The guy who invented a single wheel was an idiot
The guy who put four wheels together to form a cart was a genius
Design for a desired purpose is the name of the game
Analysis is the backbone of design

Frequency Sensitive Circuits – Filters

There are four distinct topics related to filters:

1. **Preliminaries:** Terminology, Bandwidth computations, Scaling
2. **Frequency behavior of transfer functions:** Bode Plots
3. **Determination of transfer functions of circuits**
4. **Filter Design**

The following are some specific items.

- Introduction and Terminology
- Background needed from Laplace Transform
- Ideal filters
- Computation of half power frequencies and bandwidth
- Magnitude and frequency scaling
- Bode plots
- Derivation of transfer functions of some filter circuits
- Cascading of filters
- Loading effects
- A simple methodology of filter design
- Design of filters by cascading the same filter circuit over and over
- Butterworth LPF design; if time permits
- Butterworth HPF design; if time permits
- BPF design by utilizing LPF and HPF
- BRF design by utilizing LPF and HPF
- Narrowband BPF and BRF

Some of the above material is in Chapters 14 and 15 of the text book by Nilsson and Riedel, but not all of it. Also, the order of the material covered in the class is different and follows the above indicated order.

Introduction and Terminology

A basic introduction to Fourier transform of a time-domain signal:

- A result due to Fourier and Laplace that revolutionized many scientific fields including Electrical Engineering is the concept that **any signal can be thought of as being composed by a number (finite or infinite) of sinusoidal signals of different frequencies, amplitudes, and phase angles**.

- For example, audio signals generated by human voice can be thought of as a sum of sinusoidal signals of frequencies ranging from 20 Hz to 5 kHz, while music signals can be thought of as a sum of sinusoidal signals of frequencies ranging from 20 Hz to 20 kHz.

- The sinusoidal signals that compose a signal $x(t)$ are called the frequency components of $x(t)$. If a signal $x(t)$ has a countably finite or infinite number of frequency components, then

$$x(t) = \sum_{k=1}^{N} X_k \cos(\omega_k t + \theta_k),$$

  where N is a finite integer or infinite.On the other hand, if the signal $x(t)$ has an uncountably infinite number of frequency components, then

$$x(t) = \int_{\omega=\omega_1}^{\omega_2} X(\omega)\cos(\omega t + \theta(\omega))d\omega$$

  where $[\omega_1\ ,\ \omega_2]$ is the range of frequencies contained in $x(t)$.

- The above concept implies that a signal $x(t)$ can be prescribed by two ways:

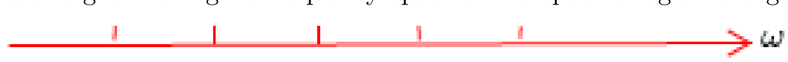
  1. Prescribing $x(t)$ directly in terms of time variable t is referred to as **Time domain** prescription.
  2. Prescribing $x(t)$ indirectly in terms of its frequency components is referred to as **Frequency domain** prescription.

Time Domain		**Frequency Domain**
$x(t)$	$\Leftrightarrow$	$X_k\ ,\omega_k\ \theta_k$ for $k = 1, 2, \cdots, N$
$x(t)$	$\Leftrightarrow$	$X(j\omega) = \lvert X(\omega)\rvert\ \underline{/\theta(\omega)}$ for $\omega_1 \le \omega \le \omega_2$.

  The frequency domain prescription of $x(t)$ is termed as the **Fourier Transform** $X(j\omega)$ of $x(t)$ (A more formal definition will be given in another course). The plot of magnitude (amplitude) X_k or $|X(\omega)|$ with respect to ω is called the **magnitude (amplitude) spectrum** of the signal $x(t)$. On the other hand, the plot of phase angle θ_k or $\theta(\omega)$ with respect to ω is called the **phase angle spectrum** of the signal $x(t)$.

- The frequency spectra of most of the practical signals occupy only a finite region along the ω axis. Such signals are said to be band-limited signals.

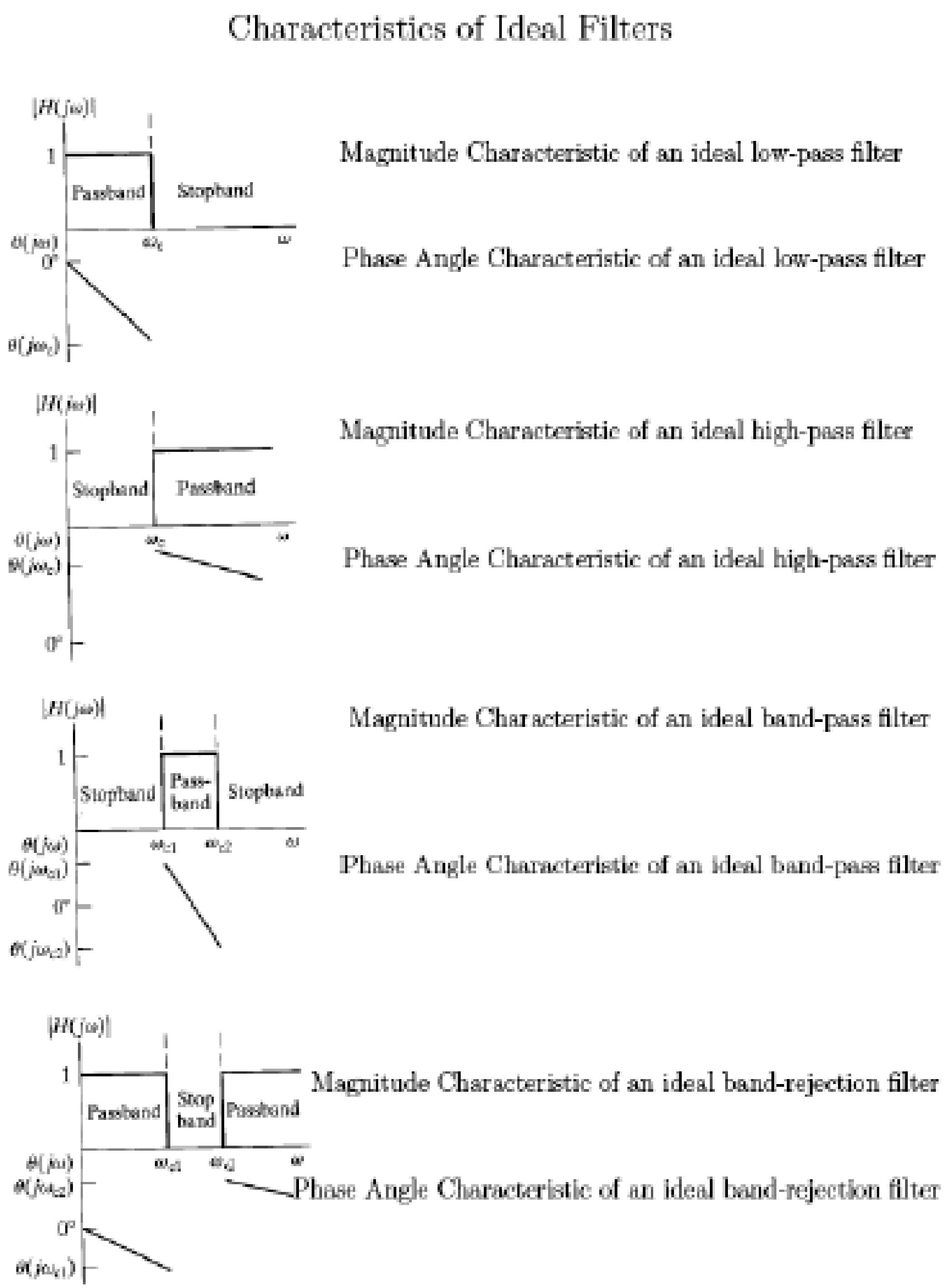
- For the purpose of **transmission** of signals, several band-limited signals can be combined to generate a composite signal, each signal having its frequency spectra in a specific region along the frequency axis.



- The above implies that the useful frequency axis can be divided into a number of channels; each channel occupying a finite and distinct region. In fact, when air is used as the medium of transmission of signals, Federal Communication Commission (FCC) divides the useful frequency axis into a number of channels, each channel corresponding to a particular type of signal transmission, say radio signal, TV signal, etc.

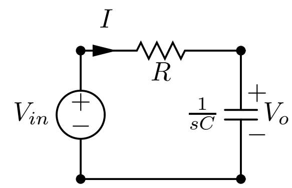
- Each channel has its own signal and the signals of all channels put together is called a composite signal. (The composite signal is formed as a result of frequency division multiplexing.) For example, WABC occupies the range 765 kHZ to 775 kHZ, while WCBC occupies the range 875 kHZ to 885 kHZ.

- A filter picks up or distills a particular signal from the composite signal. Let us expand on this. Filter is basically a circuit with having a transfer function between its input and output. The transfer function lets certain frequency components of the input pass through to the output, while it blocks all other frequency components. Depending upon what passes through, filters can be categorized into four broad categories as explained in the figure.



- An ideal filter picks up only the intended signal. However, a non-ideal filter picks up not only the intended signal but also other non-intended signals as well although the gain of non-intended signals is smaller than that of the intended signal. This causes inter channel interference (cross-talk). To minimize cross talk, it is imperative that a practical filter performance be as close as possible to that of an ideal one.

As an example of a practical filter, let us consider a simple first order RC circuit as shown on the right. By voltage division rule, it is easy to determine its transfer function as



$$H(s) = \frac{V_o}{V_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC}.$$

We can compute $H(j\omega)$ as

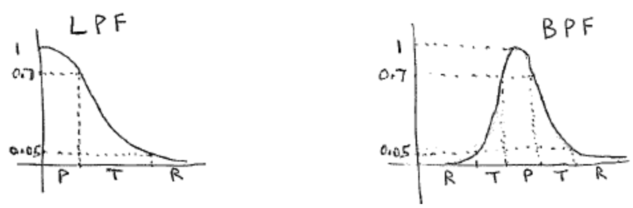
$$H(j\omega) = \frac{1}{1 + j\omega RC} = M(\omega)\angle\theta(\omega) = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}\angle -\tan^{-1}(\omega RC).$$

Thus, if the input signal $v_{in} = A\cos(\omega t)$, then the output signal

$$v_o(t) = \frac{A}{\sqrt{1 + \omega^2 R^2 C^2}}\cos(\omega t - \tan^{-1}(\omega RC)).$$

It is easy to see that the output signal $v_o(t)$ depends on frequency ω. In fact, $v_o(t)$ is maximum at $\omega = 0$ and decreases gradually as ω increases. The given circuit can be thought of as some kind of **low pass filter** in the sense that it passes low frequency signals much better than the high frequency signals.

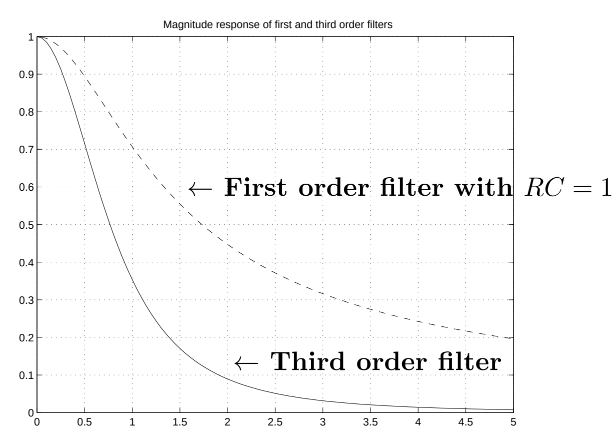
- Non-ideal filter characteristics can be divided into different bands: Pass-band (P), Transition-band (T), and Rejection-band (R), (In figure, the levels 0.7 and 0.05 can be changed depending upon the designer's choice).



- Transition-band needs to be as small as possible in order to minimize the inter channel interference (cross-talk).

- The higher the order of the filter the smaller is the transition-band. That is, the higher the order of the filter the better it approaches the ideal filter. (The order of a filter is the same as the degree of the numerator polynomial of the filter transfer function. It is also the same as the number of energy storing elements, inductances and capacitances, the filter circuit has.)

- For the same order of filter, one type of design could be better than another type of design.

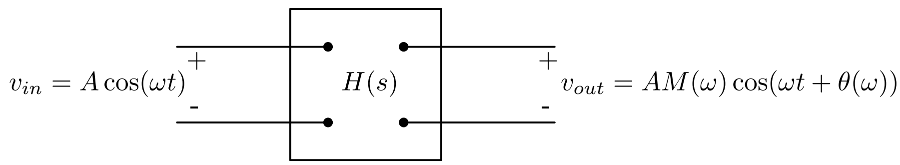


Filters – Terminology

A Block diagram representation of a filter is given below where $H(s)$ is the transfer function of the filter and

$$H(j\omega) = M(\omega)\angle\theta(\omega).$$

We note that $M(\omega)$ and $\theta(\omega)$ are together called **frequency response characteristics**. In particular, $M(\omega)$ is called **magnitude characteristic** and $\theta(\omega)$ is called **phase angle characteristic** as they respectively affect the amplitude and phase angle of the steady state output to a sinusoidal input of frequency ω.



We observe in general the following regarding the power of a signal:

$$\begin{aligned} \text{Power} \quad &\propto \quad (\text{voltage})^2 \\ \text{Power transmitted by the input to the output} \quad &\propto \quad M^2(\omega) \end{aligned}$$

At certain frequency, say ω_m, the magnitude $M(\omega)$ has a maximum possible value $M(\omega_m)$.
At that frequency ω_m, the power transmitted by the input to the output is maximum and is proportional to $M^2(\omega_m)$.

Half power frequency: is the frequency ω at which the power transmitted by the input to the output is **Half** of the maximum possible power transmission. That is, half power frequency ω is given by

$$M^2(\omega) = \frac{1}{2}M^2(\omega_m),$$

or equivalently by

$$M(\omega) = \frac{1}{\sqrt{2}}M(\omega_m) \quad \Rightarrow \quad 20\log_{10} M(\omega) = 20\log_{10} M(\omega_m) + 20\log_{10}\frac{1}{\sqrt{2}} = 20\log_{10} M(\omega_m) - 3\,\text{dB}.$$

Thus, **Half power frequency** ω is the same frequency where magnitude in dB is 3 dB below the maximum possible magnitude in dB. Hence **half power frequency** is also called 3 dB frequency. Sometimes half power frequency is also called as **cut-off frequency**.

Practical Low-pass Filters: For low-pass filters, if properly designed, there is only one half power frequency or cut-off frequency. Let us denote it by ω_c. Then 0 to ω_c in frequency domain is the pass-band, and it is also called the **band-width**.

Practical High-pass Filters: For high-pass filters, if properly designed, there is only one half power frequency or cut-off frequency. The pass-band or **band-width** of high-pass filters is infinite. In practical terms, high-pass filters do NOT exist. After certain high frequency, parasitic effects in circuits predominate and the frequency response degenarates (attenuates) rapidly. In this sense, high-pass filters are **broadband** (large band-width) band-pass filters.

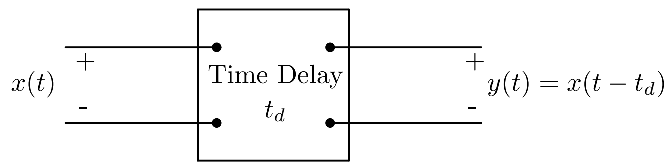
Practical Band-pass Filters: For band-pass filters, there are two half power frequencies or cut-off frequencies. The pass-band or **band-width** of band-pass filters is the distance between the two half power frequencies. The frequency at which the magnitude M is maximum is called the **center frequency or resonant frequency**. The center frequency may be a **misnomer** because it is not the arithmetic center of two half power frequencies as we shall see later on. In **narrowband** (small band-width) band-pass filters the ratio of center frequency to band-width is often called the Q factor or **quality factor**.

Band-reject Filters: For band-reject filters, there are two half power frequencies or cut-off frequencies. The reject-band or **rejection band-width** of band-reject filters is the distance between the two half power frequencies.

Filter Design

An ideal filter: Let us first simply consider the ideal case that $M(\omega)$ is a constant with respect to ω in the pass-band and zero in the rejection-band. Also, let $\theta(\omega) = 0$ in the pass-band (in this ideal case, it does not matter what $\theta(\omega)$ is in the rejection-band). In this ideal case, the output signal picks up all the intended frequencies which are in the pass-band of the input and rejects all the other frequencies which are in the rejection-band. Since the phase angle $\theta(\omega) = 0$ in the pass-band, the output signal is not shifted or delayed along the time axis. However, often in most of the applications, a time delay in receiving or transmitting a signal is acceptable. That is, a signal which is time-delayed is considered as equivalent to the original non-delayed signal.

Time delay: Consider the block-diagram shown on the right. The output $y(t)$ of the block-diagram is the time-delayed input $x(t)$.



Let us look at the relationship between the phase angle and time delay for sinusoidal signals. If $x(t) = A\cos(\omega t)$, then the time delayed signal $y(t) = x(t - t_d)$ equals

$$y(t) = A\cos(\omega(t - t_d)) = A\cos(\omega t - \omega t_d) = A\cos(\omega t + \theta),$$

where

$$\theta = -\omega t_d.$$

Thus the phase delay θ is proportional to the frequency ω. For general signals, the same is reflected in a Laplace transform property which we will learn soon. *Read this italicized sentences after we do Laplace transforms. The time-delay property of Laplace transforms says that the time delayed signal $y(t) = x(t - t_d)u(t - t_d)$ has a Laplace transform $Y(s) = X(s)e^{-st_d}$. We observe that $e^{-j\omega t_d} = \underline{\angle -\omega t_d}$ and hence it is simply a linear phase angle with respect to ω as is the case with sinusoidal signals. This implies that the time-delay simply adds a linear phase angle with respect to ω in frequency domain.*

In the context of filters, an ideal attribute is to have $M(\omega)$ as a real constant with respect to ω in the pass-band, and $\theta(\omega)$ linear with respect to ω in the pass-band. In this case, the output signal picks up all the intended frequencies which are in the pass-band of the input and then simply adds a time-delay to those components.

Roll-off rate: At the edges of the passband, the rectangular shaped idealized filter responses have infinite slopes. However, actual physically realizable filters have finite slopes outside the passband. The steeper the slope the more discriminating the filter is, that is it is a better approximation to the ideal filter. The slope of the frequency magnitude response outside the passband is often termed as the gain roll-off rate. The higher the gain roll-off rate the better it is.

Design of filters: A filter is designed to yield frequency response characteristics (magnitude $M(\omega)$ and phase angle $\theta(\omega)$) as close as possible to the desired characteristics. In most of the applications the magnitude $M(\omega)$ plays a major role than the phase angle $\theta(\omega)$. In the pass-band, the phase angle $\theta(\omega)$ is designed to have as linear a characteristic as possible with respect to ω. A linear phase angle amounts to 'time delay' in time domain as discussed above. Our exposure to filter design ignores the phase angle $\theta(\omega)$. This is justified in view of the fact that most practical applications ignore the effect of phase delay. It is known that our ears are insensitive to phase delays. Also, in narrow-band filters where pass-band is relatively small, phase characteristics are always approximately linear.

A simple methodology of filter design: Often first and second order filters are cascaded to form a composite filter (Cascade Connection is discussed soon). Thus we need to analyze common first and second order filters. We note that the **analysis is the back bone of design**. Both passive and active filter circuits are used. Passive filter circuits if properly designed have the advantage of not requiring any maintenance and are also relatively cheap. Note that when passive filter circuits are cascaded, one filter circuit acts as a load to the previous filter circuit. If the loading effect is predominant, the performance of the composite circuit degenerates. On the other hand, active filter circuits do not present any loading problems since Op-Amp circuits have very low output impedance and high input impedance. However, active filter circuits require maintenance and are also prone to parasitic effects much more so than the passive circuits.

Proto-type design: Often filters are designed with bandwidths and/or center frequencies normalized to unit values. Such filters are called Proto-type filters. A number of such proto-type filters can be found in Electrical Engineering hand-books. The proto-type filters need to be **scaled** (as we shall see later on) both regarding magnitude of impedances and frequencies of interest.

Cascaded Connection

Often several individual circuits are interconnected to form a composite circuit. This is particularly so in designing filters. Normally, first and second order filters are designed and then they are interconnected in **cascade** to form a higher order filter circuit. One method of analysis in cascaded circuits is by means of transfer function analysis. Such an analysis is valid if a later stage filter circuit does not **load** the earlier stage filter circuit. Loading of a circuit implies drawing current from the circuit. Another method of analysis which is a little more involved is by means of two port parameter analysis discussed elsewhere.

Consider two individual circuits whose transfer functions are $H_1(s)$ and $H_2(s)$ as shown in Fig. 1 and Fig. 2. Note that when transfer functions are computed, only the relationship between the input and output is taken into account and the current I_{out} supplied by the given circuit to an external circuit is considered as zero, that is, the given circuit is assumed to be NOT loaded.

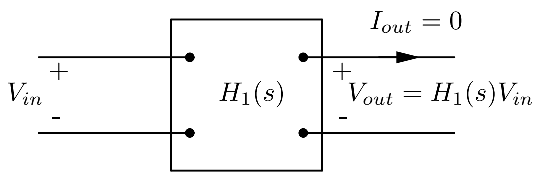


Fig. 1

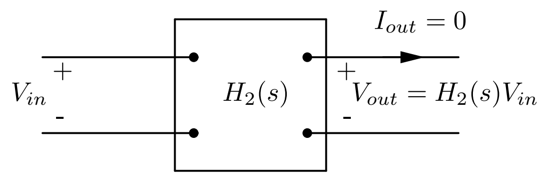


Fig. 2

Now let us consider the cascade interconnection of the above two circuits as shown in Fig. 3. Note that in a cascade connection the output V_{out-1} of the first circuit is the input V_{in-2} of the second circuit. Also, let us assume that the second circuit has a very high input impedance and as such $I_{out-1} = I_{in-2} = 0$. In this case, the conditions under which we evaluated the transfer function of the first circuit are still intact and hence

$$V_{out-1} = H_1(s)V_{in}.$$

Since the second circuit is not loaded, we have

$$V_{out} = H_2(s)V_{in-2} = H_2(s)V_{out-1} = H_2(s)H_1(s)V_{in}.$$

This shows that the over all transfer function of the composite circuit is simply the product of individual transfer functions of each cascaded part,

$$H(s) = H_1(s)H_2(s).$$

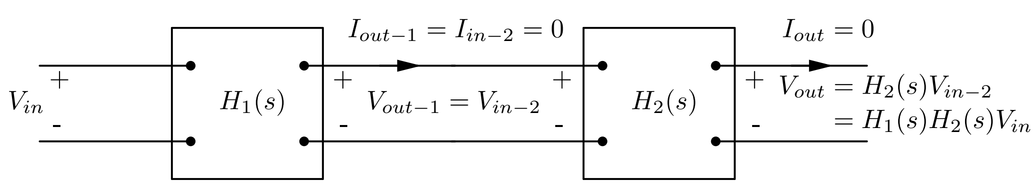


Fig. 3

Loading Effect in Cascaded Connection

When passive filters are cascaded, a later stage filter circuit always loads the previous stage filter circuit. In active filter circuits, loading effect is always negligible since most of Op-amps are almost ideal, and the ideal analysis shows that the transfer function of a filter does not depend on the current drawn by the load impedance unless the load impedance is zero or negligibly small.

Illustration of loading effect on an example: Consider first order RC filter circuits shown in Fig. 1 and Fig. 2. It is easy to see that the individual transfer functions of circuits in Fig. 1 and Fig. 2 are respectively given by

$$H_1(s) = \frac{1}{1+\frac{s}{\omega_1}} \quad \text{with} \quad \omega_1 = \frac{1}{R_1C_1} \quad \text{and} \quad H_2(s) = \frac{1}{1+\frac{s}{\omega_2}} \quad \text{with} \quad \omega_2 = \frac{1}{R_2C_2}.$$

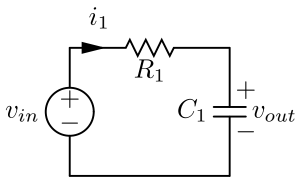


Fig. 1

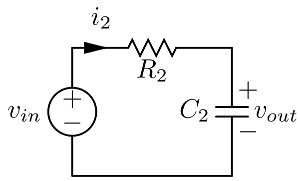


Fig. 2

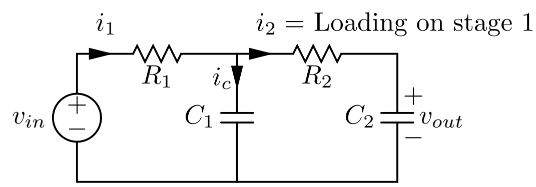


Fig. 3

If loading effect is negligible, the over all transfer function of Fig. 3 is given by

$$H_1(s)H_2(s) = \frac{1}{1+\frac{s}{\omega_1}}\,\frac{1}{1+\frac{s}{\omega_2}} = \frac{1}{1+\frac{s}{\omega_1}+\frac{s}{\omega_2}+\frac{s^2}{\omega_1\omega_2}}.$$

On the other hand, it can be shown (see next page for derivation) that the over all transfer function of Fig. 3 is given by

$$H(s) = \frac{1}{1+\frac{s}{\omega_1}+\frac{s}{\omega_2}+\frac{s^2}{\omega_1\omega_2}+\boxed{\frac{R_1}{R_2}\frac{s}{\omega_2}}}.$$

Note that the term $\frac{R_1}{R_2}\frac{s}{\omega_2}$ is an extra term in the denominator of $H(s)$ while it is absent in the denominator of $H_1(s)H_2(s)$. As R_2 takes larger larger values in comparison with R_1, this extra term will get smaller and get smaller, and thus it might be neglected. Otherwise, one has to design both stages together, individual design of each stage is not acceptable. We observe that when R_2 is increased, C_2 can be decreased so that $\omega_2 = \frac{1}{R_2C_2}$ remains constant.

Let us look at the circuit of Fig. 3 from a different perspective. The reason that the composite transfer function $H(s)$ of Fig. 3 differs from $H_1(s)H_2(s)$ can easily be traced to the current i_2 drawn by the second stage from the first stage. Although the second stage transfer function is unaffected and is given by $H_2(s)$, the first stage transfer function is different from $H_1(s)$ because it has to supply the current i_2. Consequently, $H(s)$ of Fig. 3 differs from $H_1(s)H_2(s)$. However, as R_2 increases and C_2 decreases, the input impedance of the second stage (namely, $R_2+\frac{1}{sC_2}$) increases and thus the current i_2 decreases. Hence the first stage transfer function approaches $H_1(s)$ as R_2 increases and C_2 decreases. This in turn implies that $H(s)$ approaches $H_1(s)H_2(s)$ as R_2 increases and C_2 decreases.

Derivation of transfer function of the circuit of Fig. 3

Derive the transfer function of the time-domain circuit of Fig. 3. The Laplace domain circuit of Fig. 3 is shown in Fig. 4.

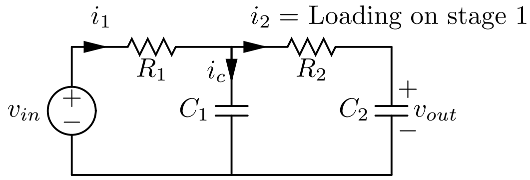


Fig. 3

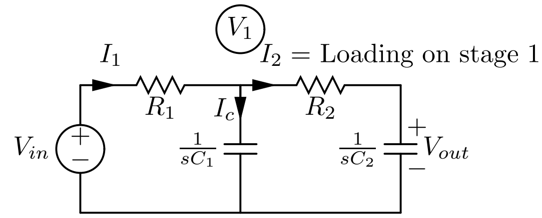


Fig. 4

Method 1, Ladder Circuit Method: It is easy to see the following:

$$I_2 = V_{out}sC_2,\;\; V_1 = V_{out}(1 + sR_2C_2),\;\; I_c = V_{out}(1 + sR_2C_2)sC_1,\;\; I_1 = V_{out}[(1 + sR_2C_2)sC_1 + sC_2].$$

Thus

$$V_{in} = I_1R_1 + V_1 = V_{out}[(1 + sR_2C_2)sC_1 + sC_2]R_1 + V_{out}(1 + sR_2C_2).$$

Simplifying, we get

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + sR_1C_1 + sR_2C_2 + s^2R_1C_1R_2C_2 + sR_1C_2}.$$

This can be rewritten as

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + \frac{s}{\omega_1} + \frac{s}{\omega_2} + \frac{s^2}{\omega_1\omega_2} + \frac{R_1}{R_2}\frac{s}{\omega_2}}.$$

Method 2, Node Voltage Method: Let the voltage at the node where R_1, R_2, and C_1 intersect be v_1 with respect to the bottom node. Then writing the node equations in Laplace domain, we get

$$\begin{aligned} \frac{V_1 - V_{out}}{R_2} + \frac{V_1 - V_{in}}{R_1} + V_1\,sC_1 &= 0 \\ \frac{V_{out} - V_1}{R_2} + V_{out}\,sC_2 &= 0. \end{aligned}$$

By solving for V_1 from the second equation, we get

$$V_1 = (1 + sR_2C_2)V_{out} = (1 + \frac{s}{\omega_2})V_{out} \quad \text{where } \omega_2 = \frac{1}{R_2C_2}.$$

Substituting for V_1 in the first equation and simplifying, we get

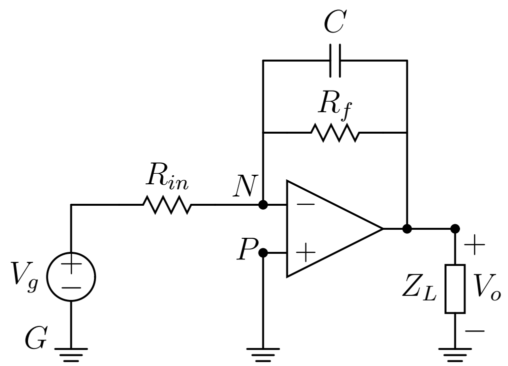
$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + \frac{s}{\omega_1} + \frac{s}{\omega_2} + \frac{s^2}{\omega_1\omega_2} + \frac{R_1}{R_2}\frac{s}{\omega_2}}.$$

Note that the transfer function of the second stage is

$$H_2(s) = \frac{V_{out}}{V_1} = \frac{1}{1 + sR_2C_2} = \frac{1}{1 + \frac{s}{\omega_2}}.$$

This second stage transfer function is unaffected. What is affected is the transfer function of the first stage which is different from $H_1(s) = \frac{1}{1+sR_1C_1} = \frac{1}{1+\frac{s}{\omega_1}}$ due to loading effect.

Transfer Function of a First-order LPF Op-Amp Circuit



Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{K\omega_o}{s + \omega_o}$$

and identify the values of K and ω_o in terms of the circuit parameters R_f, R_{in}, and C.

Let G be the reference node, and note that N is at the same potential as G, that is at zero voltage. Then the node equation at N can be written as

$$-sCV_o - \frac{V_o}{R_f} - \frac{V_g}{R_{in}} = 0.$$

Thus, we get

$$H(s) = \frac{V_o}{V_g} = -\frac{1}{R_{in}} \frac{1}{sC + \frac{1}{R_f}} = -\frac{R_f}{R_{in}} \frac{1}{1 + sR_fC} = -\frac{R_f}{R_{in}} \frac{\frac{1}{R_fC}}{s + \frac{1}{R_fC}}.$$

We note that

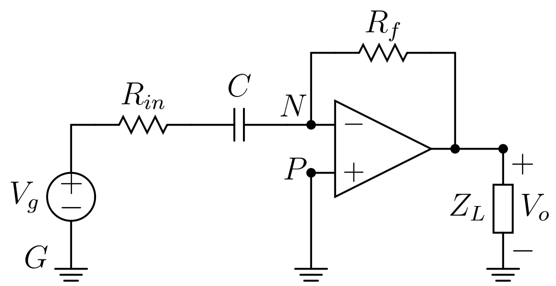
$$H(s) = \frac{K\omega_o}{s + \omega_o}, \quad \text{where} \quad K = -\frac{R_f}{R_{in}} \quad \text{and} \quad \omega_o = \frac{1}{R_fC}.$$

Note: It is clear that, at high frequencies, the capacitor impedance is zero (that is, it is basically a short circuit). This renders V_o equal to V_N which is zero. On the other hand, at low frequencies, the capacitor impedance is infinite (that is, it is basically an open circuit). Thus, the gain at low frequencies is the same as the gain of an inverting amplifier. Hence the gain at low frequencies is $K = -\frac{R_f}{R_{in}}$.

Principles of Electrical Engineering II

Transfer Function of a First-order HPF Op-Amp Circuit

This is a home-work problem which will be collected and graded.



Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{Ks}{s + \omega_o},$$

and identify the values of K and ω_o in terms of the circuit parameters R_f, R_{in}, and C. Use Node Voltage Method to do the above analysis.

Note: You can intuitively justify the behavior of the circuit by considering the following aspects of the behavior of the capacitance.

- At high frequencies, the capacitor impedance is zero (that is, it is basically a short circuit).
- At low frequencies, the capacitor impedance is infinite (that is, it is basically an open circuit).

Principles of Electrical Engineering II
RLC Circuits

A RLC series circuit can be used as a second order LPF or HPF or BPF or BRF depending upon where the output is taken. The current I in a RLC series circuit is given by

$$\begin{aligned} I &= \frac{1}{sL+R+\frac{1}{sC}}V_{in} = \frac{sC}{s^2LC+sCR+1}V_{in} \\ &= \frac{\frac{s}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}V_{in} = \frac{\frac{s}{L}}{s^2+\beta s+\omega_o^2}V_{in} \end{aligned}$$

Note that β has the dimensions of frequency.

where

$$\beta = \frac{R}{L} \text{ and } \omega_o^2 = \frac{1}{LC}.$$

The coefficient β is related to the damping coefficient ζ discussed in Chapter 8, and it equals the half power band-width of the BPF connection discussed soon. The frequency ω_o is the resonant frequency of the circuit when the capacitive impedance cancels with the inductive impedance.

LPF connection: Consider the RLC series circuit where the output is taken across the capacitance C as shown in Figure 1. We can easily determine the transfer function $H(s)$ as

$$\begin{aligned} H(s) &= \frac{V_o}{V_{in}} = \frac{I}{sC}\frac{1}{V_{in}} \\ &= \frac{\frac{1}{LC}}{s^2+\beta s+\omega_o^2} = \frac{\omega_o^2}{s^2+\beta s+\omega_o^2}. \end{aligned}$$

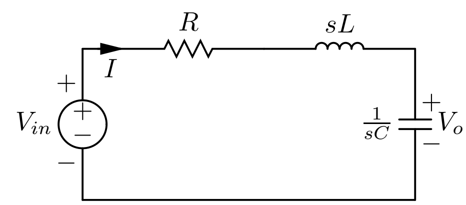


Figure 1: LPF Circuit

The magnitude of $H(j\omega)$ is given by

$$|H(j\omega)| = \frac{\omega_o^2}{|-\omega^2+j\beta\omega+\omega_o^2|} = \frac{\omega_o^2}{\sqrt{(\omega_o^2-\omega^2)^2+\beta^2\omega^2}}.$$

A simple study of $|H(j\omega)|$ reveals that the above circuit is a LPF.

HPF connection: Consider the RLC series circuit where the output is taken across the inductance L as shown in Figure 2. We can easily determine the transfer function $H(s)$ as

$$H(s) = \frac{V_o}{V_{in}} = \frac{IsL}{V_{in}} = \frac{s^2}{s^2+\beta s+\omega_o^2}.$$

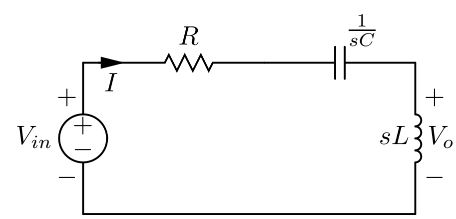


Figure 2: HPF Circuit

The magnitude of $H(j\omega)$ is given by

$$|H(j\omega)| = \frac{|-\omega^2|}{|-\omega^2+j\beta\omega+\omega_o^2|} = \frac{\omega^2}{\sqrt{(\omega_o^2-\omega^2)^2+\beta^2\omega^2}}.$$

A simple study of $|H(j\omega)|$ reveals that the above circuit is a HPF.

BPF connection: Consider the RLC series circuit where the output is taken across the resistance R as shown in Figure 3. We can easily determine the transfer function $H(s)$ as

$$\begin{aligned} H(s) &= \frac{V_o}{V_{in}} = \frac{IR}{V_{in}} \\ &= \frac{\frac{sR}{L}}{s^2+\beta s+\omega_o^2} = \frac{\beta s}{s^2+\beta s+\omega_o^2}. \end{aligned}$$

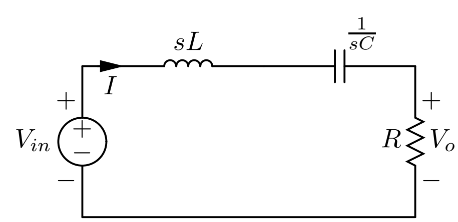


Figure 3: BPF Circuit

The magnitude of $H(j\omega)$ is given by

$$|H(j\omega)| = \frac{|j\beta\omega|}{|-\omega^2 + j\beta\omega + \omega_o^2|} = \frac{\beta\omega}{\sqrt{(\omega_o^2 - \omega^2)^2 + \beta^2\omega^2}}.$$

A simple study of $|H(j\omega)|$ reveals that the above circuit is a BPF (We will discuss this in detail soon).

BRF connection: Consider the RLC series circuit where the output is taken across both the inductance L and the capacitance C as shown in Figure 4. We can easily determine the transfer function $H(s)$ as

$$\begin{aligned} H(s) &= \frac{V_o}{V_{in}} = \frac{I\left(\frac{1}{sC} + sL\right)}{V_{in}} \\ &= \frac{s^2 + \frac{1}{LC}}{s^2 + \beta s + \omega_o^2} = \frac{s^2 + \omega_o^2}{s^2 + \beta s + \omega_o^2}. \end{aligned}$$

The magnitude of $H(j\omega)$ is given by

$$|H(j\omega)| = \frac{|\omega_o^2 - \omega^2|}{|-\omega^2 + j\beta\omega + \omega_o^2|} = \frac{|\omega_o^2 - \omega^2|}{\sqrt{(\omega_o^2 - \omega^2)^2 + \beta^2\omega^2}}.$$

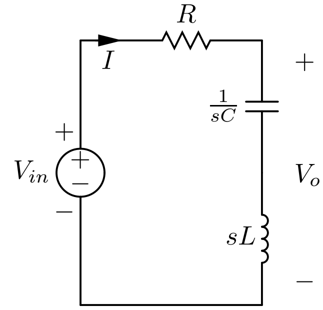


Figure 4: BRF Circuit

A simple study of $|H(j\omega)|$ reveals that the above circuit is a BRF.

A close examination of the transfer functions of BPF and BRF circuits reveals that

Transfer function of BRF = 1 − Transfer function of BPF.

This makes sense since BRF rejects a band of frequencies where as BPF passes the same band of frequencies.

Comment: All the above transfer functions are derived under the condition that no current is supplied to the load connected at the output terminals, i.e the circuit is not loaded. The loading of the circuit changes the transfer function.

BPF connection of a parallel circuit: Consider the circuit shown in Figure 5. We can show that the transfer function $H(s)$ of this circuit is given by

$$H(s) = \frac{V_o}{V_{in}} = \frac{K\beta s}{s^2 + \beta s + \omega_o^2},$$

where

$$K = R, \quad \beta = \frac{1}{RC}, \quad \text{and} \quad \omega_o^2 = \frac{1}{LC}.$$

As before, it is easy to see that the above circuit is a BPF.

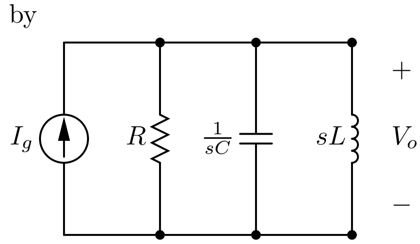


Figure 5: BPF Circuit

To derive the above transfer function, we note that $V_o = I_g Z$ where Z is the impedance of R, sL, and $\frac{1}{sC}$ connected in parallel. Thus $H(s) = \frac{V_o}{I_g} = Z$. Hence, we can determine $H(s)$ as

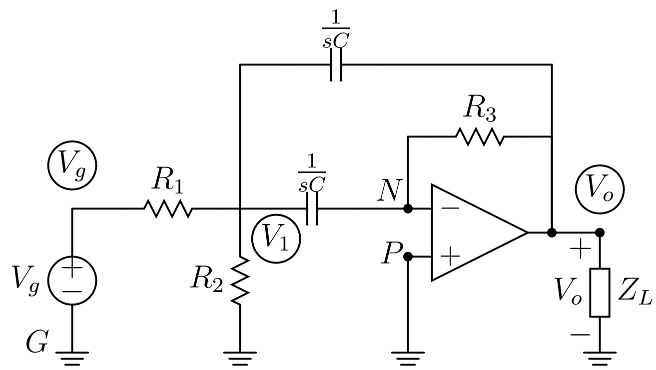
$$H(s) = Z = \frac{1}{\frac{1}{R} + \frac{1}{sL} + sC} = \frac{sRL}{s^2RLC + sL + R} = \frac{sR\frac{1}{RC}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}} = \frac{K\beta s}{s^2 + \beta s + \omega_o^2},$$

where

$$K = R, \quad \beta = \frac{1}{RC}, \quad \text{and} \quad \omega_o^2 = \frac{1}{LC}.$$

Note that the parallel circuit shown in Figure 5 is a BPF just like the series circuit given in Figure 3. The expression for ω_o^2 given here is the same as the one for the circuit of Figure 3, but the expression for β is different. The gain K does not change the shape of the frequency spectrum, it merely shifts the magnitude spectrum up or down.

Transfer Function of a BPF Op-Amp Circuit



Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{-K\beta s}{s^2 + \beta s + \omega_o^2}$$

and identify the values of K, β, and ω_o^2 in terms of the circuit parameters R_1, R_2, R_3, and C.

Let G be the reference node, and consider the node voltages as marked. Then the node equation at node V_1 can be written as

$$\frac{V_1 - V_g}{R_1} + sCV_1 + \frac{V_1}{R_2} + sC(V_1 - V_o) = 0.$$

We can write a second node equation at N as

$$\frac{-V_o}{R_3} - sCV_1 = 0.$$

From the above equation, we get

$$V_1 = -\frac{1}{sCR_3}V_o.$$

Substituting the above in the very first node equation, we get

$$\left[\frac{1}{sCR_1R_3} + \frac{1}{R_3} + \frac{1}{sCR_2R_3} + \frac{1}{R_3} + sC\right]V_o = -\frac{V_g}{R_1}.$$

Multiplying throughout by $\frac{s}{C}$ and simplifying, we get

$$\left[s^2 + \frac{2s}{R_3C} + \frac{1}{C^2R_3}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]V_o = -\frac{sV_g}{R_1C}.$$

Let

$$\omega_o^2 = \frac{1}{C^2R_3}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad \beta = \frac{2}{R_3C},$$

$$K = \frac{R_3}{2R_1} \text{ so that } K\beta = \frac{1}{R_1C}.$$

Then the transfer function is given by

$$\frac{V_o}{V_g} = H(s) = \frac{-K\beta s}{s^2 + \beta s + \omega_o^2}.$$

Note: We can intuitively justify the behavior of the circuit by considering the following aspects of the behavior of the capacitance.

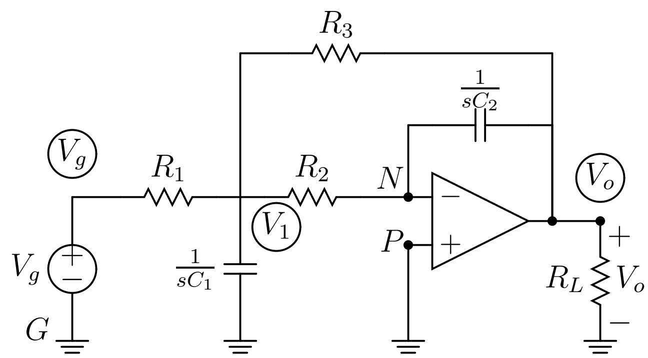
- At high frequencies, the capacitor impedance is zero (that is, it is basically a short circuit). This renders both V_o and V_1 zero.
- At low frequencies, the capacitor impedance is infinite (that is, it is basically an open circuit). This effectively blocks the input signal to reach the output. This renders V_o zero.

HW due date will be announced in the class

Student's name in capital letters:

This is a HW problem, collected and graded.

Transfer Function of a LPF Op-Amp Circuit



Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{K\omega_o^2}{s^2 + \beta s + \omega_o^2}$$

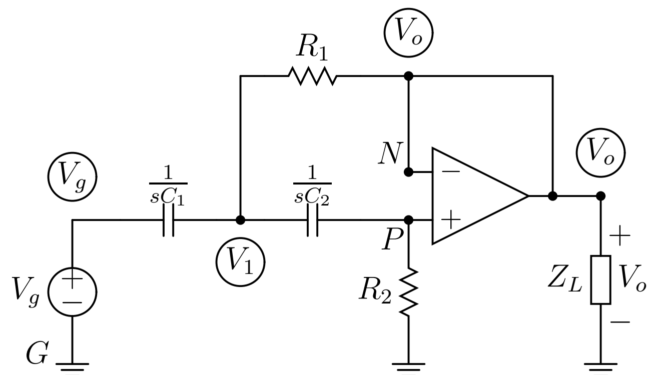
and identify the values of K, β, and ω_o^2 in terms of the circuit parameters R_1, R_2, R_3, C_1, and C_2.

HW due date will be announced in the class

Student's name in capital letters:

This is a HW problem, collected and graded.

Transfer Function of a HPF Op-Amp Circuit

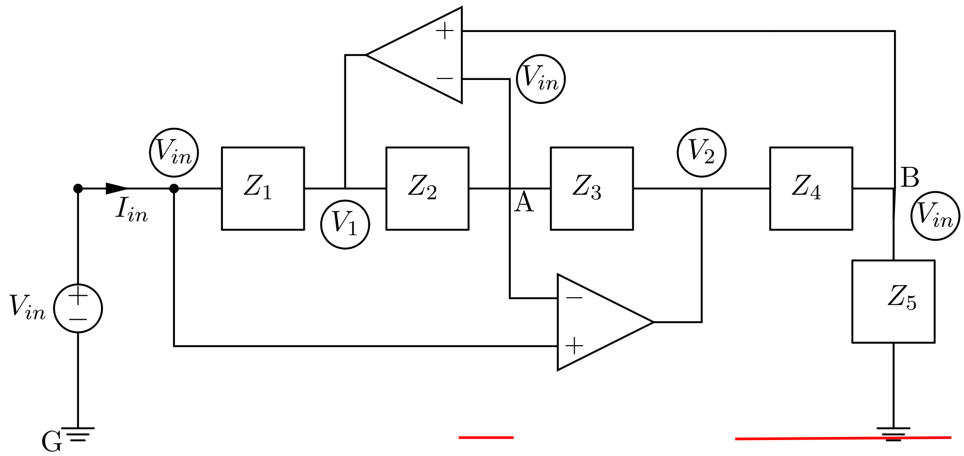


Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{Ks^2}{s^2 + \beta s + \omega_o^2}$$

and identify the values of K, β, and ω_o^2 in terms of the circuit parameters R_1, R_2, C_1, and C_2.

Emulation of Inductance via an Op-Amp Circuit - Impedance inverter



Determine the input impedance Z_{in} of the circuit shown. Obviously, in the absence of Op-Amps, $Z_{in} = Z_1 + Z_2 + Z_3 + Z_4 + Z_5$. However, because of the Op-Amps, some of the node voltages with respect to the ground are forced to equal the input voltage V_{in}, while two other node voltages are different from V_{in}. This changes the input impedance drastically.

As frequency of the signals change, inductances and capacitances behave differently, in fact quite oppositely. Such a behavior enables a circuit designer to shape the frequency response of a circuit as desired by an appropriate connection of inductances and capacitances. Passive circuits that use inductance are heavy and bulky. Miniaturization of a circuit is feasible only if active elements such as Op-Amps or transistors are used. In such active circuits, inductance effect is some how emulated by utilizing resistances and capacitances. Impedance inverter circuit given here is one such circuit.

Assuming that the Op-Amps are ideal, the node voltages are marked as shown. There are two unknown node voltages V_1 and V_2 (outputs of two Op-Amps). It is clear that the appropriate nodes where we can easily write the node equations (without introducing additional unknowns) are nodes A and B as shown. The node equation at A is given by

$$\frac{V_{in} - V_1}{Z_2} + \frac{V_{in} - V_2}{Z_3} = 0 \quad \Rightarrow \quad V_2 = V_{in}\left[1 + \frac{Z_3}{Z_2}\right] - V_1\left[\frac{Z_3}{Z_2}\right].$$

The node equation at B is given by

$$\frac{V_{in} - V_2}{Z_4} + \frac{V_{in}}{Z_5} = 0 \quad \Rightarrow \quad V_2 = V_{in}\left[1 + \frac{Z_4}{Z_5}\right].$$

In view of the above two equations, we get

$$V_{in}\left[1 + \frac{Z_4}{Z_5}\right] = V_{in}\left[1 + \frac{Z_3}{Z_2}\right] - V_1\left[\frac{Z_3}{Z_2}\right] \quad \Rightarrow \quad V_1 = \left[1 - \frac{Z_2 Z_4}{Z_3 Z_5}\right] V_{in}.$$

It is easy to see that

$$I_{in} = \frac{V_{in} - V_1}{Z_1}.$$

Substituting for V_1, we get

$$I_{in} = V_{in}\frac{Z_2 Z_4}{Z_1 Z_3 Z_5}.$$

Thus,

$$Z_{in} = \frac{V_{in}}{I_{in}} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4}.$$

For the case when $Z_1 = R_1$, $Z_2 = R_2$, $Z_3 = R_3$, $Z_5 = R_5$, and $Z_4 = \frac{1}{j\omega C}$ (impedance of a capacitance), we have

$$Z_{in} = j\omega L \quad \text{with} \quad L = \frac{R_1 R_3 R_5 C}{R_2}.$$

The impedance Z_{in} given above is that of an inductance. In other words, utilizing resistances and a capacitance, **the above Op-Amp circuit simulates an inductance**.

Operational Amplifier Circuit

Follow the following steps to determine the transfer function of the circuit given in Figure 1 where the admittance values of all passive elements are indicated. Assume that the Op-Amp is ideal.

Step 1: Take G as the reference node and mark all the unknown node voltages; there are two unknown node voltages V_1 and V_o as shown. So, we need two equations.

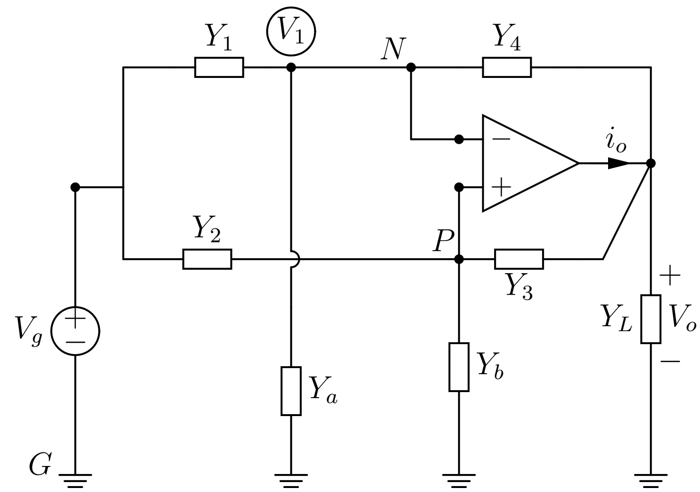


Figure 1

Step 2: Write a node equation at the node marked as N.

Step 3: Write a node equation at the node marked as P.

Step 4: Assume $Y_1 + Y_4 + Y_a = Y_2 + Y_3 + Y_b$ or equivalently assume $Y_a = Y_2 + Y_3$ and $Y_b = Y_1 + Y_4$. Solve the equation written in Steps 2 and 3 to get the ratio $\frac{V_o}{V_g}$ as the transfer function.

$\boxed{\textbf{Scaling of components } R, L, \textbf{ and } C}$

$$R' = k_m R, \quad L' = \frac{k_m L}{k_f}, \quad C' = \frac{C}{k_m k_f}.$$

The parameters R, L, and C correspond to unscaled circuit, where as R', L', and C' correspond to scaled circuit. With the scaling as given, the impedance of each parameter of the scaled circuit will increase by a factor k_m. Also, whatever used to happen at frequency ω in the unscaled circuit will happen at the frequency $k_f\omega$ in the scaled circuit.

Scaling of Transfer Functions

Frequency Scaling of a Transfer Function: The concept of frequency scaling of a transfer function is very simple. Just to make it explicitly clear, various steps are given below:

- Suppose we are given a transfer function $H(s)$.
- Let us obtain a scaled transfer function $\tilde{H}(s)$ by replacing the argument s of $H(s)$ by $\frac{s}{k}$ as given by
$$\tilde{H}(s) = H(s)\Big|_{\text{replace } s \text{ by } \frac{s}{k}} = H(\frac{s}{k}).$$
- We note that $H(j\omega_1)$ is the value of $H(s)$ for $s = j\omega_1$. Let us look at $\tilde{H}(j\omega) = H(\frac{j\omega}{k})$. Just by comparing the arguments, we observe that to get the same value $H(j\omega_1)$ for $H(\frac{s}{k})$, the frequency ω needs to be $k\omega_1$.
- Thus, $\tilde{H}(s)$ is the scaled version of $H(s)$. Whatever happens at the frequency ω in $H(s)$ will happen at the frequency $k\omega$ in $\tilde{H}(s)$.

Example: Let us take a very simple example to illustrate this. Consider a first order low pass filter with the cut-off frequency ω_0,
$$H(s) = \frac{\omega_0}{s + \omega_0}.$$
The scaled transfer function $\tilde{H}(s)$ is given by
$$\tilde{H}(s) = H(s)\Big|_{\text{replace } s \text{ by } \frac{s}{k}} = \frac{\omega_0}{\frac{s}{k} + \omega_0} = \frac{k\omega_0}{s + k\omega_0}.$$
Note that $\tilde{H}(s)$ is a first order low pass filter with the cut-off frequency $k\omega_0$.

Conversion of a LPF Transfer Function to HPF Transfer Function and vice versa:

- Suppose we are given a transfer function $H(s)$.
- Let us obtain a new transfer function $\check{H}(s)$ by replacing the argument s of $H(s)$ by $\frac{1}{s}$ as given by
$$\check{H}(s) = H(s)\Big|_{\text{replace } s \text{ by } \frac{1}{s}} = H(\frac{1}{s}).$$
- Note that the arguments of $H(s)$ and $\check{H}(s)$ are reciprocals of each other.
- Because of the reciprocal nature of the arguments, whatever happens at the low frequencies in $H(s)$ will happen at the high frequencies in $\check{H}(s)$ and vice versa.

Home work Example: Suppose we are given a transfer function $H(s)$. What does a new transfer function $H(\frac{k}{s})$ represent? If $H(s)$ is a LPF with a cut-off frequency ω_0, justify that $H(\frac{k}{s})$ represents a HPF with a cut-off frequency $\frac{k}{\omega_0}$. You may verbally explain it or show it on an example.

Example of Scaling

A fourth-order Butterworth lowpass filter is shown in Fig. 14.48(a). The filter is designed such that the cutoff frequency $\omega_c = 1$ rad/s. Scale the circuit for a cutoff frequency of 50 kHz using 10- kΩ resistors.

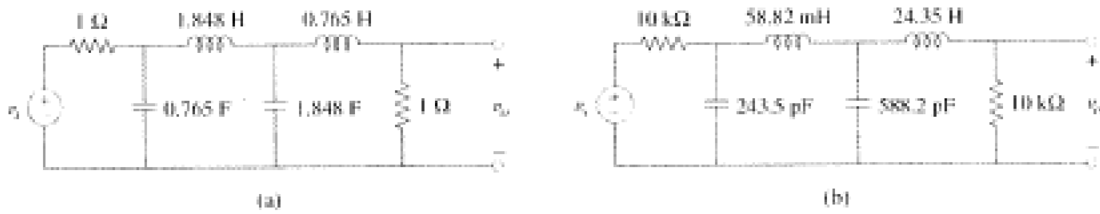


Figure 14.48
For Example 14.14: (a) Normalized Butterworth lowpass filter, (b) scaled version of the same lowpass filter.

Solution:
If the cutoff frequency is to shift from $\omega_c = 1$ rad/s to $\omega_c' = 2\pi(50)$ krad/s, then the frequency scale factor is

$$K_f = \frac{\omega_c'}{\omega_c} = \frac{100\pi \times 10^3}{1} = \pi \times 10^5$$

Also, if each 1-Ω resistor is to be replaced by a 10- kΩ resistor, then the magnitude scale factor must be

$$K_m = \frac{R'}{R} = \frac{10 \times 10^3}{1} = 10^4$$

Using Eq. (14.86),

$$L_1' = \frac{K_m}{K_f} L_1 = \frac{10^4}{\pi \times 10^5}(1.848) = 58.82 \text{ mH}$$

$$L_2' = \frac{K_m}{K_f} L_2 = \frac{10^4}{\pi \times 10^5}(0.765) = 24.35 \text{ mH}$$

$$C_1' = \frac{C_1}{K_m K_f} = \frac{0.765}{\pi \times 10^9} = 243.5 \text{ pF}$$

$$C_2' = \frac{C_2}{K_m K_f} = \frac{1.848}{\pi \times 10^9} = 588.2 \text{ pF}$$

The scaled circuit is shown in Fig. 14.48(b). This circuit uses practical values and will provide the same transfer function as the prototype in Fig. 14.48(a), but shifted in frequency.

Determination of half power frequency of a band-pass second order filter

Let us study in detail the band-pass second order transfer function as expressed below:

$$\text{Transfer function } = H(s) = \frac{\beta\, s}{s^2 + \beta\, s + \omega_0^2}.$$

The magnitude of $H(j\omega)$ can be computed easily as

$$\begin{aligned}
|H(j\omega)| &= \left|\frac{\beta\, j\omega}{\omega_0^2 - \omega^2 + \beta\, j\omega}\right| \\
&= \frac{\beta\,\omega}{\sqrt{(\omega_0^2-\omega^2)^2 + \beta^2\,\omega^2}} = \frac{\beta\,\omega}{\sqrt{(\omega_0^2-\omega^2)^2\frac{\beta^2\,\omega^2}{\beta^2\,\omega^2} + \beta^2\,\omega^2}} \\
&= \frac{1}{\sqrt{\frac{(\omega_0^2-\omega^2)^2}{\beta^2\,\omega^2} + 1}} = \frac{1}{\sqrt{(\frac{\omega_0^2}{\beta\,\omega} - \frac{\omega}{\beta})^2 + 1}}.
\end{aligned}$$

When we study $|H(j\omega)|$ with respect to ω, we have the following simple observations:

- The value of $H(j\omega)$ is real at $\omega = \omega_0$, in fact $H(j\omega_0) = 1$.
- **Maximum value of $|H(j\omega)|$:** By differentiating $|H(j\omega)|$ with respect to ω and then setting it to zero, we can show that $|H(j\omega)|$ has a maximum value at $\omega = \omega_0$. However, we can see this easily from the above simplified expression for $|H(j\omega)|$. Being a square, the expression $(\frac{\omega_0^2}{\beta\,\omega} - \frac{\omega}{\beta})^2$ has the least value of zero, and this occurs when $\omega = \omega_0$. As such, the denominator expression of $|H(j\omega)|$ has the least value when $\omega = \omega_0$, and thus $|H(j\omega)|$ has a maximum value when $\omega = \omega_0$. This implies that the so called *center frequency* of the band pass filter is ω_0.

Determination of Half Power Frequencies: Since the maximum of $|H(j\omega)|$ is one, we see that the half power frequencies are those frequencies when $|H(j\omega)|$ takes a value of $\frac{1}{\sqrt{2}}$. Clearly, in view of the last expression for $|H(j\omega)|$, we see that $|H(j\omega)|$ equals $\frac{1}{\sqrt{2}}$ whenever

$$\frac{\omega_0^2}{\beta\,\omega} - \frac{\omega}{\beta} = \pm 1.$$

The half power frequencies are then those frequencies which satisfy the above equation. We can rewrite the above equation as

$$\omega^2 \pm \beta\,\omega - \omega_0^2 = 0.$$

The positive root of $\omega^2 + \beta\,\omega - \omega_0^2 = 0$ is

$$-\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}.$$

Similarly,the positive root of $\omega^2 - \beta\,\omega - \omega_0^2 = 0$ is

$$\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}.$$

Thus there are two half power frequencies, ω_1 and ω_2, and these are respectively given by

$$\omega_1 = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2} \quad \text{and} \quad \omega_2 = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}\ .$$

We can see easily that the center frequency ω_0 does **not** equal the arithmetic mean of ω_1 and ω_2, namely $\frac{\omega_1+\omega_2}{2}$. (If ω_0 were to be the arithmetic mean of ω_1 and ω_2, it would have been the center point of points ω_1 and ω_2 along the ω axis). This is expected since a second-order band pass filter cannot in general have a perfect symmetrical frequency response around the center frequency ω_0. This is because the frequency range as ω takes values from zero to ω_0 is finite while the frequency range as ω takes values from ω_0 to infinity is indeed infinite. (Nevertheless, by selecting the order of the filter as high as necessary, the frequency response of a band pass filter can be made as close as needed to be a symmetrical response around ω_0.)

It is easy to verify that the product $\omega_1\omega_2 = \omega_0^2$. Hence, $\omega_0 = \sqrt{\omega_1\omega_2}$. That is, ω_0 is the geometric mean of ω_1 and ω_2.

Determination of Half Power Band-width: The Half Power Band-width or otherwise called the 3 dB band-width of a second order filter is given by

$$\omega_2 - \omega_1 = \beta.$$

Thus, the parameter β of the second order filter has the meaning of being the half power band-width of the filter.

The quality factor Q of a filter is defined as $$Q = \frac{\omega_0}{\beta}.$$

The quality factor Q of a filter is the inverse of frequency normalized band-width of the filter. The narrower is the band-width the higher is the quality factor and hence it shows the frequency selective property or nature of the filter.

Determination of half power frequency of a band-pass second order filter – Direct method: The magnitude of the transfer function $H(s) = \frac{\beta s}{s^2+\beta s+\omega_o^2}$ is given by

$$|H(j\omega)| = \frac{\beta\omega}{\sqrt{(\omega_o^2 - \omega^2)^2 + \beta^2\omega^2}}.$$

The magnitude $|H(j\omega)|$ is maximum at $\omega = \omega_o$ and has a value of 1. That is, the pass-band gain is 1. Thus, the half power frequencies are the roots of the equation

$$\frac{\beta^2\omega^2}{(\omega_o^2 - \omega^2)^2 + \beta^2\omega^2} = \frac{1}{2}.$$

This equation simplifies to $$\omega^4 - (2\omega_o^2 + \beta^2)\omega^2 + \omega_o^4 = 0.$$

The above is a quadratic equation in the variable ω^2. This can be easily seen by denoting ω^2 as x and rewriting the above equation as $$x^2 - (2\omega_o^2 + \beta^2)x + \omega_o^4 = 0.$$

Roots of the above quadratic equation are given by

$$x = \frac{2\omega_o^2 + \beta^2 \pm \sqrt{(2\omega_o^2 + \beta^2)^2 - 4\omega_o^4}}{2}.$$

Let ω_2 and ω_1 be the upper and lower half power frequencies. Then

$$\omega_2^2 = \frac{2\omega_o^2 + \beta^2 + \sqrt{(2\omega_o^2 + \beta^2)^2 - 4\omega_o^4}}{2} \quad \text{and} \quad \omega_1^2 = \frac{2\omega_o^2 + \beta^2 - \sqrt{(2\omega_o^2 + \beta^2)^2 - 4\omega_o^4}}{2}.$$

Since ω_2^2 and ω_1^2 are the roots of the quadratic equation $(\omega^2)^2 - (2\omega_o^2 + \beta^2)\omega^2 + \omega_o^4 = 0$, it can easily be verified that

$$\omega_2^2 + \omega_1^2 = 2\omega_o^2 + \beta^2 \quad \text{and} \quad \omega_1^2\omega_2^2 = \omega_o^4 \quad \text{or equivalently} \quad \omega_1\omega_2 = \omega_o^2.$$

The above equations can be combined to yield

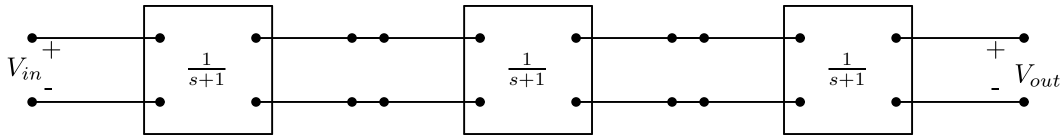
$$\omega_2^2 + \omega_1^2 - 2\omega_2\omega_1 = \beta^2.$$

This intern implies that

$$(\omega_2 - \omega_1)^2 = \beta^2 \quad \text{and thus} \quad \omega_2 - \omega_1 = \beta.$$

Cascade of First Order Filters

Cascaded low-pass filter: Cascading of several filters improves the frequency response of the resulting composite filter provided there is no loading effect. We can demonstrate this easily. Consider a cascade of three filters each having a transfer function $\frac{1}{s+1}$ and a half power frequency of 1.

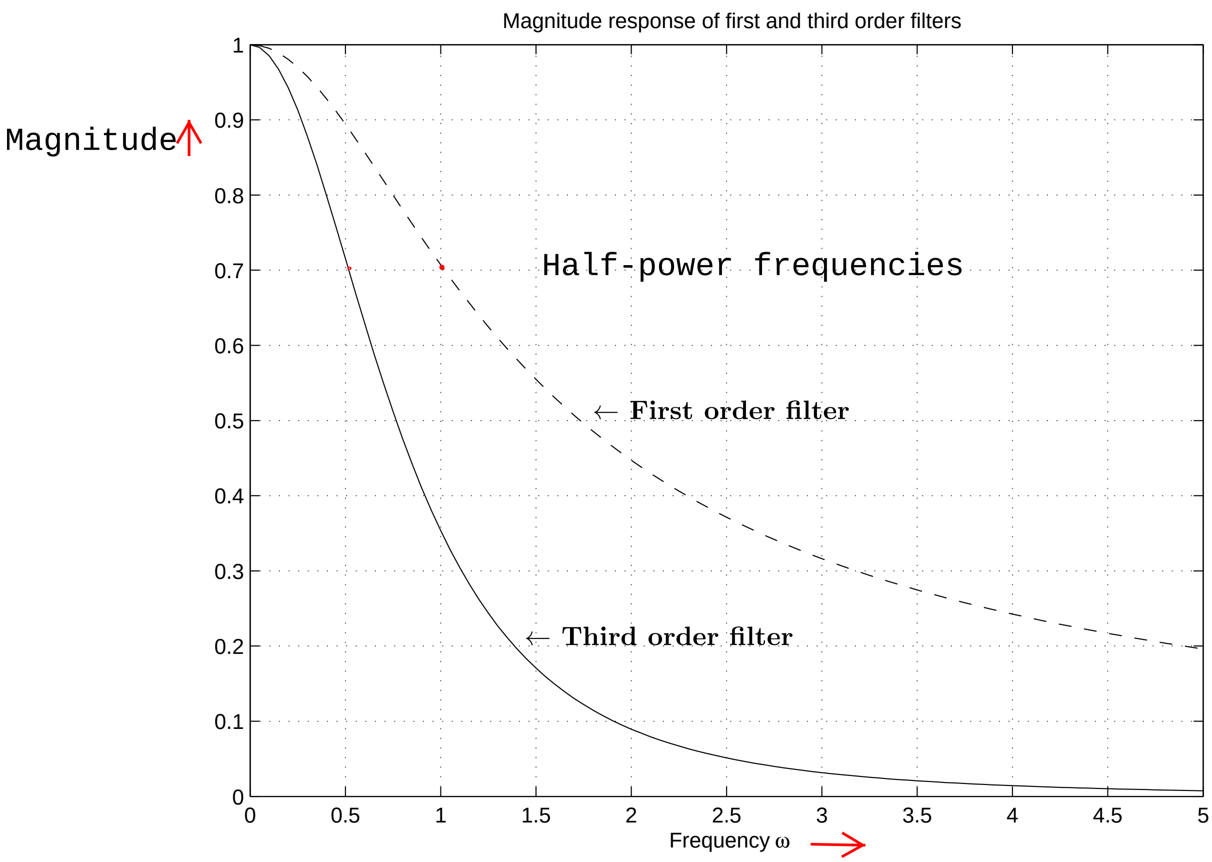


Cascade of three first order filters

The transfer function of the cascaded filter is

$$H(s) = \frac{1}{(s+1)^3}.$$

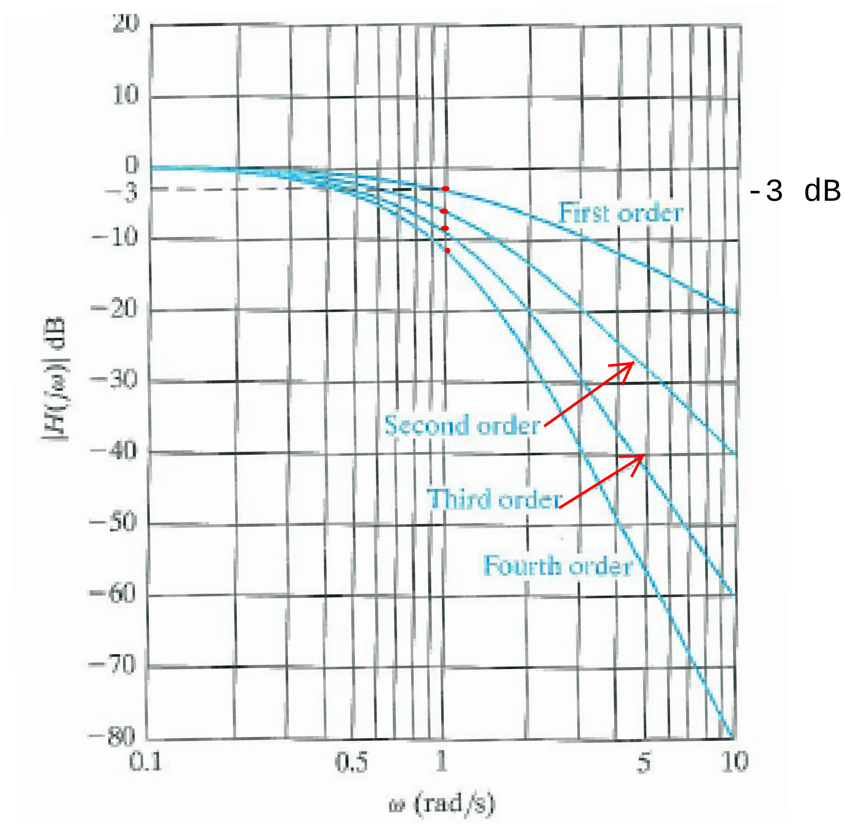
The **magnitude response** of first and cascaded third order filters is shown in the graph. It is clear that the cascaded third order filter is more discriminating in rejection band, i,e., it is much closer to the ideal filter than the first order filter. However, cascading also changes the half-power frequency (3 dB frequency). For the first order filter, the half-power frequency is 1 while for the third order filter it is 0.51 as we will observe soon. However, we can utilize frequency scaling to change the half-power frequency of any filter.



Cascading several first order filters to decrease or shrink the transition band: We can cascade n first order filters to obtain an n-th order filter,

$$H(s) = \frac{1}{(s+1)^n}.$$

The figure given below shows the Bode Magnitude characteristics of first, second, third, and fourth order filters. The negative asymptotic slopes of first, second, third, and fourth order filters are respectively 20dB/Dec, 40dB/Dec, 60dB/Dec, and 80dB/Dec. As the slope increases, the transition band shrinks.



Determination of half power frequency – cascaded low-pass filter: Our interest now is to determine the half-power frequency (3 dB frequency) of the cascaded filter. We note that the maximum of $|H(j\omega)|$ with respect to ω occurs at $\omega = 0$ and the maximum magnitude equals 1. Let us note that the half-power frequency (3 dB frequency) of $|H(j\omega)|$ is the frequency ω_c when $|H(j\omega_c)|$ equals $\frac{1}{\sqrt{2}}$. This yields

$$\left[\frac{1}{\sqrt{1+\omega_c^2}}\right]^3 = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \left[\frac{1}{1+\omega_c^2}\right]^{\frac{3}{2}} = \left[\frac{1}{2}\right]^{\frac{1}{2}}$$

By taking the $\frac{2}{3}$ power on both sides of the above equation, we get

$$\frac{1}{1+\omega_c^2} = \left[\frac{1}{2}\right]^{\frac{1}{3}} = \frac{1}{2^{\frac{1}{3}}}.$$

This implies further that

$$1 + \omega_c^2 = 2^{\frac{1}{3}}.$$

Hence

Third-order cascade

$$\omega_c^2 = 2^{\frac{1}{3}} - 1 \quad \Rightarrow \quad \omega_c = \sqrt{2^{\frac{1}{3}} - 1} = \sqrt{1.26 - 1} = \sqrt{0.26} = \boxed{0.51}.$$

Let us consider next a general n-th order cascade,

$$H(s) = \frac{1}{(s+1)^n}.$$

The square of its magnitude is given by

$$|H(j\omega)|^2 = \frac{1}{(1+\omega^2)^n}.$$

This implies that the half-power frequency ω_c is given by

$$(1+\omega_c^2)^n = 2 \quad \Rightarrow \quad 1 + \omega_c^2 = 2^{\frac{1}{n}}.$$

Then, we can see easily that the half-power frequency (3 dB frequency) is given by

$$\omega_c = \sqrt{2^{\frac{1}{n}} - 1}.$$

Although the half-power frequency of a cascaded filter changes to a new location, we can translate it to a different location by **frequency scaling**.

Example: Determine the half-power frequency of the 4-th order cascade,

$$H(s) = \frac{1}{(s+1)^4}.$$

Solution: We have

Fourth order cascade

$$\omega_c = \sqrt{2^{\frac{1}{4}} - 1} = \boxed{0.435 \text{ rad/sec.}}$$

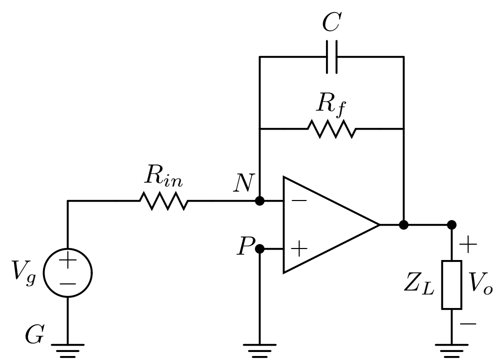
Example: Design a fourth order cascaded low-pass filter to achieve a $\boxed{\text{pass-band gain of 10}}$ and a cut-off frequency of 500 Hz. Utilize only 1μF capacitors and appropriate values of resistances.

Solution: Let us first design a proto-type low-pass filter with a transfer function,

$$H(s) = \frac{-1}{(s+1)}.$$

We can use the filter shown whose transfer function is

$$H_1(s) = \frac{K\omega_o}{s+\omega_o}, \quad K = -\frac{R_f}{R_{in}} \quad \text{and} \quad \omega_o = \frac{1}{R_f C}.$$



Let $R_f = R_{in} = 1\Omega$ and $C = 1$ F to get $H_1(s) = \frac{-1}{(s+1)}$. A cascade of four such low-pass filters will give us a transfer function $H_4(s) = \frac{1}{(s+1)^4}$. By cascading an inverting Op-Amp circuit with $\frac{R_f}{R_{in}} = 10$, we can get a pass-band gain of 10. This yields us a over-all transfer function as

$$H(s) = \frac{-10}{(s+1)^4}.$$

As determined earlier, the above filter has a $\boxed{\text{half-power frequency of 0.435 rad/sec}}$. In terms of Hz, the $\boxed{\text{half-power frequency is } \frac{0.435}{2\pi} = 0.06923 \text{ Hz.}}$

Scaling: To translate the half-power frequency from 0.06923 Hz to 500 Hz, we need a scaling factor

$$k_f = \frac{500}{0.06923} = 7.2223 \times 10^3.$$

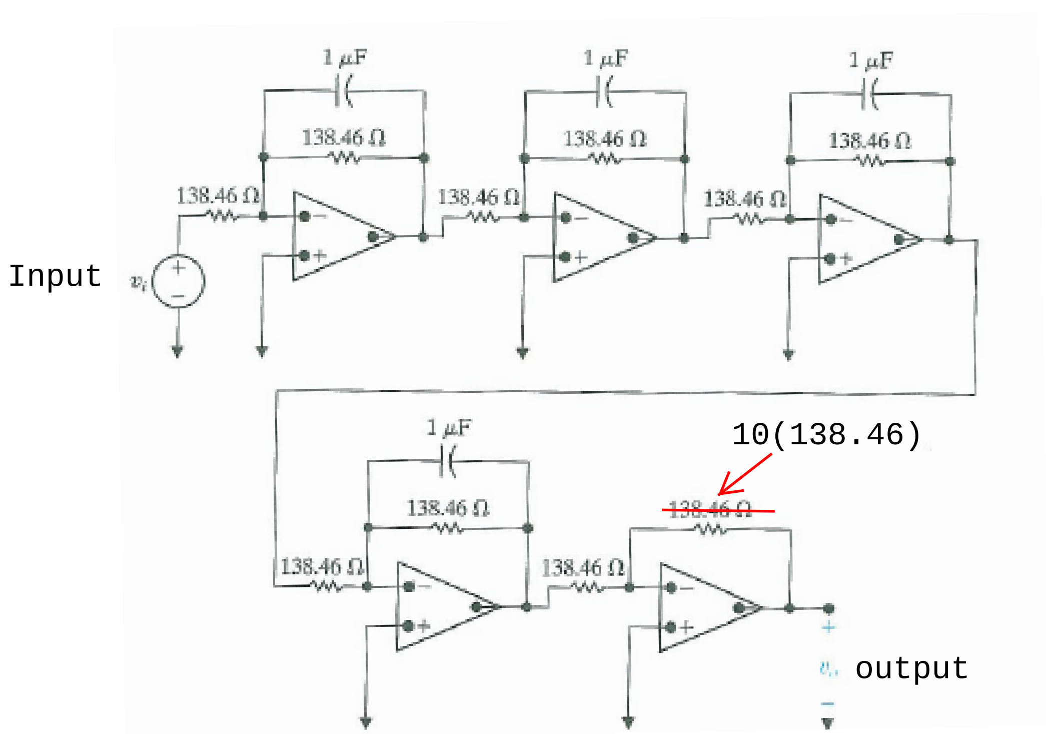
Let us now recall the formulae between the unscaled and scaled values of resistances and capacitances,

$$\boxed{R' = k_m R, \quad C' = \frac{C}{k_m k_f}.}$$

The parameters R and C correspond to the unscaled circuit, where as R' and C' correspond to the scaled circuit. Here k_m is the impedance magnitude scaling factor and k_f is the frequency scaling factor. The design specification calls for capacitances of 1μF only. In order to get a C' of 1μF, we need a k_m given by

$$10^{-6} = \frac{C}{k_m k_f} = \frac{1}{k_m(7.2223 \times 10^3)} \quad \Rightarrow \quad k_m = 138.46.$$

The over-all cascaded filter circuit is as given below.



The resulting magnitude characteristic is shown below.

Half-power frequency is 500 Hz

Drawback of first-order cascaded filters: Clearly, as seen from the above figure, the pass-band gain is not constant. It droops drastically near the half-power frequency. There are other low-pass filter designs. The scope of this course prevents a complete discussion of all such design methods. However, one method called Butterworth design method can easily be developed. The so called Butterworth filters are a cascade of second order filters for even order filters, for odd order filters in addition to second order filters, one first order filter is added.

In order to distinguish the cascade of n first order filters and the n-th order Butterworth filter, let us recall below transfer functions and their respective magnitude functions.

$$\boxed{H(j\omega) = \frac{1}{1+\frac{j\omega}{\omega_c}} \qquad |a+jb| = \sqrt{a^2+b^2}}$$

$$\text{Transfer function of a general unity gain first order filter} \ = H(s) = \frac{\omega_c}{s+\omega_c} = \frac{1}{1+\frac{s}{\omega_c}}.$$

$$\text{Magnitude Characteristic of a unity gain first order filter} \ = |H(j\omega)| = \frac{\omega_c}{\left[\sqrt{\omega^2+\omega_c^2}\right]} = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^2}}.$$

$$\text{Transfer function of a cascade of } n \text{ first order filters} \ = H(s) = \left[\frac{\omega_{cn}}{s+\omega_{cn}}\right]^n = \frac{1}{\left[1+\frac{s}{\omega_{cn}}\right]^n}.$$

$$\text{Magnitude Characteristic of a cascade of } n \text{ first order filters} \ = |H(j\omega)| = \frac{\omega_{cn}^n}{\left[\sqrt{\omega^2+\omega_{cn}^2}\right]^n} = \frac{1}{\left[\sqrt{1+\left(\frac{\omega}{\omega_{cn}}\right)^2}\right]^n}.$$

Half power frequency depends on n.
This causes the drooping characteristic.

We will be developing the transfer function of a so called n-th order Butterworth filter. However, we can quote below its magnitude Characteristic,

$$\text{Magnitude Characteristic of a } n\text{-th order Butterworth filter} \ = |H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_{cn}}\right)^{2n}}}.$$

Half power frequency does not depend on n.

An important observation: Magnitude Characteristics of a cascade of n first order filters and the n-th order Butterworth filter look alike, **but** are not the same. Observe the difference.

Two Methods of Cascaded Filter Design

Method 1

- Start with a known circuit.
- Derive its transfer function $H(s)$.
- Determine its magnitude characteristic $M(\omega)$.
- Cascade n such filter circuits and chose n to satisfy the design specifications.
- **Drawback:** There is no control on magnitude characteristic $M(\omega)$ except for the choice of n.
- Consider a cascade of n first order filters, $H(s) = \frac{\omega_c^n}{(s+\omega_c)^n}$. It has a magnitude characteristic

$$M(\omega) = \frac{1}{\left(\sqrt{1 + \left(\frac{w}{\omega_c}\right)^2}\right)^n}$$

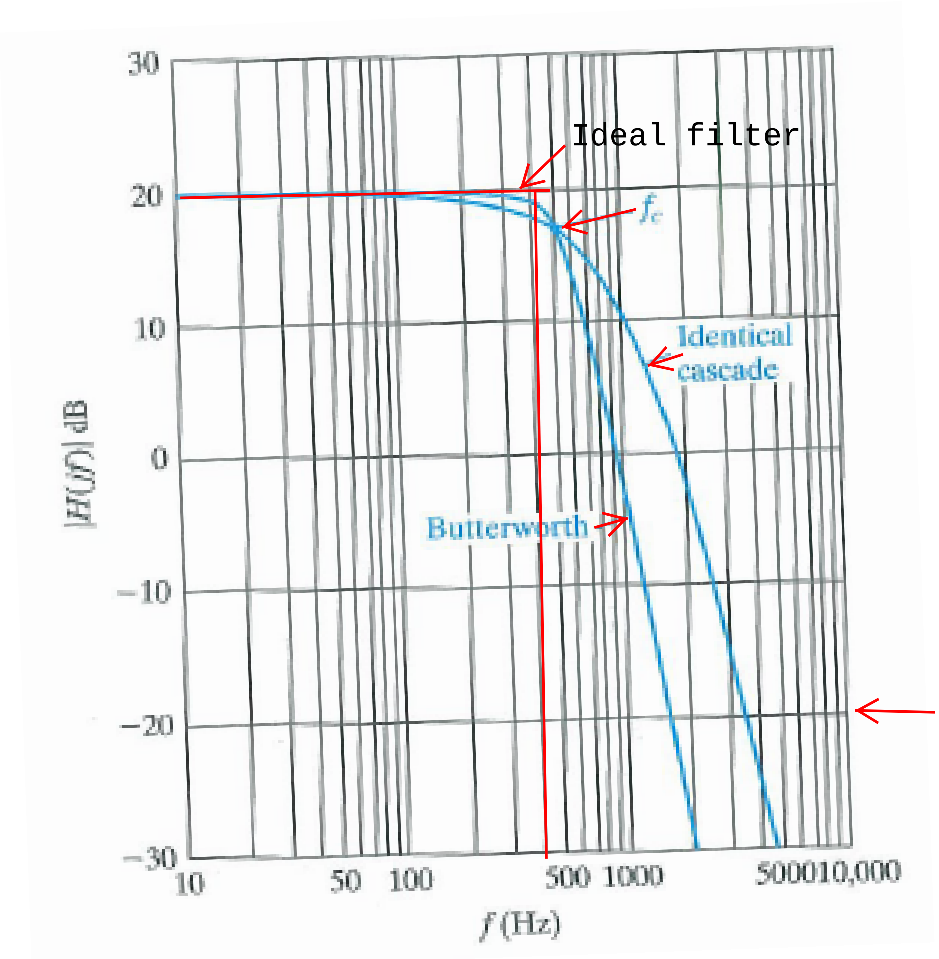
Half power frequency changes as n changes.

Method 2

- Start with an analytically known and acceptable magnitude characteristic $M(\omega)$ of an n-th order filter (Examples of such known characteristics include Butterworth filters and Cheybyshev filters).
- Determine the transfer function $H(s)$ that gives such a magnitude characteristic $M(\omega)$.
- Determine the circuit that yields such a transfer function $H(s)$.
- The magnitude characteristic of n-th order Butterworth filter is

$$M(\omega) = \frac{1}{\sqrt{1 + \left(\frac{w}{\omega_c}\right)^{2n}}}$$

Half power frequency is always ω_c whatever n is.



Comparison of fourth order cascade of identical filters with fourth order Butterworth filter.

Asymptotic slopes are the same.

Steps in the Development of Butterworth Low-Pass Filters

- Magnitude function of n-th order Butterworth Low-Pass Filter,

$$M(\omega) = \left|H(j\omega)\right| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}} \quad \Rightarrow \quad M(\omega)^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}.$$

It is easy to see that the 3 dB frequency or half power frequency of $M(\omega)$ is ω_c. Also, the slope of high frequency asymptote of $M(\omega)$ in its Bode plot is $-20n$ dB as in the case of a cascade of n first order filters.

- Let $\omega_c = 1$ so that we can work with the normalized frequency. Then,

Product of a complex number and its conjugate

$$\boxed{M(\omega)\angle\theta(\omega)\, M(\omega)\angle\theta(-\omega) = M(\omega)^2}$$

$$M(\omega)^2 = H(j\omega)H(-j\omega) = \frac{1}{1+(\omega)^{2n}}.$$

$$\boxed{\begin{aligned} s &= j\omega \\ \omega &= \frac{s}{j} \end{aligned}}$$

- We proceed to determine the transfer function $H(s)$ of n-th order Butterworth Low-Pass Filter. By replacing ω by $\frac{s}{j}$, we get

$$H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}} = \frac{1}{1+\left(\frac{1}{j}\right)^{2n} s^{2n}} = \frac{1}{1+(-1)^n s^{2n}}.$$

$$\boxed{\begin{aligned} &1+(-1)^n s^{2n} \\ &= N(s)N(-s) \end{aligned}}$$

- Suppose $1+(-1)^n s^{2n}$ can be written as $N(s)N(-s)$ where $N(s)$ has all its roots in the negative half complex-plane and $N(-s)$ has all its roots in the positive half complex-plane. Then,

$$H(s) = \frac{1}{N(s)}.$$

All the roots of $N(s)$ being in the negative half complex-plane is important for the stability of the filter. Once $H(s)$ is known, we can look for a filter circuit whose transfer function is $H(s)$.

- In order to determine $N(s)$, we proceed as follows:

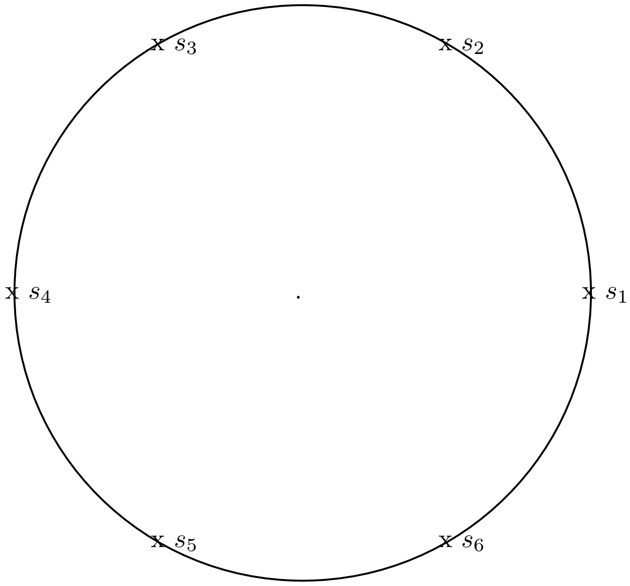
  Determine the roots of the polynomial $1+(-1)^n s^{2n} = 0$. It turns out that all the $2n$ roots of $1+(-1)^n s^{2n} = 0$ lie on the unit circle in complex-plane and are spread out uniformly with an angular distance of $\frac{360^0}{2n}$. Let $r_1, r_2, \cdots, r_n$, be the roots on the left half of the unit circle. Then,

$$N(s) = (s-r_1)(s-r_2)\cdots(s-r_n).$$

  The polynomial $N(s)$ is called the Butter-worth polynomial. If n is **an even number**, all the roots of $N(s)$ are complex, the first root being at $1\angle 90^0/n$. On the other hand if n is **an odd number**, one root lies at -1 and another root at 1 while all the other roots are complex.
  Consider a pair of complex conjugate roots. Let they be z and z^*. Then $(s-z)(s-z^*)$ yields a quadratic polynomial of the form $s^2+\beta s+1$ for some real value of β. In fact, each pair of complex conjugate roots yields one quadratic polynomial. Thus, if n is even, we get $N(s)$ as a product of quadratic polynomials; on the other hand if n is odd, $N(s) = (s+1)N_1(s)$ where $N_1(s)$ is a product of some quadratic polynomials.

- Butter-worth polynomial has always a factor $s+1$ for n an odd integer.
- A first order transfer function $\frac{1}{s+1}$ can be realized by a first order filter while a second order transfer function $\frac{1}{s^2+\beta s+1}$ can be realized by a second order filter.
- Given a low-pass filter transfer function, a high-pass filter transfer function can be obtained by replacing s with $\frac{1}{s}$. Similarly, given a high-pass filter transfer function, a low-pass filter transfer function can be obtained by replacing s with $\frac{1}{s}$.
- Given a a low-pass filter (or a high-pass filter) circuit containing only R and C elements and op-amps, a high-pass filter (or respectively a low-pass filter) circuit can be obtained by replacing R by C and C by R. Of course, one has to select values of R and C appropriately.

Example Let us choose $n = 3$, and determine the roots of $1 + (-1)^n s^{2n} = 1 - s^6 = 0$. Determine $H(s)$ such that $H(s) = \frac{1}{N(s)}$ with $N(s)$ having all its roots in the left half complex plane.



We need to compute the roots of $1 - s^6 = 0$. The six roots of $1 - s^6 = 0$ are given by

$$s^6 = 1\underline{/0^{\circ}} = 1\underline{/\,360\,m^{\circ}}\,,$$

where m is an integer. This implies that

$$s = 1 \text{ at an angle of } (360\,m^{\circ}/6) = 1\underline{/\,60m^{\circ}}$$

for $m = 0$ to 5. Therefore, the six roots are given by

$$\begin{aligned}
s_1 &= 1\underline{/0^{\circ}} = 1 \\
s_2 &= 1\underline{/60^{\circ}} = 0.5 + j0.866 \\
s_3 &= 1\underline{/120^{\circ}} = -0.5 + j0.866 \\
s_4 &= 1\underline{/180^{\circ}} = -1 \\
s_5 &= 1\underline{/240^{\circ}} = 1\underline{/\,-120^{\circ}} = -0.5 - j0.866 \\
s_6 &= 1\underline{/360^{\circ}} = 1\underline{/\,-60^{\circ}} = 0.5 - j0.866
\end{aligned}$$

The roots are shown on the unit circle in the figure given above. The roots s_3, s_4, and s_5 are in the left-half s plane and correspond to stable roots of $N(s)$. Hence $H(s)$ is given by

$$H(s) = \frac{1}{N(s)} = \frac{1}{(s+1)(s+0.5-j0.866)(s+0.5+j0.866)} = \frac{1}{(s+1)(s^2+s+1)}.$$

Example Let us choose $n = 4$, and determine the roots of $1 + (-1)^n s^{2n} = 1 + s^8 = 0$. Determine $H(s)$ such that $H(s) = \frac{1}{N(s)}$ with $N(s)$ having all its roots in the left half complex plane. We need to compute the roots of $1 + s^8 = 0$. The eight roots of $1 + s^8 = 0$ are given by $\boxed{s^8 = -1}$

$$s^8 = 1\angle\, 180^0 + 360\, m^0 \,,$$

where m is an integer. This implies that

$$s = 1 \text{ at an angle of } 22.5^0 + (360\, m^0/8)$$

for $m = 0$ to 7.

The roots are

$$s_1 = 1\angle\, 22.5^0, \quad s_2 = 1\angle\, 67.5^0, \quad s_3 = 1\angle\, 112.5^0, \quad s_4 = 1\angle\, 157.5^0,$$
$$s_5 = 1\angle\, -157.5^0, \quad s_6 = 1\angle\, -112.5^0, \quad s_7 = 1\angle\, -67.5^0, \quad \text{and} \quad s_8 = 1\angle\, -22.5^0.$$

The roots that are on the negative side (left side) of the complex plane are

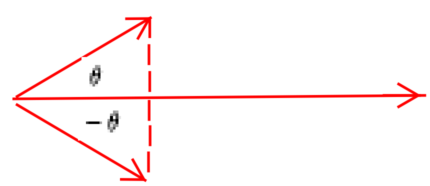
$$s_3 = 1\angle\, 112.5^0, \qquad s_4 = 1\angle\, 157.5^0,$$
$$s_5 = 1\angle\, -157.5^0, \quad \text{and} \quad s_6 = 1\angle\, -112.5^0.$$

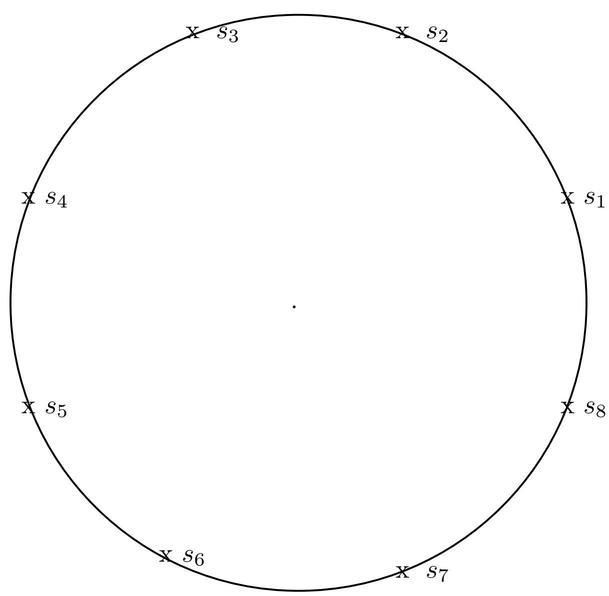
Thus

$$\begin{aligned} N(s) &= (s - 1\angle\, 112.5^0)(s - 1\angle\, -112.5^0)(s - 1\angle\, 157.5^0)(s - 1\angle\, -157.5^0) \\ &= (s^2 - 2\cos(112.5^0)s + 1)(s^2 - 2\cos(157.5^0)s + 1) \\ &= (s^2 + 0.765s + 1)(s^2 + 1.848s + 1). \end{aligned}$$

In simplifying the above, we used the fact that

$$(s - 1\angle\,\theta^0)(s - 1\angle\, -\theta^0) = s^2 - (1\angle\,\theta^0 + 1\angle\, -\theta^0)s + 1 = s^2 - 2\cos(\theta)s + 1.$$





Example Let us choose $n = 5$, and determine the roots of $1 + (-1)^n s^{2n} = 1 - s^{10} = 0$. Determine $H(s)$ such that $H(s) = \frac{1}{N(s)}$ with $N(s)$ having all its roots in the left half complex plane. We need to compute the roots of $1 - s^{10} = 0$. The eight roots of $1 - s^{10} = 0$ are given by

$$s^{10} = 1\angle 360\,m^{0},$$

where m is an integer. This implies that

$$s = 1 \text{ at an angle of } (360\,m^{0}/10)$$

for $m = 0$ to 9.

The roots are

$$\begin{array}{lllll} s_1 = 1, & s_2 = 1\angle 36^{0}, & s_3 = 1\angle 72^{0}, & s_4 = 1\angle 108^{0}, & s_5 = 1\angle 144^{0}, \\ s_6 = -1, & s_7 = 1\angle -144^{0}, & s_8 = 1\angle -108^{0}, & s_9 = 1\angle -72^{0}, & s_{10} = 1\angle -36^{0}. \end{array}$$

The roots that are on the negative side of the complex plane are

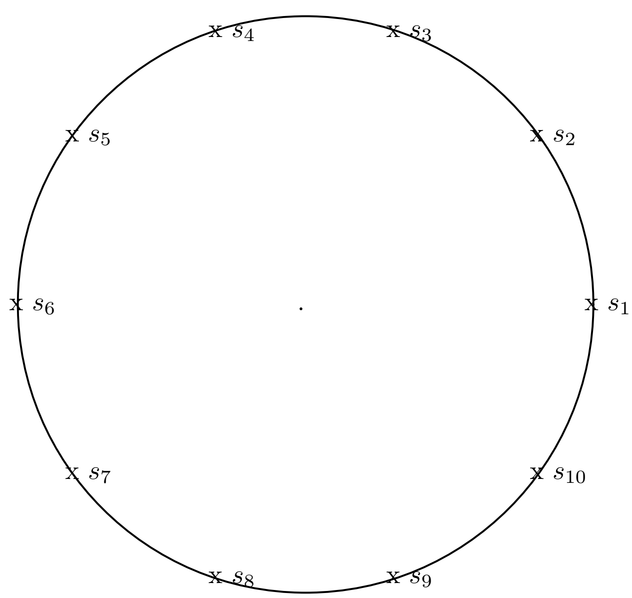
$$\begin{array}{ll} s_4 = 1\angle 108^{0}, & s_5 = 1\angle 144^{0}, \\ s_8 = 1\angle -108^{0}, & s_7 = 1\angle -144^{0} \quad \text{and} \quad s_6 = -1. \end{array}$$

Thus

$$\begin{aligned} N(s) &= (s+1)(s - 1\angle 108^{0})(s - 1\angle -108^{0})(s - 1\angle 144^{0})(s - 1\angle -144^{0}) \\ &= (s+1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1). \end{aligned}$$

In simplifying the above, we used the fact that

$$(s - 1\angle \theta^{0})(s - 1\angle -\theta^{0}) = s^2 - (1\angle \theta^{0} + 1\angle -\theta^{0})s + 1 = s^2 - 2\cos(\theta)s + 1.$$

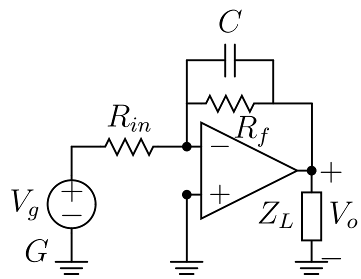


A table of n-th order Butterworth polynomials upto $n = 8$ is given below. For $n > 8$, you can Google it online.

TABLE 15.1 Normalized (so that $\omega_c = 1$ rad/s) Butterworth Polynomials up to the Eighth Order

n	nth-Order Butterworth Polynomial $N(s)$
1	$(s + 1)$
2	$(s^2 + \sqrt{2}s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$
5	$(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + \sqrt{2} + 1)(s^2 + 1.932s + 1)$
7	$(s + 1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$
8	$(s^2 + 0.390s + 1)(s^2 + 1.111s + 1)(s^2 + 1.6663s + 1)(s^2 + 1.962s + 1)$

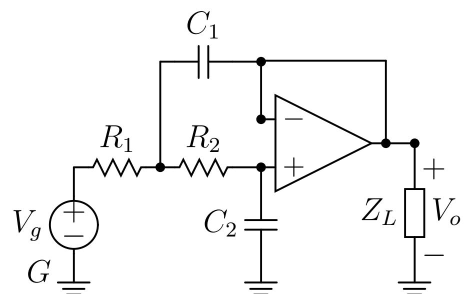
First-order LPF Op-Amp Circuit



$$H(s) = \frac{K\omega_o}{s+\omega_o}$$

$$K = -\frac{R_f}{R_{in}} \quad \text{and} \quad \omega_o = \frac{1}{R_f C}.$$

Second-order LPF Op-Amp Circuit

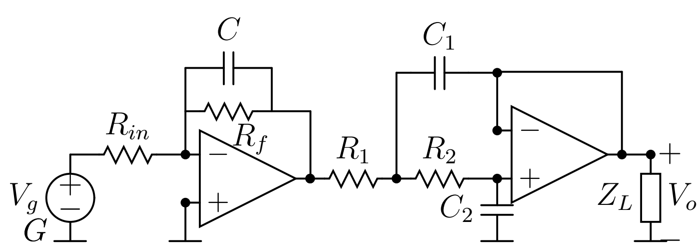


$$H(s) = \frac{\omega_0^2}{s^2+\beta s+\omega_0^2}$$

For $R = R_1 = R_2 = R$,
$\beta = \frac{2}{RC_1}$, $\omega_0^2 = \frac{1}{R^2 C_1 C_2}$

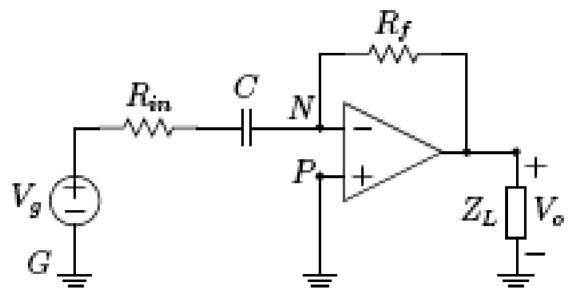
Third-order BW LPF Op-Amp Circuit

A first and second order LPF filter circuits are cascaded to form a third order filter circuit as shown.



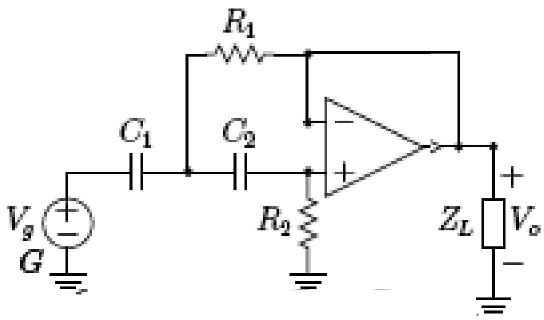
A fourth order filter is a cascade of two second order filters

First-order HPF Op-Amp Circuit



$$H(s) = \frac{Ks}{s+\omega_o} = \frac{K\frac{s}{\omega_o}}{1+\frac{s}{\omega_o}}, \quad \text{where} \quad K = -\frac{R_f}{R_{in}} \quad \text{and} \quad \omega_o = \frac{1}{R_{in}C}$$

Second-order HPF Op-Amp Circuit

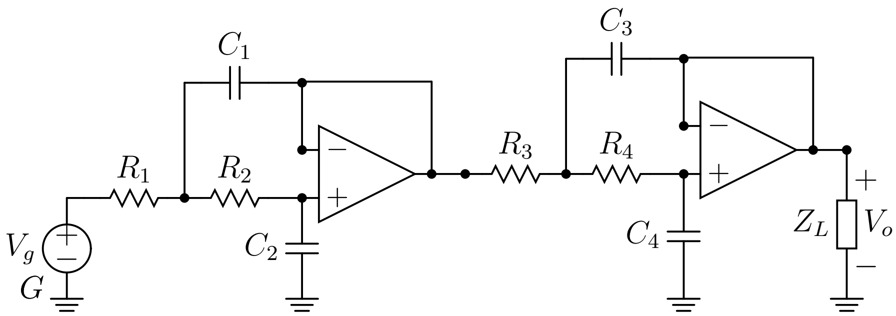


$$H(s) = \frac{s^2}{s^2+\beta_1 s+\omega_1^2}$$

$$\beta_1 = \frac{2}{R_2C_1}, \quad \omega_1^2 = \frac{1}{C_1^2R_1R_2}$$

for $C_1 = C_2$

Fourth Order LPF Filter Design:



Two second order LPF filter circuits are cascaded to form a fourth order filter circuit as shown.

Given info: When $R_1 = R_2$ and $R_3 = R_4$, the transfer function of the cascaded filter is given by

$$H(s) = \frac{\omega_1^2}{s^2 + \beta_1 s + \omega_1^2} \, \frac{\omega_2^2}{s^2 + \beta_2 s + \omega_2^2},$$

where

$$\beta_1 = \frac{2}{R_1 C_1}, \quad \omega_1^2 = \frac{1}{R_1^2 C_1 C_2}, \quad \text{and } \beta_2 = \frac{2}{R_3 C_3}, \quad \omega_2^2 = \frac{1}{R_3^2 C_3 C_4}.$$

Fourth order Butterworth polynomial is given by

$$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1).$$

This is a proto-type Butterworth Polynomial.

(a) **Proto type design:** Using the above info, determine the values of R_1, C_1, C_2, and R_3, C_3, C_4 so that the 3 dB frequency of proto-type Butterworth LPF filter equals **one radian/sec**.

We like to have $\beta_1 = 0.765$, $\beta_2 = 1.848$, and $\omega_1 = \omega_2 = 1$. This implies that

$$\beta_1 = \frac{2}{R_1 C_1} = 0.765, \quad \beta_2 = \frac{2}{R_3 C_3} = 1.848, \quad \omega_1 = \frac{1}{\sqrt{R_1^2 C_1 C_2}}, \quad \omega_2 = \frac{1}{\sqrt{R_3^2 C_3 C_4}}.$$

The above equations can be satisfied by a number of ways. One solution is as given below:

$$R_1 = R_2 = 2\Omega \qquad R_3 = R_4 = 2\Omega$$

$$C_1 = \tfrac{1}{0.765} = 1.307\,F \qquad C_3 = \tfrac{1}{1.848} = 0.541\,F$$

$$C_2 = 0.25(0.765) = 0.191\,F \qquad C_4 = 0.25(1.848) = 0.462\,F$$

(b) Using the above proto type design, determine the new scaled values of R_1, C_1, C_2, and R_3, C_3, C_4 so that the 3 dB frequency equals 10^5 **radians/sec**. Use resistances of value $10^4\Omega$ or multiples of it.

We can use an impedance scaling k_m of 10^4. Obviously, we need a frequency scaling k_f of 10^5. Thus the new parameter values designated by prime $'$ are

$$R_1' = R_2' = (20)\,10^3\Omega \qquad R_3' = R_4 = (20)\,10^3\Omega$$

$$C_1' = (1.307)\,10^{-9}\,F \qquad C_3' = (0.541)\,10^{-9}\,F$$

$$C_2' = (0.191)\,10^{-9}\,F \qquad C_4' = (0.462)\,10^{-9}\,F$$

$$R' = k_m R$$

$$C' = \frac{C}{k_m k_f}.$$

Scaling formulae

Determining the needed order of Butterworth Filters

The order n of Butterworth Filters can be determined in two ways:

- Suppose the magnitude at some frequency ω_r in rejection band is prescribed to be less than some M_r. Then,

$$\frac{1}{\sqrt{1+\left(\frac{\omega_r}{\omega_c}\right)^{2n}}} < M_r \qquad \Rightarrow \qquad 1+\left(\frac{\omega_r}{\omega_c}\right)^{2n} > \left(\frac{1}{M_r}\right)^2.$$

  Therefore

$$\left(\frac{\omega_r}{\omega_c}\right)^{2n} > \left(\frac{1}{M_r}\right)^2 - 1 \qquad \Rightarrow \qquad n > \frac{\log_{10}\left(\left(\frac{1}{M_r}\right)^2 - 1\right)}{2\log_{10}\left(\frac{\omega_r}{\omega_c}\right)}.$$

- Slope of a straight line can be determined when we know any two points on it. Suppose A_p and A_s are two magnitude values in dB at two frequencies ω_p and ω_s. Then,

$$A_p = 20\log_{10}\left(\frac{1}{\sqrt{1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}}}\right) = -10\log_{10}\left(1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right)$$

  This implies that

$$10^{-0.1A_p} = 1+\left(\frac{\omega_p}{\omega_c}\right)^{2n} \qquad \Rightarrow \qquad 10^{-0.1A_p} - 1 = \left(\frac{\omega_p}{\omega_c}\right)^{2n}.$$

  Similarly

$$10^{-0.1A_s} - 1 = \left(\frac{\omega_s}{\omega_c}\right)^{2n}.$$

  Then, we see that

$$\left(\frac{\omega_s}{\omega_p}\right)^{2n} = \frac{10^{-0.1A_s} - 1}{10^{-0.1A_p} - 1} = R.$$

  Now that we know R, we get

$$n = \frac{\log_{10}(R)}{2\log_{10}\left(\frac{\omega_s}{\omega_p}\right)}.$$

Example: The magnitude of an n-th order Butterworth low pass filter is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

where ω_c is the cutoff frequency (3 dB frequency). Determine the order n of the filter if the needed magnitude of $|H(j\omega)|$ at $\frac{\omega}{\omega_c} = 2$ has to be less than or equal to 10^{-4}.

We need

$$\frac{1}{\sqrt{1 + (2)^{2n}}} \leq 10^{-4}.$$

The above implies that

$$1 + 2^{2n} \geq 10^8.$$

Thus by taking the logarithm of the above equation (neglect 1 in comparision with 2^{2n}), we get

$$2n \log_{10} 2 \geq 8.$$

This yields

$$2n \geq \frac{8}{\log_{10} 2} = 13.288 \quad \Rightarrow \quad n \geq 14.$$

Student's name in capital letters:

This is a HW problem, collected and graded.

Design a fifth order Butterworth LPF filter with a low frequency gain of 20 and a cutoff (3 dB) frequency of 5000 rad/sec . Use your judgement to determine appropriate scale factors k_m **and** k_f**.** Draw clearly the filter circuit and mark all the resistance and capacitance values.