Chapter 1

Introduction to Signals and Systems

The goal of this chapter is to introduce the concepts of **signal** and **system**, and to provide a foundation for the later chapters of this course, in which we look at these concepts in considerable detail.

1.1 Signals

Informally a signal is a quantity that varies with time.

Example 1.1.1 Typical examples of signals:

(a) The voltage at the terminals of a transmission or receiving antenna.

(b) The atmospheric pressure at a given location on the earth.

(c) The monthly reports on the national rate of inflation.

(d) The output voltage of an analogue-to-digital converter in a C.D. player.

In Example 1.1.1(a), (b), the relevant quantity (voltage, atmosheric pressure) is given for *all* values of time t, and these signals are called *continuous-time signals.* In Example 1.1.1(c), (d), the relevant quantity (rate of inflation, voltage) is given at only *discrete* instants of time t, and these signals are called *discrete-time signals.* We can now formalize these notions as follows:

Definition 1.1.2 A **continuous-time** signal is a real or complex valued function defined for all values of t over the range $-\infty < t < +\infty$. See Fig. 1.1.

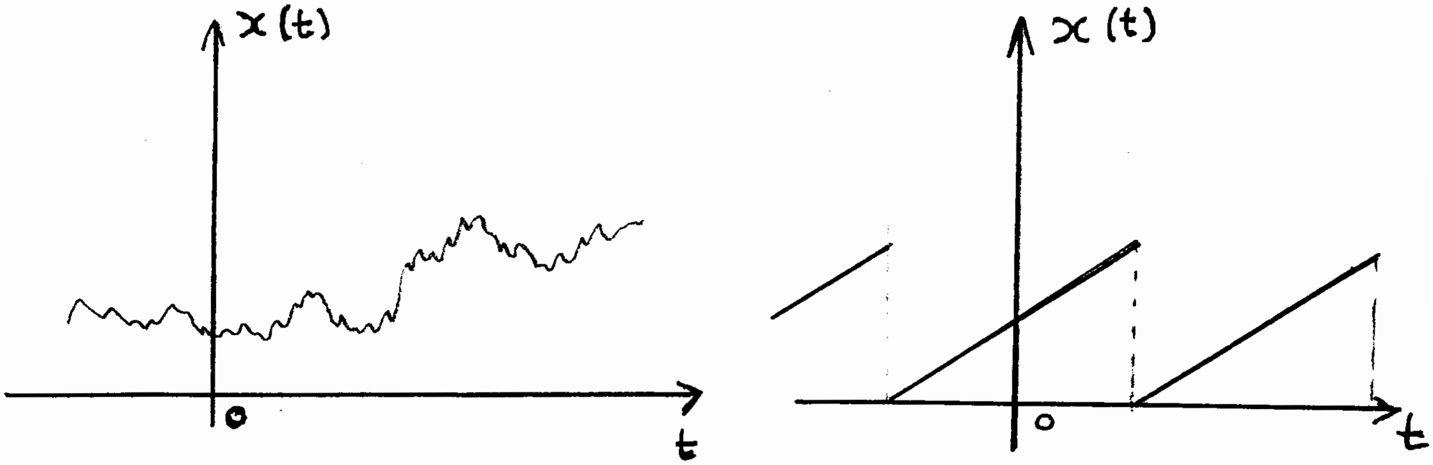


Fig. 1.1.

Definition 1.1.3 A **discrete-time** signal is a real or complex valued function defined over a discrete set of instants $\{t_k\}$. Throughout this course we shall assume that t_k is of the form $t_k = k\gamma$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$, where $\gamma > 0$ is a constant called the **intersample interval**. See Fig. 1.2.

Remark 1.1.4 We write $x(t)$ for the value of a continuous-time signal at instant t, and write $x[k]$ (rather than $x(k\gamma)$) for the value of a discrete-time signal at instant $k\gamma$.

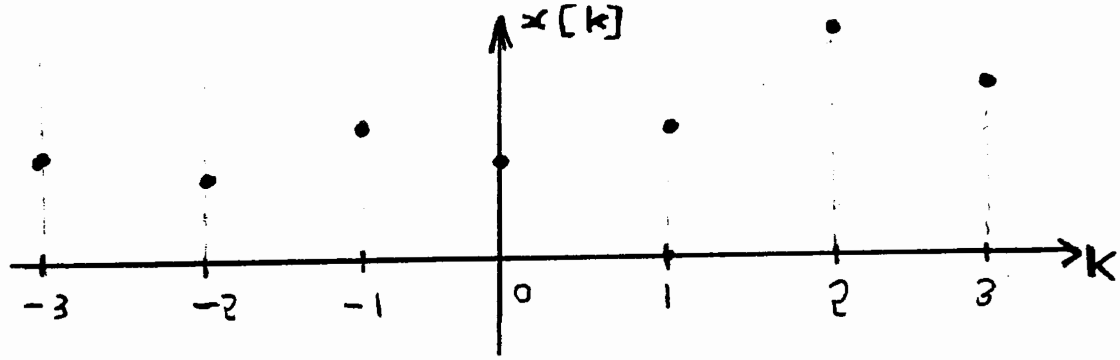


Fig. 1.2.

Remark 1.1.5 Perhaps the very simplest signal is the **zero signal** whose value is identically zero over the whole range of definition. That is, a continuous-time signal $x(t)$ is called a zero signal when $x(t) = 0$ for all t, and a discrete-time signal $x[k]$ is called a zero signal when $x[k] = 0$ for all k.

We are now going to introduce certain *standard signals* (besides the zero signal) which will be essential for this course. These signals are:

Step signals

Periodic signals

Sinusoidal signals

Generalized sinusoidal signals

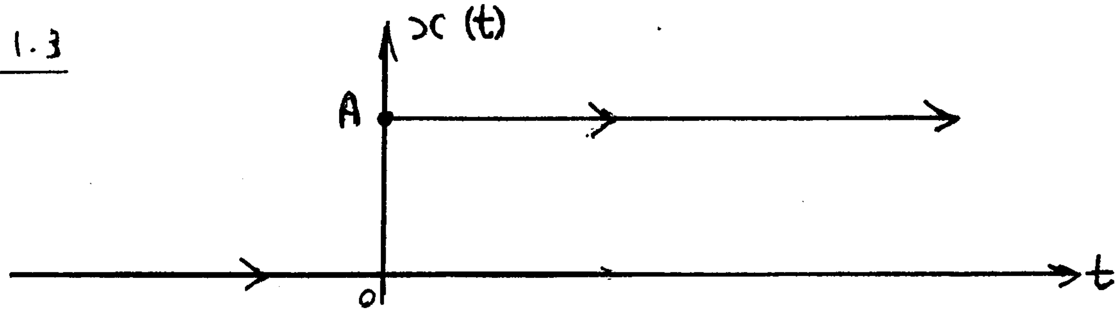
1.1.1 Step Signals

Definition 1.1.6 A continuous-time **step signal** $x(t)$ has the form

$$x(t) = \begin{cases} 0 & \text{when } t < 0, \\ A & \text{when } t \geq 0, \end{cases}$$

where A is a real or complex constant. See Fig. 1.3. In the special case of $A = 1$ we call this signal a **unit step function** and denote it by $u(t)$.

Fig 1.3



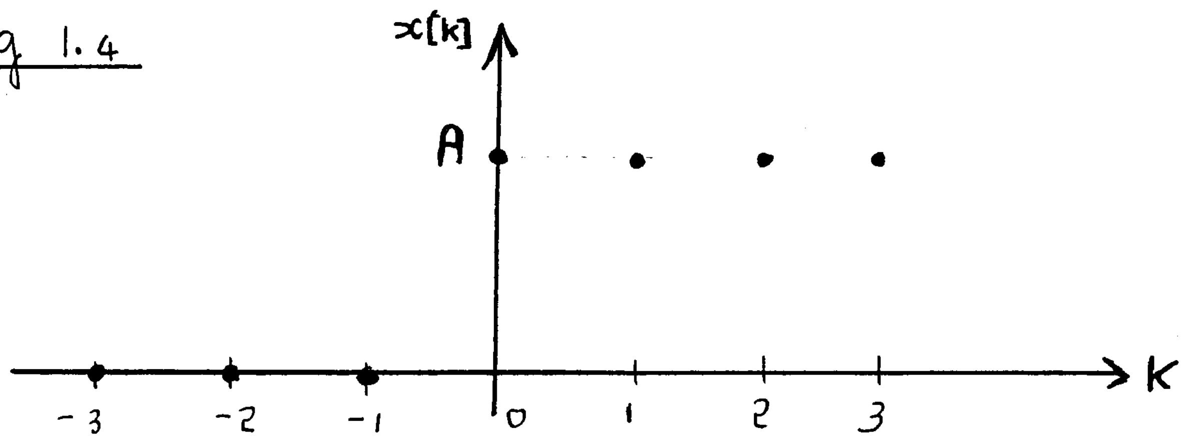
The discrete-time analogue is formulated in the same way:

Definition 1.1.7 A discrete-time **step signal** $x[k]$ has the form

$$x[k] = \begin{cases} 0 & \text{when } k < 0, \\ A & \text{when } k \geq 0, \end{cases}$$

where A is a real or complex constant. See Fig. 1.4. In the special case of $A = 1$ we call this signal a **unit step function** and denote it by $u[k]$.

Fig 1.4



1.1.2 Periodic Signals

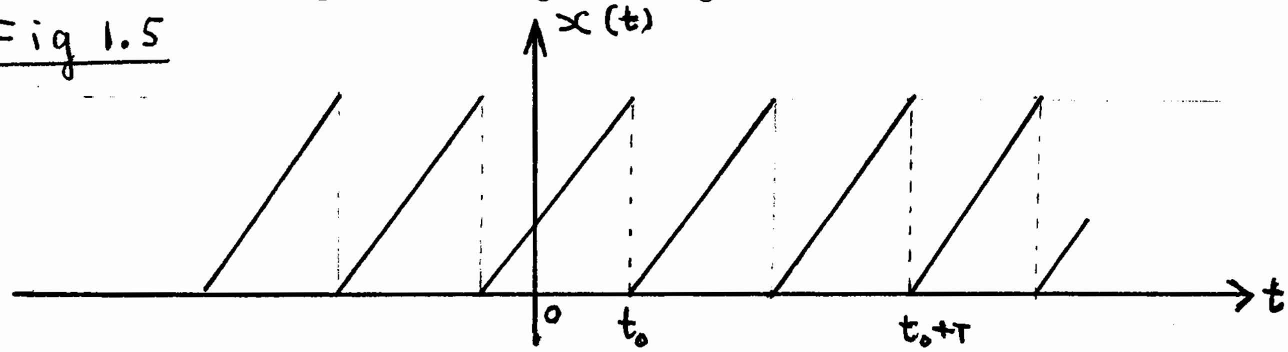
Periodic signals are essentially *repetitive* signals. We make this idea precise as follows:

Definition 1.1.8 A continuous-time signal $x(t)$ is called **periodic** when the following relation holds for some constant $T > 0$:

$$x(t) = x(t+T), \quad \text{for all real } t. \tag{1.1}$$

The constant T is called a **period** of the signal. See Fig. 1.5.

Fig 1.5



Remark 1.1.9 If $x(t)$ is periodic with a period T then clearly

$$x(t) = x(t+T) = x(t+2T) = \ldots = x(t+nT), \quad \text{for all real } t,$$

for each positive integer n. Thus, if a continuous-time signal $x(t)$ is periodic with period T, then it is also periodic with periods $2T$, $3T$, $4T, \ldots$ The *smallest* period is called the **fundamental period** of $x(t)$.

An important question is the following: Is the sum of two continuous-time periodic signals also periodic? In general the answer is NO, but we do have the following result:

Fact 1.1.10 Suppose that $x_1(t)$ is a continuous-time periodic signal with period T_1, and $x_2(t)$ is a continuous-time periodic signal with period T_2. If T_1/T_2 is a rational number (i.e. the ratio of two integers) then $x(t) \triangleq x_1(t) + x_2(t)$ is a continuous-time periodic signal.

Proof: Since T_1/T_2 is rational we can write

$$\frac{T_1}{T_2} = \frac{n_1}{n_2}$$

where n_1 and n_2 are integers, thus

$$n_2 T_1 = n_1 T_2. \tag{1.2}$$

By Remark (1.1.9) we have

$$(1.3) \qquad x_1(t) = x_1(t + n_2T_1), \qquad \text{and} \qquad x_2(t) = x_2(t + n_1T_2), \qquad \text{for all real } t.$$

Let T denote the common value in (1.2). Then, adding the expressions in (1.3), gives

$$x_1(t) + x_2(t) = x_1(t+T) + x_2(t+T), \qquad \text{for all real } t,$$

or

$$x(t) = x(t+T), \qquad \text{for all real } t.$$

This shows that the sum $x(t)$ is periodic with a period T. ∎

Remark 1.1.11 Let $x(t)$ be continuous-time periodic signal with a period T. A useful fact is that

$$(1.4) \qquad \int_{t_0}^{t_0+T} x(t)\,dt = \int_0^T x(t)\,dt, \qquad \text{for all } t_0.$$

To see this write

$$(1.5) \qquad \int_{t_0}^{t_0+T} x(t)\,dt = \int_0^T x(t)\,dt + \int_T^{t_0+T} x(t)\,dt - \int_0^{t_0} x(t)\,dt.$$

Now, taking $\tau = t - T$, gives

$$(1.6) \qquad \int_T^{t_0+T} x(t)\,dt = \int_0^{t_0} x(\tau + T)\,d\tau.$$

Since $x(t)$ is periodic with period T, we have

$$x(\tau + T) = x(\tau), \qquad \text{for all } \tau,$$

thus

$$\int_0^{t_0} x(\tau + T)\,d\tau = \int_0^{t_0} x(\tau)\,d\tau$$

thus, from (1.6), we get

$$(1.7) \qquad \int_T^{t_0+T} x(t)\,dt = \int_0^{t_0} x(\tau)\,d\tau.$$

By (1.7) we can cancel the last two terms on the right side of (1.5) to get (1.4).

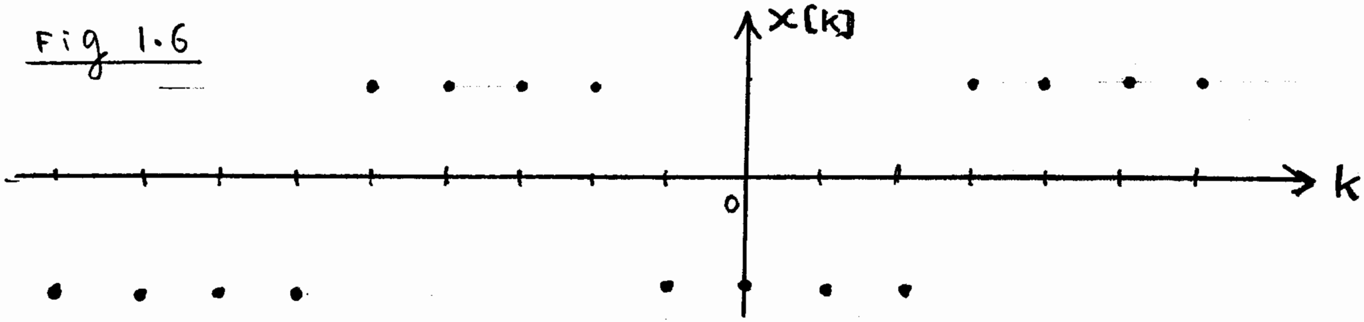
We can also formulate the notion of periodicity for a discrete-time signal:

Definition 1.1.12 A discrete-time signal $x[k]$ is called **periodic** when the following relation holds for some positive integer constant T:

$$x[k] = x[k+T], \quad \text{for all integers } k. \tag{1.8}$$

The constant T is called a **period** of the signal. See Fig. 1.6.

Fig 1.6



Remark 1.1.13 Exactly as in Remark 1.1.9 we see that when a discrete-time signal $x[k]$ is periodic with a period T, then it is also periodic with periods $2T$, $3T, \ldots$ The *smallest* period is called the **fundamental period** of $x[k]$.

As in the continuous-time case, we can ask the following question: Is the sum of two discrete-time periodic signals also periodic? In this case the periods of the two signals are positive integers, hence their ratio is a *rational number*, and hence, exactly as in Fact 1.1.10, we can establish:

Fact 1.1.14 Suppose that $x_1[k]$ is a discrete-time periodic signal with period T_1 and $x_2[k]$ is a discrete-time periodic signal with period T_2. Then $x[k] \triangleq x_1[k] + x_2[k]$ is a discrete-time periodic signal.

Thus, the sum of two discrete-time periodic signals is always periodic, in contrast to the continuous-time case, where the sum of periodic signals need not be periodic.

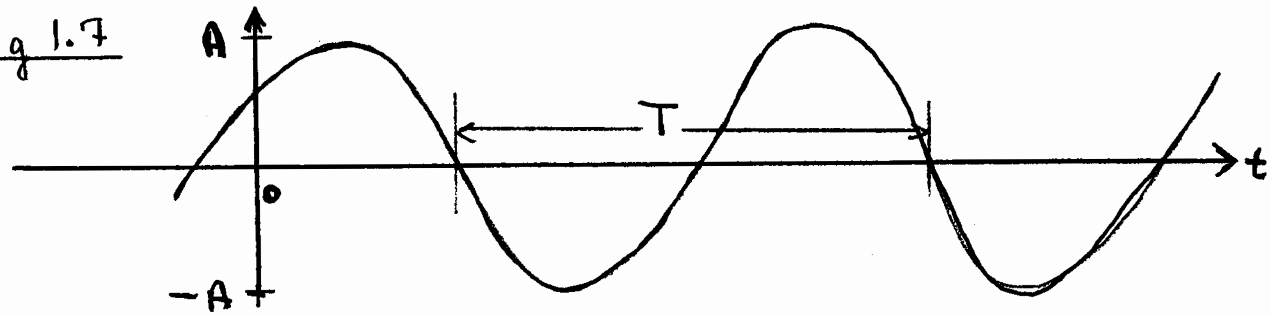
1.1.3 Sinusoidal Signals

A particularly important class of continuous-time periodic signals are the **sinusoids**, which have the following general form:

$$x(t) = A\cos(\omega t + \theta). \tag{1.9}$$

See Fig. 1.7. Here, $A > 0$ is a real constant called the **amplitude**; ω is a real constant called the

Fig 1.7



angular frequency, with units of *radians/sec*; and θ is a real constant called the **phase**, with units of *radians*. Also, put

$$f \triangleq \frac{\omega}{2\pi}, \qquad T \triangleq \frac{1}{f}.$$

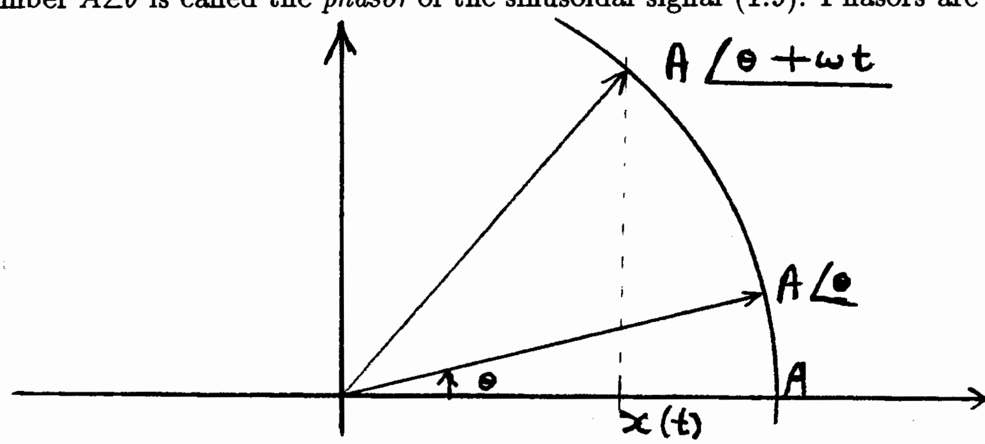
Then f is called the **frequency** with units of $\sec^{-1}$ (or *Hertz*); and T has the units of *seconds*. We see from (1.9) that

$$x(t) = x(t+T), \qquad \text{for all real } t,$$

so that $x(t)$ is periodic with period T. There is a useful geometric interpretation of $x(t)$. Visualize the complex number $A\angle\theta$. If this complex number is rotated by ωt in a counterclockwise direction, then $x(t)$ is the projection of the resulting complex number onto the real axis. See Fig. 1.8.

The complex number $A\angle\theta$ is called the *phasor* of the sinusoidal signal (1.9). Phasors are not used

Fig 1.8



much in this course, although they are extremely important for the analysis of a.c. circuits. Of much greater importance is the exponential form of the sinusoid (1.9). To establish this, recall the this, recall the *Euler identity*

$$e^{j\alpha} = \cos\alpha + j\sin\alpha, \tag{1.10}$$

which holds for angles α *expressed in radians.* From (1.10) we then get the standard identities

$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}, \qquad \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}, \tag{1.11}$$

which again holds for angles α *expressed in radians.* Then, from (1.9) and (1.11), we get

$$\begin{aligned} x(t) &= \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \\ &= \frac{A}{2}e^{j\theta}e^{j\omega t} + \frac{A}{2}e^{-j\theta}e^{-j\omega t} \end{aligned}$$

thus

$$x(t) = De^{j\omega t} + D^{*}e^{-j\omega t}, \tag{1.12}$$

where

$$D \triangleq \frac{A}{2}e^{j\theta} \qquad \text{and} \qquad D^{*} \triangleq \frac{A}{2}e^{-j\theta} \tag{1.13}$$

is the *complex conjugate* of D. We call (1.12) the **exponential form** of the sinusoid (1.9). The usefulness of the exponential form (1.12) derives from the fact that the exponential functions in (1.12) are much easier to handle than the cosine function in (1.9).

1.1.4 Generalized Sinusoids

There is an important generalization of the sinusoid, called the **generalized sinusoid**, which has the following form:

$$x(t) = Ae^{\sigma t}\cos(\omega t + \theta). \tag{1.14}$$

See Fig. 1.9.

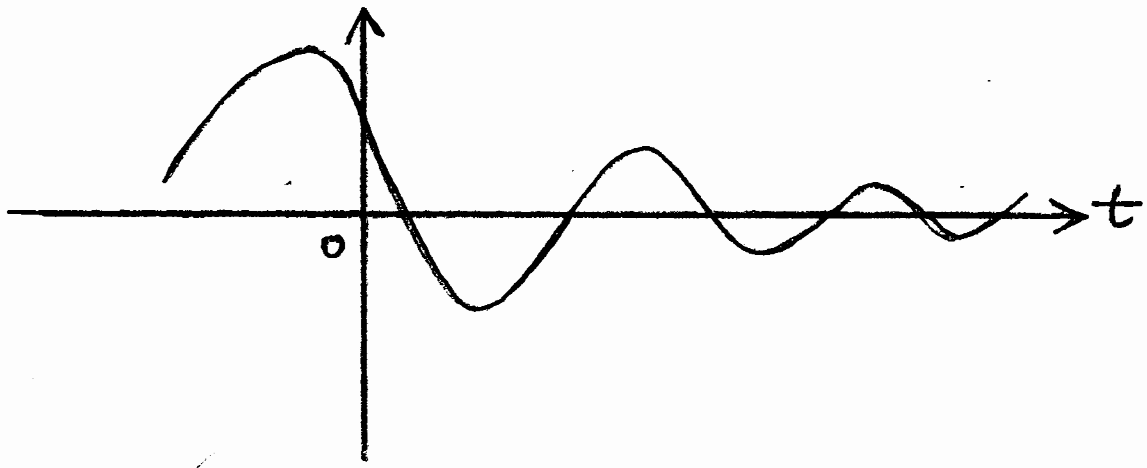


Fig. 1.9.

Here, A, ω and θ are just as in (1.9), and σ is a real constant. Clearly $x(t)$ cannot be periodic unless $\sigma = 0$, in which case the generalized sinusoid (1.14) reduces the the sinusoid (1.9). Combining (1.14) and (1.12) we clearly get

$$x(t) = e^{\sigma t}\left[De^{j\omega t} + D^*e^{-j\omega t}\right],$$

thus

$$x(t) = De^{st} + D^*e^{s^*t}, \tag{1.15}$$

where $s \triangleq \sigma + j\omega$ is called the *complex frequency*, and D, D^*, are given by (1.13). This is called the **exponential form** of the generalized sinusoid (1.14), and will be particularly useful when we look at *Laplace Tranformations* of continuous-time signals.

1.1.5 Energy and Average Power of Signals

Definition 1.1.15 Suppose that $x(t)$ is a continuous-time signal. Then the quantity

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

is called the **energy** of the signal. When $E < \infty$ then $x(t)$ is called an **energy** signal.

A precisely analogous definition can be formulated in the discrete-time case:

Definition 1.1.16 Suppose that $x[k]$ is a discrete-time signal. Then the quantity

$$E \triangleq \sum_{k=-\infty}^{\infty} |x[k]|^2$$

is called the **energy** of the signal. When $E < \infty$ then $x[k]$ is called an **energy** signal.

Remark 1.1.17 The quantity E often has very little to do with energy in the physical sense of the word. For example, if the discrete-time signal $x[k]$ represents the monthly rate of inflation, then the quantity E in Definition 1.1.16 clearly has nothing to do with energy at all! Throughout this course, the word "energy" in the preceding definitions will be interpreted as a quantity which measures the *size* of a signal, and will usually not have any physical significance.

Example 1.1.18 Determine the energy of the following signals:

$$(i) \quad x(t) = e^{-t}, \qquad \text{and} \qquad (ii) \quad x(t) = \begin{cases} 0 & \text{when } t < 0, \\ e^{-t} & \text{when } t \geq 0. \end{cases}$$

(i) Clearly

$$E = \int_{-\infty}^{\infty} e^{-2t} \, dt = +\infty.$$

(ii) Here

$$E = \int_{0}^{\infty} e^{-2t} \, dt = \frac{1}{2}.$$

Definition 1.1.19 Suppose that $x(t)$ is a continuous-time signal. Then the quantity

$$P(a) \triangleq \frac{1}{2a} \int_{-a}^{a} |x(t)|^2 \, dt, \qquad (a > 0),$$

is the **power** of the signal over the interval $-a \leq t \leq a$. Moreover, if the limit

$$P \triangleq \lim_{a \to \infty} P(a)$$

exists, then P is called the **average power** of the signal, and $x(t)$ is called a **power signal** when $0 < P < \infty$.

Definition 1.1.20 Suppose that $x[k]$ is a discrete-time signal. Then the quantity

$$P(n) \triangleq \frac{1}{2n+1} \sum_{k=-n}^{n} |x[k]|^2, \qquad (n = 1, 2, \ldots)$$

is the **power** of the signal over the interval $-n \leq k \leq n$. Moreover, if the limit

$$P \triangleq \lim_{n \to \infty} P(n)$$

exists, then P is called the **average power** of the discrete-time signal, and $x[k]$ is called a **power signal** when $0 < P < \infty$.

Example 1.1.21 Calculate the energy and average power of the continuous-time signal

$$x(t) = Au(t).$$

Then

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{0}^{+\infty} A^2 \, dt = +\infty.$$

For the average power we have

$$P(a) = \frac{1}{2a} \int_{-a}^{a} |x(t)|^2 \, dt = \frac{1}{2a} \int_{0}^{a} A^2 \, dt = \frac{A^2}{2},$$

thus

$$P \triangleq \lim_{a \to \infty} P(a) = \frac{A^2}{2}.$$

Example 1.1.22 Find the energy and average power of the continuous-time signal

$$x(t) = Ae^{-\alpha t}u(t),$$

where A and $\alpha > 0$ are real constants. For the energy:

$$E = \int_{-\infty}^{+\infty} x^2(t) \, dt = \int_{0}^{+\infty} A^2 e^{-2\alpha t} \, dt = \frac{A^2}{2\alpha}.$$

For the average power we have

$$P(a) = \frac{1}{2a} \int_{-a}^{a} x^2(t)\, dt = \frac{A^2}{2a} \int_{0}^{a} e^{-2\alpha t}\, dt = \frac{A^2}{4a\alpha} \left[1 - e^{-2a\alpha}\right].$$

Thus,

$$P = \lim_{a \to \infty} \frac{A^2}{4a\alpha} \left[1 - e^{-2a\alpha}\right] = 0.$$

Remark 1.1.23 The average power for a periodic continuous-time signal $x(t)$ with period T is always given by

$$P = \frac{1}{T} \int_0^T |x(t)|^2\, dt. \tag{1.16}$$

To see this, note that, since $x(t)$ is periodic with period T, it follows that $|x(t)|^2$ is also periodic with period T, hence periodic with period nT, for each $n = 1, 2, \ldots$ (recall Remark 1.1.9). Thus, from Remark 1.1.11, we have

$$\int_{-nT/2}^{nT/2} |x(t)|^2\, dt = \int_0^{nT} |x(t)|^2\, dt = \sum_{k=1}^{n} \int_{(k-1)T}^{kT} |x(t)|^2\, dt = n \int_0^T |x(t)|^2\, dt, \tag{1.17}$$

where the last equality in (1.17) follows because $|x(t)|^2$ is periodic with period T, so that

$$\int_{(k-1)T}^{kT} |x(t)|^2\, dt = \int_0^T |x(t)|^2\, dt, \quad k = 1, 2, \ldots, n.$$

From (1.17) we get

$$\frac{1}{nT} \int_{-nT/2}^{nT/2} |x(t)|^2\, dt = \frac{1}{T} \int_0^T |x(t)|^2\, dt, \quad n = 1, 2, \ldots,$$

that is, for $a(n) \triangleq nT/2$, one has

$$\frac{1}{2a(n)} \int_{-a(n)}^{a(n)} |x(t)|^2\, dt = \frac{1}{T} \int_0^T |x(t)|^2\, dt, \quad n = 1, 2, \ldots, \tag{1.18}$$

Since $a(n) \to \infty$ as $n \to \infty$, we see from Definition 1.1.19 and (1.18) that the average power is given by

$$P = \lim_{n \to \infty} \frac{1}{2a(n)} \int_{-a(n)}^{a(n)} |x(t)|^2\, dt = \frac{1}{T} \int_0^T |x(t)|^2\, dt,$$

as required.

Example 1.1.24 Find the energy and power of the sinusoid

$$x(t) = A\cos(\omega t + \theta).$$

For the energy we have

$$E = \int_{-\infty}^{+\infty} x^2(t)\, dt = A^2 \int_{-\infty}^{+\infty} \cos^2(\omega t + \theta)\, dt = +\infty.$$

For the average power, observe that the period of $x(t)$ is $T = 2\pi/\omega$. Thus, by Remark 1.1.23:

$$\begin{aligned} P &= \frac{1}{T}\int_0^T x^2(t)\, dt \\ &= \frac{A^2\omega}{2\pi}\int_0^{2\pi/\omega} \cos^2(\omega t+\theta)\, dt \\ &= \frac{A^2}{2\pi}\int_\theta^{2\pi+\theta} \cos^2(\tau)\, d\tau \qquad \text{for } \tau = \omega t + \theta \text{, thus } d\tau = \omega dt \\ &= \frac{A^2}{2\pi}\int_0^{2\pi} \cos^2(\tau)\, d\tau \end{aligned} \tag{1.19}$$

where the final equality follows by Remark 1.1.11 and that fact that $\cos^2(\tau)$ is periodic with a period 2π. Now, recalling the useful identity

$$\cos^2(\tau) = \frac{1}{2}\left[1 + \cos(2\tau)\right]$$

we get

$$\int_0^{2\pi} \cos^2(\tau)\, d\tau = \frac{1}{2}\int_0^{2\pi} \left[1+\cos(2\tau)\right]\, d\tau = \pi, \tag{1.20}$$

where the second equality follows since

$$\int_0^{2\pi} \cos(2\tau)\, d\tau = 0.$$

Putting (1.20) into (1.19) gives

$$P = \frac{A^2}{2}.$$

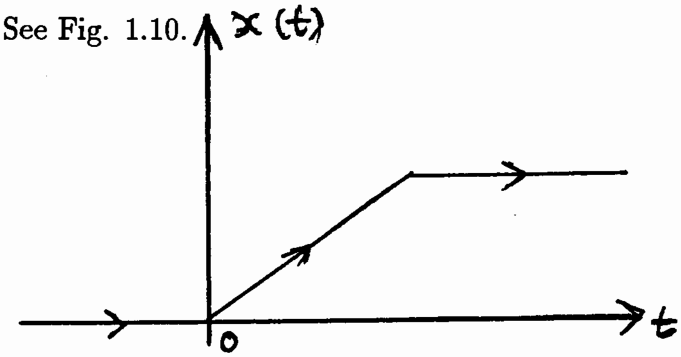
1.1.6 Transformations of the Time Variable

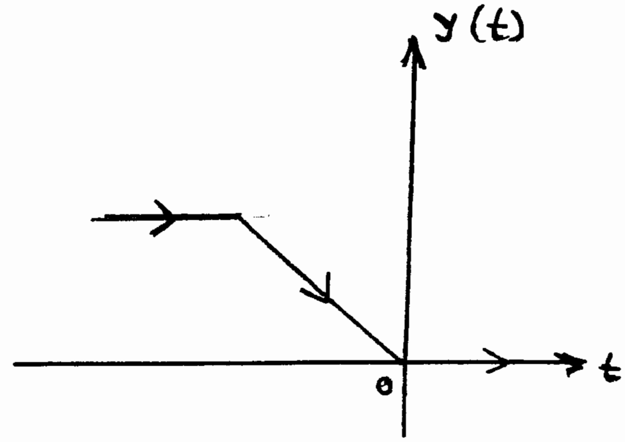
Let $x(t)$ be a continuous-time signal. We can transform this into a variety of other continuous-time signals as follows:

(a) Time Inversion: Put

$$y(t) \triangleq x(-t).$$

See Fig. 1.10.

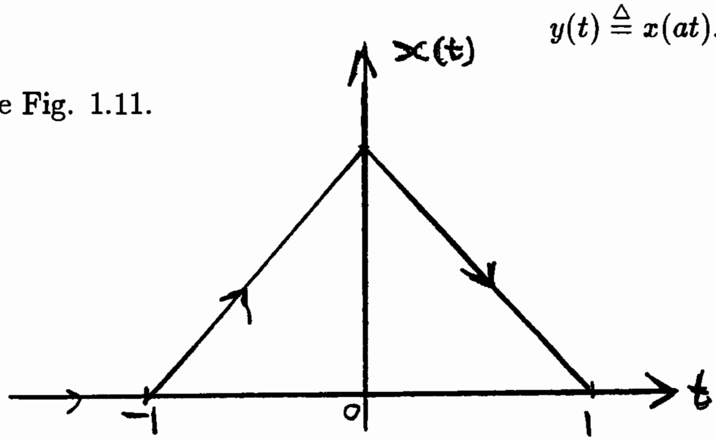


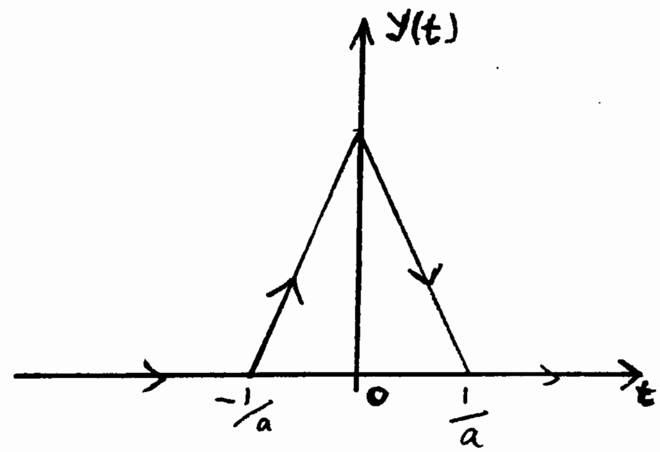


(b) Time scaling: For a real constant $a > 0$ put

$$y(t) \triangleq x(at).$$

See Fig. 1.11.

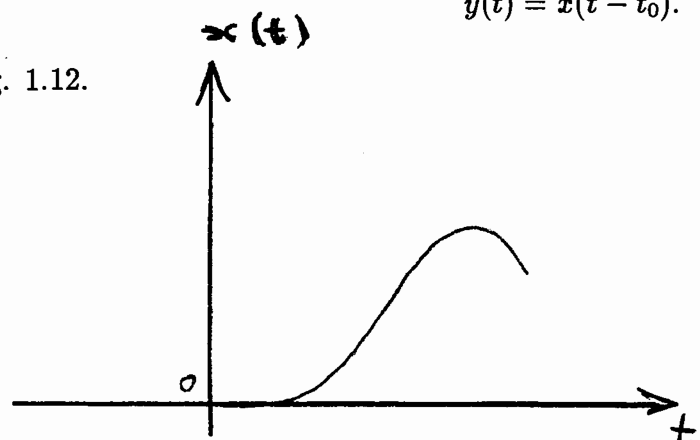


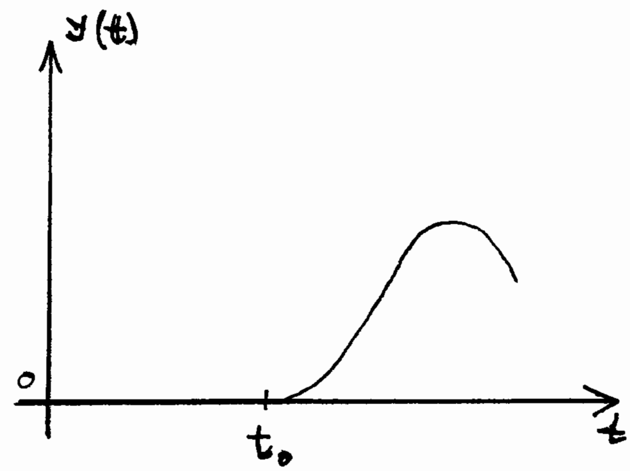


(c) Time shift: For a real constant t_0 put

$$y(t) \triangleq x(t - t_0).$$

See Fig. 1.12.





Remark 1.1.25 Suppose that $x(t)$ is the audio signal on a tape-recording. Then $y(t)$ given by the time inversion (a) is the signal when the tape is played *backwards.* If $a = 2$ in the time scaling (b) then $y(t)$ is the signal when the tape is played forward at *twice* the normal speed, while if $a = 1/2$ then $y(t)$ is the signal when the tape is played at *half* the normal speed.

One can formulate precisely analogous ideas for a discrete-time signal $x[k]$:

(a) Time Inversion: Put

$$y[k] \triangleq x[-k].$$

(b) Time scaling: For an integer constant $a > 0$ put

$$y[k] \triangleq x[an].$$

(c) Time shift: For an integer constant k_0 put

$$y[k] \triangleq x[k - k_0].$$

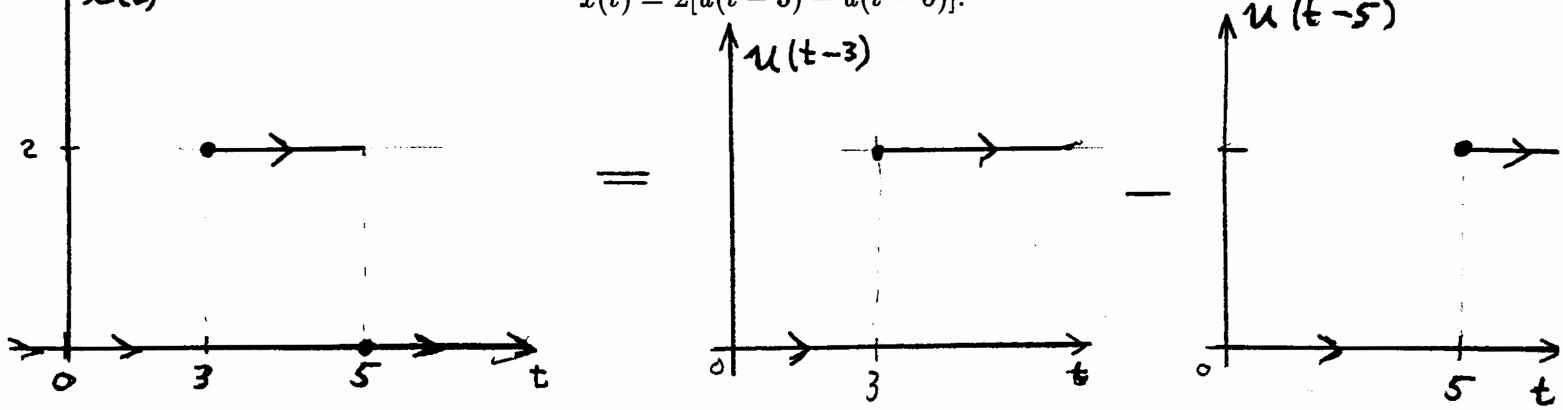
Using transformations of the time variable enables us to use simple signals such as step functions to synthesize more complicated signals. The next examples show this:

Example 1.1.26 Suppose that $x(t)$ is given by

$$x(t) \triangleq \begin{cases} 2 & \text{when } 3 \le t < 5, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$x(t) = 2[u(t-3) - u(t-5)].$$

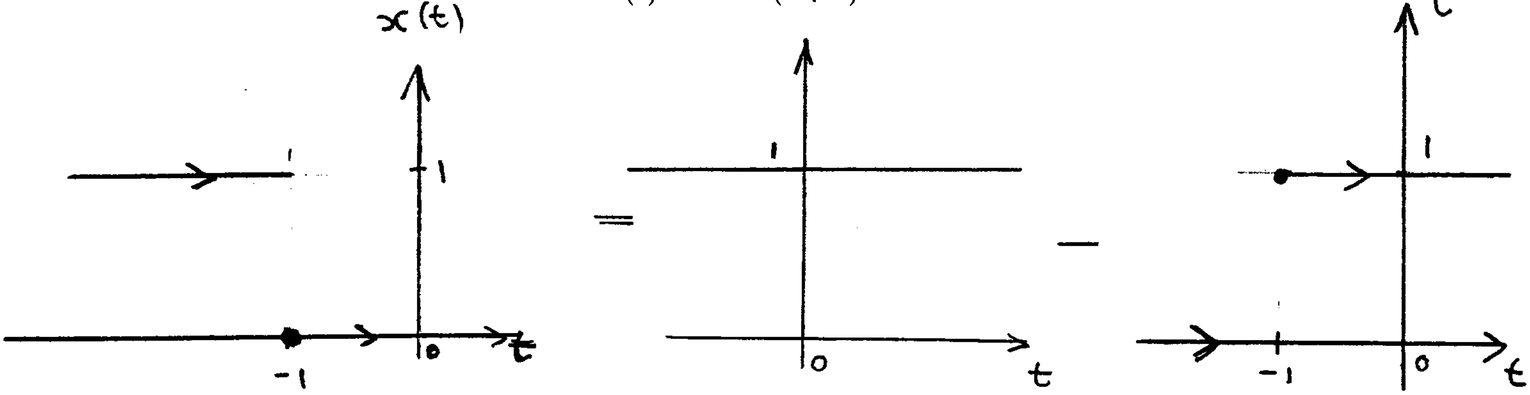


Example 1.1.27 Suppose that $x(t)$ is given by

$$x(t) \triangleq \begin{cases} 1 & \text{when } t < -1, \\ 0 & \text{otherwise.} \end{cases}$$

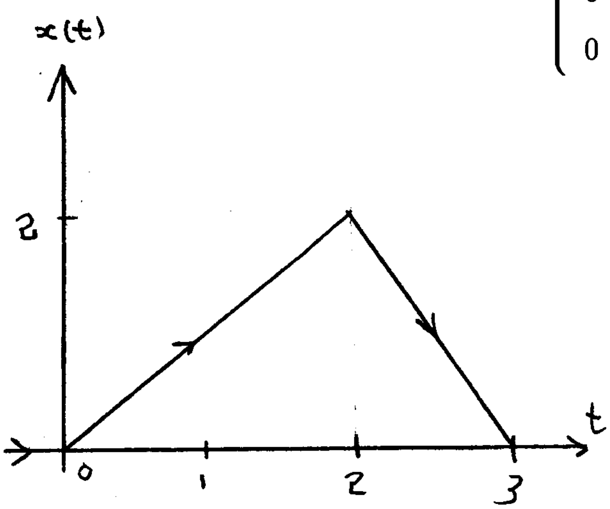
Then

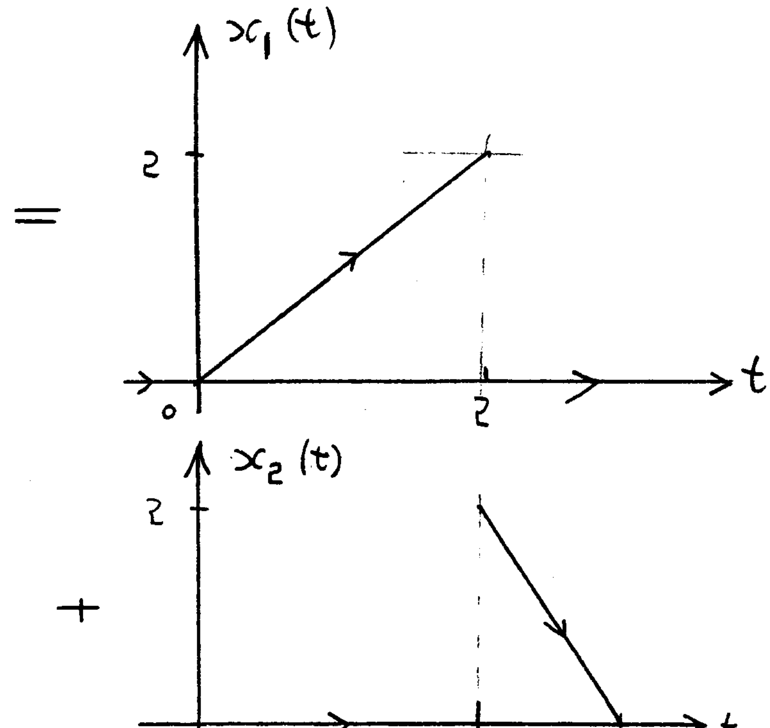
$$x(t) = 1 - u(t+1).$$



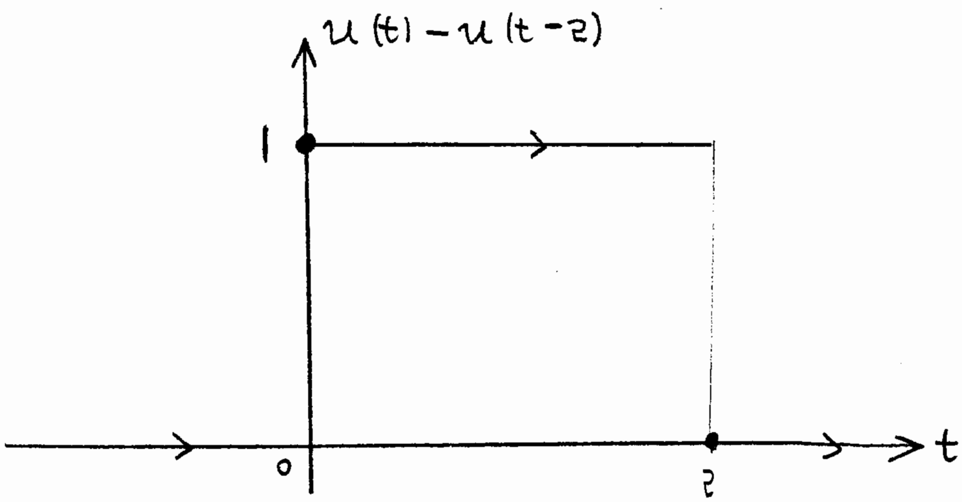
Example 1.1.28 Suppose that $x(t)$ is given by

$$x(t) \triangleq \begin{cases} 0 & \text{when } t < 0, \\ t & \text{when } 0 \leq t < 2, \\ 6-2t & \text{when } 2 \leq t < 3, \\ 0 & \text{when } t \geq 3 \end{cases}$$



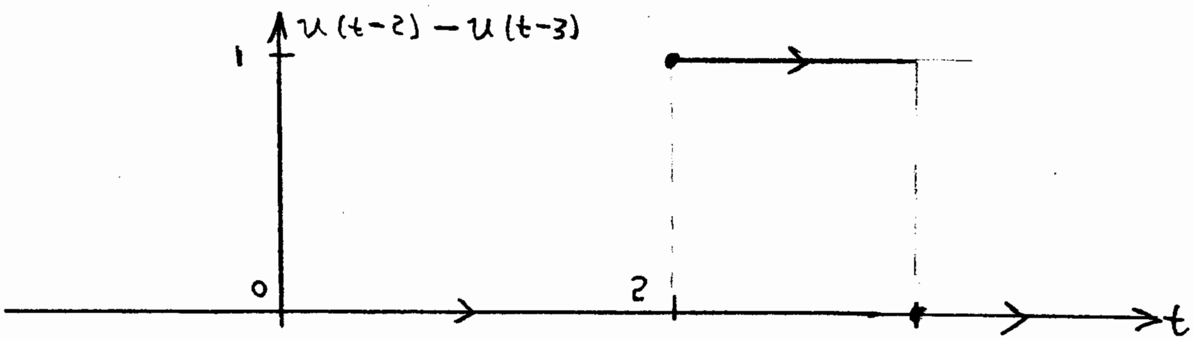


Now clearly $x_1(t)$ is the product of the signal t with the following signal: Thus



$$x_1(t) = t[u(t) - u(t-2)].$$

Similarly, $x_2(t)$ is the product of the signal $6 - 2t$ with the following signal: Thus



$$x_2(t) = (6 - 2t)[u(t-2) - u(t-3)].$$

It follows that

$$x(t) = t[u(t) - u(t-2)] + (6 - 2t)[u(t-2) - u(t-3)].$$

1.1.7 Even and Odd Functions

Definition 1.1.29 A continuous-time signal $x(t)$ is an **even function** when

$$x(-t) = x(t),$$

and is an **odd function** when

$$x(-t) = -x(t).$$

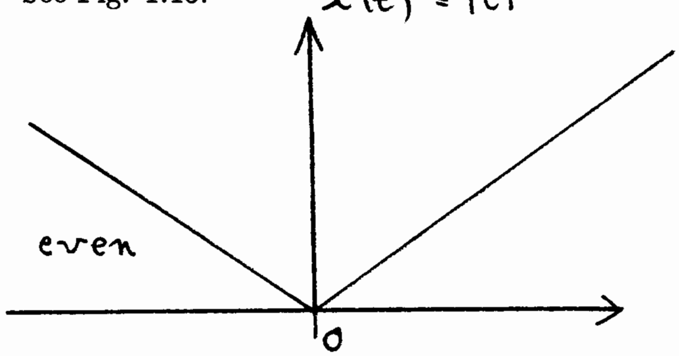
Likewise, a discrete-time signal $x[k]$ is an **even function** when

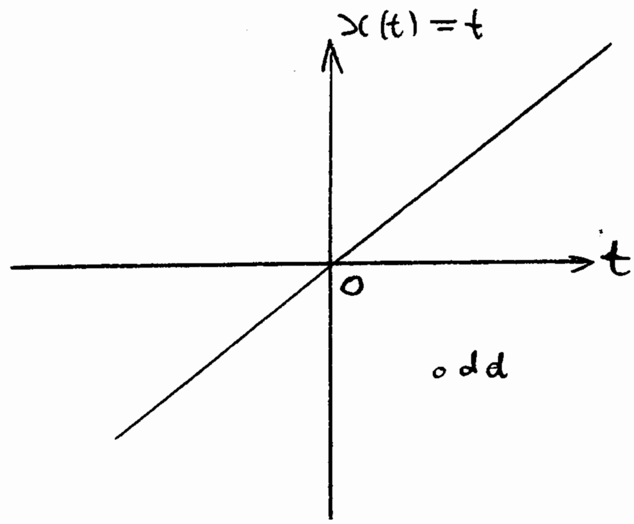
$$x[-k] = x[k],$$

and is an **odd function** when

$$x[-k] = -x[k].$$

See Fig. 1.13.





Remark 1.1.30 The signals

$$x(t) = t^2 + 1 \qquad \text{and} \qquad x(t) = \cos(\omega t)$$

are clearly examples of even functions, while the signals

$$x(t) = t \qquad \text{and} \qquad x(t) = \sin(\omega t)$$

are examples of odd functions, and the signals

$$x(t) = 1 + t \qquad \text{and} \qquad x(t) = \cos(\omega t + \pi/4)$$

are neither even nor odd functions.

Remark 1.1.31 If $x(t)$ is a continuous-time odd signal then clearly we have $x(0) = 0$. Similarly if $x[k]$ is a discrete-time odd signal.

Remark 1.1.32 Every continuous-time signal $x(t)$ can be written as the sum of an even continuous-time signal and an odd continuous-time signal. Indeed, we can always put

$$x(t) = x_e(t) + x_o(t),$$

where

$$x_e(t) \triangleq \frac{1}{2}\left[x(t) + x(-t)\right], \qquad x_o(t) \triangleq \frac{1}{2}\left[x(t) - x(-t)\right],$$

and it is trivially seen that $x_e(t)$ is an even function and $x_o(t)$ is an odd function. Likewise, every discrete-time signal $x[k]$ can be written as the sum of an even function and an odd function, namely

$$x[k] = x_e[k] + x_o[k],$$

where

$$x_e[k] \triangleq \frac{1}{2}\left[x[k] + x[-k]\right], \qquad x_o[k] \triangleq \frac{1}{2}\left[x[k] - x[-k]\right],$$

are even and odd functions respectively.

Remark 1.1.33 For even functions $x(t)$ we have

$$\int_{-a}^{a} x(t)dt = 2\int_{0}^{a} x(t)dt,$$

and for odd functions we have

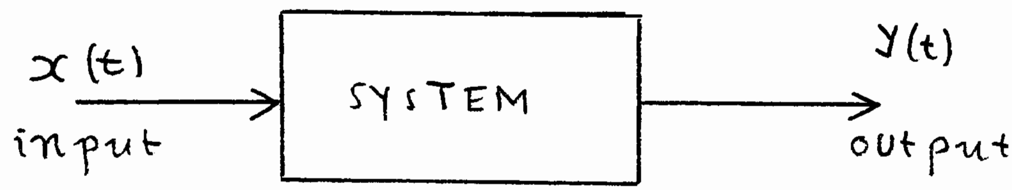
$$\int_{-a}^{a} x(t)dt = 0.$$

These simplifying properties will be useful when we deal with Fourier series.

1.2 Systems

Having formulated the notion of signals in the preceding section, we come now to the other main idea of this course, namely a system:

Definition 1.2.1 A **system** is a device which transforms an **input signal** into an **output signal**, and the output signal is called the *response of the system* to the input signal. The system is called



a **continuous-time system** when it transforms a continuous-time input signal into a continuous-time output signal, and is called a **discrete-time system** when it transforms a discrete-time input signal into a discrete-time output signal.

Remark 1.2.2 There is an important class of systems, called **hybrid systems**, which either transform a continuous-time signal into a discrete-time signal (e.g. A/D converters) or transform a discrete-time signal into a continuous-time signal (e.g. D/A converters). In this course we mainly study continuous-time systems and discrete-time systems.

Example 1.2.3 (a) The system is a resistor R, the continuous-time input signal is the current $i(t)$ through R, and the continuous-time output signal is the voltage $v(t)$ across R:

$$v(t) = Ri(t).$$

This is a continuous-time system.

(b) The system is a capacitor C, the continuous-time input signal is the current $i(t)$ through C, and the continuous-time outout signal is the voltage $v(t)$ across C:

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau.$$

This is a continuous-time system.

(c) The system is a "moving average filter" with discrete-time input signal $x[k]$ and discrete-time output signal $y[k]$ given by

$$y[k] = \frac{1}{3}\left(x[k] + x[k-1] + x[k-2]\right).$$

This is a discrete-time system.

The concept of a system is enormously broad. In order to be able to study systems we must take advantage of various simplifying properties which many systems have. These properties are

Linearity

Time invariance

Causality.

We next formulate these properties precisely:

1.2.1 Linear Systems

An especially important class of systems has the following linearity or superposition property:

Definition 1.2.4 A continuous-time system is **linear** when the following **superposition** property holds: If $y_1(t)$ is the response of the system to the input signal $x_1(t)$ and $y_2(t)$ is the response of the system to the input signal $x_2(t)$ then:

1. $y_1(t) + y_2(t)$ is the response of the system to the input signal $x_1(t) + x_2(t)$.
2. $cy_1(t)$ is the response of the system to the input signal $cx_1(t)$ for any complex constant c.

The notion of linearity for discrete-time systems is formulated in exactly the same way.

Example 1.2.5 Let $x(t)$ be the input signal and $y(t)$ be the output signal of a continuous-time system. Then the following systems are clearly linear:

$$y(t) = 2x(t), \qquad\qquad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$

On the other hand, the following system *fails* to be linear:

$$y(t) = x^2(t).$$

One can just as easily come up with similar examples in the discrete-time case.

Remark 1.2.6 Systems which fail to be linear are called **nonlinear** systems.

Remark 1.2.7 Linear systems have the important property that the response to an input which is the *zero signal* (recall Remark 1.1.5) is likewise the zero signal. To see this consider the case of a continuous-time linear system, and suppose that $y(t)$ is the response to the input $x(t)$ which is the zero signal. Then, by linearity, $2y(t)$ is the response to the input signal $2x(t)$. But $x(t)$ and $2x(t)$ are clearly identical (since $x(t)$ is the zero signal), thus the corresponding responses $y(t)$ and $2y(t)$ are also identical, namely

$$y(t) = 2y(t) \qquad \text{for all real } t.$$

This shows that $y(t) = 0$ for all t, as required. An identical argument works for discrete-time systems.

Example 1.2.8 Consider the continuous-time system for which the input $x(t)$ and output $y(t)$ are related by

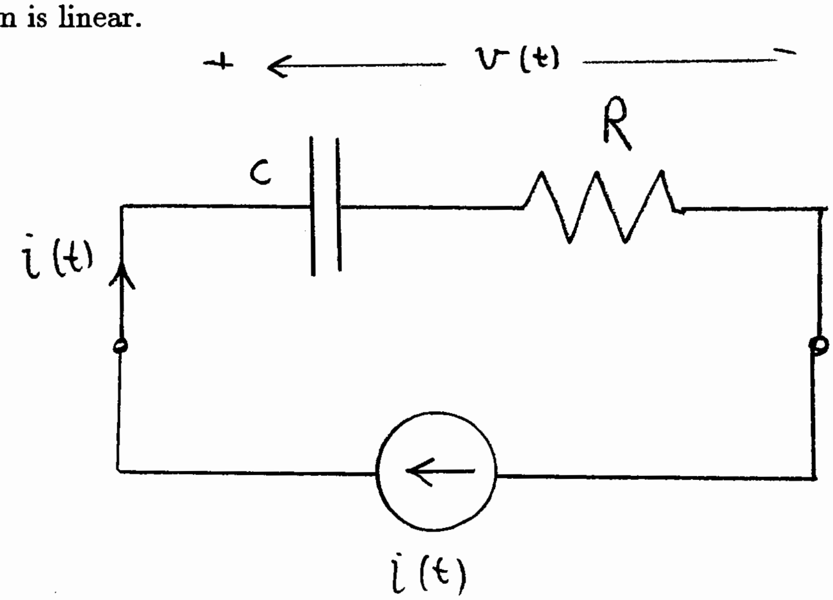
$$y(t) = 1 + x(t).$$

Clearly this *cannot* be a linear system since the response to the zero input signal is the *non-zero* output signal $y(t) = 1$ for all t.

Example 1.2.9 A system is a series combination of a resistor R and a capacitor C. The input signal is the current $i(t)$ through the series combination of R and C, and the output signal is the voltage $v(t)$ across the series combination of R and C. Clearly

$$v(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau,$$

and the system is linear.



1.2.2 Time Invariance

Informally a system is time invariant when its properties do not change with time. We make this idea precise as follows:

Definition 1.2.10 A continuous-time system is **time invariant** when a time-shift in the input signal causes the same time-shift in the output signal. Specifically, if $y_1(t)$ is the system response to an input signal $x_1(t)$ then, for any real t_0, the response of the system to the input signal $x_2(t)$ given by

$$x_2(t) \triangleq x_1(t - t_0)$$

is the signal $y_2(t)$ given by

$$y_2(t) = y_1(t - t_0).$$

Likewise, a discrete-time system is **time invariant** when, if $y_1[k]$ is the system response to an input signal $x_1[k]$ then, for any integer k_0, the response of the system to the input signal $x_2[k]$ given by

$$x_2[k] \triangleq x_1[k - k_0]$$

is the signal $y_2[k]$ given by

$$y_2[k] = y_1[k - k_0].$$

Example 1.2.11 Consider the continuous-time nonlinear system given by

$$y(t) = x^2(t).$$

To check for time invariance let $y_1(t)$ be the response to an input signal $x_1(t)$, thus

$$y_1(t) = x_1^2(t). \tag{1.21}$$

Fix some t_0 and define the input signal

$$x_2(t) \triangleq x_1(t - t_0).$$

Then the corresponding output is the signal

$$y_2(t) = x_2^2(t) = x_1^2(t - t_0) \tag{1.22}$$

Moreover from (1.21) we have

$$y_1(t - t_0) = x_1^2(t - t_0). \tag{1.23}$$

Comparing (1.22) with (1.23) shows that $y_2(t) = y_1(t - t_0)$, thus the system is time invariant.

Example 1.2.12 Consider the discrete-time system given by

$$y[k] = kx[k].$$

Let $y_1[k]$ be the response to the input signal $x_1[k]$ namely

$$y_1[k] = kx_1[k]. \tag{1.24}$$

Fix some integer k_0 and let $y_2[k]$ be the response to the input signal given by

$$x_2[k] \triangleq x_1[k - k_0],$$

namely

$$y_2[k] = kx_2[k] = kx_1[k - k_0]. \tag{1.25}$$

From (1.24) we have

$$y_1[k - k_0] = (k - k_0)x_1[k - k_0] \neq kx_1[k - k_0],$$

thus from (1.25) we see $y_1[k - k_0] \neq y_2[k]$. It follows that the system fails to be time invariant.

Remark 1.2.13 Systems which fail to be time invariant are called **time varying.**

Example 1.2.14 Consider the linear continuous-time system seen in Example 1.2.9 and let $v_1(t)$ be the response to the input signal $i_1(t)$, namely

$$v_1(t) = Ri_1(t) + \frac{1}{C}\int_{-\infty}^{t} i_1(\tau)\, d\tau. \tag{1.26}$$

Fix some t_0 and let $v_2(t)$ be the response to the input signal $i_2(t)$ defined by

$$i_2(t) \triangleq i_1(t - t_0),$$

thus

$$\begin{aligned} v_2(t) &= Ri_2(t) + \frac{1}{C}\int_{-\infty}^{t} i_2(\tau)\, d\tau \\ &= Ri_1(t - t_0) + \frac{1}{C}\int_{-\infty}^{t} i_1(\tau - t_0)\, d\tau. \end{aligned} \tag{1.27}$$

Taking $v = \tau - t_0$ we see that

$$\int_{-\infty}^{t} i_1(\tau - t_0) d\tau = \int_{-\infty}^{t-t_0} i_1(v)\, dv,$$

and therefore from (1.27),

$$v_2(t) = Ri_1(t - t_0) + \frac{1}{C}\int_{-\infty}^{t-t_0} i_1(\tau)d\tau. \tag{1.28}$$

Also, from (1.26) we must have

$$v_1(t - t_0) = Ri_1(t - t_0) + \frac{1}{C}\int_{-\infty}^{t-t_0} i_1(\tau)\, d\tau. \tag{1.29}$$

Comparing (1.28) and (1.29) gives $v_2(t) = v_1(t - t_0)$, showing that the system is time invariant.

Remark 1.2.15 A useful consequence of linearity and time invariance is the following: suppose that $y_u(t)$ is the response of a continuous-time linear time invariant system to a unit step input $u(t)$, and we must find the response of the system to an input signal $x(t)$ given by

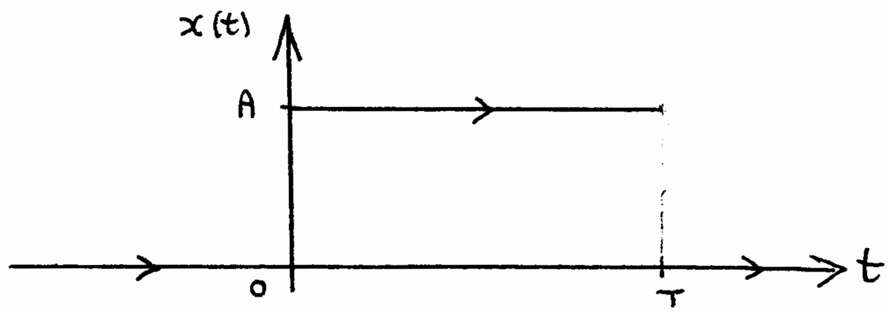
$$x(t) \triangleq \begin{cases} 0 & \text{when } t < 0, \\ A & \text{when } 0 \le t \le T, \\ 0 & \text{when } t \ge T. \end{cases}$$

Clearly we have

$$x(t) = A[u(t) - u(t - T)].$$

See Fig. 1.14. By time invariance of the system we know that $y_u(t-T)$ is the response to the input

Fig 1.14



signal $u(t - T)$, hence by linearity we know that $-y_u(t - T)$ is the response to the input signal $-u(t-T)$. Thus, again by linearity, we see that $y_u(t) - y_u(t-T)$ is the response to the input signal $u(t) - u(t - T)$. Again by linearity it follows that

$$A[y_u(t) - y_u(t - T)]$$

is the response to the input signal $x(t)$.

1.2.3 Causality

Informally a system is causal when it does not "look ahead" in time and anticipate values of the input signal *after* instant t when deciding the value of the output signal at the instant t. This informal idea is precisely formulated as follows:

Definition 1.2.16 A continuous-time system is **causal** under the following conditions: when $x_1(t)$ and $x_2(t)$ are input signals with corresponding outputs $y_1(t)$ and $y_2(t)$, and such that

$$x_1(t) = x_2(t), \qquad \text{for all } t \leq t_0,$$

for a given instant t_0, it follows that

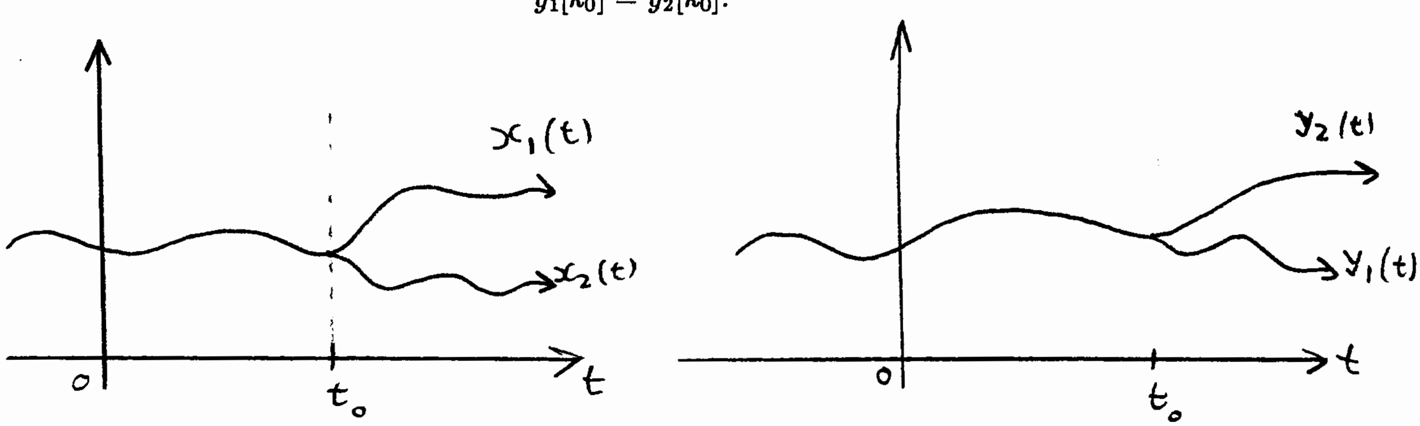
$$y_1(t_0) = y_2(t_0).$$

See Fig. 1.15. Likewise, a discrete-time system is **causal** under the following conditions: when $x_1[k]$ and $x_2[k]$ are input signals with corresponding outputs $y_1[k]$ and $y_2[k]$, and such that

$$x_1[k] = x_2[k], \qquad \text{for all } k \leq k_0,$$

for a given instant k_0, it follows that

$$y_1[k_0] = y_2[k_0].$$



Remark 1.2.17 Thus, causality in the continuous-time context means that whenever two input signals agree over their entire history up until some instant t_0, then the corresponding outputs agree at the same instant t_0 and hence of course also agree at all earlier instants $t \leq t_0$. This in turn means that the output of a causal system depends at instant t_0 *only* on the input signal $x(t)$ for $t \leq t_0$, and *in no way* depends on the input signal $x(t)$ for $t > t_0$. An exacty analogous interpretation is valid for the discrete-time case.

Example 1.2.18 Consider the RC system of Example 1.2.9 where the output signal (the voltage $v(t)$) is realted to the input signal (the current $i(t)$) by

$$v(t_0) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t_0} i(\tau)\, d\tau.$$

It is trivially clear that this is a causal system.

Example 1.2.19 Consider a discrete-time system with the output signal $y[k]$ related to the input signal $x[k]$ by

$$y[k] = x[k] + x[k+1] + x[k+2] + x[k+3] + x[k+4].$$

This is clearly *not* a causal system. Systems such as this which fail to be causal are called **noncausal** or **anticipative** systems.

Remark 1.2.20 Both causal and noncausal systems occur widely in practice. Systems which arise in the area of feedback control systems are usually causal, while on the other hand noncausal systems occur frequently in signal processing. In this course we shall encounter causal and noncausal systems, although the main emphasis will be on causal systems.

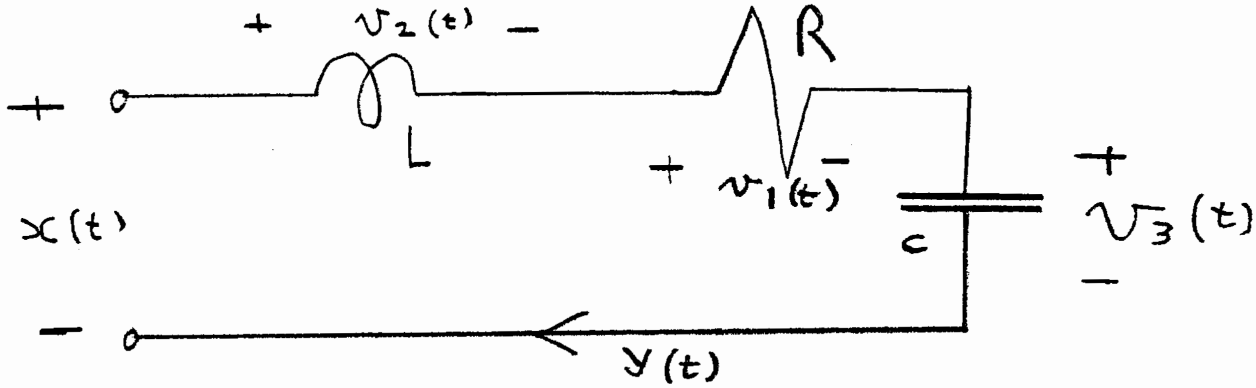
1.2.4 System Models: Electrical Networks

We have dealt with the notion of a system rather abstractly as a device which converts an input signal into an output signal. The question arises: how does one progress from a given physical system, such as an electrical network, a mechanical system, or an economic system, to a mathematical relation which defines how the system transforms the input signal into the output signal? Such a relation is called a **system model**. Here we are going to derive system models for some *electrical networks*. The basic tools for deriving these models are the Kirchhoff current and voltage laws, and the terminal characteristics of devices such as resistors, capacitors, inductors, diodes, and transistors etc. In the following examples we shall derive system models in the form of differential equations:

Example 1.2.21 In the electrical network shown the input signal is the applied voltage $x(t)$ and the output signal is the the current $y(t)$. By KVL around the circuit:

$$x(t) = v_1(t) + v_2(t) + v_3(t).$$

Now by the terminal relations for resistor, inductor and capacitor:



$$\begin{aligned} v_1(t) &= Ry(t), \\ v_2(t) &= L\frac{dy(t)}{dt}, \\ v_3(t) &= \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau, \end{aligned}$$

thus

$$x(t) = Ry(t) + L\frac{dy(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau,$$

and upon taking derivatives:

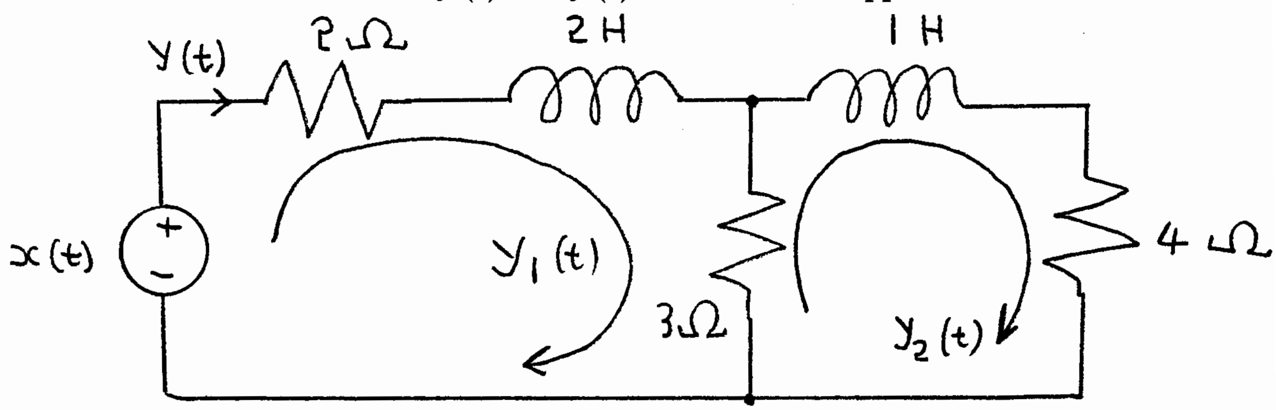
$$L\frac{d^2y(t)}{dt^2} + R\frac{dy(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}.$$

In deriving the system model from a given physical system we often end up with system variables besides the input signal and output signal, and these variables must be eliminated to get a system model which involves *only* the input and output signals. A useful technique for doing this when dealing with differential equations is to introduce the D-operator by writing

$$Dz(t) \triangleq \frac{dz(t)}{dt}$$

for a given signal $z(t)$, and treating D as if it were a *number*. An example best illustrates the idea:

Example 1.2.22 In the network shown the input signal is the applied voltage $x(t)$ and the output signal is the current $y(t)$ through the 2-ohm resistor. A system model relating $x(t)$ with $y(t)$ must be derived. Introduce mesh currents $y_1(t)$ and $y_2(t)$ as shown. It is apparent that



$$y(t) = y_1(t).$$

By KVL around loop for y_1:

$$x(t) = 2y_1(t) + 2Dy_1(t) + 3(y_1(t) - y_2(t)),$$

thus

$$(2D+5)y_1(t) - 3y_2(t) = x(t). \tag{1.30}$$

By KVL around loop for y_2:

$$4y_2(t) + 3(y_2(t) - y_1(t)) + Dy_2(t) = 0,$$

thus

$$-3y_1(t) + (D+7)y_2(t) = 0. \tag{1.31}$$

The input signal is $x(t)$ and the output signal is $y(t) = y_1(t)$. To relate $x(t)$ to $y(t)$ we must eliminate the signal $y_2(t)$ from (1.30) and (1.31) by algebraically manipulating these equations, a somewhat lenghty and error-prone method. In fact it is more efficient to procede exactly *as if D were a number* and write (1.30) and (1.31) in matrix form:

$$\begin{bmatrix} (2D+5) & -3 \\ -3 & (D+7) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \end{bmatrix}.$$

Again, regarding D as a number, we can use Cramer's rule to get $y_1(t)$ in terms of $x(t)$:

$$y_1(t) = \frac{\begin{vmatrix} x(t) & -3 \\ 0 & (D+7) \end{vmatrix}}{\begin{vmatrix} (2D+5) & -3 \\ -3 & (D+7) \end{vmatrix}}$$

thus

$$y_1(t) = \frac{(D+7)x(t)}{2D^2 + 19D + 26},$$

or

$$(2D^2 + 19D + 26)y_1(t) = (D+7)x(t).$$

Since $y_1(t) = y(t)$ this can be written as

$$2\frac{d^2y(t)}{dt^2} + 19\frac{dy(t)}{dt} + 26y(t) = \frac{dx(t)}{dt} + 7x(t).$$