28. Jointly Distributed Random Variables

The bivariate normal distribution [Ross S6.5]

Two random variables X and Y are **jointly Gaussian** (normal) or **bivariate Gaussian** (normal) with parameters:

$$\mu_X, \mu_Y, \sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1$$

when

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\} \tag{28.1}$$

It is customary to denote

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

then

$$f_{XY}(x,y) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

$\boldsymbol{\mu}$ is called the **mean vector** of (X,Y).

Σ is called the **covariance matrix** of (X,Y).

We say that the pair $(X,Y) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

Note: Σ is symmetric and +ve definite.

Marginal Distributions

To find the marginal

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

and from (28.1) + lots of algebra:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}} \tag{28.2}$$

So, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.

Likewise, $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$.

Conditional distribution

To get the conditional $f_{X|Y}(x|y)$, from (28.1) + (28.2) + lots of algebra:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \underbrace{\frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_X} \exp\left\{ \frac{-1}{2\sigma_X^2(1-\rho^2)} \left[x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y) \right) \right]^2 \right\}}_{K_1 \exp\left\{\frac{-1}{2K_2}[x-K_3]^2\right\}} \end{aligned}$$

So we recognize $f_{X|Y}(x|y)$ is the pdf of a Gaussian when X has mean

$$K_3 = \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$$

and variance $K_2 = \sigma_X^2(1 - \rho^2)$.

Note that we have

$$f_{XY}(x, y) = f_X(x) f_Y(y) \Leftrightarrow f_{X|Y}(x|y) = f_X(x)$$

and the latter happens when $\rho = 0$.

So for bivariate normal X and Y: X and Y are independent $\Leftrightarrow \rho = 0$.

Remark: ρ is called the **correlation coefficient** between X and Y.