

## 13. Random Variables (rv)

### Mean and Variance of Poisson [Ross S4.7]

Intuition: Say  $X \sim \text{Binomial}(n, p)$  with  $\lambda = np$ ,  $n$  large, so  $p$  small

Then:

$$\begin{aligned}E[X] &= np = \lambda \\Var[X] &= np(1 - p) \\&= \lambda(1 - p) \\&\approx \lambda\end{aligned}$$

Exact: Let  $X \sim \text{Poisson}(\lambda)$ . Then

$$\begin{aligned}E[X] &= \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\&= \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\&= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} \\&= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\&= \lambda \sum_{\ell=0}^{\infty} \frac{\lambda^{\ell}}{\ell!} e^{-\lambda} & \ell = k - 1 \\&= \lambda\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=1}^{\infty} \frac{k \lambda^k}{(k-1)!} e^{-\lambda} \\
&= \sum_{\ell=0}^{\infty} \frac{(\ell+1) \lambda^{\ell+1}}{\ell!} e^{-\lambda} & \ell = k-1 \\
&= \lambda \left( \underbrace{\sum_{\ell=0}^{\infty} \frac{\ell \lambda^{\ell}}{\ell!} e^{-\lambda}}_{\lambda} + \underbrace{\sum_{\ell=0}^{\infty} \frac{\lambda^{\ell}}{\ell!} e^{-\lambda}}_1 \right) \\
&= \lambda(1 + \lambda)
\end{aligned}$$

So

$$\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \lambda(1 + \lambda) - (\lambda)^2 \\
&= \lambda
\end{aligned}$$

**Example 13.1:** A radioactive substance with a large # of atoms emits 3.2 alpha particles per second on average. What is the probability that no more than 2 alpha particles are emitted in a 1 second interval?

*Solution:*

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### D) The geometric random variable [Ross 4.8.1]

Consider an infinite sequence of independent Bernoulli( $p$ ) trials.

Let  $X$  be trial # of first outcome that is a 1.

$X$  is called **geometric** with parameter  $p$ , denoted  $X \sim \text{Geometric}(p)$

$$p_X(k) = P[(k-1) \text{ zeros followed by a one}] \quad \text{for } k = 1, 2, \dots$$

$$= \begin{cases} (1-p)^{k-1}p & k \geq 1 \\ 0 & \text{else} \end{cases}$$

**Example 13.2:** A bag contains 2 white balls and 3 black balls. Balls are randomly drawn until a black ball is drawn. The selected ball is replaced before the next draw.

a) What is the probability that exactly  $n$  draws are needed?

b) What is the probability that at least  $k$  draws are needed?

*Solution:*

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## Mean and Variance

If  $X \sim \text{Geometric}(p)$ , then:

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

$$= \dots$$

$$= \frac{1}{p}$$

[see Ross example 4.8b]

$$E[X^2] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

$$= \dots$$

[see Ross example 4.8c]

$$= \frac{2-p}{p^2}$$

$$\begin{aligned} \Rightarrow \quad Var[X] &= E[X^2] - (E[X])^2 \\ &= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\ &= \frac{1-p}{p^2} \end{aligned}$$