QUANTUM ESSAY ASSIGNMENT

Ziyao Xiong

CID:02225789

Essay 1 word count: 1487

Essay 2 word count: 1490

Research & Writing Methods:

In essay 1, my primary focus was on ensuring clarity and comprehension for the intended audience. I selected a fundamental topic that aligns with our Year 1 curriculum, aiming to build a strong foundation for further learning.

In Essay 2, I explored a particular phenomenon that has always piqued my interest and delved into its practical applications. To ensure accuracy and depth in my writing, I consulted both library resources, including a textbook on nuclear physics, and conducted online research to gather relevant literature.

I wrote and edited both essays in Word, I wrote the equations in Latex, and I drew the pictures in Goodnotes.

Quantitative Description of the State of a Quantum System

As the 19th century waned, the curtain began to rise on a series of enigmatic experimental phenomena that classical physics could not explain. Notable among these were the photoelectric effect and blackbody radiation, which revealed the discrete random motion of microscopic particles, defying precise spatial localization. These discoveries necessitated a revolutionary new framework for understanding the microscopic world.

This essay delves into the establishment of this new system and elucidates its fundamental differences from classical physics. At its core, this new quantum framework is rooted in statistics, offering an approach to quantifying the motion and states of quantum entities.

I. The Statistical Explanation of De Broglie's Particle Waves

After Einstein proposed the light quantum hypothesis, suggesting that light consists of discrete energy packets known as photons, de Broglie boldly assumed the concept of particle-waves. That is to say, **not only photons but all particles exhibit both particle and wave behavior when in motion**, satisfying:

$$p = \frac{h}{\lambda} = \hbar k$$

$$E = hf = \hbar\omega$$

$$\hbar = \frac{h}{2\pi}, \qquad h = Planck's\ constant \approx 6.626\ x\ 10^{-11}\ Js$$

In these equations, the quantities representing particle-like properties are energy E and momentum p, whereas the quantities representing wave-like properties are angular frequency ω and wavenumber k. De Broglie's significant contribution was his proposal should be **universally valid**, applicable not only to photons but also to electrons, protons, neutrons, atoms, etc.

This mix of 'wave' and 'particle' views is one of the most perplexing issues in quantum mechanics, continuously baffling people since its inception. According to classical theory, 'wave-like' and 'particle-like' are entirely incompatible concepts.

Classically, particles possess localized, atomic properties with intrinsic mass and charge, interacting as a whole. They follow precise trajectories, a concept rooted in Newtonian mechanics. Classical particles are defined by physical quantities with definite, continuous values, obeying Newton's second law.

Classical waves are periodic disturbances propagable in space, like water and electromagnetic waves. They represent periodic variations of physical quantities, described by frequency, wavelength, and wavenumber. Wave phenomena include interference and diffraction, resulting from wave superposition.

The classical framework views particle-like and wave-like properties as incompatible. Yet, microscopic particles exhibit both, challenging classical understanding. Feynman stated that **an electron is neither a classical particle nor a classical wave** [1]. Essentially, it exhibits both particle and wave properties, but not in the traditional classical sense. Its particle-like aspect reflects **only** the atomicity and 'wholeness' of classical particles, existing with specific mass and charge. The wave-like nature of an electron is characterized **solely** by the superposition principle of waves.

Experiments demonstrate that even a single electron can produce interference and diffraction phenomena. Feynman, in his book "The Character of Physical Law [2]," vividly describes the electron interference experiment. In an idealized experiment, an electron released in a specific direction passes through a slit and hits a screen, producing a spot, say at point A. Under identical experimental conditions, another electron would also be expected to hit point A, following classical understanding. However, repeated experiments with single electrons show an accumulation of spots on the screen, not just behind the slits. As more electrons reach the screen, a pattern of regular interference fringes emerges. Similar results are observed in diffraction experiments, where numerous electrons create regular diffraction rings on the screen. If many electrons are released simultaneously in these experiments, interference or diffraction patterns still appear on the screen. This suggests that the appearance of interference and diffraction is **linked to the behavior of many electrons or the repeated action of a single electron**, as these phenomena are not observable otherwise. In these numerous events, electron motion exhibits statistical regularities, with the distribution of electron impacts on the screen **resembling the intensity distribution of waves with appropriate wavelengths**. The brightest spots on the screen correspond to the highest electron impact concentration, and the darkest spots to areas electrons do not reach, interpreted as maximum or minimum wave intensities.

Based on experimental data analysis, the German physicist Max Born proposed in 1927 a **statistical interpretation of particle waves**. The intensity of a matter wave in space is directly proportional to the probability of finding a particle there, correlating with position probability. Quantum mechanics was established and developed on the basis of the **particle wave hypothesis and its statistical interpretation**.

III. Quantum States

In classical mechanics, the mechanical state of a particle, is characterized by all its physical quantities (position, momentum, energy, etc.) and their variations over time. These quantities take **definite** values. In contrast, quantum theory's qualitative and quantitative descriptions of these physical quantities differ markedly from classical theory. To understand this difference, consider the electron diffraction experiment through a metal foil. Electrons from source S, with kinetic energy T = eU determined by the accelerating voltage U, form a parallel beam in a certain direction. Before scattering with foil C, electrons have definite kinetic energy and momentum, with momentum p = $p = \sqrt{2meU}$ and direction set by the aperture. The electron's position is not definite, as the associated wave is approximately a plane monochromatic wave with uniform intensity distribution. According to Born's statistical interpretation, the probability of the electron appearing at any point in space covered by this wave is equal.

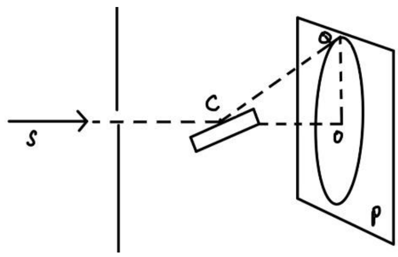

Fig.1. Illustration of the electron scattering experiment. C represents a metal foil and P represents a detector screen. The electrons show a interference pattern on the screen.

After scattering with C, it is scattered by the atoms within the crystal, altering its wavelength and direction. Therefore, the electron's momentum becomes uncertain. The wave associated with the electron post-scattering is no longer a plane monochromatic wave but a complex wave composed of several plane monochromatic components, each corresponding to a momentum. It's challenging to determine the electron's exact momentum value. Each impact of the electron on the screen has a definite location, but quantum mechanics cannot predict each impact's exact location; it can only provide the probability distribution of the electron's coordinates on the screen based on the intensity of the particle wave.

In quantum mechanics, the concept of the probability of a physical quantity's value aligns with the conventional notion of probability [3]. Suppose a physical quantity is measured N times (where N is a large number), and it is found that the value L1 occurs N1 times, L2 occurs N2 times, and Ln occurs Nn times. Ln (n=1,2,…) are considered the possible values of the physical quantity, and $W(Ln) = Nn/N$ is defined as the probability of the physical quantity assuming the possible value Ln. Since $N=N1+N2+...+Nn$, $\sum_n W(Ln) = 1$.

IV. The Wave Function

In classical theory, states are quantitatively described by the trajectory function r = r(t), which predicts all mechanical quantities of a particle and their temporal evolution. Similarly, to quantitatively describe states in quantum systems, the state-describing function should predict the probability distribution of all mechanical quantities and their changes over time. In quantum theory, **this state-describing function is known as the wave function**.

Taking free motion as an example, the particle wave associated with a freely moving electron is a plane monochromatic wave. Theoretically, a plane monochromatic wave can be expressed in the form of a complex function:

$$\boldsymbol{\Psi}(\boldsymbol{r},t) = a exp[-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} + \varphi)]$$

From de Broglie's relationship, we know:

$$\omega = \frac{E}{\hbar}; \boldsymbol{k} = \frac{\boldsymbol{p}}{\hbar}$$

The complex wave function can be therefore written as:

$$\boldsymbol{\Psi}(\boldsymbol{r},t) = A exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Et)]$$

This describes the form of the particle wave associated with a free particle. According to Born's statistical interpretation, the probability distribution of the particle's position is proportional to the square of the wave's amplitude, denoted as a^2. From the relationship:

$$|\boldsymbol{\Psi}(\boldsymbol{r},t)|^2 = \boldsymbol{\Psi}^*(\boldsymbol{r},t)\,\boldsymbol{\Psi}(\boldsymbol{r},t) = a^2,$$

the position distribution of particles is proportional to $|\boldsymbol{\Psi}(\boldsymbol{r},t)|^2$.

The wave function encapsulates all available information about a quantum system. However, it doesn't directly give us the position of a particle; instead, it provides the probability density of finding a particle in the space **r** at time t [4].

V. Calculation of Position Distribution

Let $dW(r,t)$ be the probability of finding a particle in a small volume component dτ around **r**, we know that:

$$d\boldsymbol{W}(\boldsymbol{r},t) = K|\boldsymbol{\Psi}(\boldsymbol{r},t)|^2 d\tau,$$

where K indicates a constant. The probability density that describes the probability distribution of position of particles is therefore:

$$\boldsymbol{W}(\boldsymbol{r},t) = \frac{d\boldsymbol{W}(\boldsymbol{r},t)}{d\tau} = K|\boldsymbol{\Psi}(\boldsymbol{r},t)|^2.$$

The probability of finding a particle in the entire space is 1. Therefore,

$$1 = \int_{-\infty}^{\infty} d\boldsymbol{W}(\boldsymbol{r},t) = \int_{-\infty}^{\infty} K|\boldsymbol{\Psi}(\boldsymbol{r},t)|^2 d\tau,$$

$$K = \frac{1}{\int_{-\infty}^{\infty} |\boldsymbol{\Psi}(\boldsymbol{r},t)|^2 d\tau}.$$

As a result, he equation of probability density is:

$$\boldsymbol{W}(\boldsymbol{r},t) = \frac{|\boldsymbol{\Psi}(\boldsymbol{r},t)|^2}{\int_{-\infty}^{\infty} |\boldsymbol{\Psi}(\boldsymbol{r},t)|^2 d\tau}.$$

In defining the concepts of probability and probability density, it is essential to recognize their distinction. Probability density is expressed in units of probability per unit length, area, or volume, contingent on the dimensionality of the analysis [5]. This density function, evaluated at any specific point in the sample space (the set of potential values assumed by the random variable), indicates the relative likelihood of the random variable corresponding to that point. Consequently, to ascertain the specific probability of a particle existing within a defined region R in a three-dimensional space, one can calculate this by integrating the probability density function over region R:

$$\boldsymbol{P}(\boldsymbol{x},t) = \iiint_{x1,y1,z1}^{x2,y2,z2} \boldsymbol{W}(\boldsymbol{r},t)\ dx\ dy\ dz.$$

In summary, quantum mechanics hinges on a fundamental concept — the wave function. This mathematical construct, rooted in statistical principles, serves as our window into the intricate realm of quantum entities.

REFERENCES

[1] R. P. Feynman, R. B. Leighton, and M. L. Sands, The Feynman Lectures on Physics. Uttar Pradesh, India: Pearson India Education Services Pvt Ltd, 2013.

[2] R. P. Feynman and F. Wilczek, The Character of Physical Law. Cambridge, MA: The MIT Press, 2017.

[3] M. Foulkes, Quantum Physics--An Introduction to Quatum Theory. London: Imperial College London, 2023.

[4] F. Dowker, Second Year Quantum Mechanics Lecture Notes. London: Imperial College London, 2023.

[5] R. P. Feynman, R. B. Leighton, and M. L. Sands, Quantum Mechanics. New York: Basic Books, 2010.

The Tunnel Effect and Nuclear Fusion

Imagine particles performing a feat that defies the very laws of classical physics, embarking on a journey through seemingly impenetrable barriers — welcome to the intriguing world of the tunnel effect. This cornerstone of quantum mechanics topples our everyday understanding of obstacles and energy. It's a universe where particles, such as electrons, can miraculously 'teleport' through barriers they're not energetically equipped to scale, like ghosts passing through walls. The warm, life-giving sunlight that bathes our planet is, in fact, a direct consequence of the quantum tunnel effect So, every time we feel the warmth of the Sun or see the light of day, we're experiencing the after-effects of countless quantum events happening 93 million miles away. The tunnel effect, a quantum marvel, is thus not just an abstract concept confined to the realm of theoretical physics, but a real, tangible phenomenon that plays a crucial role in sustaining life on our planet [1].

I. Particles passing a rectangular potential barrier

In order to understand the tunnel effect, we first look at the mathematical procedure of solving the Schrödinger equation to prove this phenomenon.

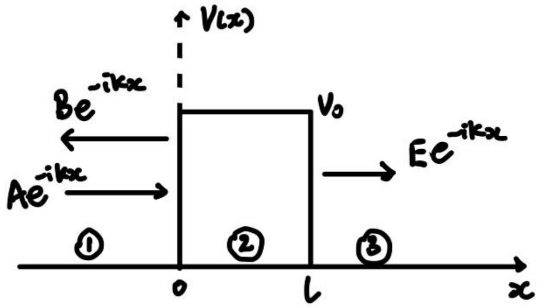

Fig.1. A potential barrier with width l and height V_0.

As shown in Figure 1, the potential function varies along the x-axis in a one-dimensional potential well, creating three distinct regions, described by:

$$V(x) = \begin{cases} 0, \ \ x < 0\,, x > l & regions \text{ ①③} \\ V_0, \qquad 0 \le x \le l & regions \text{②} \end{cases}$$

For these three regions, there are three scenarios: ① $E-V_0 < 0$; ② $E-V_0=0$; ③ $E-V_0>0$. The tunnel effect describes specifically the first scenario, since in classical Newtonian theory a particle overcomes a potential barrier only if its energy E is greater than the potential V_0. Therefore, the time-independent Schrödinger equations in this case can be written as:

$$\begin{cases} -\dfrac{\hbar^2}{2m}\nabla^2\Phi(x) = E\Phi(x), & x < 0, x > l \\ -\dfrac{\hbar^2}{2m}\nabla^2\Phi(x) = [E - V_0]\Phi(x), & 0 \le x \le l \end{cases}.$$

Solving these Schrödinger equations, we are able to obtain the solutions of the wave function in three regions respectively:

$$\begin{cases} \Phi(x) = Ae^{i\sqrt{\frac{2mE}{\hbar^2}}x} + Be^{-i\sqrt{\frac{2mE}{\hbar^2}}x}, & x < 0 \\ \Phi(x) = Ce^{-\sqrt{\frac{2mE}{\hbar^2}}x} + De^{-i\sqrt{\frac{2mE}{\hbar^2}}x}, & 0 \le x \le l \\ \Phi(x) = Ee^{i\sqrt{\frac{2mE}{\hbar^2}}x}, & x > l \end{cases}.$$

Interpreting these solutions in quantum mechanics requires an understanding of particle behavior as represented by wave functions, which depict probabilities. This perspective introduces three key scenarios [2]:

1. Approaching Particle: The term Ae^{ikx} in the solution for region 1 represents a particle approaching from the left infinity.
2. Reflected Particle: The term Be^{-ikx} in the same region signifies a particle reflected by the potential barrier, a consequence of the particle's wave-like motion encountering elastic reflection.
3. Transmitted Wave Beam: Ee^{ikx} represents a transmitted wave beam, consistent with wave properties and the phenomenon of quantum tunneling.

To demonstrate the non-zero probability of particle transmission through the barrier, one must apply boundary conditions of continuity at x=0 and x=l. This allows for the definition of R and T, coefficients representing reflection and transmission, respectively:

$$\begin{cases} R = \frac{|B|^2}{|A|^2} = \frac{(k^2+K^2)^2 sinh^2 Kl}{(k^2+K^2)^2 sinh^2 Kl + 4k^2K^2} \\ T = \frac{|E|^2}{|A|^2} = \frac{4k^2K^2}{(k^2+K^2)^2 sinh^2 Kl + 4k^2K^2} \sim \frac{16E(V_0-E)}{{V_0}^2} e^{-\frac{2l}{\hbar}\sqrt{2m(V_0-E)}} \end{cases},$$

where

$$\begin{cases} k^2 = \frac{2m}{\hbar^2} \\ K^2 = \frac{2m}{\hbar^2}(V_0 - E) \end{cases}.$$

The analysis of probability density in region 3 of a quantum system, as indicated by the non-zero magnitude of $|E|^2$, reveals a crucial insight: there exists a tangible probability for particles to surmount the potential barrier and manifest on its opposite side. The transmission coefficient's exponential decay, which reflects the wave function's exponential decay within region 2, particularly as the width of the potential barrier increases, as shown in Figure 2.

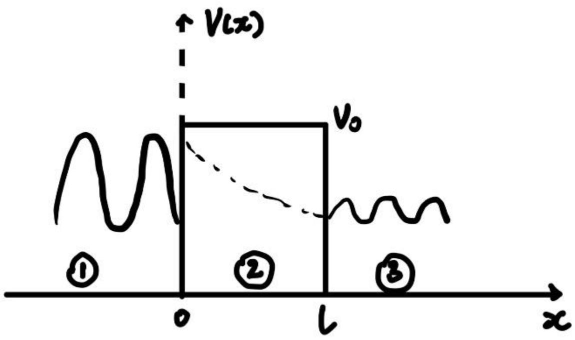

Fig.2. Illustration of the tunnel effect. The incident wave is same as the reflected wave, and the transmitted wave is a wave with attenuated amplitude.

Additionally, a fundamental aspect of quantum mechanics is illuminated through the simple relationship R+T=1. This equation encapsulates the principle that the probabilities of a particle being transmitted through the barrier and being reflected by it collectively sum up to unity. This inherent characteristic underscores the probabilistic nature of wave functions in quantum mechanics, highlighting the core concept that these functions represent probabilities, not certainties.

II. Nuclear Fusion in Classical Physics

The heart of a star is a cauldron of thermonuclear fusion, where the cosmos cooks up energy in a process more intricate and fascinating than any alchemist's dream. The energy source within stars primarily comes from thermonuclear fusion reactions [3]. This process involves the fusion of two nucleons (protons and neutrons) to form a new nucleus. The new nucleus has a slightly lower mass than the sum of the original nucleons, resulting in a mass deficit. This mass loss, in accordance with Einstein's mass-energy equivalence principle, releases a significant amount of energy.

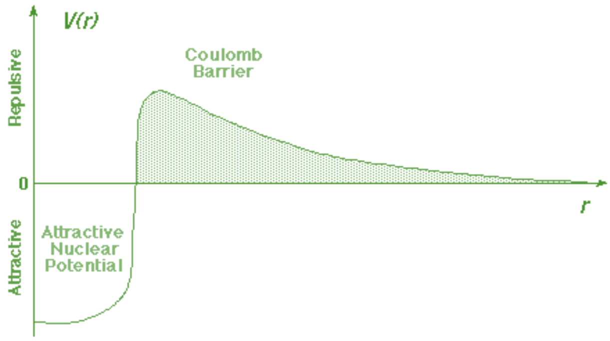

Fig.3. Illustration of the Coulomb barrier. Nucleons exhibit repulsive forces, unless the nucleons are sufficiently close, such that their potential energy exceeds the maximum of the potential barrier.[4]

Delving deeper into the stellar forge, we find that to initiate nuclear fusion, nucleons must be coaxed into an incredibly intimate proximity, typically around 10^{-15} meters, a scale where the strong nuclear force takes the stage [5], overpowering the repulsive Coulomb force between the positively charged protons, as shown in Figure 3. The primary source of energy capable of overcoming the Coulomb barrier in stars is the kinetic energy of nuclei, a result of thermal motion. The central temperature of stars, typically around 10^7 Kelvin [5], supports the assumption of an ideal gas model, allowing for the calculation of the average kinetic energy of a nucleus:

$$KE = \frac{3}{2}kT \sim 1keV,$$

where k is the Boltzmann constant. The barrier's height when strong nuclear forces become significant can be calculated by the Coulomb barrier equation:

$$E_{cb} = \frac{Z_1 Z_2 e^2}{4\pi \epsilon_0 r} \sim 1MeV,$$

where Ze represents the charge of each nucleus. The energy required to overcome this barrier is nearly a million electron volts. Inside stars like the Sun, which can be considered classical ideal gases obeying the Maxwell-Boltzmann distribution, particles have a range of energies, with some moving slowly at low energies and others moving rapidly at high energies. The probability of the nucleus having an energy to the order of 1MeV is:

$$N \propto e^{-\frac{E_{cb}}{KE}} \sim e^{-1000} \sim 10^{-434}.$$

Accordingly, the probability of particles in a star reaching energies as high as 1 MeV is exceedingly low, according to the classical distribution. Even when considering all nucleons in the observable universe ($\sim 10^{80}$) [6], it is insufficient for sustaining the nuclear reactions at the core of the Sun. The exponential factor of this distribution is extremely small, indicating a minuscule likelihood of particles reaching MeV energy levels under these conditions. Therefore, an additional mechanism beyond thermal kinetics is required to make thermonuclear reactions probable.

III. Nuclear Fusion in Quantum Physics

Quantum tunneling plays a crucial role in enabling thermonuclear reactions within stars, particularly during periods of quiescent burning. Although the thermal motion within a star's interior is insufficient to directly induce these reactions, it is high enough to bring colliding nuclei within proximity, creating a significant probability density behind the Coulomb barrier. This sets the stage for quantum tunneling, a phenomenon where nuclei can penetrate the Coulomb barrier even when they lack the kinetic energy normally required to overcome it.

Researchers have determined that the tunnel effect in nuclear interactions is observed when the distance between nuclei is on the same order as their de Broglie wavelength, which is relative to their kinetic energies:

$$KE = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2},$$

$$r \sim \lambda = \frac{h}{p}.$$

Again, the kinetic energy needs to overcome the Colomb barrier at a new interaction radius λ, by means of:

$$E_{cb} = \frac{Z_1 Z_2 e^2}{4\pi \in_0 \lambda} \leq KE,$$

$$\lambda \leq 2\pi \left(\frac{h}{e}\right)^2 \frac{\in_0}{m} \sim 10^{-12} m.$$

Here we consider the simplest case of the reaction between two Hydrogen nuclei, thus we take $Z_1=Z_2=1$. If we assume the kinetic energy of the nuclei is equal to the Colomb barrier, the temperature of sun can be calculated by:

$$T = \frac{e^2}{6\pi \in_0 k\lambda} \sim 10^7 K.$$

This temperature is consistent with the temperature of the sun in real life. The calculations regarding stellar thermonuclear reactions, though preliminary and not accounting for factors such as reaction rates and the variety of elements involved in fusion, which indicates the existence of quantum effect for the occurrence of thermonuclear reactions inside stars. The above calculation also suggest that the fusion of heavier elements requires higher kinetic energy. This is a key reason why such fusion processes predominantly occur in the cores of stars, since the fusion of heavier elements, which necessitates overcoming higher Coulomb barriers in atomic nuclei, requires higher temperatures [7]. Achieving these elevated core temperatures is dependent on the star's initial mass. Stars with greater initial masses can attain higher core temperatures, facilitating the fusion of heavier elements. This relationship between a star's mass, core temperature, and fusion capabilities is a fundamental aspect of stellar physics [8].

In summary, quantum tunneling emerges as a critical catalyst in the cosmic drama of star formation and sustenance. This quantum mechanical phenomenon defies classical barriers, enabling thermonuclear reactions under the comparatively modest energy conditions found in stars like our Sun. Without this subtle yet powerful quantum effect, the nuclear fusion that fuels stars and, by extension, supports life on planets in insolation habitable zones, would not occur. Quantum tunneling, therefore, transcends its role as a mere quantum oddity, standing at the forefront of celestial processes that forge and maintain life-supporting environments across the universe. This fascinating interplay of quantum mechanics and astrophysics not only underscores the elegance of the cosmos but also highlights the intricate dependencies that govern the emergence and evolution of complex life in the vast expanse of the universe.

REFERENCES

[1] R. P. Feynman, R. B. Leighton, and M. L. Sands, The Feynman Lectures on Physics. Uttar Pradesh, India: Pearson India Education Services Pvt Ltd, 2013.

[2] M. Foulkes, Quantum Physics--An Introduction to Quatum Theory. London: Imperial College London, 2023.

[3] E. Böhm-Vitense, Introduction to Stellar Astrophysics. Cambridge: Cambridge University Press, 1989.

[4] J. Ruth, “Coulomb Barrier,” Why is the overcoming the Coulomb or Electromagnetic Force by the Strong Force considered an almost insurmountable problem in Fusion , https://www.quora.com/Why-is-the-overcoming-the-Coulomb-or-Electromagnetic-Force-by-the-Strong-Force-considered-an-almost-insurmountable-problem-in-Fusion-when-Physics-still-quote-the-Coupling-Constant-of-the-Coulomb-Force-as-1-137th-of (accessed Jan. 18, 2024).

[5] F. Trixler, “Quantum tunnelling to the origin and evolution of life,” Current Organic Chemistry, vol. 17, no. 16, pp. 1758–1770, 2013. doi:10.2174/13852728113179990083

[6] F. Trixler, “Quantum tunnelling to the origin and evolution of life,” Current Organic Chemistry, vol. 17, no. 16, pp. 1758–1770, 2013. doi:10.2174/13852728113179990083

[7] S. Budnik, “Quantum fusion system in nuclear engineering,” SciFed Journal of Nuclear Science, vol. 1, no. 2, 2017. doi:10.23959/sfjns-1000012

[8] J. R. Newton et al., “Gamow Peak in thermonuclear reactions at high temperatures,” Physical Review C, vol. 75, no. 4, 2007. doi:10.1103/physrevc.75.045801