33. Properties of Expectations

Conditional Expectation [Ross S7.5]

Recall that for 2 discrete random variables X and Y with $P[Y = y] > 0$:

$$\begin{aligned} p_{X|Y}(x|y) &= P[X = x|Y = y] \\ &= \frac{p_{XY}(x, y)}{p_Y(y)} \end{aligned}$$

We can define the **conditional expectation**:

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

Similarly, if X and Y are continuous, then provided $f_Y(y) > 0$:

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

where $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$.

Example 33.1: Say X and Y have joint pdf [see Example 27.3]

$$f_{XY}(x, y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & x > 0, y > 0 \\ 0 & \text{else} \end{cases}$$

Find $E[X|Y = y]$.

Solution:

Note: Conditional expectations satisfy all the properties of ordinary expectation, e.g.,

$$E[g(X) \mid Y = y] = \begin{cases} \sum_x g(x) p_{X|Y}(x|y) & \text{discrete case} \\ \\ \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx & \text{continuous case} \end{cases}$$

and

$$E\left[\sum_{i=1}^{n} X_i \,\middle|\, Y = y \right] = \sum_{i=1}^{n} E[X_i | Y = y]$$

Computing Expectations by Conditioning

$E[X|Y = y]$ is a function of y, say $g(y)$.
Let $E[X|Y]$ be $g(Y)$, i.e., in Example 33.1:

$$E[X|Y = y] = y$$

So, $$E[X|Y] = Y$$

Proposition 33.1 $E[X] = E[\ E[X|Y]\ ]$, *i.e.,*

$$E[X] = \sum_y E[X|Y = y]p_Y(y) \qquad \textit{[discrete case]}$$

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)dy \qquad \textit{[continuous case]}$$

Why? [Continuous Case]

$$\begin{aligned}
\int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)dy &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx\right] f_Y(y)dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X|Y}(x|y)f_Y(y)\ dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x, y)\ dxdy \\
&= E[X]
\end{aligned}$$

Example 33.2: You are in a room with 3 doors.

The 1st door exits the building after 3 min of travel.

The 2nd door returns to room after 5 min.

The 3rd door returns to room after 7 min.

Each time you enter the room, you pick from the 3 doors with equal probability, independently of past picks. What is the expected time until you leave the building?

Solution:

Example 33.3: The number of people N that enter a store in a day is random with mean 50.

The amount spent by each person is iid with mean $\$8$, and independent of the number of people that enter.

What is the expected amount spent in the store in one day? [Hard]

Solution: