Convex Hull Pricing for the AC Optimal Power Flow Problem

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***Abstract*—We study an abstract myopic electricity market subject to general non-convex power flow models and convex generator models (without discrete commitment variables) in a setting where there may be no market equilibrium that guarantees revenue adequacy of the Independent System Operator (ISO). Convex Hull Prices (CHPs) are proposed and are defined to solve a novel multi-objective minimum uplift problem that captures the trade-off between generator side-payments and potential congestion revenue shortfall. A convex primal counterpart of this multi-objective minimum uplift problem, termed the *primal CHP problem*, is formulated in terms of the convex hull of the set of feasible net power injections. Indeed the term *Convex Hull Price* derives from the result that the CHPs are equivalent to the optimal Lagrange multipliers of the *primal CHP problem*. However, depending on the chosen model of the transmission network, the convex hull of the feasible set of net power injections may be intractable to evaluate. In this case CHPs are approximated using state-of-the-art convex relaxations that are efficiently solvable. This is the first proposed method of approximating CHPs in polynomial-time that is general enough to accommodate the non-linear transmission constraints in the AC OPF problem. In our abstract myopic market setting we show that tight relaxations of the AC OPF problem can be used to effectively approximate CHPs that decrease potential congestion revenue shortfall significantly with little effect to side-payments.**

I. Introduction

The inherent non-convexity of the Alternating Current (AC) model of the transmission network prevents its implementation into the electricity market. This non-convexity presents two problems. The first problem, computing a social welfare maximizing dispatch, has been studied extensively in the form of the AC Optimal Power Flow (OPF) problem. On the other hand, little attention has been given to a second problem that arises pertaining to market design in the absence of a market equilibrium that guarantees revenue adequacy of the Independent System Operator (ISO). In fact, reference [1] identifies the problem of pricing non-convexities that arise through the AC transmission network as an emerging challenge in electricity markets. To address this, we propose the use of Convex Hull Prices (CHPs) that solve a novel multi-objective minimum uplift problem that balances a trade-off between *generator uplift* and *Financial Transmission Right (FTR) uplift*. For the first time, this paper presents a method of approximating CHPs in polynomial-time using a transmission network model that is general enough to accommodate the AC OPF problem.

Over the past decade a significant amount of research has focused on developing algorithms that identify a social welfare maximizing dispatch by solving the AC OPF problem. Although some algorithms are capable of optimally solving special instances of the AC OPF problem [2], it is generally NP-hard and there does not exist an algorithm that efficiently solves this problem without qualifying conditions [3], [4]. With this in mind, many recent works have developed relaxations to tightly approximate the AC OPF problem, e.g. [2], [5–11]. Although these methods identify a dispatch that is not guaranteed to be feasible, a feasible dispatch can be recovered via primal-dual interior point methods that is nearly optimal.

The problem of designing an energy market in the absence of a market equilibrium is well studied in the context of the day-ahead market, which centers around a Mixed Integer Program (MIP) known as the Unit Commitment (UC) problem [12], assuming linearized DC power flow approximations. Similar to the AC OPF problem, the UC problem with DC power flow is non-convex, is NP-hard and is typically solved using heuristic algorithms that perform well in practice but do not identify dispatch values with optimality guarantees. Furthermore, there rarely exist uniform nodal prices that support the optimal dispatch in the day-ahead market. To overcome this problem CHPs have been proposed, also known as extended locational marginal prices, along with side-payments that cover lost opportunity costs of the market participants [13], [14].

CHPs represent a solution of an optimization problem that minimizes various *uplift* quantities including the aforementioned side-payments, which are not directly funded by another revenue stream of the ISO and are typically referred to as *generator uplift*. Reference [15] points out that the ISO may differentiate between generator uplift and all other types of uplift, which they aggregate into a single quantity termed the *settlement residual*, and thus a minimum uplift formulation should be modeled as having multiple competing objectives. Our work studies a recommended extension from reference [15] by incorporating *FTR uplift* into a multi-objective minimum uplift formulation that generalizes the standard minimum uplift formulation by introducing a weight constant representing the value of FTR uplift relative to generator uplift. We also refer to FTR uplift as *Potential Congestion Revenue Shortfall* (PCRS) because it represents the worst possible shortfall of congestion revenue in covering FTR payoffs. We will not address other types of uplift in this work, including all reserve related uplift, pointing to this avenue as a sensible extension.

The *Simultaneous Feasibility Condition* (SFC) states that the FTR allocation must represent feasible net power injections in the transmission network. The SFC is typically defined using a linear model of the transmission network and is a sufficient condition for ensuring FTR payoffs are fully covered by congestion revenue [16]. Under certain assumptions, reference [17] extends this traditional congestion revenue adequacy guarantee to the case where the SFC is defined by a general non-linear model of the transmission network. Traditionally, *congestion revenue shortfalls* occur only in the event of transmission line outages, in which case the SFC would not

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accurately represent the transmission network at the time of the market clearing. However, reference [18] illustrates that there may be congestion revenue shortfall without the occurrence of a transmission line outage if the SFC is defined by a non-linear model of the transmission network due to a non-convex feasible set of net power injections. Our work contributes to this literature by relating PCRS to locational prices using a general definition of the SFC and by comparing the standard Locational Marginal Prices (LMPs) to CHPs.

Computing CHPs is generally difficult because the uplift as a function of the locational prices is computationally burdensome to evaluate and is non-smooth. Special-purpose algorithms for computing CHPs have focused on the simple linear transmission constraints that are typically used to formulate the UC problem. For example, references [19–23] either use linear transmission constraints or neglect transmission constraints altogether. Although references [13] and [14] provide analysis of CHPs with general non-linear transmission constraints, they later restrict their scope by linearizing these constraints to develop computational methods. We motivate the inclusion of non-linear constraints into this literature by reiterating that computational research pertaining to the AC OPF problem has the ultimate goal of being implemented into ISO market software [24]. Encouragingly, the electricity market in New Zealand has implemented a simple transmission model that is non-linear in the generation dispatch [25].

Despite utilizing simple transmission models, the aforementioned methods of computing CHPs do not guarantee convergence in polynomial-time. On the other hand, reference [26] frames CHPs as optimal Lagrange multipliers of a polynomially-solvable convex primal counterpart of the minimum uplift problem that we refer to as the *primal CHP problem*. To date this approach of computing CHPs has only considered linear models of the transmission network. This paper extends this work by considering a multi-objective minimum uplift problem along with general non-linear models of the transmission network. Unfortunately, the associated primal CHP problem is expressed in terms of the convex hull of the feasible set of net power injections, which may not be tractable to evaluate for general transmission network models. In this case we suggest using state-of-the-art relaxations initially developed for the AC OPF problem that result in a *relaxed primal CHP problem* that can be approximately solved in polynomial-time. In an initial effort to do so, we study an abstract myopic market that considers only a single time period and defines Financial Transmission Right (FTR) payoffs by locational prices. Specifically, we focus on the non-convexity of interest by analyzing CHPs associated with the AC OPF problem, which does not incorporate integer decision variables representing generator commitment statuses. However, similar relaxation techniques can be used to approximate a solution of a multi-objective minimum uplift problem in a more general setting that includes generator commitment, see remark 4.

In the context of the AC OPF problem, we show that standard LMPs always result in zero generator uplift and non-negative FTR uplift because the generator models are convex and the transmission network model is non-convex. Using LMPs the total FTR uplift reaches over 32% of the total operating cost for an IEEE test case with 162 buses and over 13% of the total operating cost for a NESTA test case with 2224 buses [27]. This implies that the consideration of non-linear transmission constraints may be significant in the context of FTR uplift. Since generator uplift is zero when using LMPs, CHPs will actually increase generator uplift as compared to LMPs, which is starkly different from the typical observations made in the context of the UC problem. We also find that CHPs can be effectively approximated using the Semi-Definite Programming (SDP) relaxation. In fact, if the relative value of FTR uplift to generator uplift is lower than 0.72, then approximate CHPs for the 162 bus test case reduce the total uplift by over 30% of the total operating cost without introducing any generator uplift. This promising result is dampened by noting that the SDP relaxation is unable to be solved efficiently for large test cases with over 1000 buses. We emphasize the difficulty of solving large SDPs and point out that SDPs are only approximately solvable in polynomial-time, with arbitrarily small approximation error [28]. Two other state-of-the-art convex relaxations are also analyzed that are more computationally efficient, but tend to result in higher uplift payments for the test cases analyzed.

The remainder of the paper is organized as follows. Section II explains the proposed market design and formulates the CHP optimization problem as a multi-objective minimum uplift problem. We additionally define a *revenue adequate market equilibrium* and prove that CHPs support a revenue adequate market equilibrium if such prices exist. Section III then derives the primal CHP problem from a general form of the AC OPF problem and CHPs are proven to be the optimal Lagrange multipliers of the primal CHP problem. A general relaxation technique is then proposed to approximate the primal CHP problem. Throughout the main body of the paper, the model of the transmission network is left general. In fact, many of the results may be general enough to apply to other systems that are subject to non-linear transportation constraints, provided the respective relaxations well approximate the feasible regions. Section IV then focuses on the fully detailed AC model of the transmission network, provides examples where a revenue adequate market equilibrium does not exist and illustrates many of the concepts discussed. Section V concludes the paper.

II. Pricing Problem Formulation

This section describes an energy market in which financial settlements are made based on a locational pricing scheme. To neutralize a generator's incentive to deviate from dispatch instructions, side-payments are required to cover any lost opportunity cost of a generator that follows dispatch instructions. Furthermore, we study an abstract myopic market where FTR payoffs are defined by the locational prices. FTRs are assumed to satisfy the Simultaneous Feasibility Condition (SFC) and we assume no line outages occur so that the SFC accurately represents the transmission network at the time of market clearing. A connection is then drawn between FTRs and market equilibria via the notion of congestion revenue adequacy. An optimization problem is formulated to determine prices in a way that minimizes the weighted sum of generator side-payments and PCRS.

We begin by introducing notation. Lower case subscripts are used to index elements of matrices/vectors. For example, M_{ij} denotes the element in the i^{th} row and j^{th} column of matrix M. $\mathbb{R}^n$ denotes the set of n-dimensional real vectors. Furthermore, the transmission network is modeled as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n nodes (buses) and m edges (transmission lines). Associated with each node $i \in \mathcal{V}$ is a uniform price π_i.

The market participants include *generators*, *system demand* and *FTR holders* and they will now be described in detail. Subsequently, a revenue adequate market equilibrium will be defined and the pricing problem will be formulated.

A. Modeling System Demand

The *system demand* is modeled as constant real power extraction D_i at each node $i \in \mathcal{V}$. The demand at node $i \in \mathcal{V}$ is charged for its consumption in the amount $\pi_i D_i$. The demand is inelastic and is thus not modeled as a profit maximizer.

B. Modeling Generators

There is one generator located at each bus in the network indexed by $i \in \mathcal{V}$. The profit of *generator* $i \in \mathcal{V}$ is the difference between its total energy payment and its cost of producing energy. Generator $i \in \mathcal{V}$ generates an amount of real-power denoted G_i and is modeled as having a convex and increasing cost function $C_i(\cdot)$. The constraints of an individual generator i, termed as *private constraints*, are represented by the set $\mathcal{X}^i$ and enforce simple generation limits $\mathcal{X}^i := \{G_i : G_i^{\min} \leq G_i \leq G_i^{\max}\}$. Each generator is modeled as a profit maximizer whose maximum profit is a function of its corresponding locational price and is expressed as follows:

$$\Pi_i(\pi_i) := \max_{G_i \in \mathcal{X}^i} \left(\pi_i G_i - C_i(G_i)\right). \tag{1}$$

1) Generator Side-payments: In the absence of a market equilibrium, some generators may be dispatched at levels that do not maximize their profit. If the dispatched generation for generator $i \in \mathcal{V}$, denoted G_i^d, does not maximize the generator's profit, eg. $\Pi_i(\pi_i) \neq \left(\pi_i G_i^d - C_i(G_i^d)\right)$, then the generator has an incentive to deviate from its dispatched value. This problem can be overcome by introducing side-payments that cover their lost opportunity cost. A side-payment is given to a generator under the condition that they follow the dispatched generation and aims to neutralize their incentive to deviate. The required side-payments are also referred to as *generator uplift* and are expressed as follows:

$$C_i^o(\pi_i, G_i^d) := \Pi_i(\pi_i) - \left(\pi_i G_i^d - C_i(G_i^d)\right). \tag{2}$$

These side-payments are private, out-of-market payments and are undesirable for a number of reasons highlighted by a recent order of the Federal Energy Regulatory Commission [29]. For example, side-payments reduce transparency in the market making it difficult for generator investment decisions to be made. Additionally, to remain revenue neutral, the ISO must distribute the cost of these side-payments among market participants introducing an allocation problem.

C. Modeling Financial Transmission Rights

FTRs are financial contracts entitling FTR holders to a specific revenue stream that results from price differences at buses. FTRs are allocated and sold to FTR holders before energy market prices are cleared. We will use the term *allocation* to refer to both FTRs that are allocated and to those that are sold to FTR holders. There are typically two types of FTRs, Point-to-Point (PTP) obligations and PTP options [17]. For sake of brevity, we will only consider PTP obligations.

Let Γ denote the set of all FTR holders. In its most general form, a PTP obligation for FTR holder $\gamma \in \Gamma$ can be represented by an *FTR allocation vector* $f^{(\gamma)} \in \mathbb{R}^n$ where each element $f_k^{(\gamma)}$ represents a megawatt value injected into the transmission network at node k. Each FTR holder receives an FTR payoff in the amount of $-\pi^T f^{(\gamma)}$. The *total FTR payoff* to all FTR holders is in the amount of $-\pi^T f$, where $f := \sum_{\gamma \in \Gamma} f^{(\gamma)}$ is the sum of all FTR allocation vectors.

Different energy markets may choose the allocation vectors $f^{(\gamma)}$ in different ways. These allocation vectors are determined through an FTR auction process that is beyond the scope of this paper, see reference [17]. A salient feature of the FTR allocations is, however, that FTR allocation vectors are chosen to satisfy the *Simultaneous Feasibility Condition* (SFC) for reasons pertaining to revenue adequacy that will soon become clear. For the remainder of this paper, we assume the FTR allocation satisfies the SFC, which requires the sum of all FTR allocation vectors to represent a feasible net power injection in the transmission network. We additionally assume no line outages occur so the SFC accurately represents the transmission network at the time of market clearing. The set of feasible net power injections, denoted $\mathcal{P} \subset \mathbb{R}^n$, represents constraints on the net power injections imposed by the transmission network that are referred to as *coupling constraints* as they ultimately relate the demand and generation at each node to one another. At this point we leave this set general, encompassing both linear and non-linear models of the transmission network. In section III-A we will also use the set $\mathcal{P}$ to formulate the general AC OPF problem, which is used to set prices.

Definition 1. The FTR allocation satisfies the *simultaneous feasibility condition* (SFC) if the sum of all FTR allocation vectors lies in the feasible set of net power injections, e.g., $f \in \mathcal{P}$.

Remark 1. The feasible set of net power injections $\mathcal{P}$ is left general in the main body of the paper. The definition of this set will include implicit variables such as voltages, reactive power injections at buses and power flows on transmission lines. Physical constraints in the transmission network are enforced, which may include reactive power injection and voltage magnitude limits at each bus $i \in \mathcal{V}$ as well as real, reactive or apparent power flow limits on each transmission line $\ell \in \mathcal{E}$. This set does not limit real power injections at buses, effectively allowing FTRs to be allocated to any bus in the network even if no device is connected to that bus. □

1) Potential Congestion Revenue Shortfall (PCRS): Once the generators (demand) are paid (charged) for producing (consuming) energy, the ISO is left with additional revenue, called *congestion revenue*. Let's denote the dispatched net power injections as $P^d := G^d - D \in \mathcal{P}$. The congestion revenue is in the amount $\pi^T(D - G^d)$ and is used to fund the FTR payoffs. The congestion revenue is said to be *adequate* if it is larger than the total FTR payoffs. It is important to design a market that encourages congestion revenue adequacy. For this reason the prices should be chosen such that the congestion revenue covers the worst case FTR allocation. The maximum FTR payoff $\Psi(\pi)$ for given prices π can be written as follows:

$$\Psi(\pi) := \max_{f \in \mathcal{P}} \quad -\pi^T f. \tag{3}$$

The Potential Congestion Revenue Shortfall (PCRS), also referred to as *FTR uplift*, represents the maximum possible shortfall of congestion revenue and is written as follows:

$$C^s(\pi, G^d) := \Psi(\pi) - \pi^T \left(D - G^d\right). \tag{4}$$

This value can be interpreted as the maximum FTR payoff $\Psi(\pi)$ less the congestion revenue and is non-negative. It is important to recognize that the PCRS $C^s(\pi, G^d)$ represents a worst case shortfall over all possible FTR allocations. In fact the realized FTR allocation may allow congestion revenue to cover FTR payoffs even if the PCRS is positive. In the event that congestion revenue is unable to cover FTR payoffs the ISO must distribute the shortfall among market participants introducing an allocation problem. Proper allocation of congestion revenue shortfall has been a point of controversy [30].

D. A Revenue Adequate Market Equilibrium

In accordance with reference [13], a *decentralized market equilibrium* refers to a set of prices, side-payments and dispatch values for which generators have no incentive to deviate from their dispatch values. As explained in section II-B1, our proposed side-payments always ensure a decentralized market equilibrium is realized, whereas payments on the basis of energy prices alone do not have this property. This paper will expand upon this notion of a decentralized market equilibrium by additionally requiring revenue adequacy to be guaranteed. We will refer to such a decentralized market equilibrium as a *revenue adequate market equilibrium*. Note that our definition of a revenue adequate market equilibrium is consistent with the *Competitive Equilibrium Model (CEM) 1* from reference [31]. Specifically, we will define a revenue adequate market equilibrium to be a price dispatch pair (π, G) that results in zero side-payments and zero PCRS. Intuitively, a revenue adequate market equilibrium guarantees revenue adequacy in the form of side-payments and FTR payoffs.

Definition 2. A *revenue adequate market equilibrium* is defined to be a price/dispatch pair (π, G) that result in zero side-payments and zero PCRS. e.g. $C^s(\pi, G) = 0$ and $C_i^o(\pi_i, G_i) = 0 \ \forall i \in \mathcal{V}$

Reference [31] shows that a revenue adequate market equilibrium may not always exist when defining the feasible set of net power injections using the fully detailed AC model of the transmission network. Recognizing that the side-payments and PCRS are always non-negative, the lack of a revenue adequate market equilibrium implies that at least one of these values are positive, and thus additional costs accrue in the form of generator opportunity costs and/or FTR underfunding that may be difficult to evaluate or bound when designing the market.

E. Pricing Problem Formulation

To reduce deficit it is in the interest of the ISO to have low side-payments and PCRS. CHPs are defined as an optimal solution to a pricing problem that minimizes the weighted sum of these values and is referred to as the *CHP problem* or the *multi-objective minimum uplift problem.*

Definition 3. The *Convex Hull Prices* (CHPs) minimize the weighted sum of potential congestion revenue shortfall and total side-payments, are denoted $\pi^\star$, and are defined using a positive weight parameter $\alpha > 0$ that represents the value of PCRS relative to generator side-payments.

$$\pi^\star \in \operatorname*{argmin}_{\pi \in \mathbb{R}^n} \left(\alpha C^s(\pi, G^d) + \sum_{i \in \mathcal{V}} C_i^o(\pi_i, G_i^d) \right) \tag{5}$$

Recalling that side-payments and PCRS are always non-negative and recognizing that the weight parameter α is defined to be positive, it is apparent that CHPs must form a revenue adequate market equilibrium, as in definition 2, with any given dispatch values G^d if such prices exist. Furthermore, the ISO may prefer PCRS over side-payments because the PCRS only represents a potential shortfall whereas the side-payments represent an actual shortfall. For this reason, the weight parameter will likely be chosen to be less than one.

The CHP problem (5) is similar to the multi-objective minimum uplift problem from reference [15], but explicitly incorporates FTR uplift into the formulation as suggested in their future work section. Our formulation also includes non-linear transmission constraints, which has not been investigated in a multi-objective setting. When the weight parameter is set to $\alpha = 1$ the CHP problem (5) is consistent with the formulation in references [13] and [14], which provide a high-level analysis that is capable of accommodating non-linear transmission models. However, the special-purpose algorithm designed to solve the minimum uplift problem in references [13] and [14] is restricted to linear transmission models.

The objective function of the CHP problem (5) is intuitively convex in the price variables π because it is the sum of individual functions that represent the maximum of affine functions in π. However, despite convexity, this problem is still difficult to solve in general because its objective function is difficult to evaluate and is non-smooth. In fact, references [3] and [4] show that it is generally NP-hard to identify a feasible point of the maximum FTR payoff problem from equation (3), which must be solved to evaluate the PCRS function from equation (4) denoted $C^s(\pi, G^d)$.

III. Main Results

The previous section defined CHPs as an optimal solution to the CHP problem, which is generally difficult to solve. This section provides a method of approximating CHPs in polynomial-time. We begin by formulating a general AC OPF problem and explaining its relation to LMPs. We then formulate a convex primal version of the CHP problem termed the *primal CHP problem*. CHPs are proven equivalent to optimal Lagrange multipliers of the convex primal CHP problem. We then explain how CHPs can be approximated using a general relaxation of the primal CHP problem. This relaxed primal CHP problem can be approximately solved using polynomial-time algorithms initially developed to approximately solve relaxations of the AC OPF problem.

A. *General AC Optimal Power Flow Problem and LMPs*

The *general AC OPF problem* incorporates the dispatched net power injections $P^d := G^d - D$ and the weight parameter α into the power balance constraints. The *general AC OPF problem* is written as follows where all generator limits are expressed by the convex set $\mathcal{X} = \{G \in \mathbb{R}^n : G_i \in \mathcal{X}^i\}$ and the feasible set of net power injections $\mathcal{P}$ matches that used in the SFC from section II-C. We will use the term *AC OPF problem* to refer to the *general AC OPF problem* in the special case where $\alpha = 1$, in which case the dispatch values P^d drop out of the power balance constraints.

$$\min_{\substack{P\in\mathcal{P}\\ G\in\mathcal{X}}} \quad \sum_{i\in\mathcal{V}} C_i(G_i) \tag{6}$$

$$st: D_i - G_i + \alpha P_i + (1-\alpha)P_i^d = 0 \quad \forall i \in \mathcal{V} \tag{6a}$$

An optimal solution to the AC OPF problem with $\alpha = 1$, consists of a *social welfare maximizing dispatch*. By theorem 1 of reference [31] a revenue adequate market equilibrium, as in definition 2, can only occur if the dispatched generation is a social welfare maximizing dispatch. Unfortunately, this solution may be difficult to compute due to a non-convex feasible set. As a result, we will not assume that dispatch values, denoted G^d, represent a social welfare maximizing dispatch and thus the market may not be operating in a revenue adequate equilibrium. In this context additional costs may accrue in the form of PCRS and generator side-payments that are not considered in the objective of the AC OPF problem.

Many concepts associated with price setting are closely related to the Lagrange multipliers of the power balance constraints (6a). Reference [32] uses uniform prices that we term *KKT prices* and denote by $\hat{\pi} \in \mathbb{R}^n$. These prices are set by first identifying a solution of the AC OPF problem (6) with $\alpha = 1$ that satisfies the Karush-Kuhn-Tucker (KKT) conditions and then setting the prices to be the Lagrange multipliers of the power balance constraints (6a). For the remainder of this paper we will assume that the dispatched generation, denoted G^d, solves the KKT conditions of the AC OPF problem with $\alpha = 1$, see remarks 2 and 3. If a revenue adequate market equilibrium exists, the dispatched generation represents a global minimizer of the AC OPF problem, the global minimizer of the AC OPF problem solves the KKT conditions with unique Lagrange multipliers, and the AC OPF problem satisfies certain constraint qualifications [33], then KKT prices exist and represent the marginal cost of serving load at each location in the network. For this reason, we will refer to KKT prices as Locational Marginal Prices (LMPs), as is standard practice. However, it is important to note that this is a slight abuse of terminology, as LMPs do not represent the marginal cost of serving load when PCRS and/or generator side-payments are positive because the objective function of problem (6) does not consider these costs. Furthermore, LMPs are specific to the identified solution of the AC OPF problem and as a result different algorithms may produce different LMPs. Below is a general definition of a KKT price/dispatch pair that only introduces Lagrange multipliers for the power balance constraints and leaves the sets $\mathcal{P}$ and $\mathcal{X}$ general.

Definition 4. A *KKT price/dispatch pair* $(\hat{\pi}, \hat{G}) \in \mathbb{R}^n \times \mathcal{X}$ are such that constraint (6a) holds with $\alpha = 1$ for some $\hat{P} \in \mathcal{P}$ along with the following generalized stationarity conditions.

$$\begin{aligned} &-\hat{\pi} \in N_{\mathcal{P}}(\hat{P}) \quad \text{and} \\ &0 \in \partial\left(C_i(G_i) - \hat{\pi}_i G_i\right)|_{\hat{G}_i} + N_{\mathcal{X}^i}(\hat{G}_i) \quad \forall i \in \mathcal{V} \end{aligned} \tag{7}$$

where $N_{\mathcal{P}}(\hat{P})$ is the normal cone of the set $\mathcal{P}$ at the point $\hat{P}$ and $N_{\mathcal{X}^i}(\hat{G}_i)$ is the normal cone of the set $\mathcal{X}^i$ at the point $\hat{G}_i$. The subdifferential of a general function $g(x)$ evaluated at a point $\hat{x}$ is denoted $\partial(g(x))|_{\hat{x}}$.

Remark 2. We emphasize that a KKT price/dispatch pair may not exist in our general framework because constraint qualifications may not be satisfied within the set $\mathcal{P}$. It is also possible that an identified solution satisfying the KKT conditions could represent a saddle point, local maximum, or local minimum. However, in practice a local minimum to the AC OPF problem satisfying the KKT conditions is almost always attainable using standard off-the-shelf software, as is the case for each test case in this paper. Furthermore, our results regarding KKT prices hold if the solution represents a saddle point, local maximum, or local minimum.

Remark 3. Although we do not prove this directly, any dispatch $\hat{G}$ that satisfies the traditional KKT conditions for the AC OPF problem with $\alpha = 1$ will also satisfy the conditions from definition 4 for some Lagrange multipliers $\hat{\pi}$. In fact, there may be multiple such Lagrange multipliers $\hat{\pi}$ that satisfy the KKT conditions for generation dispatch $\hat{G}$. Our results do not require such Lagrange multipliers to be unique.

The following theorem states that a KKT price/dispatch pair always results in zero side-payments to generators, which is a byproduct of the private constraint sets $\mathcal{X}^i$ being convex. This highlights a fundamental difference between LMPs and CHPs in the context of the AC OPF problem. While CHPs minimize the sum of side-payments and PCRS, LMPs result in zero side-payments. As a result CHPs tend to increase side-payments and decrease PCRS as compared to LMPs. This tendency of CHPs in the context of the AC OPF problem is starkly different from from the tendency that has been observed in previous work. Specifically, CHPs tend to lower side-payments to generators in the context of the Unit Commitment (UC) problem with a linear model of the transmission network, as in references [23] and [26]. The difference is due to the fact that the private constraints $\mathcal{X}^i$ are the source of non-convexity in the UC problem, whereas the network constraints are the source of the non-convexity in the AC OPF problem.

Theorem 1. *A KKT price/dispatch pair results in zero side-payments to generators, eg.* $C_i^o(\hat{\pi}_i, \hat{G}_i) = 0 \; \forall i \in \mathcal{V}$.

Proof: The generalized stationarity conditions (7) include necessary conditions for optimality of each profit-maximizing generation problem (1). Since each problem (1) is convex, these conditions are also sufficient for global optimality. □

B. Primal Formulation of the CHP Problem

As suggested, CHPs are closely related to the Lagrange multipliers of the power balance constraints (6a). This can be seen by dualizing the power balance constraints. This yields the partial Lagrangian dual function with implicit constraints:

$$\mathcal{D}(\pi) := \sum_{i\in\mathcal{V}} \min_{G_i\in\mathcal{X}^i} (C_i(G_i) - \pi_i G_i) + \alpha \min_{P\in\mathcal{P}} \pi^T P + \pi^T\Big((1-\alpha)P^d + D\Big). \quad (8)$$

The following theorem 2 establishes CHPs as a maximizer of this partial Lagrangian dual function. Though similar to results in other papers regarding CHPs, e.g. [23], this result accounts for weight constant α and general non-linear transmission models assumed by the set $\mathcal{P}$. If the weight parameter is set to $\alpha=1$ then the partial Lagrangian dual function is independent of the dispatched generation G^d, and thus CHPs do not depend on the dispatched generation. As opposed to LMPs, this is a potential benefit using CHPs with $\alpha = 1$. It should be noted that there are analogous observations in the context of the UC problem, where CHPs are also independent of, for example, sub-optimality of the UC solution.

Theorem 2. *A maximizer of the partial Lagrangian dual function $\mathcal{D}(\pi)$ from equation (8) is also a minimizer of the CHP problem (5) and thus represents CHPs.*

Proof: A maximizer of the partial Lagrangian dual function (8) is unchanged by adding and subtracting the term $\sum_{i\in\mathcal{V}} \pi_i P_i^d$ and subtracting the constant $\sum_{i\in\mathcal{V}} C_i(G_i^d)$. Noting that $P_i^d = G_i^d - D_i$, we obtain the following expression:

$$\mathcal{D}(\pi) - \sum_{i\in\mathcal{V}} C_i(G_i^d) = \sum_{i\in\mathcal{V}}\Big(\pi_i G_i^d - C_i(G_i^d) - \Pi_i(\pi_i)\Big) - \alpha(\pi^T P^d + \Psi(\pi)).$$

By equations (4) and (2), this expression is equivalent to the negative of the objective function of the CHP problem (5). The result directly follows. □

A key insight that helps in computing CHPs is that any optimization problem with a linear objective function and a compact feasible set has the same optimal value after relaxing the feasible set to its convex hull [34]. Under the assumption that $\mathcal{P}$ is compact we have the following equivalence:

$$\min_{P\in\mathcal{P}} \pi^T P = \min_{P\in\text{conv}(\mathcal{P})} \pi^T P \quad (9)$$

where conv($\mathcal{P}$) is the convex hull of $\mathcal{P}$. This fact allows for the formulation of a convex problem with a partial Lagrangian dual function that matches $\mathcal{D}(\pi)$ from equation (8). The *primal CHP problem* is written as follows:

$$\min_{\substack{P\in\text{conv}(\mathcal{P})\\ G\in\mathcal{X}}} \sum_{i\in\mathcal{V}} C_i(G_i) \quad (10)$$

$$st: D_i - G_i + \alpha P_i + (1-\alpha)P_i^d = 0 \quad \forall i \in \mathcal{V} \quad (10a)$$

As summarized in the following theorem, a maximizer of the partial Lagrangian dual function $\mathcal{D}(\pi)$ from equation (8) can be recovered as the optimal Lagrange multipliers of constraints (10a). Indeed the term *Convex Hull Price* is derived from this key result, which is similar to results in other work, e.g. [26].

Theorem 3. *Optimal Lagrange multipliers of constraints (10a) minimize the CHP problem (5) and thus represent CHPs.*

Proof: Equation (9) implies that the function $\mathcal{D}(\pi)$ from equation (8) represents the partial Lagrangian dual function for problem (10). Furthermore, strong duality holds for problem (10) when dualizing only linear constraints because it is a convex problem with a non-empty feasible set. The result follows from theorem 2. □

C. Approximating CHPs

Unfortunately, the convex hull of the set of feasible power injections conv($\mathcal{P}$) may be intractable to evaluate, as is the case when using the fully detailed AC model of the transmission network [35]. In this case the primal CHP problem must be relaxed using a conservative convex set relax($\mathcal{P}$) $\supseteq$ conv($\mathcal{P}$) that can be expressed in closed form. The *relaxed primal CHP problem* is written as follows and the *approximate CHPs*, denoted $\bar{\pi}$, are found as the optimal Lagrange multipliers of the nodal power balance constraint (11a).

$$\min_{\substack{P\in\text{relax}(\mathcal{P})\\ G\in\mathcal{X}}} \sum_{i\in\mathcal{V}} C_i(G_i) \quad (11)$$

$$st: D_i - G_i + \alpha P_i + (1-\alpha)P_i^d = 0 \quad \forall i \in \mathcal{V} \quad (11a)$$

The Lagrange multipliers of constraint (11a) can be recovered from a slightly reformulated problem that represents a general convex relaxation of the AC OPF problem. To see this, reformulate the relaxed primal CHP problem by dividing the power balance constraint by α to yield an equivalent problem that can be interpreted as a general convex relaxation of the AC OPF problem where the demand vector is represented as $\tilde{D}_i = \frac{1}{\alpha}D_i + \frac{1-\alpha}{\alpha}P_i^d$. In this context the generators must be redefined to have a generation amount of $\tilde{G}_i = \frac{1}{\alpha}G_i$, a cost function of $\tilde{C}_i(\tilde{G}_i) = C_i(\alpha\tilde{G}_i)$, and private constraints of $\tilde{\mathcal{X}}^i := \{\tilde{G}_i : \frac{1}{\alpha}G_i^{\min} \leq \tilde{G}_i \leq \frac{1}{\alpha}G_i^{\max}\}$. The resulting reformulation will produce Lagrange multipliers that differ from those of constraint (11a) by a factor of α. This allows CHPs to be approximated using algorithms initially developed to approximately solve convex relaxations of the AC OPF problem that are proven to converge in polynomial-time.

Intuitively, approximate CHPs minimize a function that represents an upper bound on the weighted sum of side-payments and PCRS. The conservative nature of this upper bound is dependent on the how well the convex relaxation represents the convex hull and on the magnitude of the weight parameter α. The following theorem makes this result explicit.

Theorem 4. *Approximate CHPs, denoted $\bar{\pi}$ and defined as the optimal Lagrange multipliers of constraint (11a), satisfy:*

$$\bar{\pi} \in \underset{\pi\in\mathbb{R}^n}{argmin}\Big(\alpha C^s(\pi, G^d) + \sum_{i\in\mathcal{V}} C_i^o(\pi_i, G_i^d) + \alpha\left(\Psi^r(\pi) - \Psi^c(\pi)\right)\Big) \quad (12)$$

$$where \quad \Psi^r(\pi) := \max_{f\in relax(\mathcal{P})} -\pi^T f \quad (13)$$

$$and \quad \Psi^c(\pi) := \max_{f\in conv(\mathcal{P})} -\pi^T f \quad (14)$$

Proof: From the same process as the proof in theorem 2, the prices $\bar{\pi}$ maximize the following equation.

$$\sum_{i\in\mathcal{V}} (\pi_i G_i^d - C_i(G_i^d) - \Pi_i(\pi_i)) - \alpha(\pi^T P^d + \Psi^r(\pi)) \quad (15)$$

By substituting the expression for generator uplift $C_i^o(\pi_i, G_i^d)$, adding and subtracting the term $\alpha\Psi(\pi)$, and noting that equation (9) implies $\Psi(\pi) = \Psi^c(\pi)$ we attain the following:

$$\sum_{i\in\mathcal{V}} C_i^o(\pi_i, G_i^d) - \alpha(\pi^T P^d + \Psi(\pi)) + \alpha(\Psi^c(\pi) - \Psi^r(\pi))$$

This is the negative of the objective function of (12). □

From theorem 4, the objective function minimized by approximate CHPs is similar to that of CHPs from definition 3. When the relaxed set relax($\mathcal{P}$) accurately represents the convex hull conv($\mathcal{P}$), CHPs match approximate CHPs because $\Psi^r(\pi) = \Psi^c(\pi)$. Also notice that the upper bound being minimized will be tighter if the weight parameter $\alpha > 0$ is smaller. Fortunately, as we discussed in section II-E, this weight parameter should be less than one. Of course approximate CHPs may match CHPs even if the relaxed set of feasible injections does not match the convex hull. Future work will identify sufficient conditions for exactness.

Remark 4. Further studies should generalize theorems 2, 3, and 4 to accommodate the UC problem over a 24 hour period. Such a generalization requires problems (10) and (11) to be restated such that the private constraint set of each generator $\mathcal{X}_i$ is replaced by its convex hull conv($\mathcal{X}_i$) and the cost function of each generator is replaced by its convex envelope, both of which are explicitly characterized in reference [26].

IV. Examples

This section uses a feasible set of net power injections $\mathcal{P}$ defined by the AC transmission network from reference [36], which models real and reactive power as well as voltage magnitude using quadratic constraints. We will begin with a simple 3-bus example that computes approximate CHPs by applying the standard Semi-Definite Programming (SDP) relaxation to obtain the convex relaxed set relax($\mathcal{P}$). We then move onto larger, more realistic networks, to which we apply three different relaxations including the SDP relaxation, the Quadratic Convex (QC) relaxation, and the Second-Order Cone (SOC) relaxation. Standard LMPs are compared to approximate CHPs for each relaxation. The weight parameter α from definition 3 is set to one for all examples except the final example that illustrates the effect of varying α for a test case with 162 buses.

Electricity markets often use an OPF problem with a loss-less DC approximation of the feasible set of net power injections denoted $\mathcal{P}_{\text{DC}}$. This section will additionally analyze prices, termed DCLMPs, that are found as the Lagrange multipliers of the power balance constraint of the DC OPF problem, which is equivalent to problem (6) with $\mathcal{P}_{\text{DC}}$ defined by the loss-less DC approximation in reference [37].

The optimal dispatch as determined by the DC OPF problem should be dispatched along with DCLMPs; however, this dispatch may not be feasible for the true transmission network, whose feasible set of net power injections is represented by the set $\mathcal{P}$. For this reason control action must be taken on fast time scales to attain a feasible dispatch. This section will assume that fast control action adjusts generator outputs to attain the optimal dispatch from the AC OPF problem $P^\star$. This assumption idealizes fast time scale control, which typically does not minimize cost.

In this section the SFC is defined using the true set of feasible net power injections $\mathcal{P}$; however, when using the DC OPF problem the SFC is typically defined using the DC approximation $\mathcal{P}_{\text{DC}}$. This alternative definition of the SFC would not allow all financial transactions to achieve a full hedge because it does not consider losses. This is out of the scope of this paper, see [30]. To avoid confusion the remainder of this section will not specify PCRS when using DCLMPs.

A. Simple 3-bus Transmission Network

This section studies a simple 3-bus transmission network similar to that in reference [38] that can be visualized by the one-line diagram shown in figure 1. The power injection at buses 1, 2, and 3 are denoted P_1, P_2, and P_3. The reactive power injections at buses 1 and 2 are unbounded. The voltage magnitudes at buses 1 and 2 are fixed to 1 p.u. and 1.21 p.u. respectively. Bus 3 is a zero injection bus whose real and reactive power injection are fixed to zero and has no voltage magnitude constraint. A real power flow limit of 3.5 p.u. is placed on the line connecting bus 1 and bus 2. The p.u. impedance of a line connecting bus i to bus j is denoted z_{ij}.

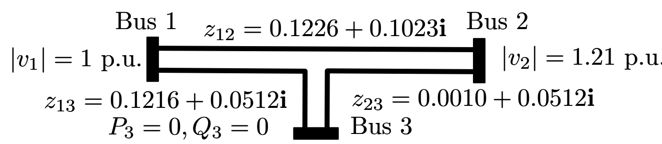


Fig. 1: One-line diagram of the 3-bus test case.

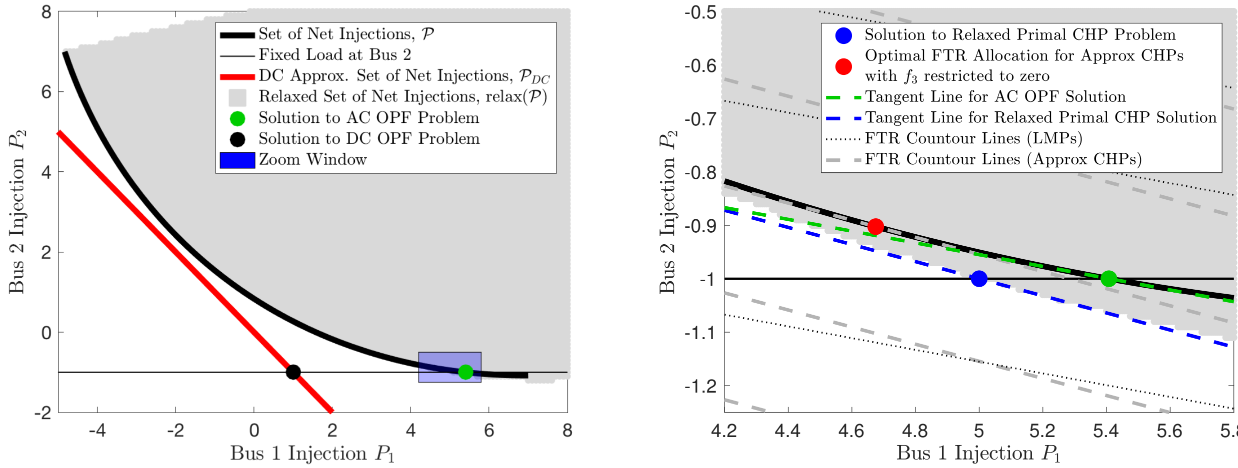
The feasible set of net power injections at buses 1 and 2 is non-convex and forms the curved black line illustrated in figure 2. Notice that the feasible set of net power injections $\mathcal{P}$ is now three dimensional; however, the curved black line in figure 2 represents a slice of this set at the plane where $P_3 = 0$. As explained in remark 1, the feasible set of net power injections does not enforce constraints on the real-power injections at buses. (Note: these constraints are accommodated by the feasible set of each generator $\mathcal{X}^i$ in the AC OPF problem (6)). This means that FTRs can be allocated to bus 3 even though its net injection is physically restricted to zero.

The demand at bus 2 is fixed to 1p.u., so the net injection at bus two is $P_2{=}{-}1$p.u. In this case the only feasible point is the green dot in figure 2. This green dot will represent the solution to the AC OPF problem $(P_1^\star, P_2^\star, P_3^\star) \approx (5.40, -1, 0)$. Notice this operating point accrues large line losses of approximately 4.5p.u. Bus 1 consists of one generator whose cost in dollars is represented by the piece-wise linear function:

$$C_1(P_1) = \begin{cases} 0.5P_1 & \text{if } P_1 \le 2 \\ P_1 & \text{if } P_1 > 2 \end{cases}$$

1) DCLMPs: The loss-less DC approximation of the feasible set of net power injections $\mathcal{P}_{\text{DC}}$ is represented by the red line in figure 2a. The black dot represents the optimal dispatch produced by the DC OPF problem; however, this point is not feasible for the true transmission network. We assume that control on a fast time scale is able to adjust the generator dispatch to attain the solution to the AC OPF problem, which is represented by the green dot in figure 2.

The DCLMPs can be recovered as the Lagrange multipliers of the power balance constraints of the DC OPF problem. Since



(a) Feasible Set of Net Power Injections

(b) Zoomed version of Fig. 2a

Fig. 2: This figure illustrates the feasible set of net power injections along with its relaxed counterpart and its approximate DC counterpart. (2a) illustrates the feasible set of net power injections in black along with its SDP relaxed version in grey and its DC approximated version in red. The feasible set is shown in two dimensions because bus three is a zero injection bus. (2b) is a zoomed in version of (2a). Contours of the objective function in problem (3) are shown for approximate CHPs and LMPs.

no line limits are binding these prices are the same at each bus $\pi_1^{\mathrm{DC}} = \pi_2^{\mathrm{DC}} = \pi_3^{\mathrm{DC}} = 0.5\$/\text{p.u.}$. These prices along with the optimal dispatch of the AC OPF problem $P^\star$ result in a generator side-payment of $C_1^o(\pi_1^{\mathrm{DC}}, P_1^\star) = (1-\pi_1^{\mathrm{DC}})P_1^\star = 2.7\$$.

2) LMPs: We now identify LMPs as described in section III-A. These prices were found by solving the AC OPF problem in MATLAB using an interior point method available in the function ‘fmincon.m.’ The feasible voltages at buses 1, 2, and 3 are $v_1 = 1$, $v_2 = 1.002 - 0.677\mathbf{i}$, and $v_3 = 0.838 - 0.535\mathbf{i}$ respectively in units of p.u. The solver provided the Lagrange multipliers for the power balance constraint (11a) that solve the KKT conditions for this point. The LMPs for bus 1, 2 and 3 are $\hat{\pi}_1 = 1$, $\hat{\pi}_2 = 9.46$, and $\hat{\pi}_3 = 9.46\$/\text{p.u.}$ respectively. The congestion revenue can be easily computed as $4.05\$$. The side-payments to generators are zero as proven by theorem 1.

The contour lines represented by the objective function of the FTR payoff maximization problem (3) are shown in figure 2. The contour lines associated with the LMPs are represented by black dotted lines and are parallel to the tangent line at the green dot. The optimal FTR allocation occurs at the point where the contour lines are tangent to the feasible set of net power injections $\mathcal{P}$. This point coincides with the dispatched set point represented by the green dot. Furthermore, at this point the congestion revenue shortfall is zero. This means that there does not exist an FTR allocation in the plane plotted in figure 2 that results in congestion revenue shortfall. However, this plane restricts the FTR allocation for bus 3, f_3, to be zero. Since the set of power injections $\mathcal{P}$ is three dimensional, we need to expand our analysis to three dimensional FTR allocation vectors. With this in mind we found the PCRS to be $0.26\$$ with an optimal FTR allocation $f = [5.5929, 6.3464, -7.3892]$ in units of p.u.

3) Approximate CHPs: We now identify approximate CHPs using the SDP relaxation from reference [38]. These prices were found by solving the relaxed primal CHP problem (11) with $\alpha = 1$ using the CVX package in MATLAB [39]. The solver provides the optimal Lagrange multipliers for the power balance constraint (11a), which are equivalent to approximate CHPs. Approximate CHPs for bus 1, 2 and 3 are $\bar{\pi}_1 = 1$, $\bar{\pi}_2 = 5.89$, and $\bar{\pi}_3 = 5.92\$/\text{p.u.}$ The congestion revenue is calculated as $0.49\$$. Since $\bar{\pi}_1 = \hat{\pi}_1$ the generator side-payment remains zero.

The contour lines associated the FTR payoff maximization problem (3) when using approximate CHPs are represented by gray dashed lines in figure 2 and are parallel to the tangent line at the blue dot. The optimal FTR allocation when using approximate CHPs, represented by the red dot, occurs at the point where the contour lines are tangent to the feasible set of net power injections $\mathcal{P}$. Since the red dot does not coincide with the dispatched set point represented by the green dot, the FTR payoff at this point will be larger than the congestion revenue. We can conclude that approximate CHPs introduce the possibility of congestion revenue shortfall when restricted to the plane of FTR allocation vectors where $f_3 = 0$. As mentioned previously, this analysis should be extended to the three dimensional SFC, in which case CHPs lower PCRS by $0.05\$$ as compared to LMPs. Specifically, approximate CHPs result in a PCRS in the amount of $0.21\$$ and an optimal FTR allocation of $f = [4.7284, -9.2816, 8.3189]$ in p.u. units.

B. Examples on Standard Test Cases

We now consider much larger test cases available by NESTA [27]. We investigate three different relaxations of the feasible set of net power injections when formulating the relaxed primal CHP problem (11). The SDP relaxation is implemented using the MATPOWER toolbox in MATLAB [36]. Some of the NESTA test cases can be solved exactly using the SDP relaxation. Such test cases yield zero side-payments and zero PCRS when using LMPs. Instead, we focus on test cases that cannot be solved exactly using the SDP relaxation and yield positive PCRS when using LMPs. We also consider the QC relaxation described in reference [9] and the SOC relaxation described in reference [40]. Both the QC and SOC relaxations are implemented using the PowerModels package in Julia [41].

TABLE I: Results for NESTA test cases. All amounts are in dollars per hour. For large test cases, a penalty is added to the objective function in the SDP-relaxed problem, associated values are denoted with an asterisk. The side-payments when using DCLMPs are ideal, assuming that fast control action optimizes cost, associated values are denoted with a triangle. PCRS values are omitted for DCLMPs to avoid confusion regarding the definition of the SFC. The weight parameter is $\alpha = 1$.

Test Case	LMPs (for AC OPF)		DCLMPs (for DC OPF)		Approximate CHPs with SDP Relaxation		Approximate CHPs with QC Relaxation		Approximate CHPs with SOC Relaxation		Total Operating Cost
	Side Payments	PCRS	Side Payments	PCRS	Side Payments	PCRS	Side Payments	PCRS	Side Payments	PCRS	
162_ieee_dtc	~ 0	1,352.92	6.66$^{\triangle}$	-	0.11	42.55	127.48	26.87	127.33	28.94	4,230.23
189_edin	~ 0	1.22	4.66$^{\triangle}$	-	0.05	0.74	0.25	0.77	0.25	0.77	849.29
300_ieee	~ 0	36.87	257.40$^{\triangle}$	-	0.03	14.77	3.12	128.18	3.15	130.88	16,891.27
2224_edin	~ 0	520.76	75.92$^{\triangle}$	-	79.26*	343.47*	1,392.20	738.48	1,838.29	430.59	38,127.69
2383wp_mp	~ 0	13,681.00	289.12$^{\triangle}$	-	1,572.55*	4,023.87*	5,552.60	7,145.92	5,601.56	7,513.18	1,868,511.77
3012wp_mp	~ 0	1,815.44	6,303.82$^{\triangle}$	-	2,411.54*	1,348.35*	12,907.15	6,179.91	12,882.12	6,924.57	2,600,842.72

Table I provides a comparison of side-payments and PCRS when using LMPs, approximate CHPs, and DCLMPs for six transmission networks with 162 buses, 189 buses, 300 buses, 2224 buses, 2383 buses, and 3012 buses. When computing approximate CHPs we set the weight parameter to α=1. By varying the weight parameter $0 < \alpha < 1$ we would expect to achieve PCRS and side-payments that fall between the two extremes produced by approximate CHPs and LMPs. Computing the PCRS for a given set of prices requires solving the non-convex max FTR payoff problem (3). The provided PCRS values are computed using an interior point solver in Julia that identifies a local maximum of problem (3).

As expected LMPs result in zero side-payments to generators and positive PCRS. The PCRS may be very large with respect to the total operating cost as in test case 162_ieee_dtc (approximately 32%) or very small as in test case 3012wp_mp (approximately 0.07%). Furthermore, DCLMPs introduce a small amount of side-payments to generators, although these values will likely be much larger in practice where control on fast time scales is imperfect.

Approximate CHPs from the SDP relaxation perform well for the three smallest test cases. That is, as compared to LMPs, they tend to increase side-payments by a small amount and decrease PCRS significantly. Furthermore, approximate CHPs result in much lower side-payments as compared to DCLMPs. The MATPOWER algorithm used to solve the SDPs was unable to converge for the three largest test cases. For these cases we use a software package that employs an additional approximation by placing a penalty in the objective of the SDP-relaxed problem to encourage convergence [42] (associated quantities are denoted with an asterisk). The penalty parameters were adjusted for each individual test case. Despite using an additional approximation, the resulting approximate CHPs are still able to reduce PCRS as compared to LMPs.

Approximate CHPs from the QC and SOC relaxations result in similar PCRS and generator side-payments. These approximate CHPs result in higher generator side-payments and PCRS as compared to approximate CHPs from the SDP relaxation. In fact, these approximate CHPs increase the PCRS as compared to LMPs for test cases 300_ieee and 3012wp_mp.

To provide insight into the trade-off between PCRS and side-payments, let's analyze approximate CHPs for the test case 162_ieee_dtc using the SDP relaxation. Figure 3 shows the generator side-payments and PCRS as the weight parameter α varies from 0 to 1.5. As expected, side payments increase in α, PCRS decreases in α, and the trajectories pass through their associated values in table I when $\alpha = 1$. Interestingly, as the parameter α decreases, the side-payments reach zero at the point $\alpha = 0.72$ and cannot decrease further. At this point the PCRS remains constant at 42.67\$. In fact, choosing parameter α below 0.72 results in approximate CHPs that achieve savings in PCRS of over 30% of the total operating cost as compared to LMPs while still maintaining zero side-payments.

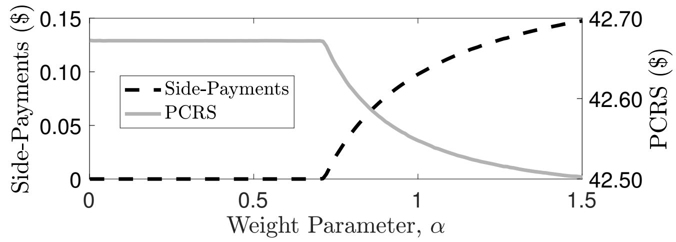


Fig. 3: Competing objectives for test case 162_ieee_dtc

V. Conclusions

This paper incorporates non-linear models of the transmission system into the existing Convex Hull Pricing framework. In an initial effort to do so, we study an abstract myopic market where FTR payoffs are defined by locational prices. In this context we theoretically prove and empirically observe the tendency of CHPs to increase generator side-payments as compared to LMPs, which is significantly different from the behavior of CHPs in the context of the UC problem with linear network models. We define CHPs as the optimal solution to a novel multi-objective minimum uplift problem that captures the trade-off between generator side-payments and PCRS. For the first time, we present a method of approximating CHPs using non-linear transmission constraints that are general enough to accommodate the AC OPF problem. Specifically, CHPs are approximated by the optimal Lagrange multipliers of the relaxed primal CHP problem, which can be approximately solved using well-known polynomial-time algorithms that approximately solve relaxed versions of the AC OPF problem. We provide a theoretical result illustrating that approximation accuracy can be improved by tightening the relaxation used or by placing more value in generator side-payments as compared to PCRS using the parameter α.

Examples show that FTR uplift may be large when using LMPs, motivating the inclusion of non-linear models of the transmission network into the CHP framework. We then approximate CHPs using SDP, QC, and SOC relaxations of the AC transmission network. The SDP relaxation is shown to effectively approximate CHPs; however, existing algorithms used to solve this relaxed SDP problem do not scale well with the size of the transmission network. In fact, using the SDP relaxation we are able to identify CHPs that significantly reduce PCRS while hardly effecting generator side-payments.

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