29. Jointly Distributed Random Variables

Joint Distribution of Functions of Random Variables [Ross S6.7]

Let X and Y have joint pdf $f_{XY}(x, y)$.

In some examples we computed the distribution of $Z = g(X, Y)$, e.g.

- in Example 23.2 we computed the cdf of $D = \sqrt{X^2 + Y^2}$
- in Example 23.3 we computed the pdf of $Z = X/Y$.

Now, consider

$$Y_1 = g_1(X_1, X_2)$$
$$Y_2 = g_2(X_1, X_2)$$

and we want the joint pdf of Y_1 and Y_2.

Assume that:

- The system of equations

  $$y_1 = g_1(x_1, x_2)$$
  $$y_2 = g_2(x_1, x_2)$$

  can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2:

  $$x_1 = h_1(y_1, y_2)$$
  $$x_2 = h_2(y_1, y_2).$$

- g_1 and g_2 have continuous partial derivates such that

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

Then, the pdf of Y_1 and Y_2 can be shown to be:

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(x_1, x_2) \left| J(x_1, x_2) \right|^{-1} \tag{29.1}$$

where

$$x_1 = h_1(y_1, y_2)$$
$$x_2 = h_2(y_1, y_2).$$

Example 29.1: Let

$$Y_1 = X_1 + X_2$$
$$Y_2 = X_1 - X_2$$

Find the joint pdf $f_{Y_1 Y_2}(y_1, y_2)$ in terms of $f_{X_1 X_2}(x_1, x_2)$.

Solution:

Example 29.2: Let R and Θ be two random variables with joint pdf $f_{R\Theta}(r, \theta)$. Consider the change of variables

$$
\begin{aligned}
X &= R\cos\Theta \\
Y &= R\sin\Theta.
\end{aligned}
$$

Find $f_{R\Theta}(r, \theta)$ in terms of $f_{XY}(x, y)$. [Hard]

Note: This is Problem T9.1; see also textbook Example 6.7b for a different tedious approach.

Solution: