

## 12. Random Variables (rv)

### Bernoulli and Binomial [Ross S4.6]

A) Let

$$p_X(k) = \begin{cases} 1 - p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}$$

with  $0 \leq p \leq 1$ .

Then  $X$  is called **Bernoulli** with parameter  $p$ , denoted  $X \sim \text{Bernoulli}(p)$ .

This random variable models binary conditions:

- coin flip outcome
  - state of a connection
  - preference for/against politician
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B) Consider  $n$  independent trials of  $\text{Bernoulli}(p)$ .

Let  $X = \#$  of ones in the  $n$  trials.

Then  $X$  is called **binomial** with parameters  $n$  and  $p$ , denoted  $X \sim \text{Binomial}(n, p)$ .

*Note:*  $\text{Bernoulli}(p) = \text{Binomial}(1, p)$ .

For  $0 \leq k \leq n$ , there are  $\binom{n}{k}$  ways to get  $k$  ones from  $n$  Bernoulli trials.

Each has probability  $p^k(1-p)^{n-k}$ . So

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & 0 \leq k \leq n \\ 0 & \text{else} \end{cases}$$

*Note:* Since  $X$  must be between 0 and  $n$ :

$$1 = \sum_{k=0}^n p_X(k) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}$$

**Example 12.1:** [Cover if time] A company sells screw in packs of 10. Each screw has a prob. 0.01 of being defective. There is a money-back guarantee if *more* than 1 screw is defective. What is the prob. that a pack will be replaced?

*Solution:*

## Moments of Binomial

Let  $X \sim \text{Binomial}(n, p)$ . Then

$$\left. \begin{aligned} E[X] &= np \\ E[X^2] &= n(n-1)p^2 + np \end{aligned} \right\} \text{Will prove these later}$$

So

$$\begin{aligned} \text{Var}[X] &= E[X^2] - (E[X])^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= np(1-p) \end{aligned}$$

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### Poisson Random Variable [Ross S4.7]

**C)** We say  $X$  is **Poisson** with parameter  $\lambda > 0$ , denoted  $X \sim \text{Poisson}(\lambda)$ , if

$$p_X(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = 0, 1, 2, \dots \\ 0 & \text{else} \end{cases}$$

Note: In Example 9.1 we saw that  $\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 1$ .

The Poisson random variable is an approximation of the binomial random variable when:

- $\lambda = np$  is fixed,
- $n$  is large enough (equivalently  $p$  small enough)

i.e.:  $\text{Poisson}(\lambda)$  is  $\text{Binomial}(n, \lambda/n)$  when  $n \rightarrow \infty$ .

Why? Let  $X \sim \text{Binomial}(n, p)$  with  $p = \lambda/n$ :

$$\begin{aligned}
 p_X(k) &= \frac{n!}{(n-k)! k!} p^k (1-p)^{n-k} \\
 &= \frac{n!}{(n-k)! k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
 &= \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{\lambda^k}{k!} \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^k}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } n \rightarrow \infty : \quad & \frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-k+1}{n} \rightarrow 1 \\
 & \left(1 - \frac{\lambda}{n}\right)^k \rightarrow 1 \\
 & \left(1 - \frac{\lambda}{n}\right)^n \rightarrow e^{-\lambda}
 \end{aligned}$$

$$\Rightarrow p_X(k) \rightarrow \frac{\lambda^k}{k!} e^{-\lambda}$$

**Example 12.2:** Say  $n = 100, p = 0.01$ . Then  $\lambda = 1$ .

$$\begin{aligned}
 \text{Then} \quad p_X(5) &= \frac{100!}{95! 5!} (0.01)^5 (0.99)^{95} \\
 &\approx 0.00290
 \end{aligned}$$

$$\text{and} \quad \frac{1^5}{5!} e^{-1} \approx 0.00306$$

If we repeat with  $n = 1000, p = 0.001$  so  $\lambda = 1$  again:

$$\text{Then} \quad p_X(5) = \frac{1000!}{995! 5!} (0.001)^5 (0.999)^{995}$$

$$\approx 0.00305$$

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Poisson should be a good approximation for:

- # of typos on a page
- # of oranges sold at a store in one day
- # of alpha particles emitted by a radioactive substance in 1 second
- # of dead pixels in an LCD display