7. Conditional Probability and Independence

Independent Events [Ross S3.4]

Definition 7.1: Events E and F are called **independent** if

$$P[EF] = P[E]P[F]$$

Two events that are not independent are called **dependent**.

From previous examples, $P[E|F]$ is not necessarily the same as $P[E]$.

But, if E and F are independent (and $P[F] > 0$):

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{P[E]P[F]}{P[F]} = P[E]$$

Example 7.1: Two 6-sided dice are rolled. Let

$$\begin{aligned} E_1 &= \{\text{sum is } 6\} \\ E_2 &= \{\text{sum is } 7\} \\ F &= \{\text{1st die is } 4\} \\ G &= \{\text{2nd die is } 3\} \end{aligned}$$

Then:

$$\begin{aligned} P[E_1 F] &= P[(4,2)] = 1/36, \quad P[E_1]P[F] = 5/36 \times 1/6 \neq 1/36 \\ P[E_2 F] &= P[(4,3)] = 1/36, \quad P[E_2]P[F] = 1/6 \times 1/6 = 1/36 \end{aligned}$$

So E_1 and F are not independent, but E_2 and F are independent.

Similarly, E_2 and G are independent.

Example 7.2: Say $EF = \emptyset$ with $P[E] > 0$ and $P[F] > 0$. Are E and F independent?

Solution:

What does this say about mutually exclusive events?

Proposition 7.1 *If E and F are independent, then E and F^c are independent*

Why?

$$\begin{aligned} P[E] &= P[EF \cup EF^c] \\ &= P[EF] + P[EF^c] \\ &= P[E]P[F] + P[EF^c] \end{aligned}$$

$$\Rightarrow \quad P[EF^c] = P[E] - P[E]P[F] = P[E](1 - P[F]) = P[E]P[F^c]$$

Example 7.3: If E is independent of F and E is independent of G, is E independent of FG?

Solution:

Definition 7.2: Events E and F are called conditionally independent given G when

$$P[EF|G] = P[E|G]P[F|G].$$

What does this mean?

$$\begin{aligned} P[E|G]P[F|G] &= P[EF|G] \\ &= \frac{P[EFG]}{P[G]} \\ &= \frac{P[E|FG] \times P[F|G] \times P[G]}{P[G]} \end{aligned}$$

So, this is equivalent to $P[E|FG] = P[E|G]$.

In words: If G is known to have occured, the additional information that F occured does not change the probability of E.

Definition 7.3: The 3 events E, F and G are said to be independent if

$$\begin{aligned} P[EFG] &= P[E]P[F]P[G] \\ P[EF] &= P[E]P[F] \\ P[EG] &= P[E]P[G] \\ P[FG] &= P[F]P[G] \end{aligned}$$

Now, E is independent of any event formed from F and G.

Example 7.4: $P[E(FG)] = P[EFG] = P[E]P[F]P[G] = P[E]P[FG]$

Example 7.5:

$$\begin{aligned}
P[E(F \cup G)] &= P[EF \cup EG] \\
&= P[EF] + P[EG] - P[EF \cap EG] \\
&= P[E]P[F] + P[E]P[G] - P[E]P[FG] \\
&= P[E](P[F] + P[G] - P[FG]) \\
&= P[E]P[F \cup G]
\end{aligned}$$

Definition 7.4: Events $E_1, E_2, \ldots, E_n$ are said to be independent if

$$P\left[\bigcap_{i \in A} E_i\right] = \prod_{i \in A} P[E_i] \tag{7.1}$$

for every $A \subset \{1, ..., n\}$.

Definition 7.5: An infinite set of events $E_1, E_2, \ldots$ is independent if every finite subset is independent.

Example 7.6: A system has n components. Each component functions/fails independently of any other. Component i has probability p_i of functioning. If at least one component functions, the system functions. What is the probability that the system functions?

Solution:

Sometimes each E_i is the outcome of one instance of a sequence of repeated sub-experiments, e.g., $E_i = \{i\text{-th coin toss is heads}\}$.

These sub-experiments are often called **trials** (or **repeated trials**).

Example 7.7: Independent trials that consist of repeatedly rolling a pair of fair dice are performed. The outcome of a roll is the sum of the dice.

What is the prob. of
$F = \{\text{an outcome of 5 eventually occurs, and there was no 7 before this}\}$?

Solution: