbb{N} := [1, \ldots, n]$ Set of nodes in graph $\mathcal{G}$ (Busses/generators)
- $\mathbb{E} := [1, \ldots, m]$ Set of edges in graph $\mathcal{G}$ (Transmission lines)
- $n,\ m$ Number of nodes and edges in graph $\mathcal{G}$ respectively
- $A \in \mathbb{R}^{m\times n}$ Branch-bus incidence matrix of the graph $\mathcal{G}$
- $B \in \mathbb{R}^{m\times m}$ Diagonal matrix of edge weights (susceptances)
- $H := \dot{A}^\mathsf{T} B A$ Reduced weighted Laplacian matrix of graph $\mathcal{G}$
- $S := B\dot{A}\mathring{H}^{-\mathsf{T}} = B\dot{A}[\mathring{A}^\mathsf{T} B \dot{A}]^{-1}$ Matrix of shift factors

Constants

- $D \in \mathbb{R}^n$ Nodal demand
- $\rho \in \mathbb{N}$ Angle reference bus
- $\sigma \in \mathbb{N}$ Price reference bus
- $\underline{F}, \bar{F} \in \mathbb{R}^m$ Line limits
- $\underline{P}, \bar{P} \in \mathbb{R}^m$ Nodal generation limits

Optimization Decision Variables

- $P \in \mathbb{R}^n$ Nodal generation dispatch
- $N \in \mathbb{R}^n$ Nodal loss allocation (Opt. B)-(Opt. E)
- $\theta \in \mathbb{R}^n$ Voltage angles (Opt. A)
- $\ell \in \mathbb{R}$ Total system losses (Opt. F)

Lagrange Multipliers

- $\pi \in \mathbb{R}$ Overall power balance constraint (Opt. B)-(Opt. E)
- $\bar{\mu}, \underline{\mu} \in \mathbb{R}^m$ Line limit constraints (Opt. B)-(Opt. E)
- $\bar{\beta}, \underline{\beta} \in \mathbb{R}^n$ Generator output constraints (Opt. B)-(Opt. E)
- $\gamma \in \mathbb{R}^n$ Nodal loss allocation constraints (Opt. B)-(Opt. C)
- $\gamma' \in \mathbb{R}^n$ Locally optimal Lagrange multipliers for the nodal loss allocation constraints (Opt. D)-(Opt. E)

Prices

- $\lambda \in \mathbb{R}^n$ Locational Marginal Prices (LMPs)
- $e \in \mathbb{R}^n$ Energy component of the LMPs
- $l \in \mathbb{R}^n$ Loss component of the LMPs
- $c \in \mathbb{R}^n$ Congestion component of the LMPs

Functions

- $C\colon \mathbb{R}^n \to \mathbb{R}$ Generation cost function
- $L : \mathbb{R}^m \to \mathbb{R}^m$ Loss function
- $\tilde{N}\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ Nodal loss allocation function
- $\nabla\tilde{N}\colon \mathbb{R}^{n-1} \to \mathbb{R}^{(n-1)\times n}$ Loss sensitivity matrix valued function
- $\Lambda_j : \mathbb{R}^n \times \left\{\mathbb{R}^{n-1}\right\}^2 \to \mathbb{R}^n$ Loss constraint function of Opt. j
- $\Phi_j\colon \left\{\mathbb{R}^n\right\}^6 \times \left\{\mathbb{R}^m\right\}^2 \times \mathbb{R} \to \mathbb{R}$ Lagrangian function of Opt. j

Other Notation

- $\mathbb{R}^d$ Set of d dimensional real numbers
- $\eta \in \mathbb{R}^n$ Loss Distribution Factors (LDFs)
- $LF \in \mathbb{R}^n$ Loss sensitivity vector
- $q \in \mathbb{R}$ Linearization offset
- $T \in \mathbb{R}^n$ Vector of net power injections
- $j \in \{B, C, D, E\}$ General reference to Opt. j
- $\alpha \in \mathbb{R}$ Scalar assumed to satisfy $\gamma' = \mathbf{1}\alpha$
- $\delta \in \mathbb{R}^{n-1}$ Perturbation to ideal base-case state
- $\Delta P \in \mathbb{R}^n$ Dispatch error with respect to $P^\star$
- $\Delta C \in \mathbb{R}$ Objective value error with respect to $C(P^\star)$
- $\mathbf{1}, \mathbf{0}$ Vector of ones and zeros with appropriate dimension
- I Identity matrix with appropriate dimension
- $\mathbf{i}$ Imaginary number $\sqrt{-1}$

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This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TPWRS.2019.2941906, IEEE Transactions on Power Systems

I. Introduction

The majority of electricity markets in North America incorporate transmission line losses into economic dispatch problems used for pricing and dispatch decisions in both real-time and day-ahead markets [1]. These markets intend to solve a non-convex economic dispatch problem that we will refer to as the *Transmission-Constrained Economic Dispatch (TCED) problem*. The TCED problem uses the traditional linear Direct Current (DC) model of the transmission system together with non-linear losses of each transmission line allocated evenly among the incident buses as "fictitious nodal demand" [1], [2]. The TCED problem is typically approximated by first linearizing the loss function around some base-case state and then introducing loss distribution factors (LDFs). The resulting linearly constrained economic dispatch (LCED) problem is termed the *common LCED problem* because it closely represents those used in practice as reported in [1] and [3]. For the first time, this paper characterizes the approximation errors associated with each of the three assumptions required to accurately recover the optimal dispatch of the non-convex TCED problem from the solution of the common LCED problem.

The economic dispatch problem that most accurately accounts for losses also captures real/reactive power coupling as well as off-nominal voltage magnitudes and is typically referred to as the Alternating Current Optimal Power Flow (AC OPF) problem [4]. The AC OPF problem is not used in electricity markets today because it is generally NP-hard [5], [6]. Instead, economic dispatch problems have traditionally used the linear DC model of the transmission system, which makes the simplifying assumption that there are no transmission losses [7]. The resulting economic dispatch problem is termed the *lossless DC OPF* problem and assumes voltage magnitudes are fixed to their nominal values, which facilitates the decoupling of real and reactive power. This lossless DC OPF problem is typically augmented with fixed losses allocated throughout the network as fictitious nodal demand [8–10]. References [9] and [10] show that introducing a fixed loss representation to the lossless DC OPF problem results in price and dispatch values that more accurately represent those from the AC OPF problem, which serves as an intuitive benchmark. However, fixed loss representations do not accurately capture the marginal effects of losses as operating conditions vary. In fact, the real-power loss of a transmission line is more accurately approximated as the product of the line resistance and the squared real-power flowing through the line, and this representation can capture the variation of marginal losses with operating conditions [11], [12]. Reference [2] derives more accurate loss approximations from first principles and establishes the proper loss allocation to distribute the losses of each line evenly among its incident buses. Incorporating these non-linear representations of line losses into the lossless DC OPF problem results in the TCED problem, which is non-convex and generally difficult to solve. To attain a tractable linearly constrained formulation, the TCED problem is typically approximated via first order Taylor expansions taken around some base-case state. A main contribution of this paper is a clear analysis of the implications of such linearizations.

Reference [1] provides a summary of different linearization techniques implemented by ISOs governed by the Federal Energy Regulatory Commission (FERC), highlighting variations among ISOs in the choice of base-case state. The *common LCED problem* derived in [1] matches that in [3] and is a linearized version of the TCED problem from [2], which accommodates a general non-linear loss function. Many recent works have attempted to improve upon this linearly constrained formulation in different ways. Reference [13] derives a linear pricing technique that does not rely on a choice of base-case state or LDFs. Reference [14] proposes a dispatch problem resulting in price components that are reference bus independent. Piece-wise linear representations of losses are accommodated by [15] and [16] via the load over-satisfaction relaxation, which is accurate under the assumption that prices are positive. Other work has focused on developing more accurate linear approximations of the AC OPF problem [17–20]. Similar linearization procedures have also been studied in the context of planning problems [21], [22]. Despite the various linearly constrained formulations suggested in the literature, the common LCED problem from [1] and [3] remains the most commonly used economic dispatch problem in practice.

Using the AC OPF problem as a benchmark, [1] illustrates that prices associated with the common LCED problem better capture the marginal effect of losses as compared to economic dispatch problems that represent losses as being fixed. However, using a simple 2-bus example, [1, section V-A] additionally identifies an issue that is missed by current practice. Specifically, using intuitive choices of base-case state and LDFs, the optimal dispatch of the common LCED problem may be far from feasible for the non-convex TCED problem from which it is derived. Previous work has addressed similar issues by use of post-processing methods [23], which, in our context, intend to identify a dispatch that is feasible for the TCED problem by slightly altering the optimal dispatch of the common LCED problem. Such post-processing methods do not provide optimality guarantees and may identify a costly dispatch, as is illustrated in [1, section V-A]. To mitigate this issue, [1] suggests a successive linearization procedure, which is similar to those from [24] and [25]. This procedure is not guaranteed to converge and requires properly tuned parameters to encourage convergence. Recognizing that the issue exhibited by the 2-bus test case in [1] is difficult to rectify, another contribution of this paper is to identify the source of this issue. Specifically, this issue occurs if the common LCED problem has multiple minimizers when the ideal choice of base-case state is used. (See Remark 11).

Despite the prevalence of the common LCED problem, no previous work has established a set of assumptions required to recover the optimal dispatch of the non-convex TCED problem from the common LCED problem. To establish such assumptions this paper derives the common LCED problem from the non-convex economic dispatch problem adopted from [2] that we term the *TCED problem with angles* or Optimization A (Opt. A). To illustrate the assumptions used to attain the common LCED problem, four intermediate problems, namely, Opt. B, C, D, and E, are also derived that are

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not individually novel but constitute a convenient route for the derivation. Figure 1 outlines the sequence of problems derived to attain the common LCED problem or Opt. F. This derivation ultimately proves that a specific dispatch that is optimal for the TCED problem with angles (Opt. A) is also optimal for the common LCED problem (Opt. F) under two key assumptions. This specific dispatch can be efficiently recovered from the common LCED problem (Opt. F) under the additional assumption that the common LCED problem has a unique optimal dispatch.

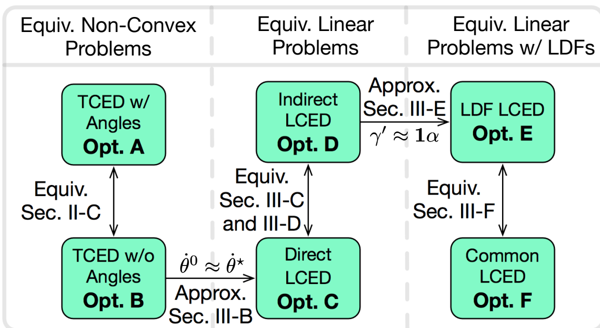

Fig. 1: Diagram outlining the derivation structure of the paper. Vertical connections with arrows pointing in both directions indicate equivalence of the two connected problems, meaning any dispatch that is optimal for one is also optimal for the other. Horizontal connections pointing in only one direction indicate that both problems share an optimal dispatch, namely the base-case dispatch, under the appropriate assumptions.

The derivation begins in Section II by eliminating voltage angles from the TCED problem with angles (Opt. A) yielding an ED problem that optimizes over generation dispatch and nodal loss allocation variables. This problem is termed the *TCED problem without angles* (Opt. B) and is shown to be equivalent to the TCED problem with angles (Opt. A), where two problems are said to be *equivalent* if any dispatch that is optimal for one is also optimal for the other.

The fundamental linearization theory in [12] uses a first order Taylor expansion of the constraints with respect to all optimization variables about the base-case state and its associated base-case dispatch. We call this *direct linearization*. Section III-B derives the *direct LCED* problem (Opt. C) by applying direct linearization to the TCED problem without angles (Opt. B). The first key assumption, termed *Assumption 1*, requires the base-case dispatch to satisfy the Karush-Kuhn-Tucker (KKT) conditions and represent an optimal dispatch of the TCED problem without angles (Opt. B). A result from [12] can then be applied to show that the same base-case dispatch is optimal for the *direct LCED* problem (Opt. C). A related practical question is how close does the base-case state need to be to a minimizer of the underlying non-convex problem for the approximate LCED problem to result in a good dispatch approximation. (See Remark 4).

Another approach is to linearize with respect to only the net power injections using total derivatives of implicit functions. We call this *indirect linearization*. In fact, the common LCED problem (Opt. F) is derived using indirect linearization as in [1] and [3]. Section III-C derives the intermediate *indirect LCED* problem (Opt. D) by applying indirect linearization to the TCED problem without angles (Opt. B). Section III-C establishes equivalence of the indirect LCED problem (Opt. D) and the direct LCED problem (Opt. C), showing any dispatch that is optimal for one of these problems is also optimal for the other. Section III-D then explains how to compute the LMPs from Lagrange multipliers of the indirect LCED problem (Opt. D), which differ from those of the direct LCED problem (Opt. C). We additionally emphasize that the indirect linearization technique requires loss sensitivities to be computed as the solution of equations with a large Jacobian matrix during each market interval, e.g. every 5-15 minutes for the real-time market, which may be computationally burdensome.

Section III-E introduces LDFs to the indirect LCED problem (Opt. D) to obtain the *LDF LCED problem* (Opt. E). Section III-E additionally explains that under Assumption 1 there do not generally exist LDFs that allow the base-case dispatch to solve the LDF LCED problem. However, under the assumption that no line limits are binding such LDFs do exist and are identified in Section III-E. The second key assumption, termed *Assumption 2*, effectively asserts that no line limits are binding.

Finally, Section III-F formulates the *common LCED problem* (Opt. F) by eliminating the loss allocation variables and introducing a single loss variable in their place. The *LDF LCED problem* (Opt. E) is shown to be equivalent to the common LCED problem (Opt. F). Furthermore, the derivation outlined in Figure 1 shows that the base-case dispatch is optimal for the common LCED problem under Assumptions 1 and 2. However, Section III-G emphasizes that the common LCED problem (Opt. F) may have multiple optimal dispatches, some of which may not be feasible for the TCED problem with angles (Opt. A). As a result, it may be difficult to recover an optimal dispatch of the TCED problem with angles (Opt. A) by solving the common LCED problem (Opt. F) using standard off-the-shelf optimization software. The third and final key assumption, termed *Assumption 3*, requires the common LCED problem (Opt. F) to have a unique optimal dispatch. Under Assumptions 1, 2, and 3 the base-case dispatch, which is optimal for the TCED problem with angles (Opt. A), is the unique optimal dispatch of the common LCED problem (Opt. F), and can be easily recovered.

Section IV provides numerical results intended to illustrate two key findings that point out significant dispatch approximation errors may occur when certain assumptions are violated. The first key finding states that very small perturbations to the ideal base-case state can result in significantly large dispatch approximation error if the common LCED problem (Opt. F) has multiple minimizers. This is illustrated in Section IV-A using an intuitive 2-bus example as well as a larger more realistic test case. The second key finding states that significant dispatch approximation error may occur when Assumption 2 is violated, even if Assumptions 1 and 3 hold. This is illustrated in Section IV-B using an intuitive 3-bus example that is highly resistive and heavily congested. However, a larger more realistic test case is used to illustrate that violating Assumption 2 typically results in insignificant dispatch approximation error.

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II. Transmission Constrained Economic Dispatch

We begin by introducing notation. Lower case subscripts are used to indicate elements of matrices. For example the element in the i^{th} row and j^{th} column of matrix M is denoted M_{ij}. The i^{th} column of matrix M is denoted M_i. The transpose of a matrix is denoted with a superscript T, for example M^{T}, and a superscript $-\mathsf{T}$ represents the inverse transpose of a matrix. The set of n dimensional real numbers is denoted $\mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is designated a column vector and its i^{th} element is denoted v_i. The element-wise absolute value of a matrix or vector is denoted $|M|$ or $|v|$ respectively. The identity matrix, the matrix of all zeros, and the matrix of all ones are denoted I, $\mathbf{0}$, and $\mathbf{1}$ respectively and are of appropriate dimension. Furthermore, the system is modeled as an undirected graph $\mathcal{G} = (\mathbb{N}, \mathbb{E})$ where $\mathbb{N}$ is the set of nodes (buses) and $\mathbb{E}$ is the set of edges (transmission lines). There are n buses and m transmission lines. The branch-bus incidence matrix of the graph $\mathcal{G}$ is denoted $A \in \mathbb{R}^{m\times n}$. Specifically, A is sparse and the row representing line k connecting bus i to bus j has element i equal to 1 and element j equal to -1. In this context bus i is arbitrarily assigned to be the *sending bus* and power flow is designated positive when flowing from bus i to bus j.

A. Transmission-Constrained Economic Dispatch with Angles

The Transmission-Constrained Economic Dispatch (TCED) problem with angles is adopted from [2]. This problem optimizes over the nodal generation dispatch vector $P \in \mathbb{R}^n$ and the vector of voltage angles excluding the known angle at the bus $\rho \in \mathbb{N}$, which will be termed the *angle reference bus*. Throughout the paper a dot over a vector represents that vector with element ρ removed. For example the vector of voltage angles is denoted $\theta \in \mathbb{R}^n$ and the vector of voltage angles excluding the angle at the angle reference bus is denoted $\dot{\theta} \in \mathbb{R}^{n-1}$. Similarly, a dot over a matrix represents that matrix with column ρ removed. For example, the matrix $\dot{A} \in \mathbb{R}^{m\times(n-1)}$ is equivalent to A with column ρ removed.

The *TCED problem with angles* (Opt. A) is written as follows. Voltage magnitudes are assumed constant and all equal to one per unit. The nodal demand is considered fixed and is represented by $D \in \mathbb{R}^n$. The cost of generation is represented by the function $C(P)$ where $C : \mathbb{R}^n \to \mathbb{R}$ is convex.

$$\min_{P\in\mathbb{R}^n, \dot{\theta}\in\mathbb{R}^{n-1}} \quad C(P) \qquad \text{(Opt. A)}$$

$$st: \tfrac{1}{2}|A|^{\mathsf{T}} L(\dot{A}\dot{\theta}) + H^{\mathsf{T}}\dot{\theta} = P - D \qquad \text{(A1)}$$

$$\underline{P} \le P \le \bar{P} \qquad \text{(A2)}$$

$$\underline{F} \le B\dot{A}\dot{\theta} \le \bar{F} \qquad \text{(A3)}$$

Constraint (A1) represents the power balance at each bus in the system as presented in [2]. The vector valued loss function is denoted $L : \mathbb{R}^m \to \mathbb{R}^m$ and maps voltage angle differences to line losses. This function is assumed to be convex on the domain of the problem and is assumed to be continuously differentiable. Half of each line's losses are assigned to its incident buses as fictitious nodal demand. In the interest of deriving a linearly-constrained economic dispatch problem, we assume a linear approximation of the mid-line power flow function described in detail in [2] and represented by $H^{\mathsf{T}}\dot{\theta}$ where $H := \dot{A}^{\mathsf{T}} B A$ is termed the *reduced weighted Laplacian matrix* and represents the weighted Laplacian matrix of the underlying system graph with row ρ removed. In the expression for H, the matrix $B \in \mathbb{R}^{m\times m}$ is full rank and diagonal where the diagonal elements represent the edge weights of the underlying system graph and can be interpreted as transmission line susceptances. Constraint (A2) enforces generator output limits and constraint (A3) enforces line limits in the form of mid-line power flow limits.

Remark 1. The *lossless DC OPF problem* is identical to the TCED problem with angles (Opt. A) if the loss function $L(\cdot)$ is replaced by the zero vector $\mathbf{0}$. This lossless DC OPF problem is linearly constrained, convex, and easy to solve.

B. General Economic Dispatch Problem without Angles

Many references, e.g. [1] and [3], analyze the energy market with respect to an economic dispatch problem that optimizes over dispatch and loss variables. We will derive four such problems that optimize over the nodal dispatch vector and a nodal loss allocation vector $N \in \mathbb{R}^n$. These four problems, Opt. B, C, D, and E respectively, can each be expressed as the following *general economic dispatch problem without angles*. The *price reference bus* (or slack bus) is designated as bus $\sigma \in \mathbb{N}$. Throughout the paper a ring over a vector represents that vector with element σ removed. Similarly, a ring over a matrix represents that matrix with column σ removed, e.g. $\mathring{A}$.

$$\min_{P\in\mathbb{R}^n, N\in\mathbb{R}^n} \quad C(P) \qquad \text{(Opt. } j\text{)}$$

$$st: \mathbf{1}^{\mathsf{T}}(P - D - N) = 0 \qquad (j1)$$

$$\Lambda_j(\mathring{P}, N; \mathring{D}) = \mathbf{0} \qquad (j2)$$

$$\underline{P} \le P \le \bar{P} \qquad (j3)$$

$$\underline{F} \le S(\mathring{P} - \mathring{D} - \mathring{N}) \le \bar{F} \qquad (j4)$$

Constraint $(j1)$ represents power balance. Constraint $(j3)$ represents generator output limits. Constraint $(j4)$ represents limits on the mid-line power flow on each transmission line expressed in terms of shift factors $S := B\mathring{A}\mathring{H}^{-\mathsf{T}}$. Note that the matrix $\mathring{H}$ is invertible under the standard assumption that the system graph is fully connected. The nodal loss allocation constraint $(j2)$ incorporates a general function $\Lambda_j(\cdot,\cdot;\mathring{D}) : \mathbb{R}^{n-1}\times\mathbb{R}^n \to \mathbb{R}^n$, which takes the constant parameter $\mathring{D} \in \mathbb{R}^{n-1}$ as an additional argument. This general function will be defined differently for each of the next four optimization problems formulated in this paper. These four problems will be differentiated by j taking values B, C, D, and E.

Remark 2. Opt. j has one more decision variable and equality constraint than Opt. A. Opt. j is also conveniently formulated because there is one linear *overall* power balance constraint and all non-convexity is concentrated to constraint $(j2)$.

The Lagrangian function is central to the First Order Necessary Conditions (FONCs) for optimality. The Lagrangian function

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of Opt. j is written as follows. The arguments of this function are partitioned into three categories, namely: primal variables, dual variables, and constant parameters. An equation number is assigned to each term for future reference.

$$\begin{aligned}
&\Phi_j(N, P; \pi, \gamma, \bar{\mu}, \underline{\mu}, \bar{\beta}, \underline{\beta}; D) := && (1)\\
&\quad C(P) && (1a)\\
&\quad + \pi \mathbf{1}^\mathsf{T}(-P + D + N) && (1b)\\
&\quad + \gamma^\mathsf{T} \Lambda_j(\mathring{P}, N; \mathring{D}) && (1c)\\
&\quad + \underline{\beta}^\mathsf{T}(\underline{P} - P) && (1d)\\
&\quad + \bar{\beta}^\mathsf{T}(P - \bar{P}) && (1e)\\
&\quad + \bar{\mu}^\mathsf{T}\left(S(\mathring{P} - \mathring{D} - \mathring{N}) - \bar{F}\right) && (1f)\\
&\quad + \underline{\mu}^\mathsf{T}\left(\underline{F} - S(\mathring{P} - \mathring{D} - \mathring{N})\right). && (1g)
\end{aligned}$$

Here the Lagrangian function is defined with: $\pi \in \mathbb{R}$ representing the Lagrange multiplier of the overall power balance constraint ($j1$); $\gamma \in \mathbb{R}^n$ representing the Lagrange multipliers of the nodal loss allocation constraint ($j2$); $(\bar{\beta}, \underline{\beta}) \in \mathbb{R}^n \times \mathbb{R}^n$ representing the Lagrange multipliers of the generator output constraint ($j3$); and $(\bar{\mu}, \underline{\mu}) \in \mathbb{R}^m \times \mathbb{R}^m$ representing the Lagrange multipliers of the line limit constraint ($j4$).

The KKT conditions for some pair of primal variables (P, N) require the existence of corresponding Lagrange multipliers that jointly satisfy the primal feasibility, dual feasibility, complementary slackness, and stationarity conditions [26]. Primal feasibility requires the primal variables to satisfy constraint ($j1$)-($j4$). Dual feasibility requires the dual variables associated with inequality constraints $\bar{\beta}, \underline{\beta}, \bar{\mu}$ and $\underline{\mu}$ to be non-negative. Complementary slackness requires each of the terms (1d)-(1g) to equate to zero. The stationarity condition requires the partial derivative of the Lagrangian function to be zero with respect to the primal variables P and N. The stationarity condition is as follows, where the partial derivatives of Φ_j with respect to N_σ, P_σ, $\mathring{N}$, and $\mathring{P}$ are represented by (2), (3), (4), and (5) respectively:

$$0 = \pi + \nabla_{N_\sigma} \Lambda_j(\mathring{P}, N; \mathring{D})\gamma, \quad (2)$$
$$0 = \nabla_{P_\sigma} C(P) - \pi + \bar{\beta}_\sigma - \underline{\beta}_\sigma, \quad (3)$$
$$\mathbf{0} = \pi \mathbf{1} + \nabla_{\mathring{N}} \Lambda_j(\mathring{P}, N; \mathring{D})\gamma + S^\mathsf{T}(\underline{\mu} - \bar{\mu}), \quad (4)$$
$$\mathbf{0} = \nabla_{\mathring{P}} C(P) - \pi \mathbf{1} + \nabla_{\mathring{P}} \Lambda_j(\mathring{P}, N; \mathring{D})\gamma - S^\mathsf{T}(\underline{\mu} - \bar{\mu}) + \mathring{\bar{\beta}} - \mathring{\underline{\beta}}. \quad (5)$$

C. Transmission-Constrained Economic Dispatch without Angles

By choosing the function Λ_j appropriately Opt. j can be made equivalent to the *TCED problem with angles* (Opt. A) in that any optimal dispatch for one of these problems is also optimal for the other. In particular, Sections II-C1-II-C3 show that Opt. A is equivalent to the following *TCED problem without angles*.

Optimization B. The *TCED problem without angles* is defined to be Opt. j with Λ_j specified by:

$$\Lambda_\mathrm{B}(\mathring{P}, N; \mathring{D}) := N - \tfrac{1}{2}\left|A\right|^\mathsf{T} L\left(\dot{A}\mathring{H}^{-\mathsf{T}}(\mathring{P} - \mathring{D} - \mathring{N})\right). \quad (6)$$

Sections II-C1 through II-C3 derive the TCED problem without angles (Opt. B) from the TCED problem with angles (Opt. A). Each step taken in the reformulation preserves the set of feasible dispatch variables P, implying any dispatch that is optimal for one of these problems is also optimal for the other. Both problems additionally have the same optimal objective value and as a result the sensitivity of the optimal objective value with respect to D is the same for both problems under the assumption that the sensitivity is well defined.

1) Loss Allocation Vector: We first introduce the nodal loss allocation vector $N \in \mathbb{R}^n$ as a decision variable in Opt. A along with the constraint:

$$N = \tfrac{1}{2}\left|A\right|^\mathsf{T} L(\dot{A}\dot{\theta}). \quad (7)$$

2) Power Balance Constraint: By (7), we can replace the first term in power balance constraint (A1) by N to obtain:

$$N + H^\mathsf{T}\dot{\theta} = P - D. \quad (8)$$

Note that $H\mathbf{1} = 0$, left multiply the left- and right-hand side of (8) by the full rank matrix $[\mathbf{1}, \mathring{I}]^\mathsf{T}$ and re-arrange to obtain:

$$\begin{aligned} 0 &= \mathbf{1}^\mathsf{T}(P - D - N), && (9)\\ \mathring{H}^\mathsf{T}\dot{\theta} &= \mathring{P} - \mathring{D} - \mathring{N}. && (10)\end{aligned}$$

Since $[\mathbf{1}, \mathring{I}]^\mathsf{T}$ is full rank, constraints (9) and (10) hold if and only if constraint (8) holds. Since $\mathring{H}$ is invertible, constraint (10) can be re-arranged to:

$$\dot{\theta} = \mathring{H}^{-\mathsf{T}}(\mathring{P} - \mathring{D} - \mathring{N}). \quad (11)$$

3) Eliminate Voltage Angles: In summary, we first introduced constraint (7) along with variable N. We then replaced the power balance constraint (A1) with equivalent constraints (11) and (9). We now substitute the expression for $\dot{\theta}$ from constraint (11) into constraints (7) and (A3). The resulting optimization problem is equivalent to Opt. B but with additional constraint (11). Since $\dot{\theta}$ is otherwise unconstrained, constraint (11) can be removed resulting in Opt. B.

Definition 1. Let the generation dispatch $P^\star$ and the nodal loss allocation $N^\star$ represent a local minimizer of the TCED problem without angles (Opt. B) with associated Lagrange multipliers $\pi^\star$, $\gamma^\star$, $\bar{\mu}^\star$, $\underline{\mu}^\star$, $\bar{\beta}^\star$, and $\underline{\beta}^\star$ that solve the KKT conditions for the TCED problem without angles (Opt. B) under the assumption that such a local minimizer exists.

D. Locational Marginal Prices

The Locational Marginal Prices (LMPs), denoted λ, are defined to be the partial derivative of the Lagrangian function with respect to the demand vector D:

$$\lambda := \nabla_D \Phi_\mathrm{B}(N^\star, P^\star; \pi^\star, \gamma^\star, \bar{\mu}^\star, \underline{\mu}^\star, \bar{\beta}^\star, \underline{\beta}^\star; D). \quad (12)$$

where Φ_B represents the Lagrangian function (1) for the TCED problem without angles (Opt. B). The LMP can be decomposed as $\lambda := e + l + c$, where e is the energy component associated with Lagrange multiplier π, l is the loss component associated with Lagrange multipliers γ, and c is the congestion component associated with Lagrange multipliers $(\bar{\mu}, \underline{\mu})$. The congestion and loss components of the LMP at the price reference bus are zero, that is $c_\sigma := 0$ and $l_\sigma := 0$. The

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remaining LMP components are explicitly written as follows:

$$e := \pi^\star \mathbf{1}, \tag{13}$$
$$\mathring{l} := \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^\mathsf{T}\nabla L(\dot{A}\dot{\theta}^\star)\,|A|\,\gamma^\star, \tag{14}$$
$$\mathring{c} := S^\mathsf{T}(\underline{\mu}^\star - \bar{\mu}^\star). \tag{15}$$

where $\dot{\theta}^\star = \mathring{H}^{-\mathsf{T}}(\mathring{P}^\star - \mathring{D} - \mathring{N}^\star)$.

Remark 3. The loss and congestion components of the LMP are zero at the price reference bus highlighting the dependency of these components of the LMP on the choice of price reference bus. On the other hand, the LMP is not dependent on the choice of price reference bus as stated in [2].

III. Linearly Constrained Economic Dispatch

The economic dispatch problems formulated in the previous section are non-convex and may be difficult to solve. For this reason it is typical for ISOs to use approximations that result in a convex problem. Today ISOs use linearization techniques to simplify non-convex economic dispatch problems into convex linearly-constrained problems. In this section we present a sequence of four linearly-constrained economic dispatch problems, respectively, Opt. C, D, E, and F, ultimately resulting in the common LCED problem (Opt. F) used in [1] and [3]. Presenting the sequence of problems highlights the nature of the approximations used in practice.

Section III-A describes linearization about a base-case state, which is fundamental to all of the linearization approaches. Section III-B derives the *direct LCED* problem (Opt. C), which serves as an approximation to the TCED problem without angles (Opt. B) as explained by a result from [12]. Section III-C introduces the *indirect LCED problem* (Opt. D) and proves that it is equivalent to Opt. C. Section III-D then explains how to compute the loss component of the LMP from Lagrange multipliers of the indirect LCED problem (Opt. D). LDFs are then introduced to the indirect LCED problem in Section III-E to obtain the *LDF LCED problem* (Opt. E). This problem is shown to be a good approximation of the indirect LCED problem under the additional assumption that no line limits are binding. The *common LCED problem* (Opt. F) is derived in Section III-F and is shown to be equivalent to the LDF LCED problem (Opt. E). Section III-G then introduces a uniqueness assumption required to efficiently recover an optimal dispatch of the TCED problem with angles (Opt. A).

A. Linearization about Base-Case State

The linearization procedure will use a first order approximation around a base-case state $\dot{\theta}^0$. Associated with the base-case state are the base-case nodal loss allocation vector $N^0 := \frac{1}{2}|A|^\mathsf{T} L(\dot{A}\dot{\theta}^0)$ and the base-case generation dispatch vector $P^0 := H^\mathsf{T}\dot{\theta}^0 + N^0 + D$. Note that by construction the base-case values $\dot{\theta}^0$ and P^0 satisfy all equality constraints in Opt. A and the base-case values N^0 and P^0 satisfy all equality constraints in Opt. B.

In the real-time market the base-case state is often chosen to match the output of the state estimator. Similarly, in the day ahead market the base-case state is often chosen by predicting the future system state based on historical data. However, other choices of base-case state are used in practice. For example, another common method constructs a base-case state from the optimal dispatch of an alternative form of the economic dispatch problem [1]. A simple example of an alternative form of the economic dispatch problem is the lossless DC OPF problem. Note that previous work has concluded that a minimizer of the lossless DC OPF problem can be used as the base-case to effectively approximate losses [27].

B. Direct Linearization

Reference [12] suggests directly linearizing the nodal loss allocation constraint $\Lambda_\mathrm{B}(\mathring{P}, N; \mathring{D}) = \mathbf{0}$ with respect to decision variables $\mathring{P}$ and N using a first order Taylor expansion. The gradient of the Left-Hand Side (LHS) with respect to $\mathring{P}$ is $-\frac{1}{2}\mathring{H}^{-1}\dot{A}^\mathsf{T}\nabla L(\dot{A}\dot{\theta}^0)|A|$ and the gradient of the LHS with respect to $\mathring{N}$ is $I + \frac{1}{2}\mathring{H}^{-1}\dot{A}^\mathsf{T}\nabla L(\dot{A}\dot{\theta}^0)|A|$. With this in mind, the direct LCED problem is defined as follows.

Optimization C. The *direct LCED problem* is defined to be Opt. j with Λ_j specified by:

$$\begin{aligned}\Lambda_\mathrm{C}(\mathring{P}, N; \mathring{D}) :=\, & N - \tfrac{1}{2}|A|^\mathsf{T} L\Big(\dot{A}\mathring{H}^{-\mathsf{T}}(\mathring{P}^0 - \mathring{D} - \mathring{N}^0)\Big) \\ & - \tfrac{1}{2}|A|^\mathsf{T}\nabla L^\mathsf{T}(\dot{A}\dot{\theta}^0)\dot{A}\mathring{H}^{-\mathsf{T}}((\mathring{P} - \mathring{P}^0) - (\mathring{N} - \mathring{N}^0)).\end{aligned} \tag{16}$$

Note that with the specification of Λ_C as in (16), Opt. C has linear constraints. With cost function $C(\cdot)$ assumed convex, Opt. C is a convex problem. If the cost function $C(\cdot)$ is linear, then Opt. C is a linear program.

The following assumption provides conditions under which an optimal dispatch of the TCED problem without angles (Opt. B) is also optimal for the direct LCED problem (Opt. C).

Assumption 1. *The base-case dispatch and nodal loss allocation vectors (P^0,N^0) represent a local minimizer of the TCED problem without angles (Opt. B) and solve the KKT conditions for the TCED problem without angles (Opt. B) along with some Lagrange multipliers π^0, γ^0, $\bar{\mu}^0$, $\underline{\mu}^0$, $\bar{\beta}^0$, and $\underline{\beta}^0$.*

We will employ this assumption for the remainder of this section. Notice that the LMPs are well defined under Assumption 1 and are expressed as in Section II-D. Under Assumption 1 reference [12] proves that (P^0, N^0) along with Lagrange multipliers π^0, γ^0, $\bar{\mu}^0$, $\underline{\mu}^0$, $\bar{\beta}^0$, $\underline{\beta}^0$ satisfy the KKT conditions for the direct LCED problem (Opt. C), implying that the base-case dispatch P^0 is optimal for the direct LCED problem, which is a convex program with linear constraints as mentioned above. This result is easy to verify by noting that these values satisfy the stationarity condition and the primal feasibility condition for both problems. Additionally note that the dual feasibility condition and complementary slackness conditions are identical for both problems.

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Remark 4. A base-case state $\dot{\theta}^0$ satisfying Assumption 1 is termed an *ideal base-case state*. However, in practice Assumption 1 will only be approximately true and the base-case values (P^0, N^0) will fall in the vicinity of a minimizer of the TCED problem without angles (Opt. B). Section IV explains the conditions under which this is a reasonable approximation. Specifically, Section IV-A studies test cases where this approximation does not work well and Section IV-B studies test cases where this approximation does work well.

C. Indirect Linearization

In this section we take another approach, similar to that of [1] and [3] by deriving loss sensitivities with respect to nodal net power injections about the base-case state. To do this we must first define the nodal loss allocation as an implicit vector valued function of the net power injections at the non-price reference buses $\tilde{N} : \mathbb{R}^{n-1} \to \mathbb{R}^n$. The vector of net power injections is denoted $T \in \mathbb{R}^n$ and is interpreted as the generation dispatch vector less the demand vector $P - D$. The function $\tilde{N}$ is defined implicitly by the relationship between N and $\mathring{P} - \mathring{D}$ in the constraint $\Lambda_{\mathrm{B}}(\mathring{P}, N; \mathring{D}) = \mathbf{0}$ from the TCED problem without angles (Opt. B), and therefore satisfies:

$$\tilde{N}(\mathring{T}) := \tfrac{1}{2}|A|^{\mathsf{T}} L\left(\dot{A}\mathring{H}^{-\mathsf{T}}(\mathring{T} - \mathring{\tilde{N}}(\mathring{T}))\right). \tag{17}$$

The *loss sensitivity matrix* is denoted $\nabla\tilde{N} : \mathbb{R}^{n-1} \to \mathbb{R}^{(n-1)\times n}$ and is defined to be the Jacobian of $\tilde{N}$ with respect to its argument. The nodal loss allocation vector can then be approximated using a simple first order Taylor expansion of the function $\tilde{N}(\mathring{T})$ evaluated at the base-case net power injections $\mathring{T}^0 := \mathring{P}^0 - \mathring{D}$. The indirect LCED problem is as follows.

Optimization D. The *indirect LCED problem* is defined to be Opt. j with Λ_j specified by:

$$\Lambda_{\mathrm{D}}(\mathring{P}, N; \mathring{D}) := N - \tilde{N}(\mathring{P}^0 - \mathring{D}) - \nabla\tilde{N}(\mathring{P}^0 - \mathring{D})^{\mathsf{T}}(\mathring{P} - \mathring{P}^0). \tag{18}$$

We derive the loss sensitivity matrix $\nabla\tilde{N}$ in Theorem 1 below. Theorem 1 allows us to prove equivalence of linear constraints $\Lambda_{\mathrm{C}}(\mathring{P}, N; \mathring{D}) = \mathbf{0}$ and $\Lambda_{\mathrm{D}}(\mathring{P}, N; \mathring{D}) = \mathbf{0}$. Specifically, these constraints are equivalent in that one constraint holds if and only if the other holds. This can be proven algebraically by substituting the expression for the loss sensitivity matrix, $\nabla\tilde{N}$, from Theorem 1 into the expression for $\Lambda_{\mathrm{D}}(\mathring{P}, N; \mathring{D})$ from (18). As a result, the direct LCED problem (Opt. C) and the indirect LCED problem (Opt. D) are equivalent in that any optimal dispatch of one of these problems is also optimal for the other. Also notice that computing the loss sensitivity matrix, $\nabla\tilde{N}$, requires the inversion of a large Jacobian matrix.

Theorem 1. *Assuming the matrix $I + \frac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}|$ is invertible, the columns of the loss sensitivity matrix evaluated at $\mathring{T}^0$ corresponding to the non-price reference buses are expressed as:*

$$\nabla\mathring{\tilde{N}}(\mathring{T}^0) = \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}|(I + \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}|)^{-1}.$$

The column of the loss sensitivity matrix evaluated at $\mathring{T}^0$ corresponding to the price reference bus is expressed as:

$$\nabla\tilde{N}_\sigma(\mathring{T}^0) = \tfrac{1}{2}(I - \nabla\mathring{\tilde{N}}(\mathring{T}^0))\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|A_\sigma|.$$

Proof: Differentiating all rows of (17) excluding the price reference bus around the point $\mathring{T}^0$ yields the following:

$$\nabla\mathring{\tilde{N}}(\mathring{T}^0) = \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}| - \tfrac{1}{2}\nabla\mathring{\tilde{N}}(\mathring{T}^0)\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}|.$$

Algebraic manipulation yields the following and the result then follows by the assumption on invertibility:

$$\nabla\mathring{\tilde{N}}(\mathring{T}^0)(I + \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}|) = \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}|.$$

Differentiating the row of (17) corresponding to the price reference bus yields the expression for $\nabla\tilde{N}_\sigma(\mathring{T}^0)$. □

D. Equivalent Loss Component of LMP

The expression for the loss component of the LMP in terms of the Lagrange multipliers of the indirect LCED problem (Opt. D) evaluated at the base-case point (P^0, N^0) is as follows:

$$\mathring{l} = \nabla_{\mathring{D}}\Lambda_D(\mathring{P}^0, N^0; \mathring{D})\gamma' = \nabla\tilde{N}(\mathring{P}^0 - \mathring{D})\gamma', \tag{19}$$

where $l_\sigma = 0$ and (P^0, N^0) along with Lagrange multipliers π^0, γ', $\bar{\mu}^0$, $\underline{\mu}^0$, $\bar{\beta}^0$, $\underline{\beta}^0$ satisfy the KKT conditions for the indirect LCED problem (Opt. D). Of course this expression is slightly different than the expression given for the loss component of the LMP from (14). This is because the Lagrange multipliers γ' corresponding to the indirect LCED problem (Opt. D) do not match the Lagrange multipliers γ^0 corresponding to the direct LCED problem (Opt. C). For the remainder of the paper γ' will denote the optimal Lagrange multiplier for the constraint $\Lambda_{\mathrm{D}}(\mathring{P}, N; \mathring{D}) = \mathbf{0}$ of the indirect LCED problem (Opt. D).

The following theorem derives a relationship between the Lagrange multipliers of the direct LCED problem (Opt. C) and the indirect LCED problem (Opt. D). By Theorem 2 below, the KKT point from Assumption 1, with γ^0 replaced by γ', solves the KKT conditions for the indirect LCED problem (Opt. D). As a result, the base-case dispatch P^0 is optimal for the indirect LCED problem (Opt. D). In addition, the expressions of the loss component of the LMP from (14) can be obtained from the expression (19) by substituting γ' and $\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})$ using the expressions from Theorems 1 and 2.

Theorem 2. *Let (P^0, N^0) along with Lagrange multipliers π^0, γ^0, $\bar{\mu}^0$, $\underline{\mu}^0$, $\bar{\beta}^0$, $\underline{\beta}^0$ satisfy the KKT conditions for the direct LCED problem (Opt. C). Then (P^0, N^0) along with Lagrange multipliers π^0, γ', $\bar{\mu}^0$, $\underline{\mu}^0$, $\bar{\beta}^0$, $\underline{\beta}^0$ satisfy the KKT conditions for the indirect LCED problem (Opt. D) where $\gamma'_\sigma = \gamma^0_\sigma$ and*

$$\mathring{\gamma}' = (I + \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|\mathring{A}|)\mathring{\gamma}^0 + \tfrac{1}{2}\mathring{H}^{-1}\dot{A}^{\mathsf{T}}\nabla L(\dot{A}\dot{\theta}^0)|A_\sigma|\gamma^0_\sigma.$$

Proof: Notice that the primal feasibility, dual feasibility, and complementary slackness conditions do not include γ and are identical for Opt. C and Opt. D. Therefore, by hypothesis these conditions also hold for Opt. D. It remains to show that the stationarity conditions (2)-(5) hold for Opt. D. Notice that the stationarity conditions for Opt. C and Opt. D differ only by their loss terms and so we need only show equivalence of the partial derivatives of the loss terms for both problems.

1) Stationarity Condition w.r.t. N_σ: For Opt. D we have $\nabla_{N_\sigma}\Lambda_{\mathrm{D}}(\mathring{P}^0, N^0; \mathring{D})\gamma' = \gamma'_\sigma$. Replacing γ'_σ with its given expression yields $\gamma^0_\sigma = \nabla_{N_\sigma}\Lambda_{\mathrm{C}}(\mathring{P}^0, N^0; \mathring{D})\gamma^0$, matching Opt. C.

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2) Stationarity Condition w.r.t. $\mathring{N}$*:* For Opt. D we have $\nabla_{\mathring{N}}\Lambda_{\mathrm{D}}(\mathring{P}^0, N^0; \mathring{D})\gamma' = \mathring{\gamma}'$. Substituting the given expression for $\mathring{\gamma}'$ results in $\nabla_{\mathring{N}}\Lambda_{\mathrm{C}}(\mathring{P}^0, N^0; \mathring{D})\gamma^0$, matching Opt. C.

3) Stationarity Condition w.r.t. $\mathring{P}$*:* For Opt. D we have

$$\nabla_{\mathring{P}}\Lambda_{\mathrm{D}}(\mathring{P}^0, N^0; \mathring{D})\gamma' = -\nabla\mathring{\tilde{N}}(\mathring{P}^0 - \mathring{D})\mathring{\gamma}' - \nabla\tilde{N}_\sigma(\mathring{P}^0 - \mathring{D})\gamma'_\sigma.$$

Substituting the expression for $\nabla\mathring{\tilde{N}}(\mathring{P}^0 - \mathring{D})$ from Theorem 1, substituting the given expressions for $\mathring{\gamma}'$ and γ'_σ, and rearranging the expression algebraically results in the following:

$$\begin{aligned}\nabla_{\mathring{P}}\Lambda_{\mathrm{C}}(\mathring{P}^0, N^0; \mathring{D})\gamma^0 = &-\tfrac{1}{2}\mathring{H}^{-1}\mathring{A}^{\mathsf{T}}\nabla L(\mathring{A}\mathring{\theta}^0)|\mathring{A}|\mathring{\gamma}^0 \\ &-\tfrac{1}{2}\mathring{H}^{-1}\mathring{A}^{\mathsf{T}}\nabla L(\mathring{A}\mathring{\theta}^0)|A_\sigma|\gamma^0_\sigma,\end{aligned}$$

again matching Opt. C. □

E. Loss Distribution Factor LCED Problem

To reduce the number of optimization variables many references, including [1] and [3], introduce *Loss Distribution Factors* (LDFs) in the form of a vector $\eta \in \mathbb{R}^n$ where $\mathbf{1}^{\mathsf{T}}\eta = 1$ and typically $\eta \in \mathbb{R}^n_+$. Each LDF represents the fraction of total system losses allocated to node i. Recognizing that the total system losses are represented by the function $\mathbf{1}^{\mathsf{T}}\tilde{N}(\mathring{P} - \mathring{D})$, the LDF LCED problem is defined as follows.

Optimization E. The *LDF LCED problem* is defined to be Opt. j with Λ_j specified by:

$$\Lambda_{\mathrm{E}}(\mathring{P}, N; \mathring{D}) = N - \eta\Big(\mathbf{1}^{\mathsf{T}}\tilde{N}(\mathring{P}^0 - \mathring{D}) + \mathbf{1}^{\mathsf{T}}\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})^{\mathsf{T}}(\mathring{P} - \mathring{P}^0)\Big). \quad (20)$$

References [1] and [3] only provide *intuitive* choices of LDFs that are not proven optimal. In fact, there may not exist LDFs that allow the base-case dispatch to be optimal for the LDF LCED problem (Opt. E). To see this notice that the constraint $\Lambda_{\mathrm{E}}(\mathring{P}, N; \mathring{D}) = \mathbf{0}$ reduces to $N^0 = \eta\mathbf{1}^{\mathsf{T}}N^0$ when evaluated at the base-case values (P^0, N^0). For this constraint to be satisfied we must have the following choice of LDFs:

$$\eta := \tfrac{1}{\mathbf{1}^{\mathsf{T}}N^0}N^0. \quad (21)$$

Remark 5. References [1] and [3] suggest the use of LDFs from (21). However, these references do not prove if or when such LDFs are optimal. Section IV-B illustrates that the LDFs from (21) are not always optimal but typically result in very small approximation error.

Unfortunately, the LDFs in (21) do not generally allow the stationarity condition for the LDF LCED problem (Opt. E) to be satisfied at the KKT point from Assumption 1. To see this notice that the stationarity conditions for the LDF LCED problem (Opt. E) and the indirect LCED problem (Opt. D) differ only by their loss terms. In order for these problems to be equivalent, the partial derivative of the loss term in the Lagrangian function from Opt. E with respect to $\mathring{P}$ must be equivalent to that of Opt. D. This partial derivative is written as follows:

$$\begin{aligned}\nabla_{\mathring{P}}\Lambda_{\mathrm{E}}(\mathring{P}, N; \mathring{D})\gamma' &= -\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})\mathbf{1}\eta^{\mathsf{T}}\gamma', \\ &= -\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})\mathbf{1}(\tfrac{1}{\mathbf{1}^{\mathsf{T}}N^0}N^0)^{\mathsf{T}}\gamma'. \quad (22)\end{aligned}$$

In general this partial derivative does not match that of the indirect LCED problem (Opt. D), which is expressed as $\nabla_{\mathring{P}}\Lambda_{\mathrm{D}}(\mathring{P}, N; \mathring{D})\gamma' = -\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})\gamma'$. A sufficient condition for the base-case dispatch to be optimal for the LDF LCED problem (Opt. E) requires that γ' be proportional to the vector of ones as embodied in Assumption 2 below. We will employ Assumption 2 for the remainder of this section and we will consider some test cases for which it is also satisfied; however, it is important to note that this assumption is not generally true.

Assumption 2. *The optimal Lagrange multipliers* γ' *associated with the indirect LCED problem (Opt. D) are uniform. E.g. there exists some* $\alpha \in \mathbb{R}$ *such that* $\gamma' = \mathbf{1}\alpha$.

To understand Assumption 2 it is useful to analyze the stationarity condition for the indirect LCED problem (Opt. D) with respect to N as defined by (2) and (4), which is as follows:

$$\pi^0 + \gamma'_\sigma = 0 \quad \text{and} \quad \mathbf{1}\pi^0 + \mathring{\gamma}' + S^{\mathsf{T}}(\underline{\mu}^0 - \bar{\mu}^0) = \mathbf{0}$$

This shows that the Lagrange multipliers $\gamma' = -\mathbf{1}\pi^0 - c$ are indeed uniform if and only if the congestion component of the LMP c is zero as defined in Section II-D, in which case $\gamma' = -\mathbf{1}\pi^0$. In turn, the congestion component of the LMP is zero if no line limits from constraint ($j4$) are binding at the base-case values (P^0, N^0), leading to the statement in Remark 6. This is because non-binding line limits require that $\underline{\mu}^0 = \bar{\mu}^0 = \mathbf{0}$ by the complementary slackness condition.

Remark 6. Assumption 2 holds if no line limits from constraint ($j4$) are binding at the base-case values (P^0, N^0).

The following theorem states that the base-case dispatch P^0 is optimal for the LDF LCED problem (Opt. E) under Assumptions 1 and 2.

Theorem 3. *Let* (P^0, N^0) *along with Lagrange multipliers* π^0, γ', $\bar{\mu}^0$, $\underline{\mu}^0$, $\bar{\beta}^0$, $\underline{\beta}^0$ *satisfy the KKT conditions for the indirect LCED problem (Opt. D) where* $\gamma' = \mathbf{1}\alpha$*. Then* (P^0, N^0) *along with the same Lagrange multipliers also satisfy the KKT conditions for the LDF LCED problem (Opt. E).*

Proof: Notice that the primal feasibility, dual feasibility and complementary slackness conditions are identical for Opt. D and Opt. E and by hypothesis these conditions therefore hold for Opt. E. It remains to show that the stationarity conditions (2)-(5) hold for Opt. E. Notice that the stationarity conditions for Opt. D and Opt. E differ only by their loss terms and so we need only show equivalence of the partial derivatives of the loss terms for both problems. First, the partial derivatives with respect to N and P_σ are equivalent for both problems:

$$\nabla_N\Lambda_j(\mathring{P}^0, N^0; \mathring{D})\mathbf{1}\alpha = \mathbf{1}\alpha \quad \text{and} \quad \nabla_{P_\sigma}\Lambda_j(\mathring{P}^0, N^0; \mathring{D})\mathbf{1}\alpha = 0.$$

Furthermore, the partial derivative with respect to $\mathring{P}$ is equivalent for both problems. This is shown as follows:

$$\begin{aligned}\nabla_{\mathring{P}}\Lambda_{\mathrm{E}}(\mathring{P}^0, N^0; \mathring{D})\mathbf{1}\alpha &= -\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})\mathbf{1}(\tfrac{1}{\mathbf{1}^{\mathsf{T}}N^0}N^0)^{\mathsf{T}}\mathbf{1}\alpha, \\ &= -\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})\mathbf{1}\alpha.\end{aligned}$$

□

F. Common LCED Problem

The commonly formulated economic dispatch problem represents the total system losses as a single decision variable $\ell \in \mathbb{R}$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TPWRS.2019.2941906, IEEE Transactions on Power Systems

as opposed to representing the loss allocation to each node as individual decision variables. This effectively eliminates $n-1$ decision variables and $n-1$ constraints, making the problem much easier to solve. Using notation similar to [1], the *common LCED problem* (Opt. F) is written as follows:

$$\min_{P\in\mathbb{R}^n,\ell\in\mathbb{R}} \quad C(P) \qquad \text{(Opt. F)}$$

$$st: \mathbf{1}^\intercal(P-D-\eta\ell)=\mathbf{0} \qquad \text{(F1)}$$

$$\ell = LF^\intercal(P-D)+q \qquad \text{(F2)}$$

$$\underline{P}\le P\le \bar{P} \qquad \text{(F3)}$$

$$\underline{F}\le S(\mathring{P}-\mathring{D}-\mathring{\eta}\ell)\le \bar{F} \qquad \text{(F4)}$$

This formulation is similar to the general economic dispatch problem without angles (Opt. j). In fact, the objective function and the generation output limit constraint are identical to those of Opt. j. The power balance constraint (F1) and line limit constraint (F4) represent constraints ($j1$) and ($j4$) after replacing the nodal loss allocation vector N with the expression $\eta\ell$. Constraint (F2) represents the total system losses where $q\in\mathbb{R}$ is an offset constant and $LF\in\mathbb{R}^n$ represents the sensitivity of the total system losses with respect to the net power injections. Notice this formulation enforces transmission line limits despite the use of Assumption 2, which effectively assumes no line limits are binding.

References [1] and [3] observe that different approximations of LF can be used and they derive three different versions of LF using total derivatives and implicit functions. With respect to our formulation the loss sensitivity vector is defined based on the loss sensitivity matrix as $\mathring{LF}=\nabla\tilde{N}(\mathring{P}^0-\mathring{D})\mathbf{1}$ and $LF_\sigma=\mathbf{0}$. Our formulation also uses a constant offset term of $q=\mathbf{1}^\intercal\tilde{N}(\mathring{P}^0-\mathring{D})+\mathbf{1}^\intercal\nabla\tilde{N}(\mathring{P}^0-\mathring{D})^\intercal(\mathring{D}-\mathring{P}^0)$. Using these definitions of LF and q the loss constraint (F2) is as follows:

$$\ell=\mathbf{1}^\intercal\tilde{N}(\mathring{P}^0-\mathring{D})+\mathbf{1}^\intercal\nabla\tilde{N}(\mathring{P}^0-\mathring{D})^\intercal(\mathring{P}-\mathring{P}^0). \qquad (23)$$

Using this proposed definition of the loss sensitivity vector LF and offset constant q, it should now be apparent that the common LCED problem (Opt. F) is equivalent to the LDF LCED problem (Opt. E) in that any optimal dispatch of one problem is also optimal for the other. This is because both problems have the same feasible set of dispatch variables P. In fact, a feasible solution of the common LCED problem (P,ℓ) can be constructed from any feasible solution of the LDF LCED problem (P,N) where $\ell=\mathbf{1}^\intercal N$. Similarly, a feasible solution of the LDF LCED problem (P,N) can be constructed from any feasible solution of the common LCED problem (P,ℓ) where $N=\eta\ell$.

Remark 7. Although we do not prove this explicitly, the base-case values P^0 and $\ell^0=\mathbf{1}^\intercal N^0$ indeed satisfy the KKT conditions of the common LCED problem (Opt. F) along with Lagrange multipliers π^0, α, $\bar{\mu}^0$, $\underline{\mu}^0$, $\bar{\beta}^0$, $\underline{\beta}^0$ under Assumptions 1 and 2. Furthermore, in agreement with (19) the loss price can now be expressed as $l=LF\alpha$. It is interesting to note that this expression of the loss price can also be derived from the common LCED problem (Opt. F) when considering the loss sensitivity matrix LF and the offset constant q to be constant parameters independent of the demand vector D. This may be counter-intuitive because LF and q are indeed (implicitly) defined in terms of the demand vector D.

G. Recovering a Locally Optimal Dispatch

Recall that, by Assumption 1, the base-case values (P^0,N^0) represent a local minimizer of the TCED problem without angles (Opt. B). Thus far we have shown that the base-case generation dispatch P^0 is locally optimal for the TCED problem with angles (Opt. A) and is globally optimal for the common LCED problem (Opt. F) under Assumptions 1 and 2. However, as mentioned in the introduction, the common LCED problem (Opt. F) may have multiple optimal dispatch vectors, some of which may not be optimal or even feasible for the TCED problem with angles (Opt. A). This is an issue because standard off-the-shelf optimization software used to solve the common LCED problem (Opt. F) only identifies one of potentially multiple optimal dispatch vectors. An additional assumption is required to guarantee the identified dispatch matches the base-case dispatch P^0.

Assumption 3. *The common LCED problem (Opt. F) has a unique minimizer* (P,ℓ).

Remark 8. The common LCED problem (Opt. F) is said to have a *unique optimal dispatch* if Assumption 3 holds true. Similarly, the common LCED problem (Opt. F) is said to have *multiple minimizers* or *multiple optimal dispatch vectors* if Assumption 3 does not hold true.

Assumption 3 ensures that the common LCED problem has a unique optimal dispatch. Under Assumptions 1, 2, and 3 the unique optimal dispatch for the common LCED problem (Opt. F) is indeed locally optimal for the TCED problem with angles (Opt. A) and represents the base-case dispatch P^0. Under these three assumptions a locally optimal dispatch for the TCED problem with angles (Opt. A) can be identified by solving the common LCED problem (Opt. F) using standard off-the-shelf optimization software. The numerical results section investigates errors associated with relaxing each of these three assumptions.

IV. Numerical Results

This section provides numerical results intended to illustrate two key findings that point out significant dispatch approximation errors may occur when certain assumptions are violated. Section IV-A illustrates the first key finding that small perturbations to the ideal base-case state can result in significantly large dispatch approximation error if Assumption 3 fails to hold. Associated errors are illustrated using an intuitive 2-bus test case as well as a realistically large test case with 2383 buses, neither of which enforce line limits so that Assumption 2 holds. Section IV-B then illustrates the second key finding that significant dispatch approximation error may occur when Assumption 2 fails to hold even if Assumptions 1 and 3 hold. Associated errors are illustrated using an intuitive 3-bus test case as well as the same

2383 bus test case as in Section IV-A but with line limits enforced. In this specific example, introducing transmission line limits to the 2383 bus test case causes the common LCED problem (Opt. F) to have a unique optimal dispatch and we show that there is no dispatch approximation error despite the presence of transmission congestion. This suggests that realistic systems may not experience significant approximation error when Assumption 2 fails to hold. Section IV-B continues to select different base-case states that introduce dispatch approximation error into the 2383 bus test case when enforcing line limits. Since the uniqueness Assumption 3 holds for this specific test case, small perturbations to the ideal base-case state result in little dispatch approximation error.

In this section the k^{th} diagonal element of B is $B_{kk} = \frac{1}{x_k}$ and the k^{th} element of the vector valued loss function is $L_k(\Theta) = \frac{r_k}{x_k^2}\Theta_k^2$ where the impedance of line $k \in \mathbb{E}$ is $r_k + \mathbf{i}x_k$ and $\Theta \in \mathbb{R}^m$ represents the voltage angle difference across each transmission line $\dot{A}\dot{\theta}$. The LDFs η are chosen as in (21). A computer with a 2.0 GHz processor is used.

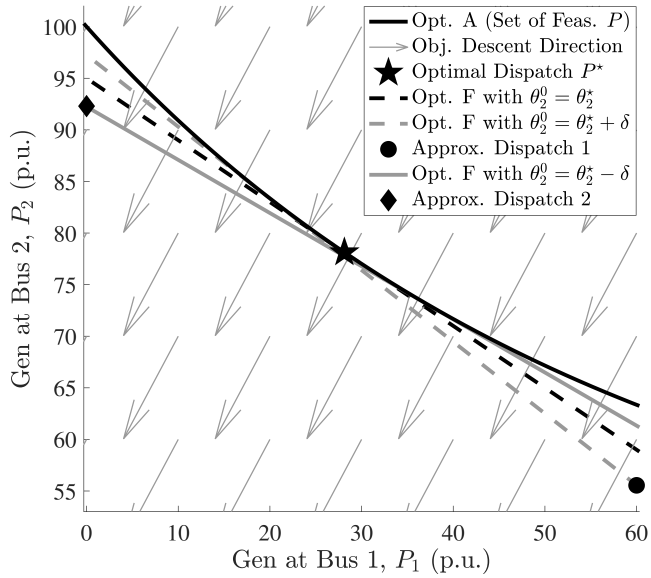

Fig. 3: Illustration of the feasible set of dispatch vectors P for various ED problems associated with the 2-bus test case.

A. Multiple Minimizers of the Common LCED Problem

This subsection studies two test cases that satisfy Assumptions 1 and 2; however, the common LCED problem (Opt. F) has multiple minimizers for these two test cases, violating Assumption 3. In this context small perturbations to the ideal base case-state are shown to result in a common LCED problem (Opt. F) with a unique optimal generation dispatch; however, this unique optimal generation dispatch differs significantly from the desired generation dispatch $P^\star$ that solves the TCED problem with angles (Opt. A), resulting in large generation dispatch approximation error. This is shown using an intuitive 2-bus test case as well as a large test case with 2383 buses. In this subsection neither test case enforces line limits so that Assumption 2 holds. (See Remark 6).

1) 2-Bus Test Case: A one-line diagram of the 2-bus test case is provided in Figure 2 along with various parameters of the test case. Notice that the transmission line is highly resistive and has no transmission limit. All system demand is located at bus 2 and is fixed to $D_2 = 100$p.u. The demand is co-located with *expensive* generation with cost function $C_2(P_2) = P_2$. The generation at bus 2 is unlimited so that $0 \le P_2 \le \infty$. *Inexpensive* generation is located remotely at bus 1 as the cost of this generator is $C_1(P_1) = 0.6P_1$. The inexpensive generation is limited as $0 \le P_1 \le 60$p.u. The total system cost $C(P)$ is the sum of the individual generator costs. Bus 1 is designated the angle reference bus and the price reference bus so that $\sigma = \rho = 1$. The resulting system state is $\dot{\theta} = \theta_2$.

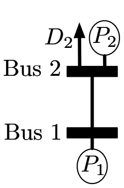

Fig. 2: 2-bus system one-line diagram. The demand is fixed at $D_2 = 100$p.u. The generators have cost functions $C_1(P_1) = 0.6P_1$ and $C_2(P_2) = P_2$. The line has per-unit impedance of $0.01 + \mathbf{i}0.01$ and has no limit so that $\bar{F}_1 = -\underline{F}_1 = \infty$.

Figure 3 plots the set of all feasible generation dispatch vectors for the TCED problem with angles (Opt. A) as a black curve. The gray arrows in this figure represent the objective descent direction. The unique optimal generation dispatch vector is $P^\star = [28.125, 78.125]^\mathsf{T}$p.u., is represented by the star in the figure, and is intuitively the feasible generation dispatch vector that is furthest downstream in the descent direction. The associated optimal vector of voltage angles is $\dot{\theta}^\star = \theta_2^\star = -0.25$ radians.

The black dashed line represents the set of all feasible generation dispatch vectors for the common LCED problem (Opt. F) when using the ideal base-case state $\dot{\theta}^0 = \dot{\theta}^\star$. Notice that the descent direction is perpendicular to the black dashed line and as a result all feasible generation dispatch vectors are optimal. Since the common LCED problem (Opt. F) has multiple minimizers, Assumption 3 does not hold. Additionally, only one of the infinitely many minimizers of Opt. F is feasible for the TCED problem with angles (Opt. A).

In practice the ideal base-case state is typically estimated. To emulate this, linearize around the base-case state $\dot{\theta}^\star + \delta$ where $\delta = 0.07$ radians represents a small perturbation. The resulting linear feasible set of generation dispatch vectors is represented by the dashed gray line. Furthermore, the common LCED problem (Opt. F) now has a unique optimal generation dispatch vector $P \approx [60, 55.559]^\mathsf{T}$p.u. that is represented by the black circle and also significantly differs from the desired generation dispatch vector $P^\star$. The dispatch approximation error vector is $\Delta P = P - P^\star \approx [31.875, -22.566]^\mathsf{T}$p.u.

Perturbing the ideal base-case state in the other direction also results in large dispatch approximation error. Specifically, linearizing around the base-case state $\dot{\theta}^\star - \delta$ results in the linear feasible set of generation dispatch vectors represented by the solid gray line. In this case the common LCED problem (Opt. F) has a unique optimal generation dispatch vector $P \approx [0, 92.242]^\mathsf{T}$p.u. that is represented by the black diamond and also significantly differs from the desired generation dispatch vector $P^\star$. The dispatch approximation error vector is $\Delta P = P - P^\star \approx [-28.125, 14.117]^\mathsf{T}$p.u.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TPWRS.2019.2941906, IEEE Transactions on Power Systems

Remark 9. Similarly large approximation error remains for an arbitrarily small perturbation $\delta > 0$. Alternatively, similarly large approximation error remains after adding slight curvature to the objective descent direction that might arise from a small positive quadratic coefficient in the cost function.

Remark 10. The lossless DC OPF problem yields a unique optimal generation dispatch vector of $P = [60, 40]^\mathsf{T}$p.u. for the 2-bus test case. In fact, the lossless DC OPF problem and the TCED problem with angles (Opt. A) both have unique optimal generation dispatch vectors; however, the common LCED problem (Opt. F) has multiple optimal generation dispatch vectors when the base-case state matches its ideal value $\dot{\theta}^\star$.

Remark 11. Reference [1, section V-A] makes a similar observation regarding a different 2-bus test case. In their example the common LCED problem (Opt. F) also has multiple minimizers when the ideal base-case state is chosen. Rather than perturbing the base-case state, as is done in this paper, they perturb the bids of the generators. Keeping the base-case state fixed, they show that a small perturbation in generator bids can result in significant dispatch approximation error. This observation is consistent with the observations made here.

2) Test Case 2383wp without Line Limits Enforced: Now consider the test case *2383wp* from the NESTA archive, which represents the Polish power system during the 1999-2000 winter evening peak [28]. Line limits are not enforced to ensure that Assumption 2 holds. The identified minimizer of the TCED problem with angles (Opt. A) is found using the interior point algorithm provided by the MATLAB function FMINCON and takes 17.10sec. to converge. The optimal objective value of this problem is $C(P^\star)$=1865459.40\$.

Let's first analyze the common LCED problem (Opt. F) with the base-case state chosen to match the identified minimizer of the TCED problem with angles (Opt. A) $\dot{\theta}^0 = \dot{\theta}^\star$ in order to satisfy Assumption 1. In this case the loss sensitivity matrix $\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})$ takes 26.13sec. to compute due to the inverse computation. There exist multiple optimal dispatch vectors P for the common LCED problem (Opt. F), some of which are not feasible for the TCED problem with angles (Opt. A). To illustrate this we use an interior point algorithm provided by Artelys KNITRO software to solve the common LCED problem (Opt. F) with different supplied initial guesses [29]. Table I compares the identified minimizers of both problems where $\Delta P \in \mathbb{R}^n$ represents the difference between $P^\star$ and the identified optimal dispatch of the common LCED problem (Opt. F) and ΔC represents the difference between $C(P^\star)$ and the optimal objective value of the common LCED problem (Opt. F).

Turning to the first row of table I, the main body of the paper proves that $P = P^\star$ and $\ell = \mathbf{1}^\mathsf{T} N^\star$ is a minimizer of the common LCED problem (Opt. F). When supplying this as the initial guess, the algorithm immediately converges to a dispatch that nearly matches $P^\star$. Notice that the associated errors, ΔP, are very small relative to the total dispatch generation $\|P^\star\|_1 = 49231.67$MW and the optimal objective values of both problems are nearly identical. However, ΔP is not identically zero due to insignificant computational error.

TABLE I: Results for test case 2383wp without line limits enforced and with the ideal base-case state $\dot{\theta}^0 = \dot{\theta}^\star$. The error quantities are denoted with Δ and represent the difference between identified optimal quantities of Opt. A and Opt. F.

Initial Guess for the Interior Point Algorithm	$\|\Delta P\|_1$ (MW)	$\|\Delta P\|_\infty$ (MW)	$\|\Delta C\|$ (\$)	Solver Time (s)
$P = P^\star$ and $\ell = \mathbf{1}^\mathsf{T} N^\star$	1.73	0.88	0.00	5.52
Typical Operating Point	21.65	10.70	0.00	5.33
Minimizer of lossless DC OPF	308.77	152.77	0.00	7.61

To demonstrate that Opt. F has multiple minimizers, two alternative initial guesses were supplied as shown in the second and third rows of table I. The first alternative was constructed from the typical operating point provided by the test case description. The second alternative was constructed from the minimizer of the lossless DC OPF problem, which was solved using the DC OPF function available in MATPOWER [30]. When using these alternative initial guesses the algorithm converges to dispatch values that do not match $P^\star$ but do attain the same optimal objective value $C(P^\star)$. Furthermore, these identified dispatch values are not feasible for the TCED problem with angles (Opt. A).

Similar to the simple 2-bus example, a small perturbation of the base-case state results in significantly large dispatch approximation error. To illustrate this we perturb the ideal base-case state $\dot{\theta}^\star$ by a perturbation vector $\delta \in \mathbb{R}^{n-1}$ that was sampled from a normal distribution with zero mean and a diagonal covariance matrix of $10^{-10} \times I$. The specific sample drawn from this distribution has the properties $\|\delta\|_\infty = 3.6 \times 10^{-5}$ radians and $\|\delta\|_1 = 0.018$ radians. The resulting common LCED problem appears to have a unique optimal dispatch because the same dispatch is identified when using any initial guess for the interior point algorithm. The dispatch approximation error is $\|\Delta P\|_1 = 314.70$ MW and $\|\Delta P\|_\infty = 156.19$ MW. This dispatch approximation error is significant compared to the small perturbation δ.

B. LDF Approximation Error with Congestion

This subsection studies test cases that satisfy Assumptions 1 and 3; however, transmission line congestion causes Assumption 2 to be violated. A highly resistive 3-bus test case with significant transmission congestion is used to illustrate the potential dispatch approximation errors associated with Assumption 2. Although associated errors are large for this extreme 3-bus test case, they are typically very small in practice as is illustrated by the 2383 bus test case with line limits enforced. In fact, this test case exhibits no error despite having transmission congestion. To illustrate error associated with Assumption 1 alternative choices of base-case state are investigated that introduce dispatch approximation error into the 2383 bus test case.

1) 3-Bus Test Case: Figure 4 provides the details of the 3-bus test case in a one-line diagram. The generation dispatch values P have no upper limit but are restricted to be non-negative, e.g. $\underline{P} = 0$. The system is highly resistive as the impedance

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of each line is $r_k + \mathbf{i}x_k = 0.01 + \mathbf{i}0.01$ in units of p.u. The cost function is given by $C(P){=}C_1(P_1){+}C_2(P_2){+}C_3(P_3)$. Bus 1 is assigned to be the slack bus and the reference bus.

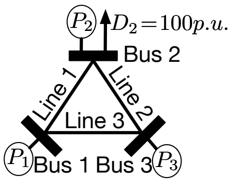

Fig. 4: 3-bus system one-line diagram. The line limits $\bar{F}{=}-\underline{F}$ are such that $\bar{F}_2{=}\bar{F}_3{=}\infty$ and $\bar{F}_1{=}11$p.u. The generators have cost functions $C_1(P_1){=}P_1^2$, $C_2(P_2){=}100P_2^2$ and $C_3(P_3){=}0.01P_3^2$.

The minimizer of the TCED problem with angles (Opt. A) is $P^\star \approx [82.38, 6.81, 123.61]^\mathsf{T}$ and $\dot{\theta}^\star \approx [-0.690, 0.110]^\mathsf{T}$ in units of p.u. and radians respectively. The ideal base-case state is chosen $\dot{\theta}^0 = \dot{\theta}^\star$ and thus the base-case dispatch is $P^0 = P^\star$. The line limit for line 1 is binding and the congestion component of the LMP is non-zero, taking the value $c{=}[0, 137.26, -137.26]^\mathsf{T}$ in units of dollars per p.u. Notice that Section III-E draws no conclusion regarding whether or not P^0 is optimal for the LDF LCED problem (Opt. E).

As expected the base-case point P^0 and $\ell^0 := \mathbf{1}^\mathsf{T}N^0$ is feasible for the common LCED problem (Opt. F); however, this point is not optimal. An alternative unique optimal dispatch P of Opt. F is identified that is not feasible for the TCED problem with angles (Opt. A). The optimal objective value for the common LCED problem (Opt. F) is \$11512.63, which is \$62.36 lower than the optimal objective value for the TCED problem with angles (Opt. A). The optimal dispatch for the common LCED problem (Opt. F) is $P{=}[86.96, 6.17, 122.41]^\mathsf{T}$p.u. and the dispatch approximation error is $P^\star{-}P{=}[-4.58, 0.64, 1.20]^\mathsf{T}$. This highly resistive network exhibits small error, suggesting this approximation is typically accurate.

2) Test Case 2383wp with Line Limits Enforced: Consider the same test case as Section IV-A but with line limits enforced. The identified minimizer of the TCED problem with angles (Opt. A) is found using the interior point algorithm provided by FMINCON and takes 18.81sec. to converge. The optimal objective value of this problem is $C(P^\star){=}1890940.57$\$. There are 4 congested lines with binding limits, which is a small amount relative to the 2896 total transmission lines.

When using the ideal base-case state $\dot{\theta}^0 = \dot{\theta}^\star$ the common LCED problem (Opt. F) appears to have a unique minimizer because the interior point algorithm converges to the same point with any choice of initial guess. Notice that test case 2383wp has the unusual property that its common LCED problem (Opt. F) has a unique minimizer when enforcing line limits but has multiple minimizers when line limits are not enforced. This property is not typical as many other test cases in the NESTA archive do not have this property.

We consider four different base-case states when formulating the common LCED problem (Opt. F). Each of the four versions of this problem appear to have a unique minimizer because the interior point algorithm converges to the same point for any choice of initial guess. Since there is a unique minimizer, we are able to take advantage of the much faster dual simplex algorithm provided by MOSEK, which does not allow for a user supplied initial guess [31]. Table II provides numerical results for each of the four choices of base-case state. Notice that the solver times are much faster and the optimal generation dispatch vector $P^\star$ is exactly recovered when setting the base-case state to its ideal value $\dot{\theta}^0 = \dot{\theta}^\star$. Note that the loss sensitivity matrix $\nabla\tilde{N}(\mathring{P}^0 - \mathring{D})$ requires approximately 28 seconds of computation time for each base-case state.

Since the common LCED problem (Opt. F) now has a unique optimal dispatch when using the ideal base-case state, small perturbations of the base-case state should result in small approximation errors. This is illustrated by using the same perturbed base-case state $\dot{\theta}^\star + \delta$ as in Section IV-A2. Notice that the dispatch approximation error is very small as compared to the error witnessed in Section IV-A2. In fact, we should expect the dispatch approximation error to disappear as the perturbation becomes smaller, e.g. $\|\delta\| \to 0$.

TABLE II: Results for test case 2383wp with line limits enforced. The error quantities are denoted with Δ and represent the difference between identified optimal quantities of Opt. A and Opt. F.

Choice of Base-Case State, $\dot{\theta}^0$	$\|\dot{\theta}^\star - \dot{\theta}^0\|_1$ (radians)	$\|\Delta P\|_1$ (MW)	$\|\Delta P\|_\infty$ (MW)	$\|\Delta C\|$ (\$)	Solver Time (s)
Local Min. of Opt. A, $\dot{\theta}^\star$	0.00	0.00	0.00	0.00	1.26
Small Perturbation, $\dot{\theta}^\star{+}\delta$	0.02	0.08	0.04	0.95	1.33
Min. of Lossless DC OPF	34.15	73.95	29.67	1061.11	1.10
Typical Operating Point	257.68	501.37	306.58	24007.79	1.37

As discussed in Section III-A, the ideal base-case state $\dot{\theta}^\star$ is not known in practice. We now consider two alternative choices of base-case state $\dot{\theta}^0$. First, is the minimizer of the lossless DC OPF problem. The second alternative is determined by solving power flow equations at the typical operating point provided by the test case description. The power flow equations and the lossless DC OPF problem are solved using MATPOWER. Table II shows that the generation dispatch error is small relative to the total generation, which satisfies $\|P^\star\|_1 = 25108.74$MW. Furthermore, the minimizer of the lossless DC OPF problem serves as a better base-case state in terms of the generation dispatch error.

V. Conclusions

For the first time, this paper characterizes the generation dispatch approximation errors associated with each of the three assumptions required to recover the optimal dispatch of the TCED problem from the common LCED problem. Four intermediate problems are used to derive the common LCED problem and identify the three key assumptions. Numerical results illustrate dispatch approximation error caused by choosing a non-ideal base-case state, using LDFs under congested conditions, and the common LCED problem having multiple minimizers. The first key finding states that the common LCED problem may have multiple minimizers when using the ideal base-case state, in which case small perturbations of the base-case state may result in significantly large dispatch approximation error. This type of approximation error is problematic in realistic systems and is illustrated using a simple 2-bus example as well as a larger more realistic test case with 2383 buses, neither of which enforce line limits. The second key finding states that under congested conditions there may not exist LDFs that allow the optimal dispatch of the TCED problem to also be optimal for the common LCED problem, even if the ideal base-case state is used. This is illustrated using an extreme 3-bus example that is highly resistive. However, the larger 2383 bus test case with line limits enforced is then used to illustrate that associated approximation error is typically insignificant.

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