18. Continuous Random Variables

2) Normal (Gaussian) random variables [Ross 5.4]

Example 18.1: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the distribution of $Z = (X - \mu)/\sigma$.

Solution:

Definition 18.1: $\mathcal{N}(0, 1)$ is called a **standard normal** or **standard Gaussian**. If $Z \sim \mathcal{N}(0, 1)$ then

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Example 18.2: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find $E[X]$ and $Var[X]$. [Var is Hard]

Solution:

CDF of Normal Random Variables

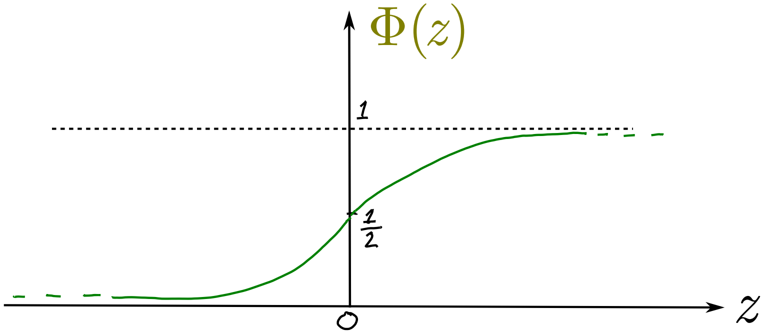
Definition 18.2: For a $Z \sim \mathcal{N}(0,1)$ distribution, we define

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du \qquad \text{[CDF of standard normal]}$$

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-u^2/2} du \qquad \text{[Q-function]}$$

$$\begin{aligned} z_\alpha &= \text{value such that: } P[Z > z_\alpha] = \alpha \qquad \text{for } 0 \le \alpha \le 1 \\ &= Q^{-1}(\alpha) = \Phi^{-1}(1-\alpha) \end{aligned}$$

Note: $\Phi(z) + Q(z) = 1; \Phi(-z) = Q(z) = 1 - \Phi(z)$.



There is also the "error function":

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv \qquad u = \sqrt{2}v, du = \sqrt{2}dv \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} du \\ &= 2 \left[\frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} du \right] \end{aligned}$$

$$= 2\left[-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-u^2/2}du + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\sqrt{2}x} e^{-u^2/2}du\right]$$
$$= 2\left[-\frac{1}{2} + \Phi(\sqrt{2}x)\right]$$
$$= 2\Phi(\sqrt{2}x) - 1$$

Table of $\Phi(z)$:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976

For Gaussian other than $\mathcal{N}(0,1)$, $\Phi(.)$ can still be used with proper transformation:

Example 18.3: Let $X \sim \mathcal{N}(3,9)$. Compute $P[2 < X < 5]$.

Solution:

$$\begin{aligned}
P[2 < X < 5] &= P\left[\frac{2-3}{\sqrt{9}} < \frac{X-3}{\sqrt{9}} < \frac{5-3}{\sqrt{9}}\right] \\
&= P\left[-\frac{1}{3} < Z < \frac{2}{3}\right] \qquad \text{where } Z \sim \mathcal{N}(0,1) \\
&= P\left[Z < \frac{2}{3}\right] - P\left[Z < -\frac{1}{3}\right] \\
&= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) \\
&= \Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right] \\
&\approx 0.37807
\end{aligned}$$

[Nearest values from the table give 0.37787]

Note: $2/3$ and $-1/3$ are called z-scores of $X = 5$ and $X = 2$ as they indicate how many σ above the mean of X these are.

Example 18.4: In finance, the Value At Risk (VaR) of an investment is the value $v > 0$ such that there is only a 1% chance the investment will lose more than v.

If the profit from an investment is $X \sim \mathcal{N}(\mu, \sigma^2)$, what is its VaR?

Solution:

The normal distribution is used (and mis-used) a lot:

- Central Limit Thm: normal is a good approximation when observation is sum of many small independent components, e.g., thermal noise
- A good model for parameter estimation errors under some conditions
- Finance (e.g., Black-Scholes option pricing)
- Is the velocity distribution of particles in an ideal gas with $\sigma^2 = kT/m$.
- Hypothesis testing
- It is the maximum entropy distribution subject to a specified variance.