332:345 – Linear Systems and Signals – Fall 2016
Convolution Examples – Set 3 – S. J. Orfanidis

The convolution between two signals $h(t)$ and $x(t)$, denoted by $y(t) = h(t) * x(t)$, is defined by the following integrals,

$$y(t) = \int_{-\infty}^{\infty} h(t')x(t-t')\,dt' = \int_{-\infty}^{\infty} h(t-t')x(t')\,dt' \tag{1}$$

In this set we discuss a number of convolution examples done by the method presented in class, that is, first determining the range of the output time t, and then, determining the proper limits of integration over t'. These limits depend on the value of t and whether one uses the left or the right expression in Eq. (1). This method usually results in a unified expression that is valid for all the output times t. One can then specialize the expression to the various time subintervals that are relevant in each problem.

The starting point of this method is to write down the two inequalities on t' and $t-t'$ that enforce the time constraints of the given functions $h(t)$ and $x(t)$. Then, one solves the inequalities for t, and then for t'. In particular, let us assume that $h(t)$ and $x(t)$ are nonzero over the intervals:

$$\begin{aligned} &h(t)\,, \quad a \le t \le b \\ &x(t)\,, \quad c \le t \le d \end{aligned}$$

where a, b, c, d may be positive or negative or even $\pm\infty$. Working with the left expression in Eq. (1) we have the following inequalities,

$$\begin{array}{c} a \le t' \le b \\ c \le t - t' \le d \end{array} \;\Rightarrow\; \begin{array}{c} a \le t' \le b \\ -d \le t' - t \le -c \end{array} \;\Rightarrow\; \begin{array}{c} a \le t' \le b \\ t - d \le t' \le t - c \end{array} \;\Rightarrow$$

$$\boxed{\begin{array}{c} a + c \le t \le b + d \\ \max(a, t-d) \le t' \le \min(b, t-c) \end{array}} \;\Rightarrow\; \boxed{y(t) = \int_{\max(a,t-d)}^{\min(b,t-c)} h(t')x(t-t')\,dt'} \tag{2}$$

where we obtained the range of the output t by adding them up, that is, $a+c \le t \le b+d$, and then we flipped the second one around and solved for t'. The two inequalities for t' must be simultaneously satisfied, hence the min/max limits. Similarly, working with the right expression in Eq. (1), we have,

$$\begin{array}{c} c \le t' \le d \\ a \le t - t' \le b \end{array} \;\Rightarrow\; \begin{array}{c} c \le t' \le d \\ -b \le t' - t \le -a \end{array} \;\Rightarrow\; \begin{array}{c} c \le t' \le d \\ t - b \le t' \le t - a \end{array} \;\Rightarrow$$

$$\boxed{\begin{array}{c} a + c \le t \le b + d \\ \max(c, t-b) \le t' \le \min(d, t-a) \end{array}} \;\Rightarrow\; \boxed{y(t) = \int_{\max(c,t-b)}^{\min(d,t-a)} h(t-t')x(t')\,dt'} \tag{3}$$

for $a+c \le t \le b+d$. The min/max functions switch from one of their arguments to the other when the arguments are equal. For the expression in Eq. (2), the max-function switches when $a = t - d$, or, $t = a + d$, and the min-function, when $b = t - c$, or, $t = b + c$. These define three subintervals, over which Eq. (2) simplifies. The switch points and subintervals are the same for Eq. (3). These operations are illustrated in the examples below.

Example 1

Determine the convolution of the signals $h(t)$ and $x(t)$ that have the following supports, i.e., the intervals over which they are non-zero,

$$h(t) = 2, \quad -1 \le t \le 1$$
$$x(t) = t, \quad 0 \le t \le 3$$

Since $h(t)$ is a simpler expression, let us work with the right form of Eq. (1). The argument t' of $x(t')$ must lie in its support interval, and similarly, $t - t'$ must lie in the support of $h(t-t')$, that is,

$$\frac{\begin{array}{c} 0 \le t' \le 3 \\ -1 \le t - t' \le 1 \end{array}}{-1 \le t \le 4} \quad \Rightarrow \quad \begin{array}{c} 0 \le t' \le 3 \\ -1 \le t' - t \le 1 \end{array} \quad \Rightarrow \quad \begin{array}{c} 0 \le t' \le 3 \\ t - 1 \le t' \le t + 1 \end{array} \qquad \text{or,}$$

$$-1 \le t \le 4$$
$$\max(0, t-1) \le t' \le \min(3, t+1)$$

The convolution integral then becomes, for $-1 \le t \le 4$,

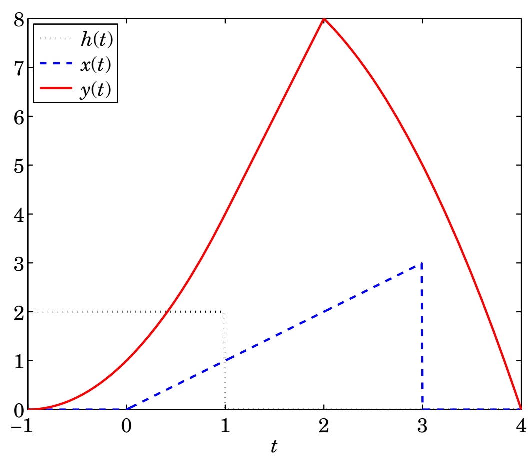
$$y(t) = \int_{\max(0,t-1)}^{\min(3,t+1)} h(t-t')x(t')\,dt' = \int_{\max(0,t-1)}^{\min(3,t+1)} 2 \cdot t'\,dt' = \left[\min(3,t+1)\right]^2 - \left[\max(0,t-1)\right]^2$$

and the switch points are at, $0 = t - 1$, or, $t = 1$, and, $3 = t + 1$, or, $t = 2$, so that the overall output interval $[-1, 4]$ is split into the subintervals, $[-1, 1]$, $[1, 2]$, $[2, 4]$. The expression for $y(t)$ then simplifies accordingly over each subinterval,

$$-1 \le t \le 1 \quad \Rightarrow \quad y(t) = (t+1)^2$$
$$1 \le t \le 2 \quad \Rightarrow \quad y(t) = (t+1)^2 - (t-1)^2 = 4t$$
$$2 \le t \le 4 \quad \Rightarrow \quad y(t) = 9 - (t-1)^2$$

The following MATLAB code evaluates and plots $h(t), x(t), y(t)$.

```
u = @(t) (t>=1);
t = linspace(-2,5,701);
h = 2*(u(t+1)-u(t-1));
x = t.*(u(t)-u(t-3));
y = min(3,t+1).^2 - max(0,t-1).^2;
plot(t,h,'k:', t,x,'b--', t,y,'r-')
```



Note that $h(t)$ corresponds to a non-causal system and, as you can see, the output $y(t)$ starts coming out before the input $x(t)$ begins!

Example 2

Determine the convolution of $h(t)$ and $x(t)$, defined as follows over the support intervals,

$$h(t) = \frac{3t}{2}\,, \qquad 0 \le t \le 1$$
$$x(t) = \frac{4}{3}\,, \qquad 2 \le t \le 5$$

Since $x(t)$ is simpler, let us work with the left form of Eq. (1). The argument t' of $h(t')$ must lie in its support interval, and similarly, $t - t'$ must lie in the support of $x(t - t')$, that is,

$$\begin{array}{c} 0 \le t' \le 1 \\ 2 \le t - t' \le 5 \\ \hline 2 \le t \le 6 \end{array} \quad \Rightarrow \quad \begin{array}{c} 0 \le t' \le 1 \\ -5 \le t' - t \le -2 \end{array} \quad \Rightarrow \quad \begin{array}{c} 0 \le t' \le 1 \\ t-5 \le t' \le t-2 \end{array} \quad \text{or, finally}$$

$$\begin{gathered} 2 \le t \le 6 \\ \max(0, t-5) \le t' \le \min(1, t-2) \end{gathered}$$

The convolution integral then becomes, for $2 \le t \le 6$,

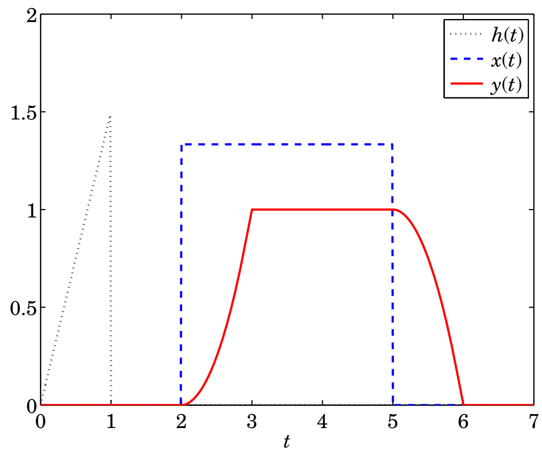
$$y(t) = \int_{\max(0,t-5)}^{\min(1,t-2)} \frac{3t'}{2} \cdot \frac{4}{3}\, dt' = \big[\min(1, t-2)\big]^2 - \big[\max(0, t-5)\big]^2$$

and the switch points are at, $0 = t - 5$, or, $t = 5$, and, $1 = t - 2$, or, $t = 3$, so that the overall output interval $[2,6]$ is split into the subintervals, $[2,3]$, $[3,5]$, $[5,6]$. The expression for $y(t)$ then simplifies accordingly over each subinterval,

$$\begin{aligned} 2 \le t \le 3 \quad &\Rightarrow \quad y(t) = (t-2)^2 \\ 3 \le t \le 5 \quad &\Rightarrow \quad y(t) = 1 \\ 5 \le t \le 6 \quad &\Rightarrow \quad y(t) = 1 - (t-5)^2 \end{aligned}$$

The following MATLAB code evaluates and plots $h(t), x(t), y(t)$.

```
u = @(t) (t>=0);
t = linspace(0,7,701);
h = 3/2*t.*(u(t)-u(t-1));
x = 4/3*(u(t-2)-u(t-5));
y = (min(1,t-2).^2 - max(0,t-5).^2) .* (u(t-2)-u(t-6));
plot(t,h,'k:', t,x,'b--', t,y,'r-')
```



Here, $h(t)$ is causal and the output begins as soon as the input is applied at $t = 2$. The input-on and input-off transients are also observed.

Example 3

Determine the convolution of $h(t)$ and $x(t)$, defined over the following support intervals by,

$$h(t) = 1\,, \qquad -2 \le t \le 1$$
$$x(t) = 1\,, \qquad 4 \le t \le 6$$

Let us work with the left form of Eq. (1). The argument t' of $h(t')$ must lie in its support interval, and similarly, $t - t'$ must lie in the support of $x(t - t')$, that is,

$$\begin{array}{c} -2 \le t' \le 1 \\ 4 \le t - t' \le 6 \\ \hline 2 \le t \le 7 \end{array} \quad \Rightarrow \quad \begin{array}{c} -2 \le t' \le 1 \\ -6 \le t' - t \le -4 \end{array} \quad \Rightarrow \quad \begin{array}{c} -2 \le t' \le 1 \\ t - 6 \le t' \le t - 4 \end{array} \qquad \text{or, finally}$$

$$\begin{array}{c} 2 \le t \le 7 \\ \max(-2, t-6) \le t' \le \min(1, t-4) \end{array}$$

The convolution integral then becomes, for $2 \le t \le 7$,

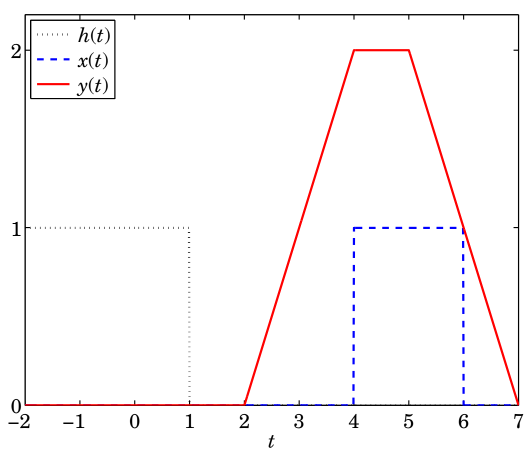
$$y(t) = \int_{\max(-2,t-6)}^{\min(1,t-4)} 1 \cdot 1\, dt' = \min(1, t-4) - \max(-2, t-6)$$

and the switch points are at, $1 = t - 4$, or, $t = 5$, and, $-2 = t - 6$, or, $t = 4$, so that the overall output interval $[2, 7]$ is split into the subintervals, $[2, 4]$, $[4, 5]$, $[5, 7]$. The expression for $y(t)$ then simplifies accordingly over each subinterval,

$$\begin{aligned} 2 \le t \le 4 \quad &\Rightarrow \quad y(t) = (t-4) - (-2) = t - 2 \\ 4 \le t \le 5 \quad &\Rightarrow \quad y(t) = (t-4) - (t-6) = 2 \\ 5 \le t \le 7 \quad &\Rightarrow \quad y(t) = 6 - (t-1) = 7 - t \end{aligned}$$

The following MATLAB code evaluates and plots $h(t), x(t), y(t)$.

```
u = @(t) (t>=0);
t = linspace(-2,7,901);
h = u(t+2)-u(t-1);
x = u(t-4)-u(t-6);
y = (min(1,t-4) - max(-2,t-6)) .* (u(t-2)-u(t-7));
plot(t,h,'k:', t,x,'b--', t,y,'r-')
```



This is also a non-causal system with output that begins before the input!

Example 4

Use the results of the previous example to determine the convolution of $h(t)$ and $x(t)$, defined over the support intervals,

$$h(t) = 1\,, \quad 0 \le t \le 3$$

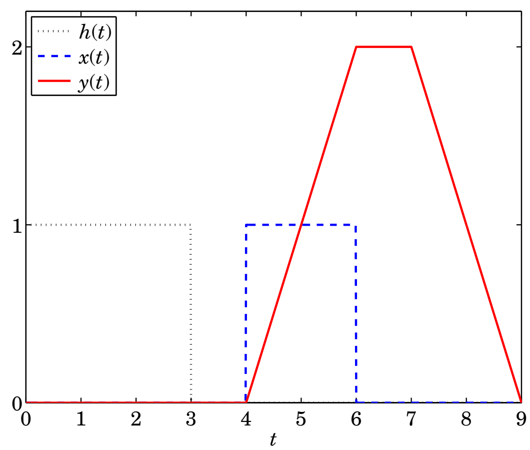
$$x(t) = 1\,, \quad 4 \le t \le 6$$

The present $h(t)$ is the *delayed* version of the previous $h(t)$ by $t = 2$ time units. Therefore, the output $y(t)$ will also be delayed by the same amount. Thus, replacing t by $t - 2$ in the previous example, we find that $y(t)$ will be nonzero over $2 \le t - 2 \le 7$, or, $4 \le t \le 9$, and within that range it will be given by,

$$y(t) = \min(1, t-2-4) - \max(-2, t-2-6) = \min(1, t-6) - \max(-2, t-8)\,, \quad 4 \le t \le 9$$

The following MATLAB code evaluates and plots $h(t), x(t), y(t)$.

```
u = @(t) (t>=0);
t = linspace(0,9,901);
h = u(t)-u(t-3);
x = u(t-4)-u(t-6);
y = (min(1,t-6) - max(-2,t-8)) .* (u(t-4)-u(t-9));
plot(t,h,'k:', t,x,'b--', t,y,'r-')
```



Now, $h(t)$ is causal and the output begins at the same time as the input, i.e., at $t = 4$.

Example 5

Use the results of the previous example to determine the convolution of $h(t)$ and $x(t)$, defined over the following support intervals by,

$$h(t) = 1\,, \quad 0 \le t \le 3$$

$$x(t) = 1\,, \quad 0 \le t \le 2$$

The present $x(t)$ is the time-advanced version of the previous one by $t = 4$ time units. Therefore, the output $y(t)$ will also be advanced by the same amount. Thus, replacing t by $t+4$ in the previous example, we find that $y(t)$ will be nonzero over $4 \le t + 4 \le 9$, or, $0 \le t \le 5$, and within that range it will be given by,

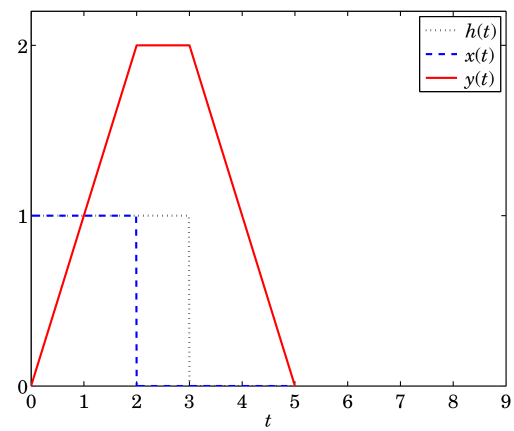
$$y(t) = \min(1, t+4-6) - \max(-2, t+4-8) = \min(1, t-2) - \max(-2, t-4)\,, \quad 0 \le t \le 5$$

this can also be written as,

$$y(t) = \min(3, t) - \max(0, t-2)\,, \quad 0 \le t \le 5$$

with MATLAB code,

```
u = @(t) (t>=0);
t = linspace(0,5,501);
h = u(t)-u(t-3);
x = u(t)-u(t-2);
y = (min(1,t-2) - max(-2,t-4)) .* (u(t)-u(t-5));
plot(t,h,'k:', t,x,'b--', t,y,'r-')
```



Example 6

The convolution, $y(t) = f_1(t) * f_2(t)$, between two unit pulses of durations T_1 and T_2,

$$f_1(t) = u(t) - u(t - T_1)\,, \quad 0 \le t \le T_1$$

$$f_2(t) = u(t) - u(t - T_2)\,, \quad 0 \le t \le T_2$$

is given as follows, where the output time t is in the range, $0 \le t \le T_1 + T_2$,†

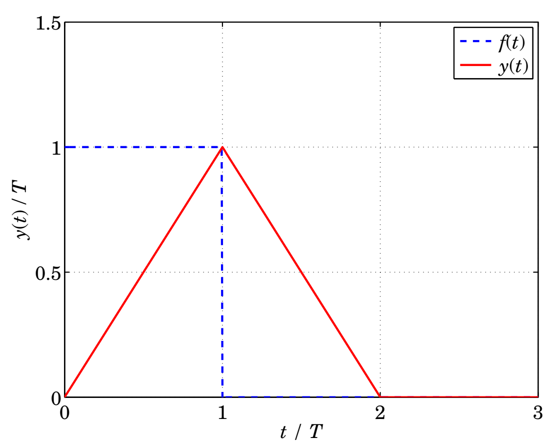
$$y(t) = \min(T_1, t) - \max(0, t - T_2) = \frac{T_1 + T_2 - |t - T_1| - |t - T_2|}{2}$$

For example, the convolution between two identical unit pulses of duration T is,

$$f(t) = u(t) - u(t - T)\,, \quad y(t) = f(t) * f(t)$$

for, $0 \le t \le 2T$,

$$y(t) = \min(T, t) - \max(0, t - T) = \begin{cases} t\,, & 0 \le t \le T \\ 2T - t\,, & T \le t \le 2T \end{cases} = \text{triangular pulse}$$



†we used the following identities, $\min(a,b) = \dfrac{a + b - |a - b|}{2}$, $\max(a,b) = \dfrac{a + b + |a - b|}{2}$, for real a, b.

Example 7

Use the previous result and the delay property to determine the convolution of the two unit pulses,

$$f_1(t) = u(t-1) - u(t-2)\,, \quad 1 \le t \le 2$$
$$f_2(t) = u(t-1) - u(t-3)\,, \quad 1 \le t \le 3$$

If we undelay them, we would have the unit pulses,

$$f_1(t) = u(t) - u(t-1)\,, \quad 0 \le t \le 1$$
$$f_2(t) = u(t) - u(t-2)\,, \quad 0 \le t \le 2$$

which according to the previous problem have convolution ($T_1 \le T_2$ case),

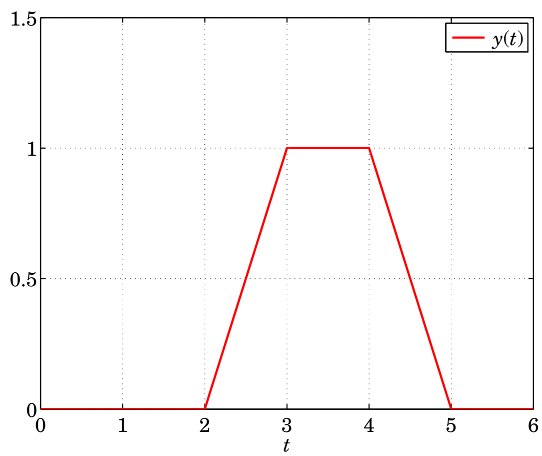
$$y(t) = \min(1, t) - \max(0, t-2)\,, \quad 0 \le t \le 1 + 2 = 3$$

Now, if we apply two delays, each by $t = 1$ time units, we would get the desired result,

$$y(t) = \min(1, t-2) - \max(0, t-4)\,, \quad 2 \le t \le 5$$

with MATLAB code,

```
u = @(t) (t>=0);
t = linspace(0,6,601);
y = (min(1,t-2)-max(0,t-4)).*(u(t-2)-u(t-5));
figure; plot(t,y,'r-','linewidth',2);
```



Example 8

If $h(t)$ or $x(t)$ consist of multiple segments, you can break them into separate parts, and add up the answers. For example, consider the convolution of the two signals,

$$h(t) = \begin{cases} t\,, & 0 \le t \le 1 \\ 1\,, & 1 \le t \le 2 \\ 0\,, & \text{otherwise} \end{cases}\,, \qquad x(t) = u(t-1) - u(t-3) = \begin{cases} 1\,, & 1 \le t \le 3 \\ 0\,, & \text{otherwise} \end{cases}$$

We can write $h(t)$ as the sum of the two parts,

$$h(t) = h_1(t) + h_2(t)\,, \qquad \begin{aligned} h_1(t) &= t\big[u(t) - u(t-1)\big] \\ h_2(t) &= u(t-1) - u(t-2) \end{aligned}$$

Then, $y(t)$ can be calculated as the sum,

$$y(t) = h(t) * x(t) = \big[h_1(t) + h_2(t)\big] * x(t) = h_2(t) * x(t) + h_2(t) * x(t) = y_1(t) + y_2(t)$$

The case, $y_2 = h_2 * x$, was worked out in Example 7. The case, $y_1 = h_1 * x$, is very similar to that of Example 2, and may be solved by setting up the inequalities,

$$\begin{array}{c} 0 \le t' \le 1 \\ 1 \le t - t' \le 3 \\ \hline 1 \le t \le 4 \end{array} \quad \Rightarrow \quad \begin{array}{c} 0 \le t' \le 1 \\ -3 \le t' - t \le -1 \end{array} \quad \Rightarrow \quad \begin{array}{c} 0 \le t' \le 1 \\ t-3 \le t' \le t-1 \end{array} \quad \text{or, finally}$$

$$\begin{gathered} 1 \le t \le 4 \\ \max(0, t-3) \le t' \le \min(1, t-1) \end{gathered}$$

The convolution integral then becomes, for $1 \le t \le 4$,

$$y_1(t) = \int_{\max(0,t-3)}^{\min(1,t-1)} t' \cdot 1\, dt' = \frac{1}{2}\big[\min^2(1, t-1) - \max^2(0, t-3)\big]$$

Combining this with the answer of the previous example, we obtain the total output, for, $1 \le t \le 5$,

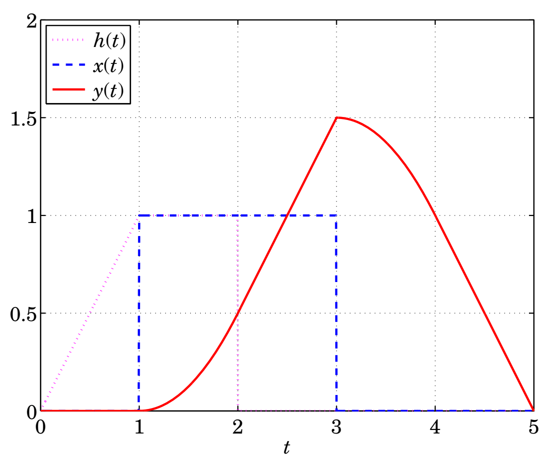
$$\begin{aligned} y(t) = y_1(t) + y_2(t) = &\frac{1}{2}\big[\min^2(1, t-1) - \max^2(0, t-3)\big]\big[u(t-1) - u(t-4)\big] \\ &+ \big[\min(1, t-2) - \max(0, t-4)\big]\big[u(t-2) - u(t-5)\big] \end{aligned}$$

The switch times are at $t = 2$, $t = 3$, and $t = 4$, and the above expression specializes as follows in the four subintervals, $[1,2]$, $[2,3]$, $[3,4]$, $[4,5]$, where $y_1(t)$ and $y_2(t)$ overlap over the subintervals $[2,3]$ and $[3,4]$,

$$\begin{aligned} 1 \le t \le 2 \quad &\Rightarrow \quad y(t) = \frac{1}{2}(t-1)^2 \\ 2 \le t \le 3 \quad &\Rightarrow \quad y(t) = \frac{1}{2} + t - 2 = t - \frac{3}{2} \\ 3 \le t \le 4 \quad &\Rightarrow \quad y(t) = \frac{1}{2} - \frac{1}{2}(t-3)^2 + 1 = \frac{3}{2} - \frac{1}{2}(t-3)^2 \\ 4 \le t \le 5 \quad &\Rightarrow \quad y(t) = 1 - (t-4) = 5 - t \end{aligned}$$

The MATLAB code is,

```
u = @(t) (t>=0);
t = linspace(0,5,1001);
h = t.*(u(t)-u(t-1)) + u(t-1)-u(t-2);
x = u(t-1)-u(t-3);
y = 1/2*(min(1,t-1).^2 - max(0,t-3).^2) .* (u(t-1)-u(t-4)) + ...
    (min(1,t-2) - max(0,t-4)).*(u(t-2)-u(t-5));
figure; plot(t,h,'k:', t,x,'b--', t,y,'r-')
```



Example 9

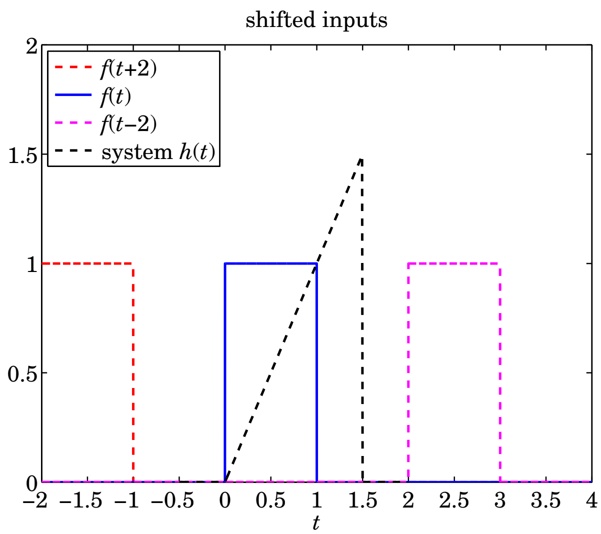
This example clarifies the calculations of Problem 2.4-29 of the text using the approach discussed in class. Consider an input signal $f(t)$ and an LTI system $h(t)$, with corresponding convolutional output,

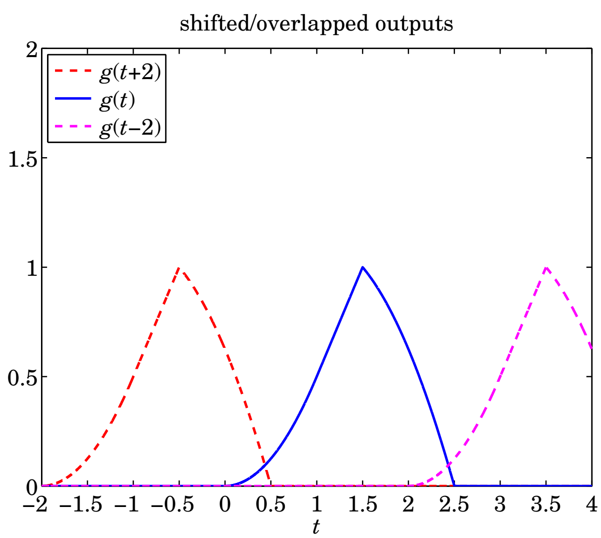
$$g(t) = h(t) * f(t) = \int_{-\infty}^{\infty} h(t')f(t-t')dt'$$

From the linearity and time-invariance of the system, it follows that if the input is periodic and expressed as a sum of shifted copies of $f(t)$ at some period T, then the output will also be periodic with period T and expressed as a sum of shifted copies of $g(t)$, that is,

$$x(t) = \sum_{p=-\infty}^{\infty} f(t-pT) \quad \Rightarrow \quad y(t) = \sum_{p=-\infty}^{\infty} g(t-pT) \tag{4}$$

If $f(t)$ has duration T, then $f(t)$ represents one period of the input. However, the output $g(t)$ will necessarily have longer length than T by an amount equal to the length of the filter $h(t)$. Therefore, the shifted copies of $g(t)$ will overlap with each other. The graphs below demonstrate this for the case of Problem 2.4-29.





The signal $f(t)$ and the system $h(t)$, shown below, are defined as follows,

$$f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}, \qquad h(t) = \begin{cases} t, & 0 \le t \le 1.5 \\ 0, & \text{otherwise} \end{cases}$$

The signal $f(t)$ is periodically replicated with period $T = 2$. The upper graph below shows three replicas of $f(t)$ shifted at period $T = 2$, that is, the copies $f(t+2), f(t), f(t-2)$. The bottom graph shows the corresponding individual outputs, $g(t+2), g(t), g(t-2)$, which partially overlap.

The duration of $g(t)$ extends over $0 \le t \le 2.5$. Therefore, its replica $g(t+2)$ that starts at $t = -2$, will extend over the time interval $-2 \le t \le 0.5$, thus, overlapping with $g(t)$ over the interval $0 \le t \le 0.5$. Therefore, over one period, say, $0 \le t \le 2$, the complete output $y(t)$ due to the periodic signal $x(t)$ will be given as follows,

$$y(t) = \begin{cases} g(t+2) + g(t), & 0 \le t <= 0.5 \\ g(t), & 0.5 \le t \le 2 \end{cases} \tag{5}$$

and this period will be replicated at multiples of $T = 2$. The individual output $g(t)$ due to $f(t)$ can be calculated with our class method, that is,

$$g(t) = \int_{-\infty}^{\infty} h(t')f(t-t')\,dt'$$

with t, t' being restricted as follows,

$$\frac{\begin{matrix} 0 \le t' \le 1.5 \\ 0 \le t - t' \le 1 \end{matrix}}{0 \le t \le 2.5} \quad \Rightarrow \quad \begin{matrix} 0 \le t' \le 1.5 \\ -1 \le t' - t \le 0 \end{matrix} \quad \Rightarrow \quad \frac{\begin{matrix} 0 \le t' \le 1.5 \\ t-1 \le t' \le t \end{matrix}}{\max(0, t-1) \le t' \le \min(1.5, t)}$$

Thus, $g(t)$ is given by the following single expression over, $0 \le t \le 2.5$,

$$g(t) = \int_{\max(0,t-1)}^{\min(1.5,t)} t'\,dt' = \frac{1}{2}\big[\min(1.5,t)\big]^2 - \frac{1}{2}\big[\max(0,t-1)\big]^2$$

The switch points are at $t = 1$ and $t = 1.5$, so that $g(t)$ specializes as follows over the subintervals,

$$\begin{aligned}
0 \le t \le 1, &\quad g(t) = \frac{1}{2}t^2 \\
1 \le t \le 1.5, &\quad g(t) = \frac{1}{2}t^2 - \frac{1}{2}(t-1)^2 = t - \frac{1}{2} \\
1.5 \le t \le 2.5, &\quad g(t) = \frac{9}{8} - \frac{1}{2}(t-1)^2 = \frac{5}{8} - \frac{1}{2}t^2 + t
\end{aligned}$$

Splitting the interval $0 \le t \le 1$ in half and adding the contribution of $g(t+2)$ over $0 \le t \le 0.5$, we obtain from Eq. (5),

$$\begin{aligned}
0 \le t \le 0.5, &\quad y(t) = \frac{9}{8} - \frac{1}{2}(t+2-1)^2 + \frac{1}{2}t^2 = \frac{5}{8} - t \\
0 \le t \le 1, &\quad y(t) = \frac{1}{2}t^2 \\
1 \le t \le 1.5, &\quad y(t) = \frac{1}{2}t^2 - \frac{1}{2}(t-1)^2 = t - \frac{1}{2} \\
1.5 \le t \le 2, &\quad y(t) = \frac{5}{8} - \frac{1}{2}t^2 + t
\end{aligned}$$

The MATLAB code for producing the above graphs was as follows:

```
u = @(t) (t>=0);
f = @(t) u(t)-u(t-1);
g = @(t) (min(t,1.5).^2 - max(0,t-1).^2).*(u(t)-u(t-2.5))/2;

t = linspace(-2,4,6001);

x = f(t+4) + f(t+2) + f(t) + f(t-2) + f(t-4);
h = t.*(u(t)-u(t-1.5));

figure; plot(t,f(t+2),'r--', t,f(t),'b-', t,f(t-2),'m--', t,h,'k--');
figure; plot(t,g(t+2),'r--', t,g(t),'b-', t,g(t-2),'m--');
```