数字图像处理第四次作业

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1. 证明拉普拉斯算子 $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ 的旋转不变性。
原坐标 (x, y) 和旋转后坐标 (x', y') 的关系如下：

$$x = x'cos\theta - y'sin\theta, y = x'sin\theta + y'cos\theta$$

$proof$：根据复合函数求偏导可得

$$\begin{aligned}
\frac{\partial f}{\partial x'} &= f_1'\frac{\partial x}{\partial x'} + f_2'\frac{\partial y}{\partial x'} = f_1'cos\theta + f_2'sin\theta \\
\frac{\partial^2 f}{\partial x'^2} &= f_{11}''cos^2\theta + 2f_{12}''cos\theta sin\theta + f_{22}''sin^2\theta \\
\frac{\partial f}{\partial y'} &= f_1'\frac{\partial x}{\partial y'} + f_2'\frac{\partial y}{\partial y'} = -f_1'sin\theta + f_2'cos\theta \\
\frac{\partial^2 f}{\partial x'^2} &= f_{11}''sin^2\theta - 2f_{12}''cos\theta sin\theta + f_{22}''cos^2\theta
\end{aligned}$$

根据拉普拉斯算子定义可知：

$$\nabla^2 f(x', y') = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = f_{11}'' + f_{22}'' = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f(x, y)$$

因此拉普拉斯算子具有旋转不变性

2. 证明傅里叶变换的旋转性：$f(r, \theta + \theta_0)$ 的傅里叶变换为 $F(\omega, \theta + \theta_0)$
$proof$：由二维傅里叶变换可知

$$F(u, v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)e^{-j2\pi(ux+vy)}dxdy$$

做极坐标变换，令 $x = rcos\theta, y = rsin\theta$，此时 $dxdy = rdrd\theta$
再做换元令 $u = \omega cos\phi, v = \omega sin\phi$，由此可以得到：

$$\begin{aligned}
F(\omega, \phi) &= \int_0^{2\pi} d\theta \int_0^{+\infty} f(rcos\theta, rsin\theta)e^{-j2\pi(\omega rcos\theta\cos\phi + \omega rsin\theta\sin\phi)}rdr \\
&= \int_0^{2\pi} d\theta \int_0^{+\infty} f(rcos\theta, rsin\theta)e^{-j2\pi\omega rcos(\theta-\phi)}rdr
\end{aligned}$$

对旋转后空域函数 $f(rcos(\theta + \theta_0), rsin(\theta + \theta_0))$ 做二维傅里叶变换可得

$$I = \int_0^{2\pi} d\theta \int_0^{+\infty} f(rcos(\theta + \theta_0), rsin(\theta + \theta_0))e^{-j2\pi\omega rcos(\theta-\phi)}rdr$$

注意到 $f(rcos\theta, rsin\theta)$ 是关于 θ 周期为 2π 的周期函数，对 θ 积分区间为 $[0, 2\pi]$，因此做换元 $\theta' = \theta + \theta_0$ 后积分区间依旧可以采用 $[0, 2\pi]$，所以可得：

$$I = \int_0^{2\pi} d\theta' \int_0^{+\infty} f(rcos\theta', rsin\theta') e^{-j2\pi\omega rcos(\theta' - (\theta_0 + \phi))} rdr = F(\omega, \phi + \theta_0)$$

综上 $f(r, \theta + \theta_0)$ 的傅里叶变换为 $F(\omega, \theta + \theta_0)$