

11. Random Variables (rv)

Examples [Ross S4.5]

Example 11.1: [Cover if time] [Friendship Paradox]

There are n people named $1, 2, \dots, n$.

Person i has $f(i)$ friends. Let $m = \sum_{i=1}^n f(i)$.

Let X be a random person, equally likely to be any of the n people.

Let $Z = f(X)$, i.e., Z is # of friends of random person X .

Then

$$E[Z] = \sum_{i=1}^n f(i) \underbrace{P[X=i]}_{1/n} = \frac{m}{n} \quad \text{[by Prop. 10.1]}$$

$$E[Z^2] = \sum_{i=1}^n (f(i))^2 P[X=i] = \frac{1}{n} \sum_{i=1}^n (f(i))^2$$

Now, each person writes the names of their friends on a sheet of paper (one sheet per friend).

There are m sheets, and one sheet is drawn at random, each sheet being equally likely to be chosen.

Let

Y = name of friend on drawn sheet

$W = f(Y)$

Now

$$P[Y = i] = \frac{f(i)}{m} \quad \left[\text{as opposed to } \frac{1}{n} \right]$$

$$\begin{aligned} E[W] &= E[f(Y)] \\ &= \sum_i f(i) P[Y = i] \\ &= \sum_i f(i) \times \frac{f(i)}{m} \\ &= \frac{n}{m} \times \frac{1}{n} \sum_i (f(i))^2 \\ &= \frac{E[Z^2]}{E[Z]} \\ &\geq E[Z] \quad [\text{since } E[Z^2] \geq (E[Z])^2] \end{aligned}$$

So:

$$\begin{aligned} &(\text{expected \# of friends of random person} = E[Z]) \\ &\leq (\text{expected \# of friends of random friend} = E[W]) \end{aligned}$$

Example 11.2: There are n days in a year.

Persons 1, 2 and 3 are independently born on day r with probability p_r , for $r = 1, 2, \dots, n$.

Let $A_{i,j} = \{\text{persons } i \text{ and } j \text{ born on same day}\}$

a) Find $P[A_{1,3}]$

b) Find $P[A_{1,3} \mid A_{1,2}]$

Solution:

Remark 11.1: We had $E[aX + b] = aE[X] + b$. What about $Var[aX + b]$?

$$\begin{aligned} Var[aX + b] &= E \left[(aX + b - E[aX + b])^2 \right] \\ &= E \left[(aX + b - aE[X] - b)^2 \right] \\ &= E \left[(aX - aE[X])^2 \right] \\ &= E \left[a^2 (X - E[X])^2 \right] \end{aligned}$$

$$\begin{aligned}
&= E[a^2 Y] && \text{where } Y = (X - E[X])^2 \\
&= a^2 E[Y] \\
&= a^2 E[(X - E[X])^2] \\
&= a^2 \text{Var}[X]
\end{aligned}$$

Remark 11.2: If X has units of, say, kg, then:

- $E[X] = \mu_X$ has units of kg,
- $\text{Var}[X] = \sigma_X^2$ has units of kg^2 .

We also define $SD[X] = \sqrt{\text{Var}[X]} = \sigma_X$, called **standard deviation**.

$SD[X]$ has units of kg again.