332:345 – Linear Systems and Signals – Fall 2016
Solved Examples – Set 1 – S. J. Orfanidis

Problem 1

Consider the following linear system, driven by the input $x(t) = 10e^{-3t}u(t)$, and subject to the initial conditions at $t = 0^-$, $y(0^-) = 0$, $\dot{y}(0^-) = -5$,

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t)\,, \quad \text{or,} \quad (D^2 + 3D + 2)y(t) = Dx(t) \tag{1}$$

where D denotes the differential operator $D = d/dt$.

(a) Determine the transfer function $H(s)$ of this system, and determine analytically (i.e., by hand) its partial fraction expansion (PFE). Then, determine the PFE again using MATLAB's **residue** function, and alternatively, using the **partfrac** function of the symbolic toolbox.

(b) Using inverse Laplace transforms, determine analytically the impulse response $h(t)$ of this system. Then, determine it again using MATLAB's symbolic toolbox.

(c) Determine $h(t)$ again using the impulse matching method described in Eq. (2.23) and Appendix 2.1 of the text. Then, implement the impulse matching method using the **ilaplace** function of the symbolic toolbox, and alternatively, using the **dsolve** function.

(d) Using Laplace transforms, determine analytically the *zero-input response* subject to the given initial conditions. Then, determine it again by working exlusively in the time domain and expressing it as a linear combination of characteristic modes, and fixing the expansion coefficients from the initial conditions. Finally, determine the solution again with MATLAB's symbolic toolbox, using the **ilaplace** function and, alternatively, the **dsolve** function.

(e) For the given input $x(t)$, determine the *zero-state response* by analytically performing the convolution operation, $y(t) = h(t) * x(t)$.

(f) Determine the above zero-state response analytically using Laplace transforms. Then, determine it again with MATLAB's symbolic toolbox, using the **ilaplace** function and, alternatively, the **dsolve** function.

(g) For the given input and initial conditions, determine the *full* solution of Eq. (1) consisting of the sum of the zero-input and zero-state responses found above. Then, determine it again analytically using Laplace transaforms and carrying out the partial fraction expansions by hand. Then, determine the full solution again using the function **ilaplace** of the symbolic toolbox.

(h) Given the above initial conditions, $y(0^-) = 0$, $\dot{y}(0^-) = -5$, what are the corresponding initial conditions at $t = 0^+$, that is, $y(0^+), \dot{y}(0^+)$? Using the conditions at $t = 0^+$, re-derive the full solution of part (g), using the "classical method" described in Section 2.5 of the text. Then, derive it again with the symbolic toolbox and the function **dsolve**.

(i) Using the built-in function **lsim**, compute the output $y(t)$ that corresponds to the given input $x(t)$ and initial conditions, $y(0^-) = 0$, $\dot{y}(0^-) = -5$, and plot it over the time interval $0 \le t \le 6$. This is a bit tricky since the initial conditions are non-zero.

Solution

(a) Taking Laplace transforms of both sides of Eq. (1) with no initial conditions, we have,

$$s^2Y(s) + 3sY(s) + 2Y(s) = sX(s) \quad \Rightarrow \quad \boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^2 + 3s + 2}}$$

For the PFE, we have,

$$H(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

where

$$A = \left.\frac{s}{s+1}\right|_{s=-2} = \frac{-2}{-2+1} = 2\,, \qquad B = \left.\frac{s}{s+2}\right|_{s=-1} = \frac{-1}{-1+2} = -1$$

so that

$$\boxed{H(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+2)(s+1)} = \frac{2}{s+2} - \frac{1}{s+1}} \tag{2}$$

Using the **residue** function we find,

```
num = [1,0]; den = [1,3,2];
[r,p] = residue(num,den)

% r =
%      2
%     -1
% p =
%     -2
%     -1
```

where the residues r_1, r_2 are the same as A, B. Using the symbolic toolbox and the function **partfrac**, we obtain the same PFE result,

```
syms s
H = s/(s^2+3*s+2);
H = partfrac(H);     % H = 2/(s + 2) - 1/(s + 1)
```

(b) Inverting the PFE in Eq. (2), we find,

$$\boxed{h(t) = 2e^{-2t}u(t) - e^{-t}u(t)}$$

where we used the basic transform pair,

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$$

Using the symbolic toolbox, we obtain the same,

```
syms s
H = s/(s^2+3*s+2);
h = ilaplace(H)       % h = 2*exp(-2*t) - exp(-t)
```

(c) From Eq. (2.23), we must first determine the solution of the all-pole problem,

$$\ddot{y}_n(t) + 3\dot{y}_n(t) + 2y_n(t) = 0\,, \quad \text{with} \quad y_n(0) = 0\,, \quad \dot{y}_n(0) = 1 \tag{3}$$

where here the initial conditions are the same at $t = 0^{\pm}$. Then, since $H(s) = P(s)/Q(s)$, with, $P(s) = s$, and $Q(s) = s^2 + 3s + 2$, we will obtain $h(t)$ from,

$$h(t) = b_0\delta(t) + \big[P(D)y_n(t)\big]u(t) = \big[\dot{y}_n(t)\big]u(t)$$

The solution of Eq. (3) and its derivative are linear combinations of characteristic modes,

$$y_n(t) = c_1e^{-t} + c_2e^{-2t}$$
$$\dot{y}_n(t) = -c_1e^{-t} - 2c_2e^{-2t}$$

The initial conditions result in two equations in the two unknowns c_1, c_2,

$$\begin{aligned} y_n(0) &= c_1 + c_2 = 0 \\ \dot{y}_n(0) &= -c_1 - 2c_2 = 1 \end{aligned} \quad\Rightarrow\quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

Thus,

$$\begin{aligned} y_n(t) &= e^{-t} - e^{-2t} \\ \dot{y}_n(t) &= 2e^{-2t} - e^{-t} \end{aligned} \quad\Rightarrow\quad h(t) = \big[\dot{y}_n(t)\big]u(t) = \big[2e^{-2t} - e^{-t}\big]u(t)$$

Using the **ilaplace** function, we obtain the same,

```
syms s t Yn
Yn = 1/(s^2 + 3*s + 2);    % 1/Q(s), denominator part of H(s)
yn = ilaplace(Yn)          % yn = exp(-t) - exp(-2*t)
h = diff(yn,t)             % h = 2*exp(-2*t) - exp(-t)
```

These operations are equivalent to the more direct approach,

```
syms s
H = s/(s^2+3*s+2);
h = ilaplace(H)        % h = 2*exp(-2*t) - exp(-t)
```

The same answers for $y_n(t)$ and $h(t)$ can also be obtained using the **dsolve** function,

```
syms t yn(t)
yn = dsolve('D2yn + 3*Dyn + 2*yn=0','yn(0)=0','Dyn(0)=1')
h = diff(yn,t)
```

Note that the **ilaplace** method requires the initial conditions at $t = 0^-$, and the **dsolve** method, the conditions at $t = 0^+$. But these are the same for the solution $y_n(t)$.

We note also that $Y_n(s) = 1/Q(s)$, the denominator part of $H(s)$. This is generally true for any denominator order, and is a consequence of the defining properties of the solution $y_n(t)$, indeed, using the Laplace transforms of the derivatives of $y_n(t)$ and incorporating the initial conditions, we have for the above 2nd order case,

$$\begin{aligned} y_n(t) &\Rightarrow Y_n(s) \\ \dot{y}_n(t) &\Rightarrow sY_n(s) - y_n(0^-) = sY_n(s) \\ \ddot{y}_n(t) &\Rightarrow s^2Y_n(s) - sy_n(0^-) - \dot{y}_n(0^-) = s^2Y_n(s) - 1 \end{aligned} \tag{4}$$

Thus, Eq. (3) transforms into,

$$\ddot{y}_n(t) + 3\dot{y}_n(t) + 2y_n(t) = 0 \quad\Rightarrow\quad s^2Y_n(s) - 1 + 3sY_n(s) + 2Y_n(s) = 0 \quad \text{or,}$$

$$Y_n(s) = \frac{1}{s^2 + 3s + 2}$$

(d) Let us solve this for arbitrary initial conditions, $y(0^-) = y_0$ and $\dot{y}(0^-) = \dot{y}_0$, and at the end set $y_0 = 0$ and $\dot{y}_0 = -5$. Using Eq. (4), the differential equation (1) with $x(t) = 0$ transforms in the s-domain into,

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0 \quad\Rightarrow\quad s^2Y(s) - sy_0 - \dot{y}_0 + 3\big(sY(s) - y_0\big) + 2Y(s) = 0$$

Solving for $Y(s)$ and performing its partial fraction expansion, we have,

$$Y(s) = \frac{sy_0 + \dot{y}_0 + 3y_0}{s^2 + 3s + 2} = \frac{sy_0 + \dot{y}_0 + 3y_0}{(s+1)(s+2)} = \frac{\dot{y}_0 + 2y_0}{s+1} - \frac{\dot{y}_0 + y_0}{s+2}$$

which gives the zero-input response in the time domain,

$$\boxed{y_{zi}(t) = (\dot{y}_0 + 2y_0)e^{-t} - (\dot{y}_0 + y_0)e^{-2t}}, \quad t \geq 0 \tag{5}$$

and in the specific case of $y_0 = 0$ and $\dot{y}_0 = -5$,

$$\boxed{y_{zi}(t) = -5e^{-t} + 5e^{-2t}}, \quad t \ge 0 \tag{6}$$

An alternative approach is to work in the time-domain and express $y(t)$ and its derivative as a linear combination of characteristic modes, and fix the expansion coefficients from the initial conditions, that is, set

$$\begin{aligned} y(t) &= c_1 e^{-t} + c_2 e^{-2t} \\ \dot{y}(t) &= -c_1 e^{-t} - 2c_2 e^{-2t} \end{aligned}$$

and at $t = 0^-$, impose the conditions,

$$\begin{aligned} y(0^-) &= c_1 + c_2 = y_0 \\ \dot{y}(0^-) &= -c_1 - 2c_2 = \dot{y}_0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \dot{y}_0 + 2y_0 \\ c_2 &= -\dot{y}_0 - y_0 \end{aligned}$$

which results in the same answer as in Eq. (5). The same expression is obtained using the **ilaplace** function of the symbolic toolbox, where `y0,dy0` stand for the constants $y_0, \dot{y}_0$,

```
syms s y0 dy0 Y
Y = solve(s^2*Y-s*y0-dy0 + 3*(s*Y-y0)+2*Y==0,Y)
Y = partfrac(Y,s)         % Y = (dy0 + 2*y0)/(s + 1) - (dy0 + y0)/(s + 2)
yzi = ilaplace(Y)         % yzi = exp(-t)*(dy0 + 2*y0) - exp(-2*t)*(dy0 + y0)
```

Alternatively, we can use the **dsolve** function,

```
syms y0 dy0
yzi = dsolve('D2y + 3*Dy+ 2*y = 0', 'y(0)=y0', 'Dy(0)=dy0')
```

(e) The convolutional expression for the zero-state output is,

$$y_{zs}(t) = \int_{-\infty}^{\infty} h(t')x(t-t')dt' \tag{7}$$

Because the input $x(t) = 10e^{-3t}u(t)$ is causal, the range of its argument in Eq. (7) must be restricted to, $t - t' \ge 0$. Similarly, because $h(t')$ is causal, its argument must be $t' \ge 0$. Combining the two inequalities, we have,

$$\begin{aligned} t - t' &\ge 0 \\ t' &\ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} t &\ge 0 \\ 0 \le t' &\le t \end{aligned}$$

Thus, $y_{zs}(t)$ must also be causal, and for $t \ge 0$, the integral in (7) simplifies into,

$$\begin{aligned} y_{zs}(t) &= \int_0^t h(t')x(t-t')dt' = \int_0^t (2e^{-2t'} - e^{-t'})10e^{-3(t-t')}dt' \\ &= 10e^{-3t}\int_0^t (2e^{-2t'} - e^{-t'})e^{3t'}dt' = 10e^{-3t}\int_0^t (2e^{t'} - e^{2t'})dt' \\ &= 10e^{-3t}\left[2(e^t - 1) - \frac{1}{2}(e^{2t} - 1)\right] = -5e^{-t} + 20e^{-2t} - 15e^{-3t} \end{aligned}$$

thus,

$$\boxed{y_{zs}(t) = \left[-5e^{-t} + 20e^{-2t} - 15e^{-3t}\right]u(t)} \tag{8}$$

The same can also be obtained from the convolution table on p.177 of the text. The integration can also be performed with the **int** function of the symbolic toolbox,

```
syms t tau
x = 10*exp(-3*(t-tau));
h =  2*exp(-2*tau) - exp(-tau);
yzs = int(h*x, tau, 0, t)       % yzs = 20*exp(-2*t) - 5*exp(-t) - 15*exp(-3*t)
```

(f) The Laplace transform of the input $x(t) = 10e^{-3t}u(t)$ is, $X(s) = 10/(s+3)$. It follows that the transform of the zero-state output will be,

$$Y(s) = H(s)X(s) = \frac{s}{s^2+3s+2} \cdot \frac{10}{s+3} = \frac{10s}{(s+1)(s+2)(s+3)}$$

with PFE,[†]

$$Y(s) = \frac{10s}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

where,

$$\begin{aligned}
A &= (s+1)Y(s)\Big|_{s=-1} = \frac{10s}{(s+2)(s+3)}\Big|_{s=-1} = \frac{10(-1)}{(-1+2)(-1+3)} = -5 \\
B &= (s+2)Y(s)\Big|_{s=-2} = \frac{10s}{(s+1)(s+3)}\Big|_{s=-2} = \frac{10(-2)}{(-2+1)(-2+3)} = 20 \\
C &= (s+3)Y(s)\Big|_{s=-3} = \frac{10s}{(s+1)(s+2)}\Big|_{s=-3} = \frac{10(-3)}{(-3+1)(-3+2)} = -15
\end{aligned}$$

Inverting the Laplace transform $Y(s)$, we obtain the time-domain zero-state response, which agrees with that of Eq. (8),

$$y_{zs}(t) = \left[-5e^{-t} + 20e^{-2t} - 15e^{-3t}\right]u(t)$$

The PFE residues can also be obtained by the function **residue**, where the outputs r_1, r_2, r_3 correspond to C, B, A, respectively,

```
[r,p] = residue([10,0], conv([1 3 2],[1 3]))

% r =
%   -15.0000
%    20.0000
%    -5.0000
% p =
%    -3.0000
%    -2.0000
%    -1.0000
```

The indicated convolution operation, `conv([1 3 2],[1 3])`, results in the coefficients, $[1,\ 6,\ 11,\ 6]$, and effectively multiplies the polynomials,

$$(s^2+3s+2)(s+3) = s^3 + 6s^2 + 11s + 6$$

The PFE and the Laplace inversions can also be accomplished with the symbolic toolbox,

```
syms s
H = s/(s^2+3*s+2);
X = 10/(s+3);
Y = H*X;                 % Y = 10*s/((s + 3)*(s^2 + 3*s + 2))
Y = partfrac(Y)          % Y = 20/(s + 2) - 5/(s + 1) - 15/(s + 3)
yzs = ilaplace(Y)        % yzs = 20*exp(-2*t) - 5*exp(-t) - 15*exp(-3*t)
```

For $t \ge 0$, we obtain from the above solution,

$$\begin{aligned}
y_{zs}(t) &= -5e^{-t} + 20e^{-2t} - 15e^{-3t} \\
\dot{y}_{zs}(t) &= 5e^{-t} - 40e^{-2t} + 45e^{-3t}
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
y_{zs}(0^+) &= -5 + 20 - 15 = 0 \\
\dot{y}_{zs}(0^+) &= 5 - 40 + 45 = 10
\end{aligned}$$

The term, "zero-state" solution refers to zero initial conditions at time $t = 0^-$. As we see above, at $t = 0^+$ the initial conditions are not zero. See part (h) for more discussion

[†]Partial fractions are reviewed in Sect. B.5 of the text.

on this issue, and on how to predict the conditions at $t = 0^+$ from those at $t = 0^-$. The symbolic toolbox solution using the function **ilaplace** requires the $t = 0^-$ conditions, whereas the solution using **dsolve**, requires the $t = 0^+$ conditions.

In the present case, since we just found $y_{zs}(0^+) = 0$, $\dot{y}_{zs}(0^+) = 10$, we can apply the **dsolve** function, noting that $\dot{x}(t) = -3\cdot 10e^{-3t}$ for $t \geq 0^+$,

```
syms t yzs(t)
yzs = dsolve('D2y+3*Dy+2*y = 10*(-3)*exp(-3*t)', 'y(0)=0', 'Dy(0)=10')
```

which results in the same solution as that of Eq. (8).

(g) Adding up the zero-input and zero-state solutions of Eqs. (5) and (8), and combining like exponential terms, we obtain the total solution of Eq. (1), which meets the arbitrary initial conditions, $y(0^-) = y_0$, $\dot{y}(0^-) = \dot{y}_0$,

$$\boxed{y(t) = (\dot{y}_0 + 2y_0 - 5)e^{-t} + (20 - y_0 - \dot{y}_0)e^{-2t} - 15e^{-3t}}\,, \quad t \geq 0^+ \tag{9}$$

and for the particular values, $y(0^-) = 0$, $\dot{y}(0^-) = -5$,

$$\boxed{y(t) = -10e^{-t} + 25e^{-2t} - 15e^{-3t}}\,, \quad t \geq 0^+ \tag{10}$$

The first two terms depend only on the characteristic modes e^{-t}, e^{-2t}, and are referred to as the “natural response” or “homogeneous solution”, whereas the last term depends only on the input $x(t) = 10e^{-3t}$ and is referred to as the “particular solution” or “forced response”,

$$y(t) = \underbrace{(\dot{y}_0 + 2y_0 - 5)e^{-t} + (20 - y_0 - \dot{y}_0)e^{-2t}}_{\text{homogeneous}} - \underbrace{15e^{-3t}}_{\text{forced}}$$

The factor -15 in the forced response can be predicted in advance using the following result: Given a system with transfer function $H(s)$ and an exponential causal input $x(t) = Ae^{-at}$, then the forced response output is simply, $y_{\text{forced}}(t) = AH(-a)e^{-at}$, where $H(-a)$ is the transfer function $H(s)$ evaluated at $s = -a$ (assuming that $s = -a$ is not a pole of $H(s)$). Thus, in our example,

$$y_{\text{forced}}(t) = 10H(-3)e^{-3t} = 10\cdot \left.\frac{s}{s^2 + 3s + 2}\right|_{s=-3} e^{-3t} = -15e^{-3t}$$

Next, we derive the total solution using Laplace transforms and partial fraction expansions. The approach is similar to that of part (d), except here the right-hand sides are not zero. For the case of arbitrary initial conditions, $y(0^-) = y_0$ and $\dot{y}(0^-) = \dot{y}_0$, the transform of the differential equation (1) is,

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) \quad \Rightarrow \quad s^2Y(s) - sy_0 - \dot{y}_0 + 3\big(sY(s) - y_0\big) + 2Y(s) = sX(s)$$

where the transform of $\dot{x}(t)$ was, $sX(s) - x(0^-) = sX(s)$, since $x(0^-) = 0$ because $x(t)$ is causal. Solving for $Y(s)$, and replacing $X(s) = 10/(s+3)$, we obtain,

$$\begin{aligned} Y(s) &= \frac{sy_0 + \dot{y}_0 + 3y_0 + sX(s)}{s^2 + 3s + 2} = \frac{(sy_0 + \dot{y}_0 + 3y_0)(s+3) + 10s}{(s^2 + 3s + 2)(s+3)} \\ &= \frac{(sy_0 + \dot{y}_0 + 3y_0)(s+3) + 10s}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \\ &= \frac{\dot{y}_0 + 2y_0 - 5}{s+1} + \frac{20 - y_0 - \dot{y}_0}{s+2} - \frac{15}{s+3} \end{aligned} \tag{11}$$

where we may verify easily,

$$A = \frac{(sy_0 + \dot{y}_0 + 3y_0)(s+3) + 10s}{(s+2)(s+3)}\bigg|_{s=-1} = \dot{y}_0 + 2y_0 - 5$$

$$B = \frac{(sy_0 + \dot{y}_0 + 3y_0)(s+3) + 10s}{(s+1)(s+3)}\bigg|_{s=-2} = 20 - y_0 - \dot{y}_0$$

$$C = \frac{(sy_0 + \dot{y}_0 + 3y_0)(s+3) + 10s}{(s+1)(s+2)}\bigg|_{s=-3} = -15$$

It follows that the inverse Laplace transform of Eq. (11) is as in Eq. (9). The same partial fraction expansion and inverse transform can be obtained easily by the symbolic toolbox,

```
syms s y0 dy0 Y
X = 10/(s+3);
Y = solve(s^2*Y-s*y0-dy0 + 3*(s*Y-y0) + 2*Y == s*X, Y)
Y = partfrac(Y,s)    % Y = (dy0+2*y0-5)/(s + 1) + (20-dy0-y0)/(s + 2) - 15/(s + 3)
y = ilaplace(Y)      % y = exp(-t)*(dy0+2*y0-5) + exp(-2*t)*(20-dy0-y0) - 15*exp(-3*t)
```

(h) We recall from the discussion in class that for a second-order system of the form,

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = b_0\ddot{x}(t) + b_1\dot{x}(t) + b_2 x(t) \quad \Rightarrow \quad \boxed{H(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}}$$

and for a causal input $x(t)$ that is differentiable at $t = 0^+$, the mapping between the initial conditions at $t = 0^-$ and the initial conditions at $t = 0^+$ is given by,

$$\boxed{\begin{aligned} y(0^+) &= y(0^-) + b_0 x(0^+) \\ \dot{y}(0^+) &= \dot{y}(0^-) + b_0\dot{x}(0^+) + (b_1 - b_0 a_1)x(0^+) \end{aligned}} \tag{12}$$

For our particular system, we have, $[b_0, b_1, b_2] = [0, 1, 0]$, so that Eqs. (12) become,

$$\begin{aligned} y(0^+) &= y(0^-) \\ \dot{y}(0^+) &= \dot{y}(0^-) + x(0^+) \end{aligned} \tag{13}$$

Thus, for the input $x(t) = 10e^{-3t}u(t)$, and initial conditions $y_0, \dot{y}_0$ at $t = 0^-$, we have,

$$\begin{aligned} y(0^+) &= y_0 \\ \dot{y}(0^+) &= \dot{y}_0 + 10 \end{aligned} \tag{14}$$

These are the conditions that must be used in applying the classical method, or the **dsolve** function. In the classical method, we construct the solution as the sum of a particular solution and a general homogeneous solution. For the particular solution, we may take the forced response, which in our example is, $y_{\text{forced}}(t) = -15e^{-3t}$. For the homogeneous solution we form a linear combination of the characteristic modes e^{-t}, e^{-2t}. Thus,

$$\begin{aligned} y(t) &= c_1 e^{-t} + c_2 e^{-2t} - 15e^{-3t} \\ \dot{y}(t) &= -c_1 e^{-t} - 2c_2 e^{-2t} + 45e^{-3t} \end{aligned} \qquad \text{(classical method)}$$

for $t \ge 0$. Imposing the $t = 0^+$ conditions (14), we have,

$$\begin{aligned} y(0^+) &= c_1 + c_2 - 15 = y_0 \\ \dot{y}(0^+) &= -c_1 - 2c_2 + 45 = \dot{y}_0 + 10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \dot{y}_0 + 2y_0 - 5 \\ c_2 &= 20 - \dot{y}_0 - y_0 \end{aligned}$$

Thus, we obtain the same solution as that in Eq. (9), for $t \ge 0^+$,

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} - 15e^{-3t} = (\dot{y}_0 + 2y_0 - 5)e^{-t} + (20 - y_0 - \dot{y}_0)e^{-2t} - 15e^{-3t}$$

Finally, the same solution can be obtained with the **dsolve** function applied with the initial conditions at $t = 0^+$ of Eq. (14),

```
syms t y0 dy0 y(t)
dy = diff(y,t); ddy = diff(dy,t);
x = 10*exp(-3*t); dx = diff(x,t);
y = dsolve(ddy + 3*dy + 2*y == dx, y(0) == y0, dy(0) == dy0+10)
```

(i) Suppose that one naively tries to use the function **lsim** to compute the system output for the given input. This can be done simply by the MATLAB code,

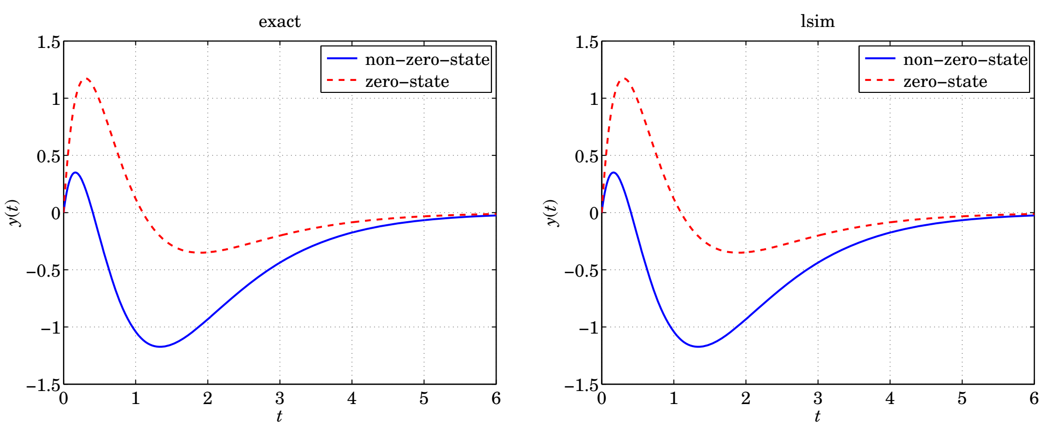
```
t = linspace(0,6,601);          % desired range of t's
x = 10*exp(-3*t);               % input signal
s = tf('s');                    % transfer function variable
H = s/(s^2+3*s+2);              % transfer function object - class(H) is tf
y = lsim(H,x,t);                % assumes zero initial conditions
```

This code, however, will generate only the zero-state part, $y_{zs}(t)$, of the correct answer. The function **lsim** can handle initial conditions, but those are for state-space realizations only. If the initial conditions are specified in terms of the output $y(t)$ and its derivatives, then one must map these initial conditions to the proper state-vector initial conditions to be used in **lsim**.

Such mapping can be accomplished by the so-called observability matrix (we'll discuss it at a later date). The built-in function **obsv** allows one to perform such mapping and thus, use **lsim** with any desired initial conditions at $t = 0^-$. The following MATLAB code illustrates the procedure.

```
y0 = 0; dy0 = -5;           % given initial conditions at t=0-
t = linspace(0,6,601);      % desired time range
x = 10*exp(-3*t);           % input signal
s = tf('s');                % transfer function variable
H = s/(s^2+3*s+2);          % transfer function object - class(H) is tf
S = ss(H);                  % S is state-space model of H - class(S) is ss
yi = [y0; dy0];             % vector of initial conditions with respect to y
xi = obsv(S) \ yi;          % map yi to initial state-vector xi
y = lsim(S,x,t,xi);         % run model S with initial state xi
yzs = lsim(S,x,t);          % run model S with zero initial state xi=0
plot(t,y, t,yzs,'--');      % compare the nonzero-state and zero-state outputs
```

The computed outputs are shown on the right graph below. Those on the left graph are the exact responses derived in Eqs. (8) and (9) . They are virtually indistinguishable from the numerically computed ones using **lsim**.



Problem 2

Repeat questions (d–i) of the previous problem, for the same system defined by Eq. (1), but with input $x(t) = 10e^{-2t}u(t)$, and initial conditions at $t = 0^-$, $y(0^-) = 2$, $\dot{y}(0^-) = -7$.

Note that parts (a–c) are the same as before. The new input has a pole that coincides with one of the characteristic modes of the system, and thus, we will have to deal with a double-pole in the Laplace inverses.

Solution

(d) Because the initial conditions are different from those of Problem 1, the zero-input solution will also be different. However, the steps are identical to those leading to Eq. (5),

$$y_{zi}(t) = (\dot{y}_0 + 2y_0)e^{-t} - (\dot{y}_0 + y_0)e^{-2t}, \quad t \geq 0 \tag{15}$$

Replacing the constants by $y_0 = 2$ and $\dot{y}_0 = -7$, we have,

$$y_{zi}(t) = 5e^{-2t} - 3e^{-t}, \quad t \geq 0 \tag{16}$$

The other methods mentioned in part (d) of Problem 1 remain the same.

(e) The zero-state output will be different here because the input is different. The impulse response $h(t)$ is the same as in Problem 1, therefore, the convolutional formula gives,

$$\begin{aligned} y_{zs}(t) &= \int_0^t h(t')x(t-t')\,dt' = \int_0^t (2e^{-2t'} - e^{-t'})10e^{-2(t-t')}\,dt' \\ &= 10e^{-2t}\int_0^t (2e^{-2t'} - e^{-t'})e^{2t'}\,dt' = 10e^{-2t}\int_0^t (2 - e^{t'})\,dt' \\ &= 10e^{-2t}\left[2t - (e^t - 1)\right] = 10e^{-2t} - 10e^{-t} + 20te^{-2t} \end{aligned}$$

so that,

$$y_{zs}(t) = \left[10e^{-2t} - 10e^{-t} + 20te^{-2t}\right]u(t) \tag{17}$$

Using the function **int** of the symbolic toolbox returns the same answer,

```
syms t tau
x1 = 10*exp(-2*(t-tau));
h1 =  2*exp(-2*tau) - exp(-tau);
yzs = int(h1*x1, tau, 0, t)
```

(f) The Laplace transform of the input $x(t) = 10e^{-2t}u(t)$ is, $X(s) = 10/(s+2)$. It follows that the transform of the zero-state output will be,

$$Y(s) = H(s)X(s) = \frac{s}{s^2+3s+2}\cdot\frac{10}{s+2} = \frac{10s}{(s+2)^2(s+1)}$$

with PFE,

$$Y(s) = \frac{10s}{(s+2)^2(s+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+1} = \frac{10}{s+2} + \frac{20}{(s+2)^2} - \frac{10}{s+1}$$

where,[†]

$$\begin{aligned} A &= \frac{d}{ds}\left[(s+2)^2Y(s)\right]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{10s}{s+1}\right]_{s=-2} = \frac{10}{(s+1)^2}\Big|_{s=-2} = 10 \\ B &= (s+2)^2Y(s)\Big|_{s=-2} = \frac{10s}{s+1}\Big|_{s=-2} = 20 \\ C &= (s+1)Y(s)\Big|_{s=-1} = \frac{10s}{(s+2)^2}\Big|_{s=-1} = -10 \end{aligned}$$

[†] see Sect. B.5-3 of the text.

Inverting the Laplace transform $Y(s)$, we obtain the time-domain zero-state response $y_{zs}(t)$, which agrees with that of Eq. (17).

$$y_{zs}(t) = \left[10e^{-2t} - 10e^{-t} + 20te^{-2t}\right]u(t)$$

The PFE residues can also be obtained by the function **residue**, which correctly accounts for the double pole, where the outputs r_1, r_2, r_3 correspond to A, B, C, respectively,

```
[r,p] = residue([10,0], conv([1 3 2],[1 2]))

% r =
%    10.0000
%    20.0000
%   -10.0000
% p =
%    -2.0000
%    -2.0000
%    -1.0000
```

The indicated convolution operation, `conv([1 3 2],[1 2])`, results in the coefficients, $[1, 5, 8, 4]$, and effectively multiplies the polynomials,

$$(s^2 + 3s + 2)(s + 2) = (s + 2)^2(s + 1) = s^3 + 5s^2 + 8s + 4$$

The PFE and the Laplace inversions can also be accomplished with the **ilaplace** function,

```
syms s
H = s/(s^2+3*s+2);
X = 10/(s+2);
Y = H*X;             % Y(s) = H(s)*X(s) = 10*s/((s + 2)*(s^2 + 3*s + 2))
Y = partfrac(Y)      % Y = 10/(s + 2) - 10/(s + 1) + 20/(s + 2)^2
yzs = ilaplace(Y)    % yzs = 10*exp(-2*t) - 10*exp(-t) + 20*t*exp(-2*t))
```

To derive the solution using the **dsolve** function, we must transform the initial conditions of the zero-state solution from their zero values at $t = 0^-$ to their values at 0^+ using the mapping of Eq. (12), or, specifically, here, Eq. (14). Thus, we have $y_{zs}(0^+) = 0$, and $\dot{y}_{zs}(0^+) = 0 + 10 = 10$. The application of **dsolve** is then,

```
syms t y(t)
dy = diff(y,t); ddy = diff(dy,t);
x = 10*exp(-2*t); dx = diff(x,t);
yzs = dsolve(ddy + 3*dy + 2*y == dx, y(0) == 0, dy(0) == 10)
```

which produces exactly the same expression as in Eq. (17).

(g) Adding up the zero-input and zero-state responses, we obtain the full solution, for $t \geq 0$,

$$\begin{aligned} y_{zi}(t) &= 5e^{-2t} - 3e^{-t} \\ y_{zs}(t) &= 10e^{-2t} - 10e^{-t} + 20te^{-2t} \\ y(t) &= 15e^{-2t} - 13e^{-t} + 20te^{-2t} \end{aligned} \tag{18}$$

The first two terms represent the "homogeneous" solution and the third, the "forced" response,

$$y(t) = \underbrace{15e^{-2t} - 13e^{-t}}_{\text{homogeneous}} + \underbrace{20te^{-2t}}_{\text{forced}}, \quad t \geq 0 \tag{19}$$

The expression $20te^{-2t}$ for the forced response can be predicted in advance. We recall from part (g) of Problem 1 that the forced response of a linear system $H(s)$ due to an exponential input of the form $x(t) = Ae^{-at}u(t)$ is given simply by $AH(-a)e^{-at}u(t)$, provided that $s = -a$ is not a pole of the system. But if $s = -a$ is a pole of the system (and that pole is assumed to be a simple pole), then, the forced response is given by the modified expression,†

$$x(t)=Ae^{-at}u(t)\;\longrightarrow\; y_{\text{forced}}(t)=ARte^{-at}u(t)\,,\quad R=(s+a)H(s)\Big|_{s=-a}$$

where the factor $(s+a)$ will cancel a similar factor in the denominator of $H(s)$. For our particular example, since $A=10$ and $s=-2$, we have,

$$AR=10\,(s+2)H(s)\Big|_{s=-2}=(s+2)\,\frac{10s}{(s+1)\,(s+2)}\Big|_{s=-2}=\frac{10s}{s+1}\Big|_{s=-2}=20$$

Next, we rederive Eq. (19) using Laplace transforms and partial fraction expansions. For the given initial conditions, $y(0^-)=2$, $\dot{y}(0^-)=-7$, the transform of the differential equation (1) is,

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\dot{x}(t)\quad\Rightarrow\quad s^2Y(s)-2s+7+3\big(sY(s)-2\big)+2Y(s)=sX(s)$$

Solving for $Y(s)$, and replacing $X(s)=10/(s+2)$, we find after some algebra,

$$\begin{aligned}Y(s)&=\frac{2s^2+13s-2}{(s+2)\,(s^2+3s+2)}=\frac{2s^2+13s-2}{s^3+5s^2+8s+4}=\frac{2s^2+13s-2}{(s+2)^2(s+1)}\\&=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{s+1}=\frac{15}{s+2}+\frac{20}{(s+2)^2}-\frac{13}{s+1}\end{aligned}\tag{20}$$

which upon inversion yields exactly Eq. (19). The PFE coefficients can be confirmed using the **residue** function,

```
[r,p] = residue([2, 13, -2], [1, 5, 8, 4])

% r =
%    15.0000
%    20.0000
%   -13.0000
% p =
%    -2.0000
%    -2.0000
%    -1.0000
```

Moreover, the solution for $Y(s)$, its PFE expansion, and inversion can be carried out simply by the symbolic toolbox,

```
syms s t Y
X = 10/(s+2);
Y = solve(s^2*Y-s*2+7 + 3*(s*Y-2) + 2*Y == s*X, Y);
Y = simplify(Y)      % Y = (2*s^2 + 13*s - 2)/((s + 2)*(s^2 + 3*s + 2))
Y = partfrac(Y,s)    % Y = 15/(s + 2) - 13/(s + 1) + 20/(s + 2)^2
y = ilaplace(Y)      % y = 15*exp(-2*t) - 13*exp(-t) + 20*t*exp(-2*t)
```

(h) The initial conditions at $t=0^+$ can be derived from Eq. (13),

$$\begin{aligned}y(0^+)&=y(0^-)=2\\\dot{y}(0^+)&=\dot{y}(0^-)+x(0^+)=-7+10=3\end{aligned}\tag{21}$$

Using these conditions, we may derive the solution of Eq. (19) by the classical method, in which we construct the solution as the sum of a particular solution and a general

†other types of forced responses are listed in Table 2.2 on p.199 of the text.

homogeneous solution. For the particular solution, we may take the forced response, which in our example is, $y_{\text{forced}}(t) = 20te^{-2t}$. For the homogeneous solution we form a linear combination of the characteristic modes e^{-t}, e^{-2t}. Thus,

$$\begin{aligned} y(t) &= c_1 e^{-t} + c_2 e^{-2t} + 20te^{-2t} \\ \dot{y}(t) &= -c_1 e^{-t} - 2c_2 e^{-2t} + 20(1-2t)e^{-2t} \end{aligned} \qquad \text{(classical method)}$$

for $t \geq 0$. Imposing the $t = 0^+$ conditions (14), we have,

$$\begin{aligned} y(0^+) &= c_1 + c_2 = 2 \\ \dot{y}(0^+) &= -c_1 - 2c_2 + 20 = 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= -13 \\ c_2 &= 15 \end{aligned}$$

Thus, we obtain the same solution as that in Eq. (19), for $t \geq 0^+$,

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + 20te^{-2t} = -13e^{-t} + 15e^{-2t} + 20te^{-2t}$$

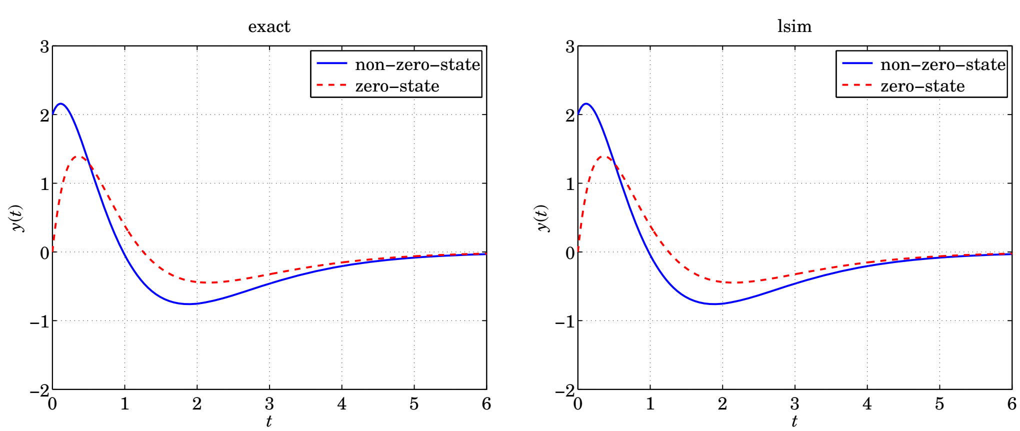
Finally, the same solution can be obtained with the **dsolve** function applied with the initial conditions at $t = 0^+$ of Eq. (21),

```
syms t y(t)
dy = diff(y,t); ddy = diff(dy,t);
x = 10*exp(-2*t); dx = diff(x,t);
yy = dsolve(ddy + 3*dy + 2*y == dx, y(0) == 2, dy(0) == 3)
```

(i) The numerical computation using the **lsim** function is carried out in exactly the same way as in part (i) of Problem 1, only the input and initial conditions are different. The MATLAB code is listed below.

```
y0 = 2; dy0 = -7;          % given initial conditions at t=0-
t = linspace(0,6,601);     % desired time range
x = 10*exp(-2*t);          % input signal
s = tf('s');               % transfer function variable
H = s/(s^2+3*s+2);         % transfer function object - class(H) is tf
S = ss(H);                 % S is state-space model of H - class(S) is ss
yi = [y0; dy0];            % vector of initial conditions with respect to y
xi = obsv(S) \ yi;         % map yi to initial state-vector xi
y = lsim(S,x,t,xi);        % run model S with initial state xi
yzs = lsim(S,x,t);         % run model S with zero initial state xi=0
plot(t,y, t,yzs,'--');     % compare the nonzero-state and zero-state outputs
```

The computed outputs are shown on the right graph below. Those on the left graph are the exact responses $y_{zs}(t), y(t)$ derived in Eq. (18). They are virtually indistinguishable from the numerically computed ones using **lsim**.



Problem 3

Repeat questions (d–i) of Problem 1, for the same system defined by Eq. (1), but with input $x(t) = \big(t^2 + 5t + 3\big)u(t)$, and initial conditions at $t = 0^-$, $y(0^-) = 2$, $\dot{y}(0^-) = 0$.

This example illustrates how to handle non-exponential inputs and how to guess the corresponding forced response. Note that parts (a–c) are the same as in Problem 1.

Solution

(d) Because the initial conditions are different from those of Problems 1 and 2, the zero-input solution will also be different. However, the steps are identical to those leading to Eq. (5),

$$y_{zi}(t) = (\dot{y}_0 + 2y_0)e^{-t} - (\dot{y}_0 + y_0)e^{-2t}\,, \quad t \ge 0 \tag{22}$$

Replacing the constants by $y_0 = 2$ and $\dot{y}_0 = 0$, we have,

$$y_{zi}(t) = 4e^{-t} - 2e^{-2t}\,, \quad t \ge 0 \tag{23}$$

(e) The impulse response $h(t)$ is the same as in Problem 1, therefore, the convolutional formula gives, for the given input,

$$\begin{aligned} y_{zs}(t) &= \int_0^t h(t')x(t-t')\,dt' = \int_0^t \big[2e^{-2t'} - e^{-t'}\big]\big[(t-t')^2 + 5(t-t') + 3\big]dt' \\ &= \int_0^t \big[2e^{-2t'} - e^{-t'}\big]\big[t^2 + t'^2 - 2tt' + 5t - 5t' + 3\big]dt' \end{aligned}$$

The integrations can be done with the help of the following integrals (set $a = 1$, $a = 2$),

$$\int_0^t e^{-at'}dt' = \frac{1 - e^{-at}}{a}\,, \qquad \int_0^t t'\,e^{-at'}dt' = \frac{1 - e^{-at}(1 + at)}{a^2}$$

$$\int_0^t t'^2 e^{-at'}dt' = \frac{2 - e^{-at}(2 + 2at + a^2t^2)}{a^3}$$

eventually resulting in,

$$y_{zs}(t) = \big(1 + t - e^{-2t}\big)u(t) \tag{24}$$

The disappearance of the e^{-t} term is explained below in part (f). Using the function **int** of the symbolic toolbox returns the same answer as Eq. (24)

```
syms t tau
x1 = (t-tau)^2 + 5*(t-tau) + 3;
h1 =  2*exp(-2*tau) - exp(-tau);
y1 = int(h1*x1, tau, 0, t)       % yzs = t - exp(-2*t) + 1
```

(f) The Laplace transform of the input $x(t) = (t^2 + 5t + 3)u(t)$ is,

$$X(s) = \frac{2}{s^3} + \frac{5}{s^2} + \frac{3}{s} = \frac{3s^2 + 5s + 2}{s^3} = \frac{(s+1)(3s+2)}{s^3}$$

Thus, accidentally, $X(s)$ contains a zero-factor $(s + 1)$, which will get cancelled by the same pole factor of $H(s)$ when computing the Laplace tranaform of the zero-state response, that is,

$$\begin{aligned} Y(s) = H(s)X(s) &= \frac{s}{(s+1)(s+2)} \cdot \frac{(s+1)(3s+2)}{s^3} = \frac{3s+2}{s^2(s+2)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s+2} \end{aligned} \tag{25}$$

where the PFE coefficients are computed in the usual manner,

$$\begin{aligned}
A &= \frac{d}{ds}\big[s^2 Y(s)\big]\Big|_{s=0} = \frac{d}{ds}\left[\frac{3s+2}{s+2}\right]_{s=0} = \frac{4}{(s+2)^2}\Big|_{s=0} = 1\\
B &= s^2 Y(s)\Big|_{s=0} = \frac{3s+2}{s+2}\Big|_{s=0} = 1\\
C &= (s+2)\,Y(s)\Big|_{s=-2} = \frac{3s+2}{s^2}\Big|_{s=-2} = -1
\end{aligned}$$

which can also be obtained by the **residue** function,

```
[r,p] = residue([3,2],[1,2,0,0])

% r =
%     -1
%      1
%      1
% p =
%     -2
%      0
%      0
```

The inversion of Eq. (25) leads directly to Eq. (24). The inversion using the symbolic toolbox leads to the same answer,

```
syms s t
H = s/(s^2+3*s+2);          % system
x = t^2 + 5*t + 3;          % input
X = simplify(laplace(x))    % X = (3*s^2 + 5*s + 2)/s^3
Y = simplify(H*X);          % Y = (3*s + 2)/(s^2*(s + 2))
Y = partfrac(Y)             % Y = 1/s +1/s^2 - 1/(s + 2)
yzs = ilaplace(Y)           % yzs = 1 + t - exp(-2*t)
```

To derive the solution using the **dsolve** function, we must transform the initial conditions of the zero-state solution from their zero values at $t = 0^-$ to their values at 0^+ using the mapping of Eq. (12), or, since now, $x(0+) = 3$, we have, $y_{zs}(0^+) = 0$, $\dot{y}_{zs}(0^+) = 0 + 3 = 3$. The application of **dsolve** is then,

```
syms t y(t)
dy = diff(y,t); ddy = diff(dy,t);
x = t^2 + 5*t + 3; dx = diff(x,t);
yzs = dsolve(ddy + 3*dy + 2*y == dx, y(0) == 0, dy(0) == 3)
```

which produces exactly the same expression as in Eq. (24).

(g) Adding up the zero-input and zero-state responses, we obtain the full solution, for $t \geq 0$,

$$\begin{aligned}
y_{zi}(t) &= 4e^{-t} - 2e^{-2t}\\
y_{zs}(t) &= 1 + t - e^{-2t}\\
y(t) &= 4e^{-t} - 3e^{-2t} + 1 + t
\end{aligned} \tag{26}$$

The first two terms represent the “homogeneous” solution and the third, the “forced” response,

$$y(t) = \underbrace{4e^{-t} - 3e^{-2t}}_{\text{homogeneous}} + \underbrace{1+t}_{\text{forced}}, \qquad t \geq 0 \tag{27}$$

The expression $1+t$ for the forced response can be worked out in advance by the following argument. Since the input is a quadratic polynomial in t, we may seek a forced response that is a similar 2nd order polynomial. However, because the input gets differentiated in the right-hand-side of Eq. (1), it will become first-order in t. Thus, we seek a forced response of the form $y(t) = \beta_0 + \beta_1 t$. Inserting this into Eq. (1) gives,

$$\ddot{y} + 3\dot{y} + 2y = \dot{x} \quad \Rightarrow \quad 0 + 3\beta_1 + 2(\beta_0 + \beta_1 t) = 2t + 5$$

Matching like powers of t, gives the two equations in β_0, β_1,

$$\begin{aligned} 5\beta_0 &= 5 \\ 2\beta_1 &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \beta_0 &= 1 \\ \beta_1 &= 1 \end{aligned}$$

Thus, the forced response to the input, $x(t) = t^2 + 5t + 3$, is $y_{\text{forced}}(t) = \beta_0 + \beta_1 t = 1 + t$. Next, we rederive Eq. (27) using Laplace transforms and partial fraction expansions. For the given initial conditions, $y(0^-) = 2$, $\dot{y}(0^-) = 0$, the transform of the differential equation (1) is,

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) \quad \Rightarrow \quad s^2 Y(s) - 2s + 3\big(sY(s) - 2\big) + 2Y(s) = sX(s)$$

Solving for $Y(s)$, and replacing $X(s) = (3s^2 + 5s + 2)/s^3$, we find after some algebra,

$$Y(s) = \frac{2s^3 + 9s^2 + 5s + 2}{s^2(s^2 + 3s + 2)} = \frac{2s^3 + 9s^2 + 5s + 2}{s^2(s+1)(s+2)} = \frac{4}{s+1} - \frac{3}{s+2} + \frac{1}{s} + \frac{1}{s^2} \tag{28}$$

where we omitted the details of the PFE expansion. The inversion of Eq. (28) leads to (27). The symbolic toolbox derivation of this result is straightforward,

```
syms s t Y
X = simplify(laplace(t^2+5*t+3))
Y = solve(s^2*Y-s*2 + 3*(s*Y-2) + 2*Y == s*X, Y)
Y = simplify(Y)       % Y = (2*s^3 + 9*s^2 + 5*s + 2)/(s^2*(s^2 + 3*s + 2))
Y = partfrac(Y,s)     % Y = 4/(s + 1) - 3/(s + 2) + 1/s + 1/s^2
yt = ilaplace(Y)      % y = 4*exp(-t) - 3*exp(-2*t) + 1 + t
```

(h) Next, we work out the full solution of Eq. (27) using the classical method applied with the $t = 0^+$ initial conditions, $y(0^+) = 2$, $\dot{y}(0^+) = 3$. Forming the sum of the forced response and a linear combination of the characteristic modes, we have,

$$\begin{aligned} y(t) &= c_1 e^{-t} + c_2 e^{-2t} + 1 + t \\ \dot{y}(t) &= -c_1 e^{-t} - 2c_2 e^{-2t} + 1 \end{aligned} \qquad \text{(classical method)}$$

for $t \ge 0$. Imposing the $t = 0^+$ conditions, we find,

$$\begin{aligned} y(0^+) &= c_1 + c_2 + 1 = 2 \\ \dot{y}(0^+) &= -c_1 - 2c_2 + 1 = 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 4 \\ c_2 &= -3 \end{aligned}$$

Thus, we obtain the same solution as that in Eq. (27), for $t \ge 0^+$,

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + 1 + t = 4e^{-t} - 3e^{-2t} + 1 + t$$

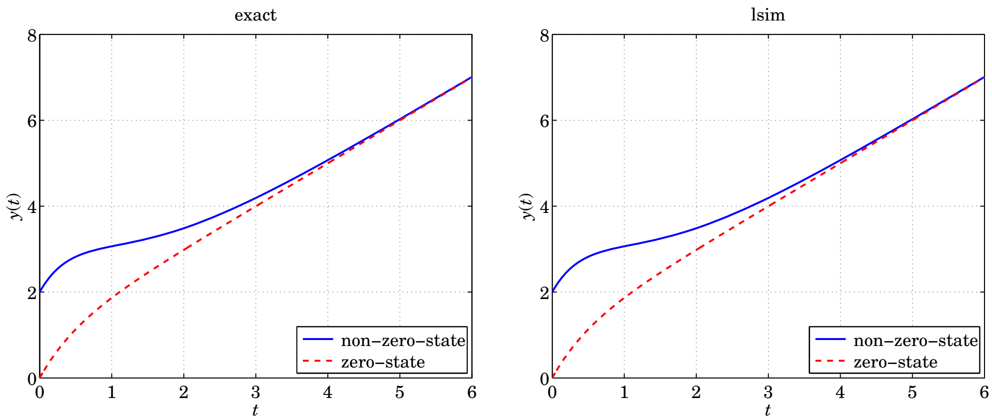
Finally, the same solution can be obtained with the **dsolve** function applied with the initial conditions at $t = 0^+$,

```
syms t y(t)
dy = diff(y,t); ddy = diff(dy,t);
x = t^2 + 5*t + 3; dx = diff(x,t);
y = dsolve(ddy + 3*dy + 2*y == dx, y(0) == 2, dy(0) == 3)
```

(i) The numerical computation using the **lsim** function is carried out in exactly the same way as in the previous problems, only the input and initial conditions are different. The MATLAB code is listed below.

```
y0 = 2; dy0 = 0;           % given initial conditions at t=0-
t = linspace(0,6,601);     % desired time range
x = t.^2 + 5*t + 3;        % input signal
s = tf('s');               % transfer function variable
H = s/(s^2+3*s+2);         % transfer function object - class(H) is tf
S = ss(H);                 % S is state-space model of H - class(S) is ss
yi = [y0; dy0];            % vector of initial conditions with respect to y
xi = obsv(S) \ yi;         % map yi to initial state-vector xi
y = lsim(S,x,t,xi);        % run model S with initial state xi
yzs = lsim(S,x,t);         % run model S with zero initial state xi=0
plot(t,y, t,yzs,'--');     % compare the nonzero-state and zero-state outputs
```

The computed outputs are shown on the right graph below. Those on the left graph are the exact responses $y_{zs}(t), y(t)$ derived in Eq. (26). They are virtually indistinguishable from the numerically computed ones using **lsim**.



Problem 4

Repeat questions (a–i) of Problem 1 for the following linear system,

$$\begin{gathered}\ddot{y}(t)+4\dot{y}(t)+3y(t)=2\ddot{x}(t)+\dot{x}(t)+x(t)\\(D^2+4D+3)y(t)=(2D^2+D+1)x(t)\,,\quad D=\frac{d}{dt}\end{gathered}\tag{29}$$

with initial conditions, $y(0^-)=2$, $\dot{y}(0^-)=-4$, and driven by the causal input,

$$x(t)=e^{-2t}u(t)\tag{30}$$

This problem illustrates how to handle systems that have numerator and denominator of the same order, so that their impulse response has a delta-function term, and also how to handle the mapping between $t=0^-$ and $t=0^+$ initial conditions in the more general case of Eq. (12).

Solution

(a) Taking Laplace transforms of both sides of Eq. (29) with no initial conditions, we have,

$$s^2Y(s)+4sY(s)+3Y(s)=2s^2X(s)+sX(s)+X(s)\quad\Rightarrow\quad\boxed{H(s)=\frac{Y(s)}{X(s)}=\frac{2s^2+s+1}{s^2+4s+3}}$$

Note the factorization,

$$s^2 + 4s + 3 = (s + 1)(s + 3)$$

Thus, the system's poles are at $s = -1$ and $s = -3$. After long division and PFE, we find,

$$H(s) = \frac{2s^2 + s + 1}{s^2 + 4s + 3} = 2 - \frac{7s + 5}{s^2 + 4s + 3} = 2 - \frac{7s + 5}{(s + 1)(s + 3)} = 2 + \frac{1}{s + 1} - \frac{8}{s + 3}$$

These can also be found with the **residue** function,

```
[r,p,k] = residue([2,1,1],[1,4,3])
% r =
%     -8
%      1
% p =
%     -3
%     -1
% k =
%      2
```

Using the symbolic toolbox, we obtain the same PFE,

```
syms s
H = (2*s^2 + s + 1)/(s^2+4*s+3);
H = partfrac(H)     % H = 1/(s + 1) - 8/(s + 3) + 2
```

(b) Inverting the PFE of $H(s)$, we find,

$$h(t) = 2\delta(t) + \left[e^{-t} - 8e^{-3t}\right]u(t)$$

where the constant term inverted into a Dirac delta. Using the symbolic toolbox, we obtain the same,

```
syms s
H = (2*s^2 + s + 1)/(s^2+4*s+3);
H = partfrac(H)
h = ilaplace(H)      % h = exp(-t) - 8*exp(-3*t) + 2*dirac(t)
```

(c) From Eq. (2.23) of the text, we must first determine the solution of the all-pole problem,

$$\ddot{y}_n(t) + 4\dot{y}_n(t) + 3y_n(t) = 0\,, \quad \text{with} \quad y_n(0) = 0\,, \quad \dot{y}_n(0) = 1 \tag{31}$$

Then, since $H(s) = P(s)/Q(s)$, with, $P(s) = 2s^2 + s + 1$, and $Q(s) = s^2 + 4s + 3$, we can obtain $h(t)$ from,

$$h(t) = b_0\delta(t) + \left[P(D)y_n(t)\right]u(t) = 2\delta(t) + \left[2\ddot{y}_n(t) + \dot{y}_n(t) + y_n(t)\right]u(t)$$

The solution of Eq. (31) and its derivative are linear combinations of characteristic modes,

$$\begin{aligned} y_n(t) &= c_1e^{-t} + c_2e^{-3t} \\ \dot{y}_n(t) &= -c_1e^{-t} - 3c_2e^{-3t} \end{aligned}$$

The initial conditions give two equations in the unknowns c_1, c_2,

$$\begin{aligned} y_n(0) &= c_1 + c_2 = 0 \\ \dot{y}_n(0) &= -c_1 - 3c_2 = 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \frac{1}{2} \\ c_2 &= -\frac{1}{2} \end{aligned}$$

Thus,

$$\begin{aligned} y_n(t) &= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \dot{y}_n(t) &= \frac{3}{2}e^{-3t} - \frac{1}{2}e^{-t} \\ \ddot{y}_n(t) &= \frac{1}{2}e^{-t} - \frac{9}{2}e^{-3t} \end{aligned} \quad \Rightarrow \quad \begin{aligned} h(t) &= 2\delta(t) + \left[2\ddot{y}_n(t) + \dot{y}_n(t) + y_n(t)\right]u(t) \\ &= 2\delta(t) + \left[e^{-t} - 8e^{-3t}\right]u(t) \end{aligned}$$

Using the **ilaplace** function, we obtain the same,

```
syms s t Yn
Yn = 1/(s^2 + 4*s + 3);   % denominator part of H(s)
yn = ilaplace(Yn)         % yn = exp(-t)/2 - exp(-3*t)/2
h = 2*dirac(t) + 2*diff(yn,t,2) + diff(yn,t) + yn
```

The same answers for $y_n(t)$ and $h(t)$ can also be obtained using the **dsolve** function,

```
syms t yn(t)
yn = dsolve('D2yn + 4*Dyn + 3*yn=0','yn(0)=0','Dyn(0)=1')
h = 2*dirac(t) + 2*diff(yn,t,2) + diff(yn,t) + yn
```

(d) For the given initial conditions, $y(0^-)=2$ and $\dot{y}(0^-)=-4$, the differential equation (29) with $x(t)=0$ transforms in the s-domain into,

$$\ddot{y}(t)+4\dot{y}(t)+3y(t)=0 \quad \Rightarrow \quad s^2Y(s)-2s+4+4\big(sY(s)-2\big)+3Y(s)=0$$

Solving for $Y(s)$ and performing its partial fraction expansion, we have,

$$Y(s)=\frac{2s+4}{s^2+4s+3}=\frac{2s+4}{(s+1)(s+3)}=\frac{1}{s+1}+\frac{1}{s+3}$$

which gives the zero-input response in the time domain,

$$y_{zi}(t)=e^{-t}+e^{-3t}\,, \quad t\ge 0 \tag{32}$$

An alternative approach is to work in the time-domain and express $y(t)$ and its derivative as a linear combination of characteristic modes, and fix the expansion coefficients from the initial conditions, that is, set

$$\begin{aligned} y(t)&=c_1e^{-t}+c_2e^{-3t}\\ \dot{y}(t)&=-c_1e^{-t}-3c_2e^{-3t}\end{aligned}$$

and at $t=0^-$, impose the conditions,

$$\begin{aligned} y(0^-)&=c_1+c_2=2\\ \dot{y}(0^-)&=-c_1-3c_2=-4\end{aligned} \quad\Rightarrow\quad \begin{aligned} c_1&=1\\ c_2&=1\end{aligned}$$

which results in the same answer as in Eq. (32). The same expression is obtained using the **ilaplace** function of the symbolic toolbox,

```
syms s Y
Y = solve(s^2*Y-s*2+4 + 4*(s*Y-2)+3*Y == 0,Y)   % Y = (2*s + 4)/(s^2 + 4*s + 3)
Y = partfrac(Y,s)                               % Y = 1/(s + 1) + 1/(s + 3)
yzi = ilaplace(Y)                               % yzi = exp(-t) + exp(-3*t)
```

Alternatively, we can use the **dsolve** function (here, the $t=0^{\pm}$ conditions are the same),

```
syms t y(t)
dy = diff(y,t); ddy = diff(dy,t);
yzi = dsolve(ddy + 4*dy + 3*y == 0, y(0) == 2, dy(0) == -4)
```

(e) The convolutional expression for the zero-state output is,

$$y_{zs}(t)=\int_{-\infty}^{\infty}h(t')x(t-t')\,dt'$$

Separating out the delta-function term of $h(t)$, we may write,

$$h(t)=2\delta(t)+g(t)\,, \quad g(t)=\big[e^{-t}-8e^{-3t}\big]u(t)$$

and the convolution integral becomes, for $t\ge 0$,

$$y_{zs}(t)=\int_{-\infty}^{\infty}\big[2\delta(t')+g(t')\big]x(t-t')\,dt'=2x(t)+\int_0^t g(t')x(t-t')\,dt'$$

and for the given input, $x(t)=e^{-2t}u(t)$, we find for $t\geq 0$,

$$
\begin{aligned}
y_{zs}(t) &= 2e^{-2t}+\int_0^t (e^{-t'}-8e^{-3t'})e^{-2(t-t')}dt' \\
&= 2e^{-2t}+e^{-2t}\int_0^t (e^{-t'}-8e^{-3t'})e^{2t'}dt' = 2e^{-2t}+e^{-2t}\int_0^t (e^{t'}-8e^{-t'})dt' \\
&= 2e^{-2t}+e^{-2t}\big[(e^{t}-1)-8(1-e^{-t})\big]
\end{aligned}
$$

thus,

$$
y_{zs}(t)=e^{-t}+8e^{-3t}-7e^{-2t}\,,\quad t\geq 0 \tag{33}
$$

(f) The Laplace transform of the input $x(t)=e^{-2t}u(t)$ is, $X(s)=1/(s+2)$. It follows that the transform of the zero-state output and its PFE will be,

$$
Y(s)=H(s)X(s)=\frac{2s^2+s+1}{s^2+4s+3}\cdot\frac{1}{s+2}=\frac{2s^2+s+1}{(s+1)(s+2)(s+3)}=\frac{1}{s+1}-\frac{7}{s+2}+\frac{8}{s+3}
$$

Inverting the Laplace transform $Y(s)$, we obtain the same zero-state response of Eq. (33). The PFE residues can also be obtained by the function **residue**,

```
[r,p] = residue([2 1 1], conv([1 4 3],[1 2]))
% r =
%     8.0000
%    -7.0000
%     1.0000
% p =
%    -3.0000
%    -2.0000
%    -1.0000
```

The PFE and the Laplace inversions can also be accomplished with the symbolic toolbox,

```
syms s
H = (2*s^2 + s + 1)/(s^2+4*s+3);
X = 1/(s+2);
Y = H*X;             % Y(s) = (2*s^2 + s + 1)/((s + 2)*(s^2 + 4*s + 3))
Y = partfrac(Y)      % Y = 1/(s + 1) - 7/(s + 2) + 8/(s + 3)
yzs = ilaplace(Y)    % yzs = exp(-t) - 7*exp(-2*t) + 8*exp(-3*t)
```

(g) Adding up the zero-input and zero-state responses, we obtain the full solution, for $t\geq 0$,

$$
\begin{aligned}
y_{zi}(t) &= e^{-t}+e^{-3t} \\
y_{zs}(t) &= e^{-t}+8e^{-3t}-7e^{-2t} \\
y(t) &= 2e^{-t}+9e^{-3t}-7e^{-2t}
\end{aligned} \tag{34}
$$

The first two terms represent the "homogeneous" solution and the third, the "forced" response,

$$
y(t)=\underbrace{2e^{-t}+9e^{-3t}}_{\text{homogeneous}}-\underbrace{7e^{-2t}}_{\text{forced}},\quad t\geq 0 \tag{35}
$$

As in Problem 1, the forced response can be predicted in advance by the rule,

$$
x(t)=e^{-at}\;\longrightarrow\; y_{\text{forced}}(t)=H(-a)\,e^{-at}
$$

where with $a=2$, we evaluate $H(-a)=-7$. Next, we rederive Eq. (35) using Laplace transforms and partial fraction expansions. For the given initial conditions, $y(0^-)=2$, $\dot{y}(0^-)=-4$, the transform of the differential equation (29) is,

$$
\begin{aligned}
&\ddot{y}(t)+4\dot{y}(t)+3y(t)=2\ddot{x}(t)+\dot{x}(t)+x(t)\quad\Rightarrow \\
&s^2Y(s)-2s+4+4\big(sY(s)-2\big)+4Y(s)=(2s^2+s+1)X(s)
\end{aligned}
$$

Solving for $Y(s)$, and replacing $X(s) = 1/(s+2)$, we find after some algebra,

$$Y(s) = \frac{4s^2 + 9s + 9}{(s+2)(s^2 + 4s + 3)} = \frac{2}{s+1} + \frac{9}{s+3} - \frac{7}{s+2} \tag{36}$$

which upon inversion yields exactly Eq. (35). The solution for $Y(s)$, its PFE expansion, and inversion can also be carried out simply by the symbolic toolbox,

```
syms s Y
X = 1/(s+2);
Y = solve(s^2*Y-s*2+4 + 4*(s*Y-2) + 3*Y == (2*s^2+s+1)*X, Y)
Y = partfrac(Y,s)
y = ilaplace(Y)
```

(h) We recall that for a second-order system of the form,

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = b_0\ddot{x}(t) + b_1\dot{x}(t) + b_2 x(t) \quad \Rightarrow \quad H(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

and for a causal input $x(t)$ that has no delta-function singularities, the mapping between the initial conditions at $t = 0^-$ and the initial conditions at $t = 0^+$ is given by,

$$\begin{aligned} y(0^+) &= y(0^-) + b_0 x(0^+) \\ \dot{y}(0^+) &= \dot{y}(0^-) + b_0\dot{x}(0^+) + (b_1 - b_0 a_1)x(0^+) \end{aligned} \tag{37}$$

For our particular system, we have, $[b_0, b_1, b_2] = [2, 1, 1]$, and $[a_1, a_2] = [4, 3]$, so that $b_0 = 2$, $b_1 = 1$, $a_1 = 4$, and Eqs. (37) become,

$$\begin{aligned} y(0^+) &= y(0^-) + 2x(0^+) \\ \dot{y}(0^+) &= \dot{y}(0^-) + 2\dot{x}(0^+) - 7x(0^+) \end{aligned} \tag{38}$$

Thus, for the input $x(t) = e^{-2t}u(t)$, and the given initial conditions at $t = 0^-$, we have, $x(0^+) = 1$ and $\dot{x}(0^+) = -2$, so that

$$\begin{aligned} y(0^+) &= 2 + 2 = 4 \\ \dot{y}(0^+) &= -4 - 4 - 7 = -15 \end{aligned} \tag{39}$$

Using these conditions, we may derive the solution of Eq. (35) by the classical method, in which we construct the solution as the sum of a particular solution and a general homogeneous solution. For the particular solution, we may take the forced response, which in our example is, $y_{\text{forced}}(t) = -7e^{-2t}$. For the homogeneous solution we form a linear combination of the characteristic modes e^{-t}, e^{-3t}. Thus,

$$\begin{aligned} y(t) &= c_1 e^{-t} + c_2 e^{-3t} - 7e^{-2t} \\ \dot{y}(t) &= -c_1 e^{-t} - 3c_2 e^{-3t} + 14e^{-2t} \end{aligned} \qquad \text{(classical method)}$$

for $t \geq 0$. Imposing the $t = 0^+$ conditions (39), we have,

$$\begin{aligned} y(0^+) &= c_1 + c_2 - 7 = 4 \\ \dot{y}(0^+) &= -c_1 - 3c_2 + 14 = -15 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 2 \\ c_2 &= 9 \end{aligned}$$

Thus, we obtain the same solution as that in Eq. (35), for $t \geq 0^+$,

$$y(t) = c_1 e^{-t} + c_2 e^{-3t} - 7e^{-2t} = 2e^{-t} + 9e^{-3t} - 7e^{-2t}$$

Finally, the same solution can be obtained with the **dsolve** function applied with the initial conditions at $t = 0^+$ of Eq. (39),

```
syms t y(t)
x = exp(-2*t);
dy = diff(y,t); ddy = diff(dy,t);
dx = diff(x,t); ddx = diff(dx,t);
y = dsolve(ddy + 4*dy + 3*y == 2*ddx+dx+x, y(0) == 4, dy(0) == -15)
```

We finish with some remarks on the 0^- and 0^+ approaches. In both cases, displayed in Eq. (40), the total solution is written as a sum of a "homogeneous" solution and a "particular" inhomogeneous solution.

$$y(t) = \underbrace{e^{-t} + e^{-3t}}_{\text{zero-input}} + \underbrace{e^{-t} + 8e^{-3t} - 7e^{-2t}}_{\text{zero-state}} = \underbrace{2e^{-t} + 9e^{-3t}}_{\text{homogeneous}} - \underbrace{7e^{-2t}}_{\text{forced}}, \qquad t \geq 0 \tag{40}$$

In the 0^- approach, the zero-state part is such a particular solution which is computable independently of the initial conditions using convolution, $h(t) * x(t)$, or by inverting, $H(s)X(s)$, while the zero-input part is a homogeneous solution whose coefficients c_1, c_2 are determined from the $t = 0^-$ conditions.

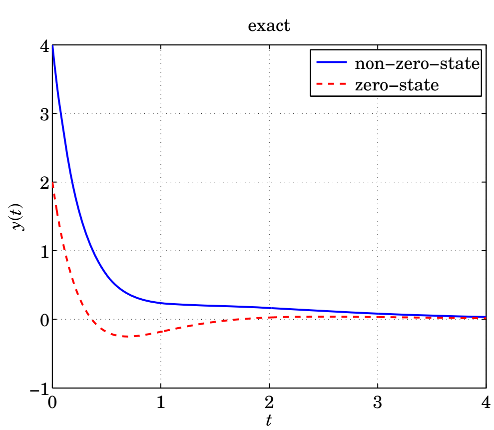
On the other hand, in the 0^+ approach the particular solution is the forced response, while the homogeneous part is determined by fixing its coefficients c_1, c_2 from the $t = 0^+$ conditions.

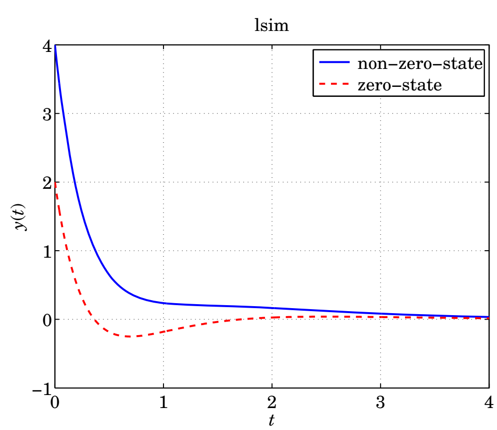
In the 0^- approach, the coefficients of the homogeneous parts arising from the zero-input and zero-state components combine to give the net homogeneous coefficients of the 0^+ approach.

Whether one uses the 0^- or the 0^+ approach depends on how the problem is posed. It is evident that the decomposition into a homogeneous and a particular solution is not unique since we can always add some arbitrary homogeneous terms to the homogeneous part while subtracting them from the particular part. However, the above two specific ways of decomposing are convenient in terms of applying the initial conditions and in terms of guessing the particular solution. These remarks are valid more generally, not just in the second-order case of the present example.

(i) The numerical computation using the **lsim** function is carried out in exactly the same way as in the previous problems. The MATLAB code is listed below.

```
y0 = 2; dy0 = -4;               % given initial conditions at t=0-
t = linspace(0,4,401);          % desired time range
x = exp(-2*t);                  % input signal
s = tf('s');                    % transfer function variable
H = (2*s^2+s+1)/(s^2+4*s+3);    % transfer function object - class(H) is tf
S = ss(H);                      % S is state-space model of H - class(S) is ss
yi = [y0; dy0];                 % vector of initial conditions with respect to y
xi = obsv(S) \ yi;              % initial state-vector
y = lsim(S,x,t,xi);             % run with initial state xi
yzs = lsim(S,x,t);              % run with zero initial state xi=0
plot(t,y, t,yzs,'--');          % compare the nonzero-state and zero-state outputs
```





Problem 5

Consider the following system identification examples. Please solve them analytically (i.e. by hand) and by using MATLAB's symbolic toolbox.

(a) The zero-state response of an unknown causal LTI system H to a unit-step input is,

$$u(t) \xrightarrow{H} e^{-t}u(t) - e^{-3t}u(t)$$

Determine the impulse response $h(t)$ by working exlusively in the time domain. Then, determine it again by working in the s-domain and inverting its transfer function $H(s)$.

(b) The input and corresponding zero-state output of an unknown causal LTI system are shown below,

$$e^{-t}u(t) \xrightarrow{H} e^{-2t}u(t) - e^{-3t}u(t)$$

Determine the system transfer function $H(s)$, and from it, the impulse response $h(t)$.

(c) An unknown signal $x(t)$ is send to the input of the system found in part (b) and the following zero-state output is observed,

$$? \xrightarrow{H} te^{-2t}u(t)$$

Determine $x(t)$. Without any further calculations, determine the input that would cause the following zero-state output and justify your answer,

$$? \xrightarrow{H} (t-5)\,e^{-2(t-5)}u(t-5)$$

Solution

(a) Working in the time domain, the zero-state output is related to the input by convolution,

$$y(t) = \int_0^t h(t')x(t-t')\,dt' = \int_0^t h(t')u(t-t')\,dt' = \int_0^t h(t')\,dt'$$

where the limits were determined by the assumed causality of the input and system, and we replaced the unit-step input, $x(t-t') = u(t-t') = 1$, because $t - t' \geq 0$. Thus, differentiating both sides, we obtain for $t \geq 0$,

$$h(t) = \frac{dy(t)}{dt} = \frac{d}{dt}\left(e^{-t} - e^{-3t}\right) = 3e^{-3t} - e^{-t}$$

Working in the s-domain, we have the Laplace transforms of the unit-step input, the output, and the transfer function,

$$X(s) = \frac{1}{s}, \quad Y(s) = \frac{1}{s+1} - \frac{1}{s+3} = \frac{2}{s^2+4s+3}$$
$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s^2+4s+3} = \frac{3}{s+3} - \frac{1}{s+1}$$

so that the causal Laplace inverse is,

$$h(t) = \left[3e^{-3t} - e^{-t}\right]u(t)$$

The symbolic toolbox calculation is as follows,

```
syms s t
X = 1/s;
y = exp(-t)-exp(-3*t);
Y = laplace(y)          % Y = 1/(s + 1) - 1/(s + 3)
H = collect(Y/X)        % H = (2*s)/(s^2 + 4*s + 3)
H = partfrac(H)         % H = 3/(s + 3) - 1/(s + 1)
h = ilaplace(H)         % h = 3*exp(-3*t) - exp(-t)
```

(b) From the Laplace transforms of the input and output, we obtain $H(s)$,

$$X(s) = \frac{1}{s+1}\,, \quad Y(s) = \frac{1}{s+2} - \frac{1}{s+3} = \frac{1}{s^2+5s+6}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)\,(s+3)} = \frac{2}{s+3} - \frac{1}{s+2}$$

with Laplace inverse,

$$h(t) = \left[2e^{-3t} - e^{-2t}\right]u(t)$$

The symbolic toolbox calculation is,

```
syms s t
x = exp(-t);
y = exp(-2*t)-exp(-3*t);
X = laplace(x)            % X = 1/(s + 1)
Y = laplace(y)            % Y = 1/(s + 2) - 1/(s + 3)
H = collect(Y/X)          % H = (s + 1)/(s^2 + 5*s + 6)
H = partfrac(H)           % 2/(s + 3) - 1/(s + 2)
h = ilaplace(H)           % h = 2*exp(-3*t) - exp(-2*t)
```

(c) From $Y(s) = H(s)X(s)$, we have, $X(s) = Y(s)/H(s)$,

$$Y(s) = \frac{1}{(s+2)^2}\,, \quad H(s) = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)\,(s+3)}$$

$$X(s) = \frac{Y(s)}{H(s)} = \frac{\dfrac{1}{(s+2)^2}}{\dfrac{s+1}{(s+2)\,(s+3)}} = \frac{s+3}{(s+1)\,(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

and inverting,

$$x(t) = \left[2e^{-t} - e^{-2t}\right]u(t)$$

The symbolic toolbox calculation is,

```
syms s t
y = t*exp(-2*t);
Y = laplace(y)           % Y = 1/(s+2)^2
H = (s+1)/(s^2+5*s+6);   % H = (s + 1)/((s + 2)*(s + 3))
X = simplify(Y/H)        % X = (s + 3)/(s^2 + 3*s + 2)
X = prod(factor(X))      % X = (s + 3)/((s + 1)*(s + 2))
X = partfrac(X)          % X = 2/(s + 1) - 1/(s + 2)
x = ilaplace(X)          % x = 2*exp(-t) - exp(-2*t)
```

Since the system is linear time-invariant, if $x(t)$ causes $y(t)$, then, $x(t-t_0)$ will cause $y(t-t_0)$, thus, since the output is delayed by $t_0 = 5$ time units, we must have the same delay at the input, that is,

$$x(t-5) = \left[2e^{-(t-5)} - e^{-2(t-5)}\right]u(t-5)$$