3. Axioms (or Laws) of Probability

[Ross S2.3, S2.4]

We wish to assign to each event E a probability, denoted $P[E]$ (or $P(E)$).

How do we determine it?

Frequentist approach: Let $n(E)$ be number of occurences of E in n repeated experiments. Then define

$$P[E] = \lim_{n\to\infty} \frac{n(E)}{n}. \tag{3.1}$$

Does this limit exist? In what sense?

Modern Approach: Instead, assume that certain rules (axioms) must hold.

[A1] $0 \leq P[E] \leq 1$

[A2] $P[S] = 1$

[A3] If $E_1, E_2, \ldots$ are disjoint (i.e., mutually exclusive), then

$$P[E_1 \cup E_2 \cup \ldots] = \sum_{i=1}^{\infty} P[E_i]$$

Consequences of axioms:

Corollary 3.1 $P[\emptyset] = 0$.

Why? Let $E_1 = S, E_2 = \emptyset, E_3 = \emptyset, \ldots$.

Then $E_1, E_2, E_3, \ldots$ are disjoint.

Hence,

$$\begin{aligned} P[E_1 \cup E_2 \cup E_3 \cup \cdots] &= P[E_1] + P[E_2] + P[E_3] + \cdots \\ &= P[S] + P[\emptyset] + P[\emptyset] + \cdots \\ &= 1 + P[\emptyset] + P[\emptyset] + \cdots \end{aligned}$$

But this sum must be ≤ 1, so $P[\emptyset] = 0$.

Corollary 3.2 *Say $E_1, E_2, \ldots, E_n$ are disjoint. Then*

$$P[\cup_{i=1}^n E_i] = \sum_{i=1}^n P[E_i]$$

Why? Take $\emptyset = E_{n+1} = E_{n+2} = \cdots$. Then

$$\begin{aligned} P[\cup_{i=1}^n E_i] &= P[\cup_{i=1}^\infty E_i] \\ &= \sum_{i=1}^\infty P[E_i] \\ &= \sum_{i=1}^n P[E_i] + \sum_{i=n+1}^\infty P[E_i] \\ &= \sum_{i=1}^n P[E_i] \end{aligned}$$

Example 3.1: If each of roulette's 38 possible outcomes are equally likely, then

$$
\begin{aligned}
&1) \quad P[00] = P[0] = P[1] \cdots = P[36] \\
&2) \quad 1 = P[\{00, 0, 1, \cdots, 36\}] = P[00] + P[0] + \cdots + P[36]
\end{aligned}
$$

Hence,

$$P[00] = P[0] = \cdots = P[36] = 1/38$$

So,

$$
\begin{aligned}
P[\text{even}] &= P[\{2, 4, \ldots, 36\}] \\
&= P[2] + P[4] + \cdots + P[36] \\
&= 18/38 = 9/19
\end{aligned}
$$

Corollary 3.3 $P[E^c] = 1 - P[E]$

Why? E and E^c are disjoint, and $E \cup E^c = S$.

$$\Rightarrow \quad 1 = P[S] = P[E \cup E^c] = P[E] + P[E^c]$$

So $P[E^c] = 1 - P[E]$

Corollary 3.4 *If $E \subset F$ then $P[E] \leq P[F]$.*

Why? Since $E \subset F$, then

- $F = SF = (E \cup E^c)F = EF \cup E^cF = E \cup E^cF$
- E and E^cF are disjoint

Then

$$
\begin{aligned}
& P[F] = P[E] + \underbrace{P[E^c F]}_{\geq 0} \\
\Rightarrow \quad & P[F] \geq P[E]
\end{aligned}
$$

Example 3.2: In roulette, $\mathsf{odd} \subset \mathsf{even}^c$, so

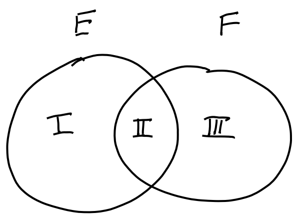
$$
\underbrace{P[\mathsf{odd}]}_{9/19} \leq \underbrace{P[\mathsf{even}^c]}_{10/19}
$$

Corollary 3.5 $P[E \cup F] = P[E] + P[F] - P[E \cap F]$

Why?

$$
\left.\begin{aligned}
I &= EF^c \\
II &= EF \\
III &= E^cF
\end{aligned}\right\} \text{disjoint}
$$

$$
\begin{aligned}
E &= I \cup II \\
F &= II \cup III \\
E \cup F &= I \cup II \cup III
\end{aligned}
$$



$$
\begin{aligned}
P[E] + P[F] &= P[I \cup II] + P[II \cup III] \\
&= P[I] + P[II] + P[II] + P[III] \\
&= P[I \cup II \cup III] + P[II] \\
&= P[E \cup F] + P[EF]
\end{aligned}
$$

Example 3.3: After 5 years, a car may need

i) new brakes with prob. 0.5
ii) new tires with prob. 0.4
iii) both with prob. 0.3

What is probability it needs neither?

Solution:

Can we generalize the $P[E \cup F]$ idea of Corollary 3.5? Yes!

$$\begin{aligned}
P[E \cup F \cup G] &= P[(E \cup F) \cup G] \\
&= P[(E \cup F)] + P[G] - P[(E \cup F)G] \\
&= P[E] + P[F] - P[EF] + P[G] - P[EG \cup FG] \\
&= P[E] + P[F] + P[G] - P[EF] \\
&\qquad - (P[EG] + P[FG] - P[EGFG]) \\
&= P[E] + P[F] + P[G] \\
&\qquad - P[EF] - P[EG] - P[FG] \\
&\qquad + P[EFG]
\end{aligned}$$

Proposition 3.1 *Inclusion/Exclusion Principle*

$$
\begin{aligned}
&P[E_1 \cup E_2 \cup \ldots \cup E_n] \\
&\quad = P[E_1] + P[E_2] + \cdots P[E_n] && \textit{include all events} \\
&\qquad - \sum_{1 \le i_1 < i_2 \le n} P[E_{i_1} E_{i_2}] && \textit{exclude intersections of pairs} \\
&\qquad + \sum_{1 \le i_1 < i_2 < i_3 \le n} P[E_{i_1} E_{i_2} E_{i_3}] && \textit{include triple intersections} \\
&\qquad\quad \vdots && \vdots \\
&\qquad + (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} P[E_{i_1} E_{i_2} \cdots E_{i_r}] && \textit{(in/ex)clude } r\textit{-way intersections} \\
&\qquad\quad \vdots && \vdots \\
&\qquad + (-1)^{n+1} P[E_1 E_2 \cdots E_n] && \textit{(in/ex)clude } n\textit{-way intersection}
\end{aligned}
$$

Proof: See textbook.