6. Conditional Probability and Independence

Baye's Theorem [Ross S3.3]

Law of Total Probability:

Let $E, F \subset S$.

Then $\quad E = ES = E(F \cup F^c) = EF \cup EF^c$

and $\quad P[E] = P[EF] + P[EF^c]$

$$= P[E|F]P[F] + P[E|F^c]P[F^c]$$

Example 6.1: The probability of an insurance claim is

- 0.4 for 30% of persons (type 1),
- 0.5 for 70% of persons (type 2).

What is the probability that a random person has a claim?

Solution:

Let $F_1, ..., F_n$ partition S.

Then $$E = ES = E\left(\bigcup_{i=1}^{n} F_i\right) = \bigcup_{i=1}^{n} (EF_i)$$

So $$P[E] = P[\cup_{i=1}^{n}(EF_i)] = \sum_{i=1}^{n} P[EF_i] = \sum_{i=1}^{n} P[E|F_i]P[F_i] \qquad \text{[Law of total probability]}$$

Example 6.2: You roll a 4-sided die. If result is ≤ 2, you roll once more, otherwise you stop. What is probability that the sum ≥ 4?

Solution:

Baye's Theorem and Inference:

Let $F_1, F_2, \ldots, F_n$ partition S.

Say we know $P[E|F_j]$ and $P[F_j]$. We want to compute $P[F_j|E]$:

$$
\begin{aligned}
P[F_j|E] &= \frac{P[EF_j]}{P[E]} \\
&= \frac{P[E|F_j]P[F_j]}{P[E|F_1]P[F_1] + P[E|F_2]P[F_2] + \cdots + P[E|F_n]P[F_n]}
\end{aligned}
\tag{6.1}
$$

This is Baye's theorem/rule.

Application to inference:

Before any partial information is revealed (i.e., observing E occurs), the probabilities are:

$P[F_1], P[F_2], \ldots, P[F_n]$ $\Big\}$ "prior probabilities"

After observing E occur, they are revised as:

$P[F_1|E], P[F_2|E], \ldots, P[F_n|E]$ $\Big\}$ "posterior probabilities"

according to (6.1).

Posterior probabilities are key to practical inference (e.g., classification, pattern recognition, detection, etc.)

Example 6.3: A 3-card deck has

- one card with red on both sides
- one card with black on both sides
- one card with red on one side + black on the other.

One side of 1 card is picked at random. It is red. What is the probability that other side is black?

Solution:

Example 6.4: A blood test has 95% prob of detecting a desease when it is present. It has a 1% false positive rate when it is not present. 0.5% of people have the desease.

a) If a random person tests positive, what is prob. that desease is present?

b) If a random person tests negative, what is prob. that desease is present?

Solution: