Bode Plots

What are Bode Plots?

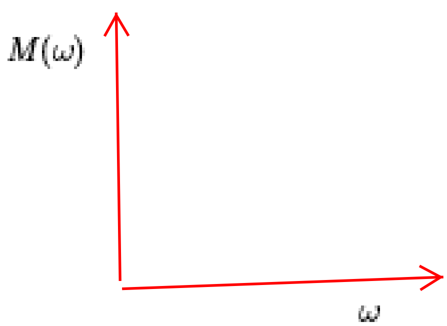
Transfer Function (TF) $H(s)$

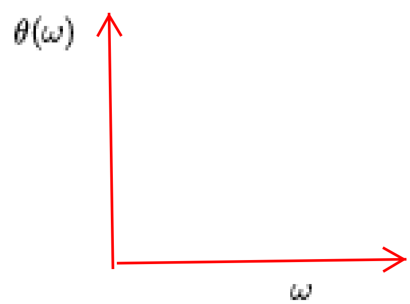
$H(j\omega) = M(\omega)\underline{/\,\theta(\omega)}$

$M(\omega)$ is the magnitude of $H(s)$ as a function of ω

$\theta(\omega)$ is the angle of $H(s)$ as a function of ω

We need to plot both $M(\omega)$ and $\theta(\omega)$ with respect to ω





Often the range of ω is from very low frequency to a high frequency.

Low frequency could be as low as 0.01 rad/sec and high frequency could be as high as 100 K rad/sec.

How to accommodate such a large range is some thing we need to look into.

It turns out linear scale for ω is not well suited.

We need to resort to **logarithmic scale**.

Magnitude $M(\omega)$ could be plotted by itself or could be converted into **deci Bells (dB)**.

What is dB?

Magnitude of M in dB $= 20\log_{10} M$.

Properties of logarithms:

$\log(ab) = \log(a) + \log(b)$

$\log_{10}(10a) = \log_{10}(10) + \log_{10}(a) = 1 + \log_{10}(a)$

$\log \frac{a}{b} = \log(a) - \log(b)$

Magnitude M	**Magnitude M in dB = 20 $\log_{10}$ M**
0	$-\infty$
1	0
$\sqrt{2}$	3
$\frac{1}{\sqrt{2}}$	-3
2	6
$\frac{1}{2}$	-6
10	20
100	40
1000	60
0.1	-20
0.01	-40
0.001	-60

We cannot plot this

M > 0

For every two fold increase, 6 dB gets added.

For every ten fold increase, 20 dB gets added.

Motivation for non-linear scaling along the frequency axis:

- First explain linear scale

- **Disadvantage of linear scale when wide-spread data exists:** Let us take an example. Consider that we are interested in a frequency range from 1 to 10,000 Hz. Suppose we can afford to have 10 CM graph along the frequency axis. Then in a linear scale each cm corresponds to 1000 HZ.

  Draw the axis

  In this scale it is hard to distinguish low frequencies, say 1 Hz from 10 Hz although there is a **tenfold** increase from 1 to 10.

- **Linear scale disregards the sense of proportionality:** To explain this, for the sake of argument, let us say we can afford a graph of 1000 CM. (I know it is not feasible.) We can perhaps distinguish all frequencies then.

  Draw the axis

  In this case, a change from 1 Hz to 10 Hz is a **tenfold** change in frequency and it is represented by 1 CM of graph. On the other hand, a change from 1000 Hz to 1010 Hz is only a 1 % change in frequency and it is still represented by 1 CM of graph. In 1 % change in frequency, the characteristic we are trying to draw would not have changed much (unless 1000 Hz happens to be a critical frequency). What this says is that a linear scale **lacks** the sense of proportionality along the frequency axis.

- A logarithmic scale takes into account such a sense of proportionality and cures both the above disadvantages.

$\log_{10}(10a) = \log_{10}(10) + \log_{10}(a) = 1 + \log_{10}(a)$

Logarithmic Scale

We are interested in plotting the magnitude and angle characteristics of $H(s)$ when $s = j\omega$ for a very **broad range** of values of ω. Linear scale along the ω axis is not feasible. We need a non-linear scale. A common non-linear scale used is the **logarithmic scale**. The following describes the nature of logarithmic scale. Note that the **representation of zero along the logarithmic axis is not feasible** since $\log_{10}(0) = -\infty$.

Linear Versus Logarithmic Scale

Linear Progression ω	Non-linear Logarithmic Progression $\log_{10}(\omega)$
0	$-\infty$ (not feasible)
1	0
2	0.30103
3	0.47712
4	0.60206
5	0.69897
6	0.77815
7	0.84510
8	0.90309
9	0.95424
10	1

$$\log_{10}(\alpha\omega) = \log_{10}(\omega) + \log_{10}(\alpha)$$

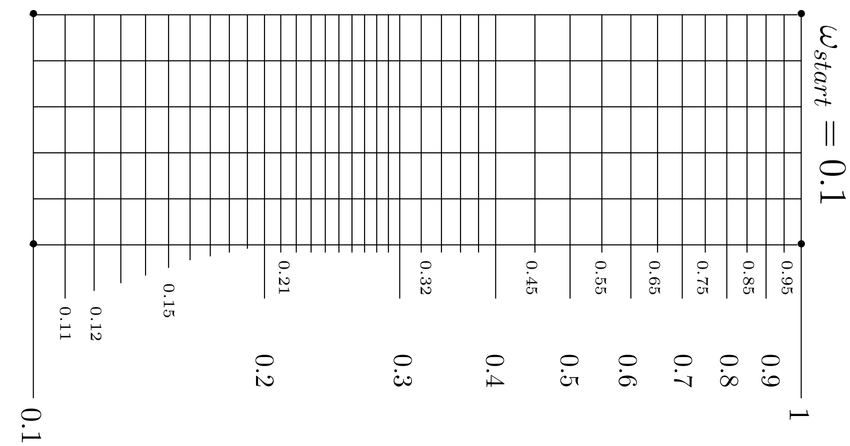
- Multiplying ω by a number α implies adding $\log_{10}(\alpha)$ to $\log_{10}(\omega)$ in logarithmic domain.
- Let $\alpha = 10$. In order to plot $\log_{10}(10\omega)$, we need to add **one unit** to $\log_{10}(\omega)$ since $\log_{10}(10) = 1$. **A ten fold increase in ω adds only one unit of length along the logarithmic axis. This means that the frequency axis is compressed in logarithmic domain.** Clearly, as seen in the graph paper on the other side, each unit along the logarithmic axis corresponds to one cycle. **Say we start at some number n. In order to reach $10n$, we need to move one unit along the logarithmic axis.**

- A range of magnitude extending from some value to **ten** times that value is called a **decade**. Thus, a **one unit** length on the logarithmic axis corresponds to **one decade**. Note that a range of magnitude extending from some value to **two** times that value is called an **octave**.
- We can start with any number ω_{start} other than zero at the beginning of first cycle. If the graph paper has six cycles, then the end of sixth cycle would be $10^6\omega_{start}$ (a multiple of 10 or a decade for each cycle).
- In **Bode plots**, either the magnitude in dB or angle of $H(j\omega)$ is plotted with respect to frequency. To do so, we use semi-log sheets. Frequency axis is the Logarithmic Axis.

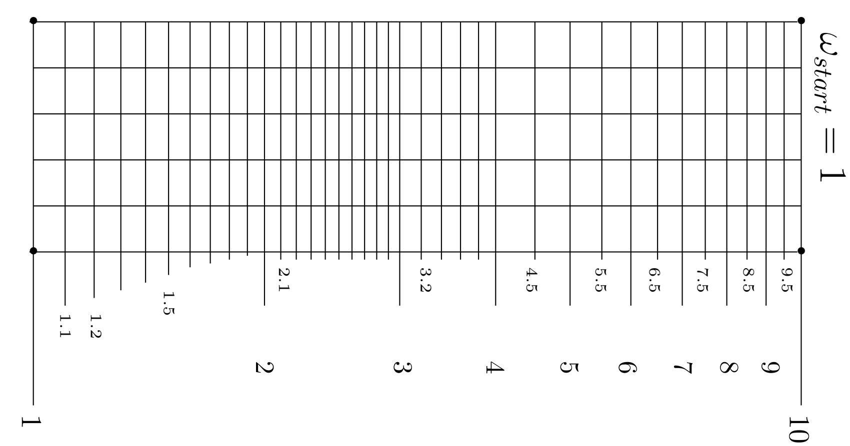
An increase or decrease of frequency by a factor 10 is referred to as a DECADE.

An increase or decrease of frequency by a factor 2 is referred to as an OCTAVE.

In actuality, octave refers to eight and not two. The reason that doubling a frequency is called an octave comes from the musical world. An octave is a doubling up frequency, but it's eight notes in the scale to go up an octave.

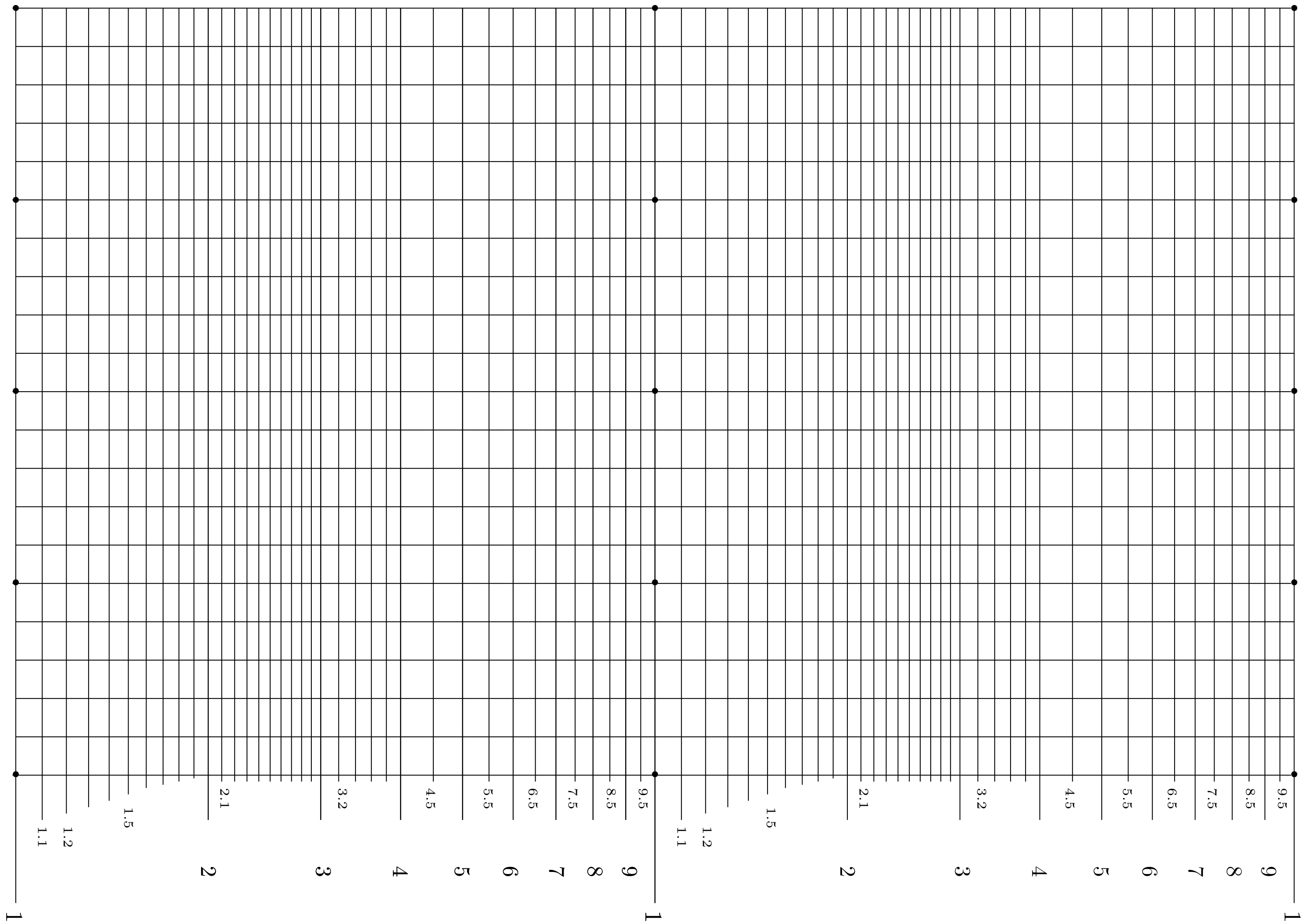


Logarithmic Axis



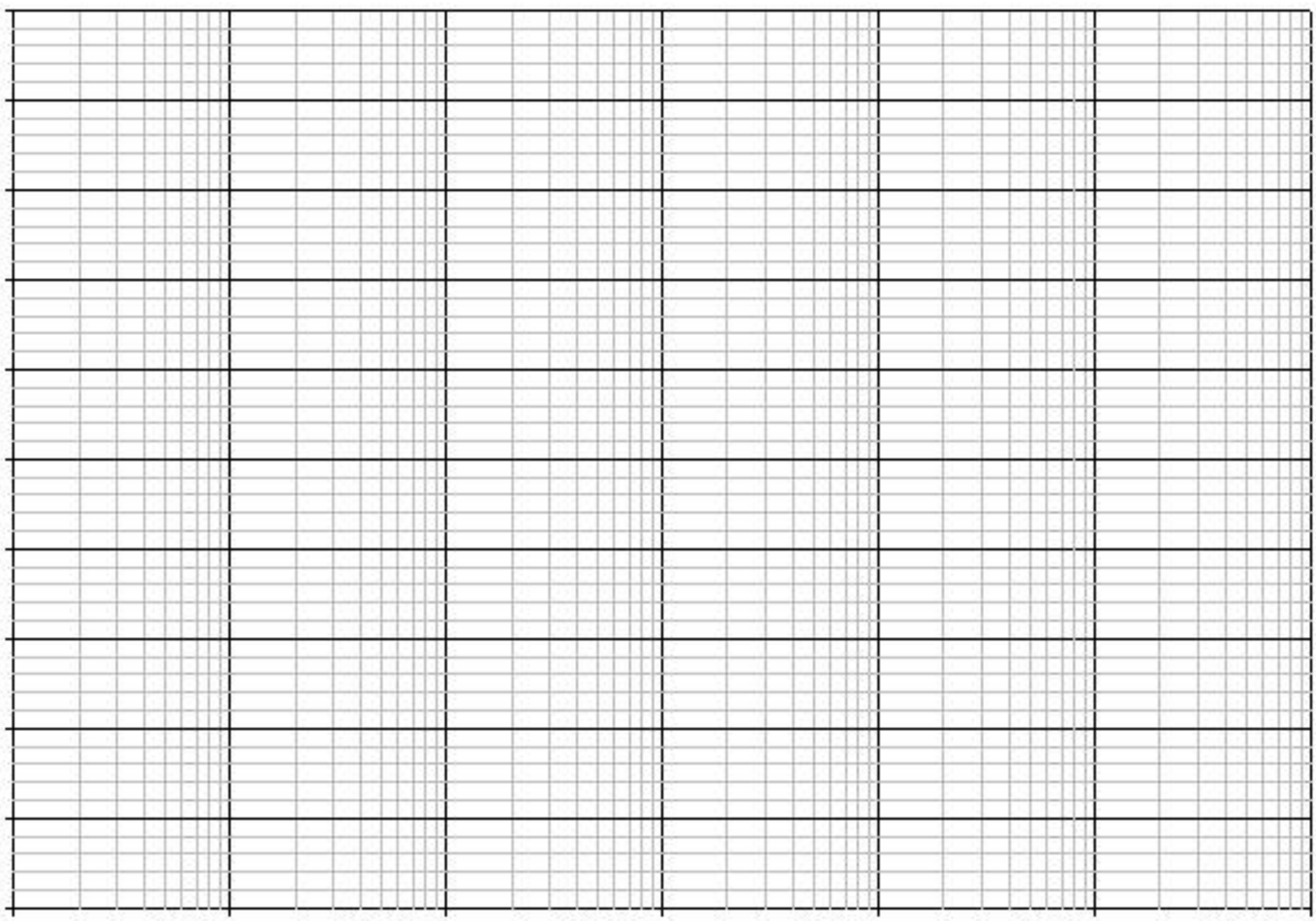
Logarithmic Axis

This is a two cycle semi-log sheet



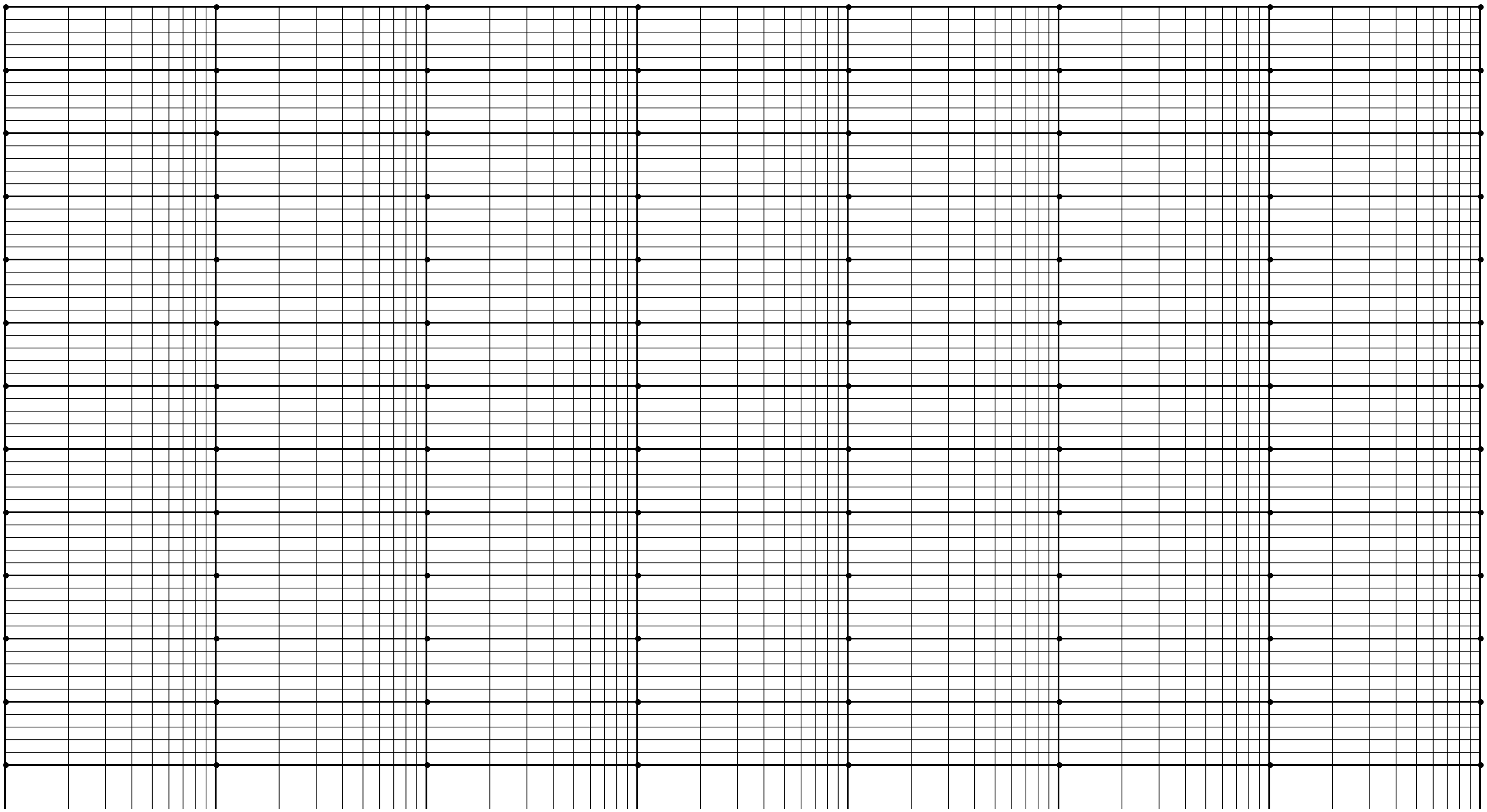
Logarithmic Axis

This is a six cycle semi-log sheet



Logarithmic Axis

This is a 7 cycle semi-log sheet



Logarithmic Axis

20 dB per decade is equivalent to 6 dB per octave.
In music, there are eight scalesto go up when the frequency is doubled.

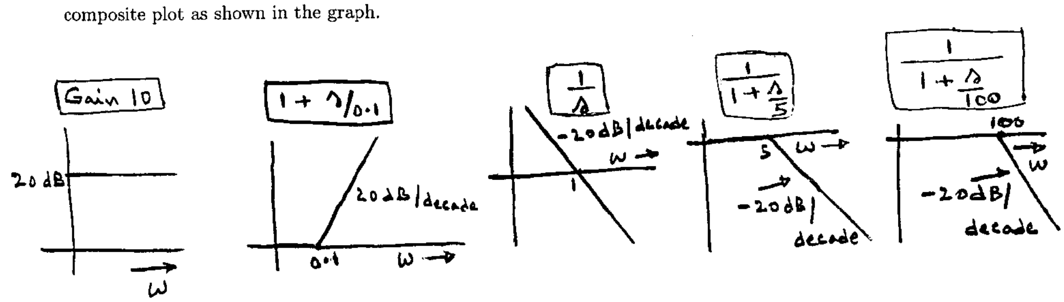
Bode Plots

Example Construct the asymptotic magnitude Bode plot for the transfer function

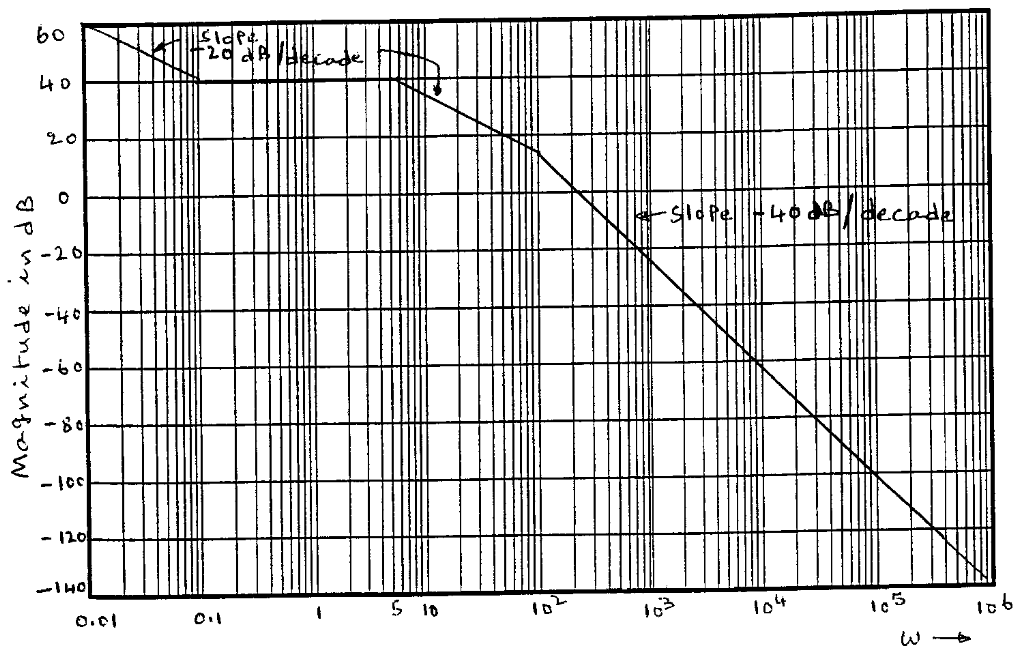
$$H(s) = \frac{5(10^4)(s+0.1)}{s(s+5)(s+100)} = \frac{10(1+\frac{s}{0.1})}{s(1+\frac{s}{5})(1+\frac{s}{100})}$$

The gain 10 contributes 20 dB **for all** ω. The term $1+\frac{s}{0.1}$ in the numerator has a corner frequency at $\omega = 0.1$, and hence it contributes 20 dB per decade asymptote starting at $\omega = 0.1$. It is critical to observe that the term s in the denominator contributes **for all** ω a linear plot with a slope of -20 dB per decade passing through 0 dB at $\omega = 1$. Thus, this term contributes to a gain of 40 dB at $\omega = 0.01$. The terms $1+\frac{s}{5}$ and $1+\frac{s}{100}$ in the denominator have corner frequencies at $\omega = 5$ and $\omega = 100$ respectively, and hence they contribute -20 dB per decade asymptotes starting at $\omega = 5$ and $\omega = 100$ respectively.

Individual plots for each term are sketched below. Addition of all the individual plots gives the over all composite plot as shown in the graph.



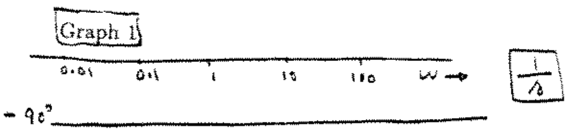
critical corner frequencies are $\boxed{0.1, 5 \text{ and } 100.}$

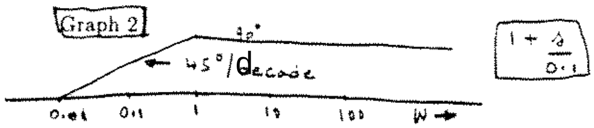


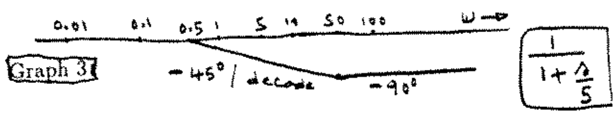
Example Construct the asymptotic phase angle Bode plot for the transfer function

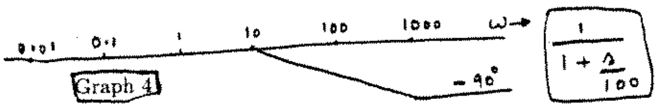
$$H(s) = \frac{5(10^4)(s+0.1)}{s(s+5)(s+100)} = \frac{10(1+\frac{s}{0.1})}{s(1+\frac{s}{5})(1+\frac{s}{100})}$$

The phase angle Bode plot for each term is shown below.









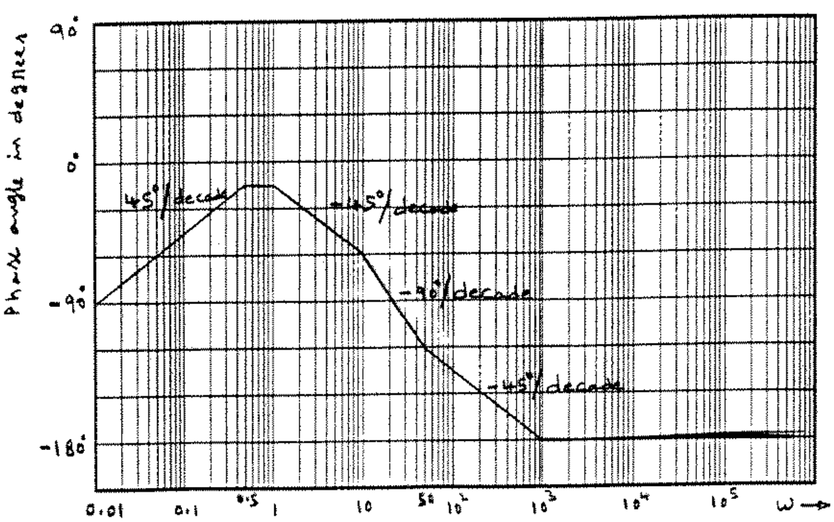
Addition of Piece-wise linear graphs

Piece-wise linear graphs consists of segments of linear graphs. The characteristic of each piece-wise linear graph changes its character at some **critical frequencies** along the frequency axis. In the graphs shown on the left, **critical frequencies** are as follows:

1. Graph 1 has no critical frequencies.
2. Graph 2 has critical frequencies at 0.01 and 1.0.
3. Graph 3 has critical frequencies at 0.5 and 50.0.
4. Graph 4 has critical frequencies at 10 and 1000.

The addition of all four graphs results in **critical frequencies** at

0.01, 0.5, 1.0, 10, 50, and 1000.



At 0.01, a positive slope starts

At 0.5, a negative slope starts

The effect of the above two renders the net slope zero from 0.5 on wards until 1.
At 1, the positive slope started at 0.01 ends. The net effect from 1 onwards until 10 is a negative slope.
At 10, another negative slope starts. the net effect is then twice the negative slope until 50 at which point the negative slope started at 0.5 disappears bringing the net slope to only one negative value.
At 1000, the negative slope started at 10 disappears rendering the net slope 0 from 1000 omwards.

Since Graph 1 starts with -90° and the rest at zero, the graph starts at -90°.

As we proceed from the first critical frequency 0.01 to the next critical frequency 0.5, the slope of 45° per decade of Graph 2 comes into picture as Graphs 3 and 4 do not yet contribute any to the total.

As we proceed from the critical frequency 0.5 to the next critical frequency 1.0, the slope of 45° per decade of Graph 2 and the slope of -45° per decade of Graph 3 cancel each other. Graph 4 in this region has no contribution.

As we proceed from the critical frequency 1.0 to the next critical frequency 10, we see that Graphs 2 and 4 have no slope, and only Graph 3 has the slope of -45° per decade. The net slope is -45° per decade.

As we proceed from the critical frequency 10 to the next critical frequency 50, we see that Graph 2 has no slope, and Graphs 3 and 4 each have the slope of -45° per decade. The net slope is -90° per decade.

As we proceed from the critical frequency 50 to the next critical frequency 1000, we see that Graphs 1, 2, and 3 have no slope, and only Graph 4 has the slope of -45° per decade. The net slope is -45° per decade.

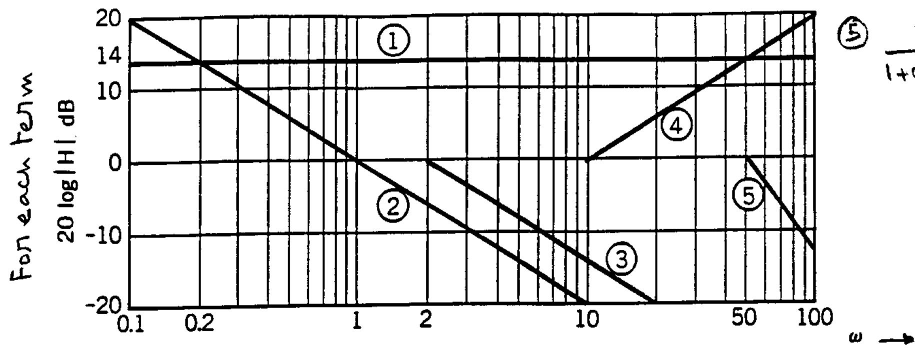
As we proceed from the critical frequency 1000 onwards, none of the graphs have any slope, so the net graph flattens.

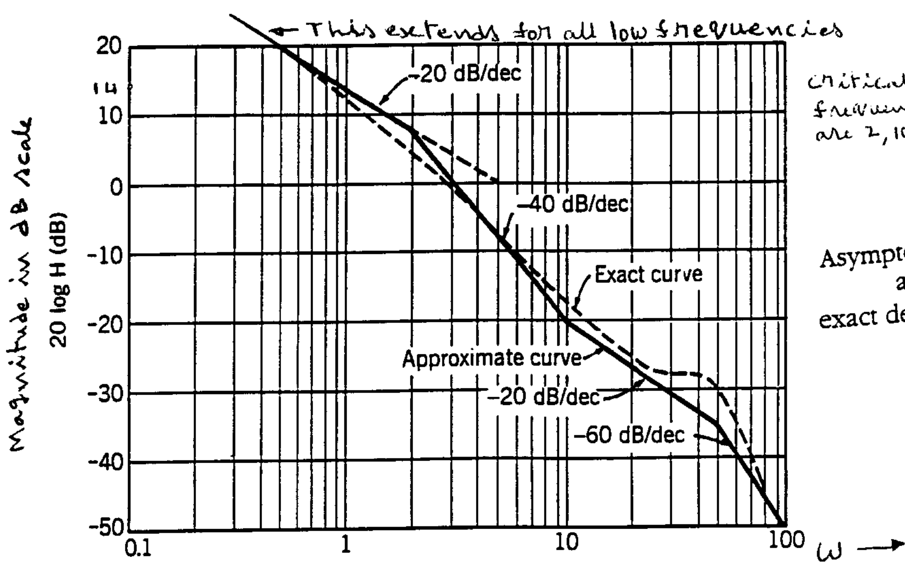
$\boxed{\text{Example with Complex Roots}}$

$$H(s) = \frac{5(1+s/10)}{s(1+s/2)\left(1+0.6\frac{s}{50}+\left(\frac{s}{50}\right)^2\right)}.$$

Term ① constant 5
② s
③ $1/(1+\frac{s}{2})$
④ $1+\frac{s}{10}$
⑤ $\dfrac{1}{1+0.6\frac{s}{50}+\left(\frac{s}{50}\right)^2}$

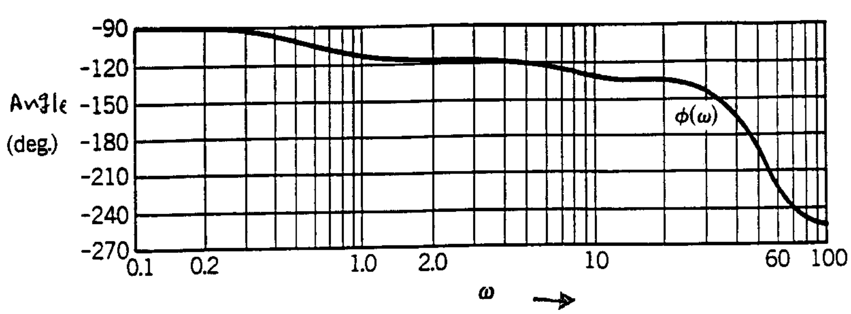
The quadratic factor with complex poles has a ζ of 0.3.





critical frequencies are 2, 10 and 50.

Asymptotic curve and exact decibel curve



phase characteristic

`HW due date will be announced in the class`

Student's name in capital letters:

This is a HW problem, collected and graded.

Problem 1: Construct both the magnitude and phase angle plots for the transfer function

$$H(s) = \frac{2000s}{(s+2)(s+10)}$$

Correct the asymptotic plots as needed. Draw the plots on semi-log sheets with appropriate scale so that the entire graph with all details fills the graph sheet.

Problem 2: Construct ~~both~~ the magnitude ~~and phase angle~~ plot~~s~~ for the transfer function

$$H(s) = \frac{100(s+10)}{(s+1)(1+s+s^2)}$$

Correct the asymptotic plots as needed. ~~Correct the asymptotic plots as needed~~. Draw the plot~~s~~ on semi-log sheet~~s~~ with appropriate scale so that the entire graph with all details fills the graph sheet.