14. Random Variables (rvs)

Expectation of sums of random variables [Ross S4.9]

Recall, a random variable X is a function $X(s)$ of the outcome s of a random experiment.

We can have two functions of the same outcome s, say $X(s)$ and $Y(s)$.

Example 14.1: Flip a coin 5 times.

Let $X = \#$ heads in first 3 flips; $Y = \#$ heads in last 2 flips.

Since X and Y are numbers, we can add them: $Z(s) = X(s) + Y(s)$.

In other words, Z is also a random variable.

Here, $Z = \#$ of heads in all 5 flips.

Now, for each $s \in S$, let $p(s) = P[\{s\}]$.

Then $P[A] = \sum_{s \in A} p(s)$

Let
$$\begin{aligned} X \in \mathcal{X} &= \{x_1, \dots, x_n\} \\ A_k &= \{s \in S \mid X(s) = x_k\} \end{aligned}$$

Then
$$
\begin{aligned}
E[X] &= \sum_{k=1}^{n} x_k P[X = x_k] \\
&= \sum_{k=1}^{n} x_k P[A_k] \\
&= \sum_{k=1}^{n} x_k \sum_{s \in A_k} p(s) \\
&= \sum_{k=1}^{n} \sum_{s \in A_k} x_k p(s) \\
&= \sum_{k=1}^{n} \sum_{s \in A_k} X(s) p(s) \\
&= \sum_{s \in S} X(s) p(s)
\end{aligned}
$$

Example 14.2: Two independent flips of a fair coin are made.

Let $X = \#$ heads.

Then
$$
\begin{aligned}
P[X = 0] &= 1/4 \\
P[X = 1] &= 1/2 \\
P[X = 2] &= 1/4
\end{aligned}
$$

So
$$E[X] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1.$$

Also, $S = \{tt, th, ht, hh\}$, and each outcome has probability 1/4.

So $$E[X] = X(tt) \times \frac{1}{4} + X(th) \times \frac{1}{4} + X(ht) \times \frac{1}{4} + X(hh) \times \frac{1}{4}$$
$$= 0 \times \frac{1}{4} \quad + 1 \times \frac{1}{4} \quad + 1 \times \frac{1}{4} \quad + 2 \times \frac{1}{4}$$
$$= 1$$

Why is this interpretation useful?

Let $Z(s) = X(s) + Y(s)$. What is $E[Z]$?

$$\begin{aligned} E[Z] &= \sum_{s \in S} Z(s)p(s) \\ &= \sum_{s \in S} (\ X(s) + Y(s)\ )p(s) \\ &= \sum_{s \in S} X(s)p(s) + \sum_{s \in S} Y(s)p(s) \\ &= E[X] + E[Y] \end{aligned}$$

This result can be generalized.

Proposition 14.1 *For random variables $X_1, X_2, \ldots, X_n$:*

$$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$$

Why?

Let $Z = X_1 + \cdots + X_n$. Then

$$
\begin{aligned}
E[Z] &= \sum_{s\in S} Z(s)p(s) \\
&= \sum_{s\in S} (\ X_1(s) + \cdots + X_n(s)\ )p(s) \\
&= \sum_{s\in S} X_1(s)p(s) + \cdots + \sum_{s\in S} X_n(s)p(s) \\
&= E[X_1] + \cdots + E[X_n]
\end{aligned}
$$

Example 14.3: Let $X_1, X_2, \ldots, X_n$ be outcomes of n independent $\mathsf{Bernoulli}(p)$ trials. Then

$$X = X_1 + \cdots + X_n$$

counts the number of 1's in the n trials. So $X \sim \mathsf{Binomial}(n, p)$.

Also:

$$
\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
&= E[X_1] + \cdots + E[X_n] && \text{[by Proposition 14.1]} \\
&= p + \cdots + p \\
&= np
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= E\left[\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{\ell=1}^{n} X_\ell\right)\right] \\
&= E\left[\sum_{k=1}^{n}\sum_{\ell=1}^{n} X_k X_\ell\right] \\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{n} E\left[X_k X_\ell\right] && \text{[by Proposition 14.1]}
\end{aligned}
$$

$$= \sum_{k=1}^{n} E\left[X_k^2\right] + \sum_{k=1}^{n} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} E\left[X_k X_\ell\right]$$

Now $\quad P[X_k^2 = 1] = P[X_k = 1] = p$

$$\begin{aligned} P[X_k X_\ell = 1] &= P[X_k = 1, X_\ell = 1] && \text{[for } k \neq \ell\text{]} \\ &= P[X_k = 1] P[X_\ell = 1] && \text{[since trials are independent]} \\ &= p^2 \end{aligned}$$

So $\quad E[X^2] = np + n(n-1)p^2$

Properties of CDFs [Ross 4.10]

Recall $F_X(x) = P[X \leq x]$

Therefore:

1) $0 \leq F_X(x) \leq 1$
2) If $a < b$ then $\{X \leq a\} \subset \{X \leq b\}$

   $\Rightarrow P[X \leq a] \leq P[X \leq b]$

   $\Rightarrow F_X(a) \leq F_X(b)$

   or, $F_X(x)$ is non-decreasing in x.

It can also be show that:

3) $\lim_{x \to \infty} F_X(x) = 1$

4) $\lim_{x \to -\infty} F_X(x) = 0$

5) $\lim_{x \downarrow b} F_X(x) = F_X(b)$

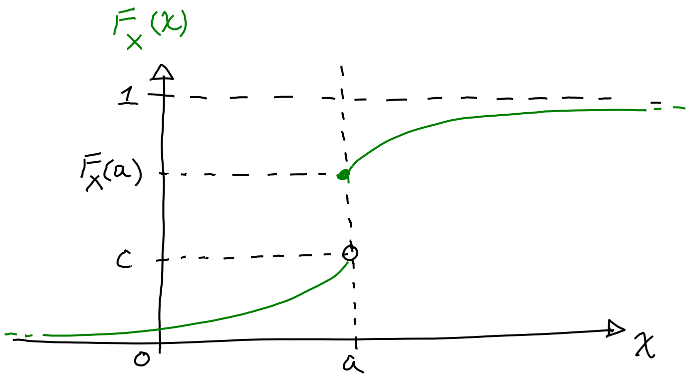
[i.e., $F_X(x)$ is continuous from the right]

6) $\lim_{x \uparrow b} F_X(x)$ exists

[i.e., $F_X(x)$ has left limits]

A function with properties 5) and 6) is called **càdlàg** [continue à droite, limite à gauche].

Example 14.4:



Here:

$$\lim_{x \downarrow a} F_X(x) = F_X(a)$$

$$\lim_{x \uparrow a} F_X(x) = c \neq F_X(a)$$