36. Properties of Expectations

Moment Generating Functions [Ross S7.7]

Definition 36.1: The **moment generating function** (MGF) $M_X(t)$ of a random variable X is

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \begin{cases} \sum_x e^{tx} p_X(x) & \text{discrete case} \\ \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)dx & \text{continuous case} \end{cases} \end{aligned}$$

Note: a closely related concept is the **characteristic function** defined as

$$\phi_X(t) = E[e^{itX}] \qquad\qquad i = \sqrt{-1}$$

$M_X(t)$ is called moment generating function because we can find the moments $E[X^n]$ from it easily:

$$\begin{aligned} M'_X(t) &= \frac{d}{dt} E[e^{tX}] && [f'(t) = \text{derivative of } f(t)] \\ &= E\left[\frac{d}{dt} e^{tX}\right] \\ &= E\left[X e^{tX}\right] \end{aligned}$$

$$M_X^{(n)}(t) = E\left[X^n e^{tX}\right] \qquad\qquad [f^{(n)}(t) = n\text{th derivative of } f(t)]$$

Hence
$$\begin{aligned} M'_X(0) &= E[X] \\ M_X^{(n)}(0) &= E[X^n] \end{aligned}$$

Example 36.1: Find $M_X(t)$ if $X \sim \mathsf{Poisson}(\lambda)$. Use this to find $E[X]$, $E[X^2]$ and $Var[X]$.

Solution:

Example 36.2: Find $M_X(t)$ if $X \sim \mathcal{N}(\mu, \sigma^2)$. Use this to find $E[X]$, $E[X^2]$ and $Var[X]$.

Solution:

MGF of Sum of Independent Random Variables [Ross S7.7]

Let X and Y be independent random variables:

$$\begin{aligned} M_{X+Y}(t) &= E\left[e^{t(X+Y)}\right] \\ &= E\left[e^{tX}e^{tY}\right] \\ &= E\left[e^{tX}\right]E\left[e^{tY}\right] \qquad \text{[since } X \text{ and } Y \text{ are independent]} \\ &= M_X(t)M_Y(t) \end{aligned}$$

Another useful fact: the distribution of X ($f_X(x)$ or $p_X(k)$) is uniquely determined by $M_X(t)$.

Your textbook has tables of MGF for different distributions.

Example 36.3: Let $X \sim \mathsf{Poisson}(\lambda_1)$ and $Y \sim \mathsf{Poisson}(\lambda_2)$ be independent. What is the distribution of $X + Y$?

Solution:

Example 36.4: Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be independent. What is the distribution of $X + Y$?

Solution:

Joint Moment Generating Functions

For random variables $X_1, X_2, \ldots, X_n$, the joint moment generating function is defined as

$$M(t_1, \ldots, t_n) = E\left[e^{t_1 X_1 + t_2 X_2 + \cdots + t_n X_n}\right]$$

Then

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, 0, \ldots, t, 0, \ldots, 0)$$

The joint MGF uniquely determines the joint pdf.

If $X_1, \ldots, X_n$ are independent then:

$$\begin{aligned} M(t_1, t_2, \ldots, t_n) &= E\left[e^{t_1 X_1 + t_2 X_2 + \cdots + t_n X_n}\right] \\ &= E\left[e^{t_1 X_1}\right] E\left[e^{t_2 X_2}\right] \cdots E\left[e^{t_n X_n}\right] \\ &= M_{X_1}(t_1) M_{X_2}(t_2) \cdots M_{X_n}(t_n) \end{aligned}$$

Since the joint MGF uniquely specifies the joint distribution, then $X_1, \ldots, X_n$ independent is equivalent to

$$M(t_1, t_2, \ldots, t_n) = M_{X_1}(t_1) M_{X_2}(t_2) \cdots M_{X_n}(t_n)$$

Example 36.5: $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\mu, \sigma^2)$ are independent. Show that $X + Y$ and $X - Y$ are independent.

Solution: