

Sinusoidal Signals – Phasors – Impedances

A sinusoidal signal is represented in time domain by

$$x(t) = A\sqrt{2}\cos(\omega t + \theta), \text{ where}$$

$A\sqrt{2}$ = Amplitude of the signal

A = RMS value of the signal

ω = Angular or Radian frequency of the signal

θ = Phase angle of the signal

$\omega = 2\pi f$, f is the frequency of the signal in Hz or cycles per second

$f = \frac{1}{T}$, T is the period of the signal.

A time domain sinusoidal signal can be transformed to what is known as a phasor domain. All systems including circuits function or work in time domain. However, for the purpose of analysis and design, time domain signals are transformed to other domains, phasor domain, Laplace domain etc. Three essential variables that define a sinusoidal signal $x(t)$ are A , θ , and ω . We often analyze a circuit with ω fixed. In this case, A and θ take prominence. On the other hand, A and θ can be viewed as a polar form of a complex number, $A\angle\theta$. In Electrical Engineering, a sinusoidal signal when represented by $A\angle\theta$, we call $A\angle\theta$ as a phasor,

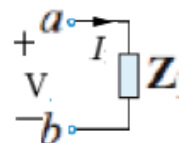
Time domain signal	RMS Phasor domain signal
$A\sqrt{2}\cos(\omega t + \theta)$	$\Rightarrow A\angle\theta$
	Cosine is used as a reference

Often, RMS value is used as the magnitude of a phasor although some times amplitude is used. Whenever a phasor domain signal is known, we can come back to time domain,

Phasor domain signal	Time domain signal
$A\angle\theta(\omega)$	$\Rightarrow A\sqrt{2}\cos(\omega t + \theta)$

Sinusoidal Steady State Circuit Analysis is done in Phasor domain. Let $v(t)$ and $i(t)$ be sinusoidal voltage across an element and current through it. Let the phasors corresponding to $v(t)$ and $i(t)$ be V and I . We define the impedance of an element as the ratio of the phasor voltage across it to the current phasor through it,

$$\text{Impedance} = \frac{\text{Phasor Voltage}}{\text{Phasor Current}} = \frac{V}{I} = Z.$$



We emphasize that notationally lower case variables are used for time domain, and upper case for Phasor domain. Thus,

Time domain signal	Phasor domain signal
$x(t)$	$\Leftrightarrow X(\omega).$

Laplace domain is a generalization of a phasor domain, and is useful for even non-sinusoidal time domain signals. We pursue this next in an elementary way just to introduce notations. For general signals not necessarily sinusoidal, one can **transform a time domain signal into a Laplace domain signal**.

$$\text{The impedance of an element in Laplace domain} = \frac{\text{Laplace Transform of its voltage}}{\text{Laplace Transform of its current}} = Z(s).$$

$$\text{The impedance of an element in phasor domain} = \frac{\text{Phasor of its voltage}}{\text{Phasor of its current}} = Z(\omega).$$

The impedances of elements, R , L , and C are given by

Element :	Resistance R	Inductance L	Capacitance C
Impedance in Laplace domain :	R	sL	$\frac{1}{sC}$
Impedance in Phasor domain :	R	jωL	$\frac{1}{j\omega C}$

For Phasor domain, the Laplace variable $s = j\omega$ where ω is the radian frequency of the sinusoidal signal.

The transfer function $H(s)$ of a circuit in Laplace domain is defined as:

$$H(s) = \text{The transfer function of a circuit} = \frac{\text{Transform of the output}}{\text{Transform of the input}}.$$

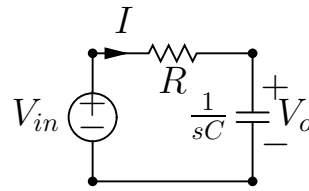
Then, the transfer function $H(j\omega)$ of a circuit in phasor domain is

$$H(j\omega) = \text{The transfer function of a circuit} = \frac{\text{Phasor of the output}}{\text{Phasor of the input}} = M(\omega) \angle \theta(\omega).$$

$$\begin{aligned} \text{Phasor of the output} &= (\text{Phasor of the input})(H(j\omega)) \\ &= (\text{Phasor of the input})(M(\omega) \angle \theta(\omega)) \end{aligned}$$

$$\begin{aligned} v_{in} = A \cos(\omega t) & \quad \begin{array}{c} \text{---} \text{+} \text{---} \\ \text{---} \text{---} \end{array} \quad \begin{array}{c} \bullet \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \begin{array}{c} \bullet \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \text{+} \text{---} \\ \text{---} \text{---} \end{array} \quad v_{out} = AM(\omega) \cos(\omega t + \theta(\omega)) \\ \text{Phasor} = \frac{A}{\sqrt{2}} & \quad \begin{array}{c} \text{---} \text{+} \text{---} \\ \text{---} \text{---} \end{array} \quad \begin{array}{c} \bullet \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \begin{array}{c} \bullet \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \text{+} \text{---} \\ \text{---} \text{---} \end{array} \quad \text{Phasor} = \frac{A}{\sqrt{2}} M(\omega) \angle \theta(\omega) \end{aligned}$$

Example 1: As a simple example, consider a RC circuit as shown on the right. By voltage division rule, it is easy to determine its transfer function as



$$H(s) = \frac{V_o}{V_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC} = \frac{1}{RC} \frac{1}{s + \frac{1}{RC}} = \frac{\alpha}{s + \alpha}$$

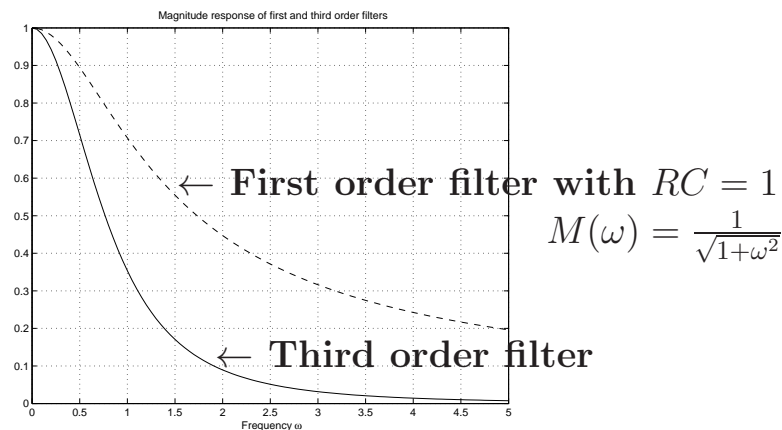
where $\alpha = \frac{1}{RC}$.

Transfer function is normally expressed in a form where the coefficient of highest power in the denominator is unity (one).

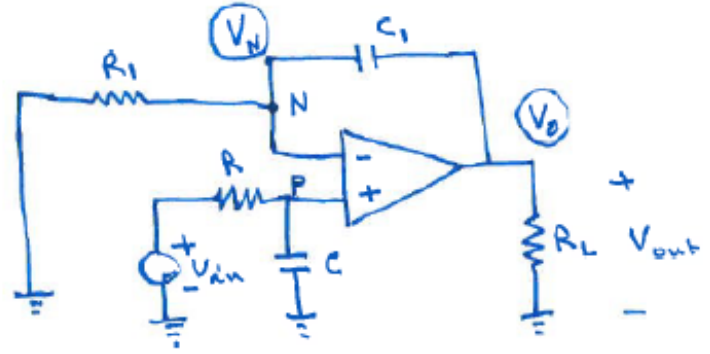
We can compute $H(j\omega)$ as

$$H(j\omega) = \frac{1}{1 + j\omega RC} = M(\omega) \angle \theta(\omega) = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \angle -\tan^{-1}(\omega RC).$$

We observe that the magnitude as well as phase angle of $H(j\omega)$ are functions of ω . In this example magnitude decreases as ω increases. This implies that the given circuit passes to the output low frequency signals better than high frequency signals. **We will explore this in depth when we begin to discuss filters.**



Example 2: Determine the transfer function of the circuit shown. Assume that the Op-Amp is ideal. The solution is simple. In what follows we show all steps clearly showing all the mathematical manipulations.



By voltage division rule,

$$V_N = V_P = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_{in} = \frac{1}{1 + sRC} V_{in} = \frac{1}{RC} \frac{1}{s + \frac{1}{RC}} V_{in} = \frac{\alpha}{s + \alpha} V_{in}$$

where $\alpha = \frac{1}{RC}$.

We can write the node equation at N as

$$(V_N - V_0)sC_1 + \frac{V_N}{R_1} = 0.$$

We can simplify the above equation as

$$V_N - V_0 + \frac{V_N}{sC_1R_1} = 0 \Rightarrow V_0 = V_N + \frac{V_N}{sC_1R_1} = V_N \left[1 + \frac{1}{sC_1R_1} \right] = V_N \frac{1 + sC_1R_1}{sC_1R_1}.$$

Thus

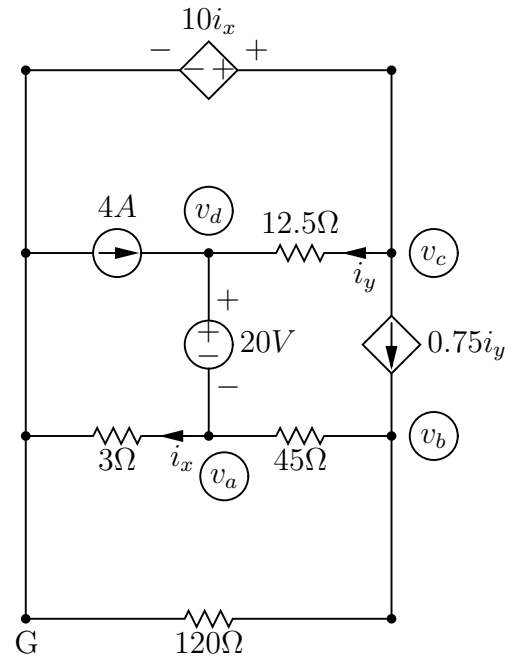
$$V_0 = V_N \frac{1 + sC_1R_1}{sC_1R_1} = V_N \frac{s + \frac{1}{C_1R_1}}{s} = V_N \frac{s + \beta}{s}$$

where $\beta = \frac{1}{C_1R_1}$.

We get

$$\text{The transfer function } = H(s) = \frac{V_0}{V_{in}} = \frac{V_0}{V_N} \frac{V_N}{V_{in}} = \frac{s + \beta}{s} \frac{\alpha}{s + \alpha} = \frac{\alpha(s + \beta)}{s(s + \alpha)}.$$

Example by Node Voltage Method: We would like to use nodal analysis to solve the circuit shown on the right. Using G as the ground or reference, the node voltages v_a , v_b , v_c , and v_d are marked in circles near the nodes. To start with, all these node voltages can be considered as unknown although values of some of them can be obtained easily. Note that we have four unknowns and hence we need to write four equations in addition to the equations that define the controlling variables. The following steps illustrate a systematic way of writing the necessary equations to determine all the node voltages.



1. There are controlling variables in the circuit. Write one equation for each one of them in terms of basic node voltages v_a , v_b , v_c , and v_d .

$$i_x = \frac{v_a}{3} \quad \text{and} \quad i_y = \frac{v_c - v_d}{12.5}.$$

2. There are voltage sources in the circuit (both independent and dependent). These voltage sources yield relationships among the basic node voltage variables. Write these relationships below. The dependent sources are controlled by certain variables, substitute for them using the results of Step 1 and re-write the equations of this step so that they depend only on node voltages v_a , v_b , v_c , and v_d .

$$v_d - v_a = 20V \quad \text{and} \quad v_c = 10i_x = 10\frac{v_a}{3}.$$

3. The above two steps yield two equations among the basic node voltage variables. In addition to these two equations, we need to write two more equations by examining appropriate nodes.

Write one KCL equation for the node whose voltage is v_b . Substitute for any controlling variables using the results of Step 1 and re-write the equation.

$$\frac{v_b}{120} + \frac{v_b - v_a}{45} = 0.75i_y = 0.75\frac{v_c - v_d}{12.5}.$$

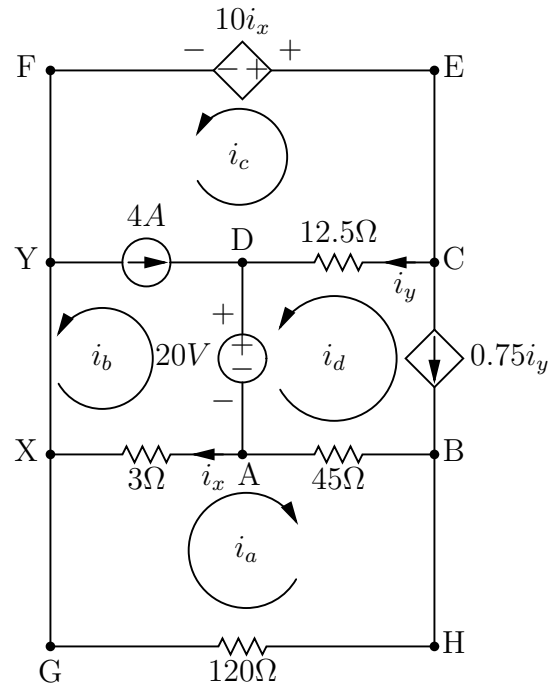
Write another KCL equation at another node or multiple of nodes (super node equation). Specify which node equation(s) you wrote.

$$\frac{v_a}{3} + \frac{v_a - v_b}{45} - 4 + \frac{v_d - v_c}{12.5} = 0.$$

Example by Mesh Current Method:

We would like to use mesh analysis to solve the circuit shown on the right.

The mesh currents i_a , i_b , i_c , and i_d are marked as shown. To start with, all these mesh currents can be considered as unknown although values of some of them can be obtained easily. Note that we have four unknowns and hence we need to write four equations in addition to the equations that define the controlling variables. The following steps illustrate a systematic way of writing the necessary equations to determine all the mesh currents:



1. There are controlling variables in the circuit. Write one equation for each one of them in terms of basic mesh currents i_a , i_b , i_c , and i_d .

$$i_x = -(i_a + i_b) \quad \text{and} \quad i_y = i_d - i_c.$$

2. There are current sources in the circuit (both independent and dependent). These current sources yield relationships among the basic mesh current variables. Write these relationships below. The dependent sources are controlled by certain variables, substitute for them using the results of Step 1 and re-write the equations of this step so that they depend only on mesh currents i_a , i_b , i_c , and i_d .

$$4 = i_c - i_b \quad \text{and} \quad 0.75i_y = 0.75(i_d - i_c) = -i_d.$$

3. The above two steps yield two equations among the basic mesh current variables. In addition to these two equations, we need to write two more equations by examining appropriate meshes.

Write one equation for the first mesh, i.e., the mesh GXABHG.

$$-3(i_a + i_b) - 45(i_a + i_d) - 120i_a = 0.$$

Write another mesh equation or multiple of mesh equations (super mesh equation). Specify which mesh equation(s) you wrote.

We write the super mesh XADCEFYX equation,

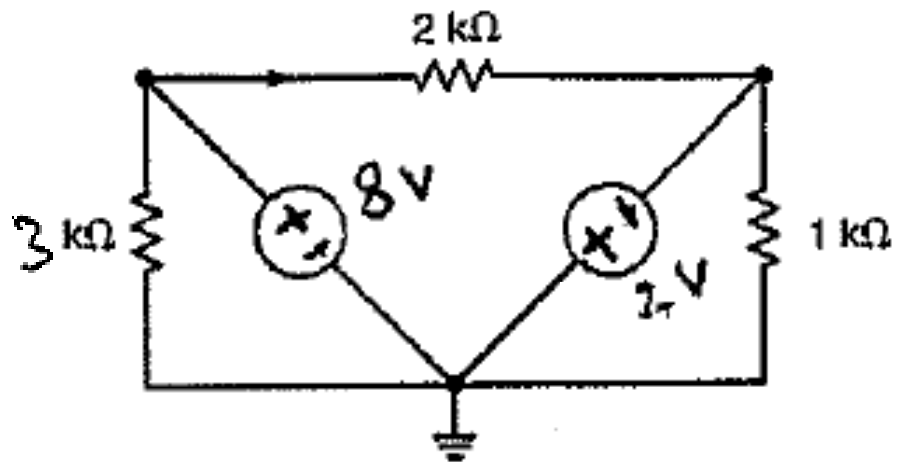
$$-3(i_a + i_b) + 20 - 12.5(i_c - i_d) - 10i_x = 0.$$

NAME IN CAPITAL LETTERS:

Determine both the magnitude and direction of current through each resistance as well as voltage across it.

Mark them on the circuit clearly.

What is the easiest method of solving this circuit?

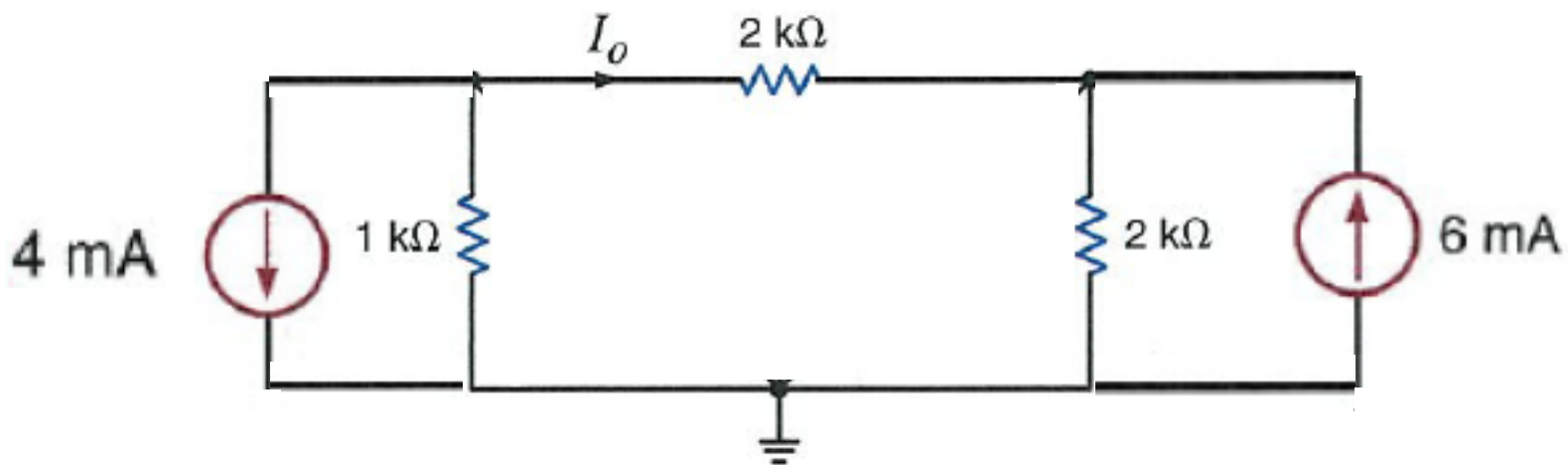
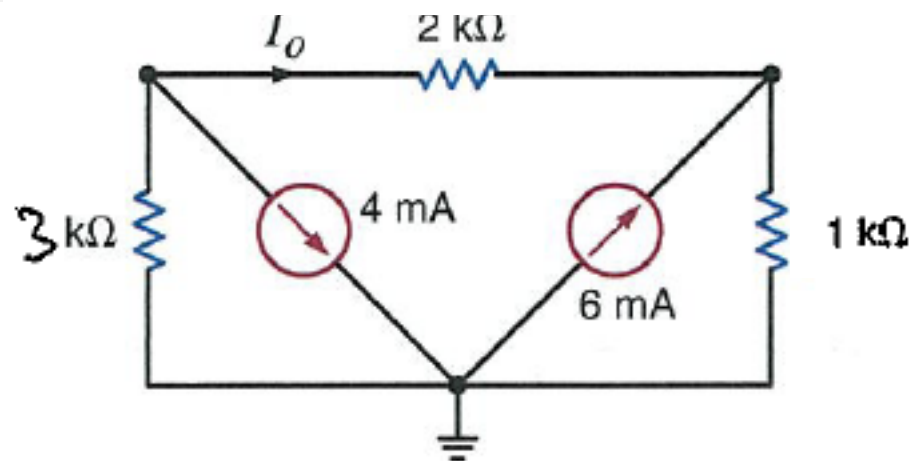


NAME IN CAPITAL LETTERS:

Determine both the magnitude and direction of current through each resistance as well as voltage across it.

Mark them on the circuit clearly.

Re-draw the circuit and observe which method is easier to solve this circuit.

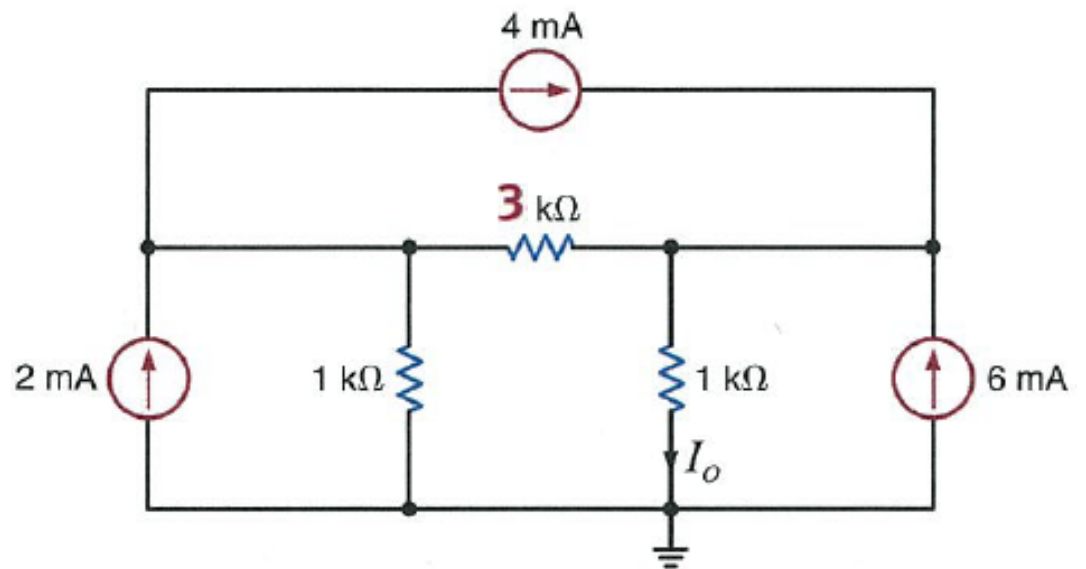


NAME IN CAPITAL LETTERS:

Determine both the magnitude and direction of current through each resistance as well as voltage across it.

Mark them on the circuit clearly.

Utilize either Node Voltage Method or Mesh Current Method whichever is easier.

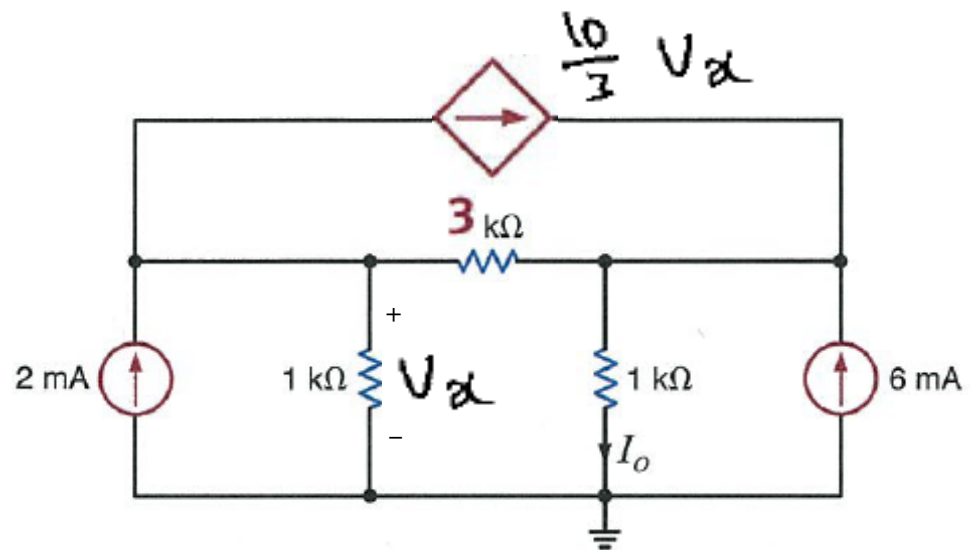


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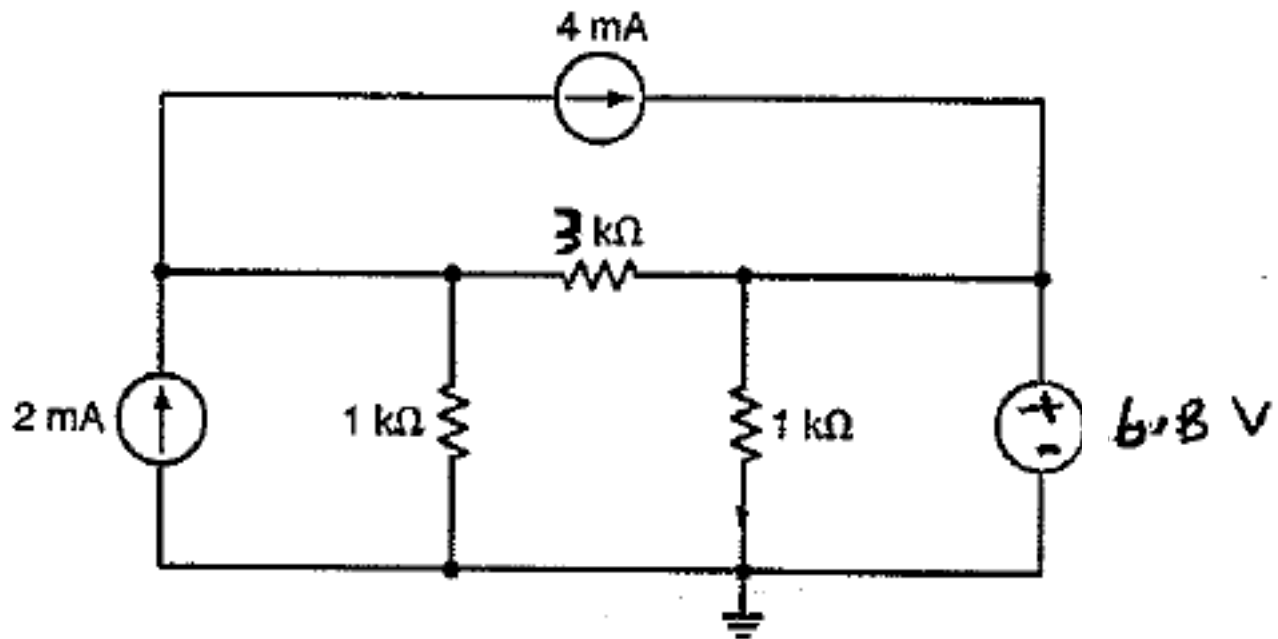


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Determine both the magnitude and direction of current through each resistance as well as voltage across it.

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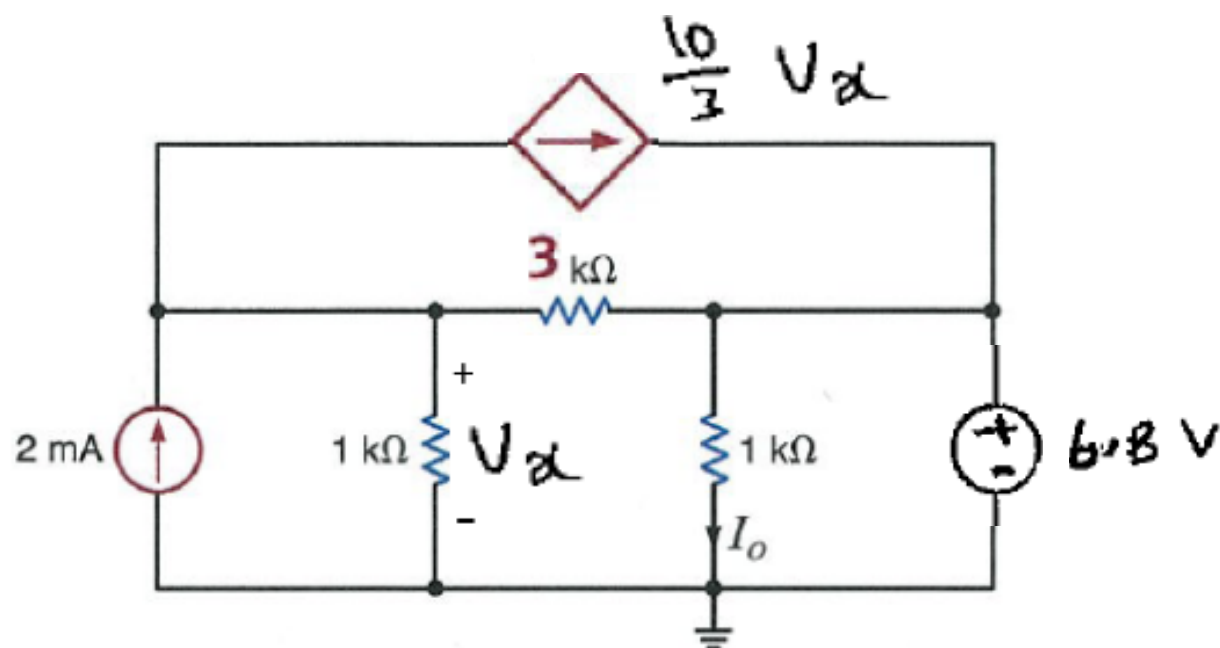


NAME IN CAPITAL LETTERS:

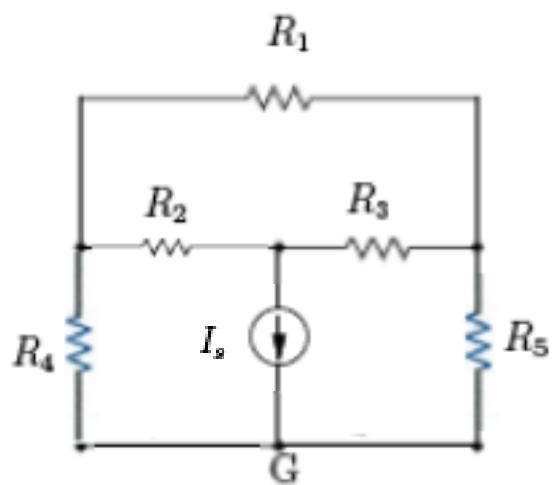
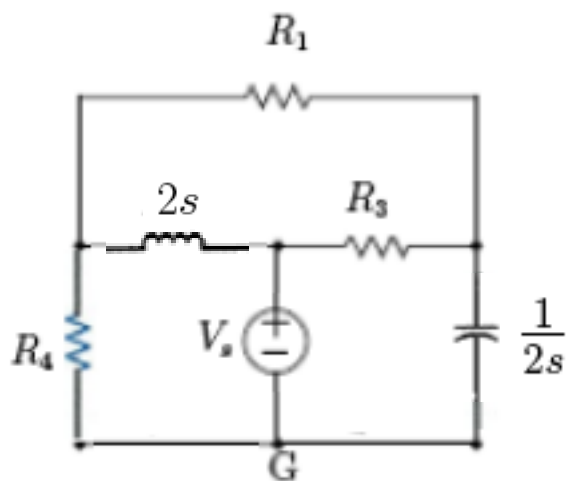
Determine both the magnitude and direction of current through each resistance as well as voltage across it.

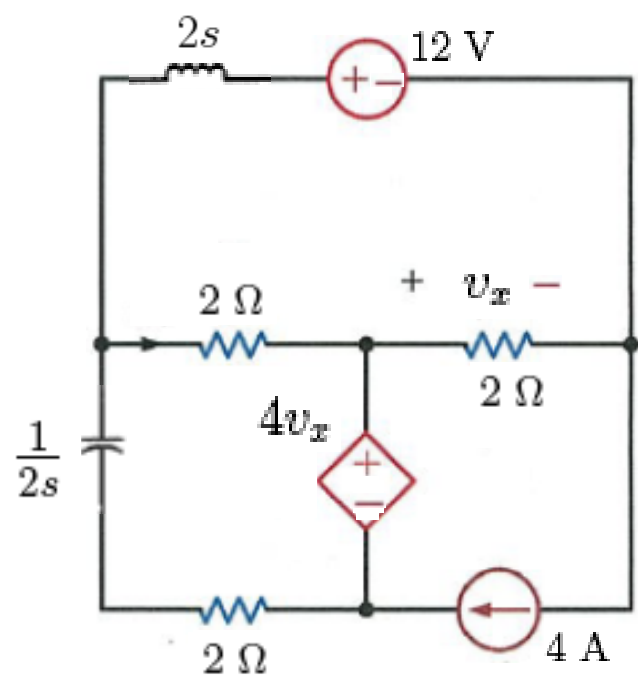
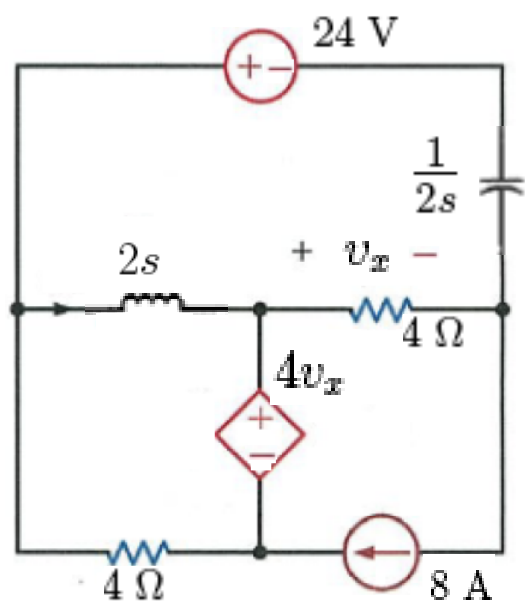
Mark them on the circuit clearly.

Utilize Node Voltage Method.

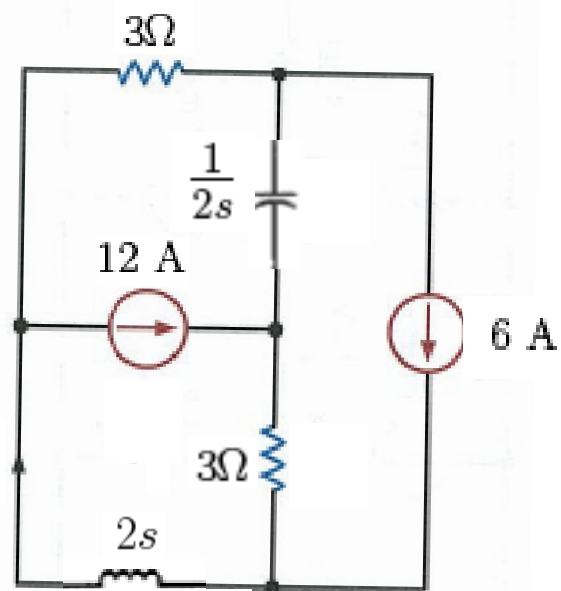
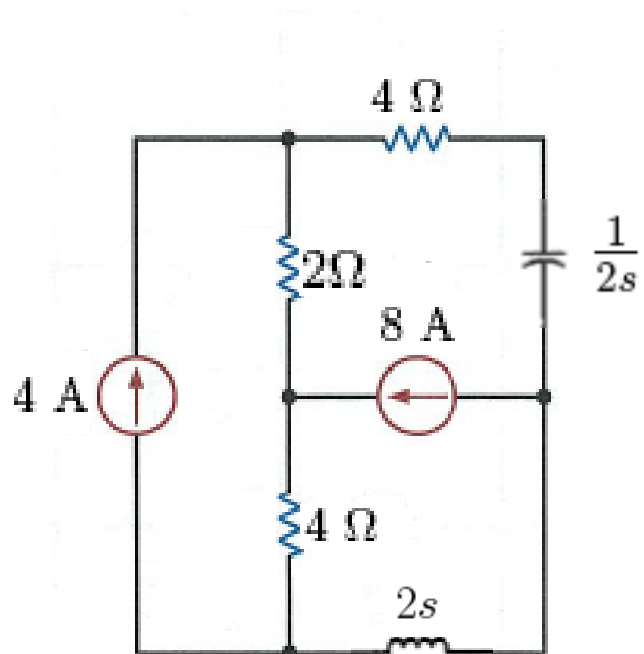


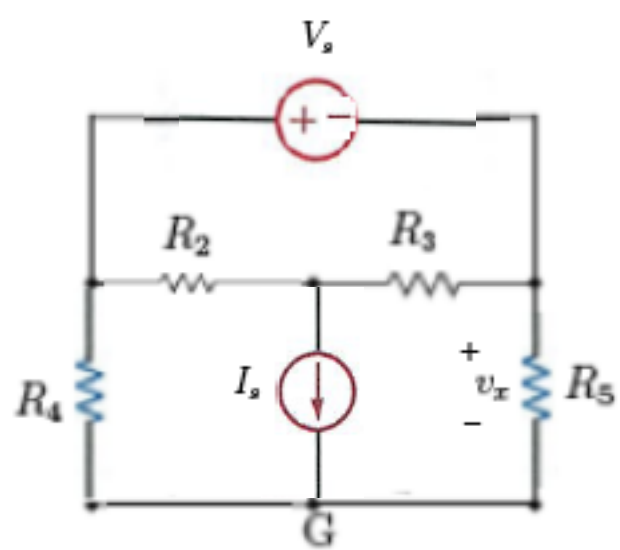
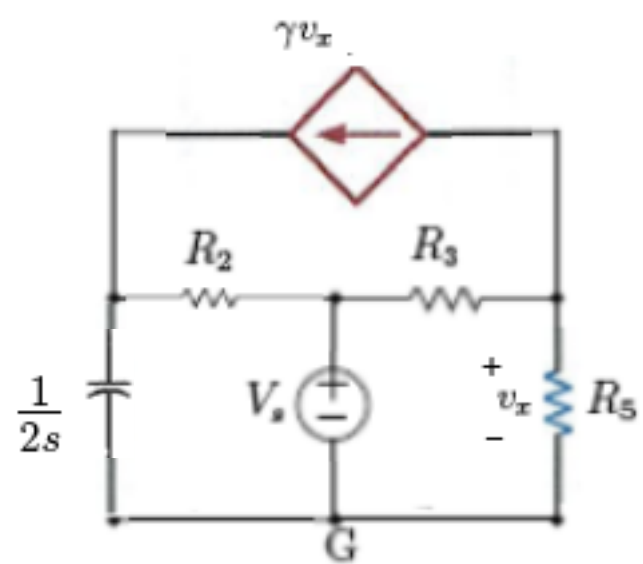
For each circuit shown below write down sufficient number of independent equations supported by
 (a) Node voltage method and (b) Mesh current method.

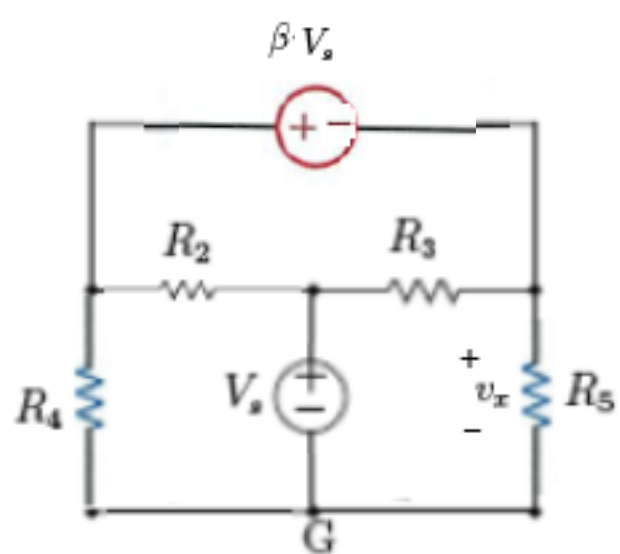
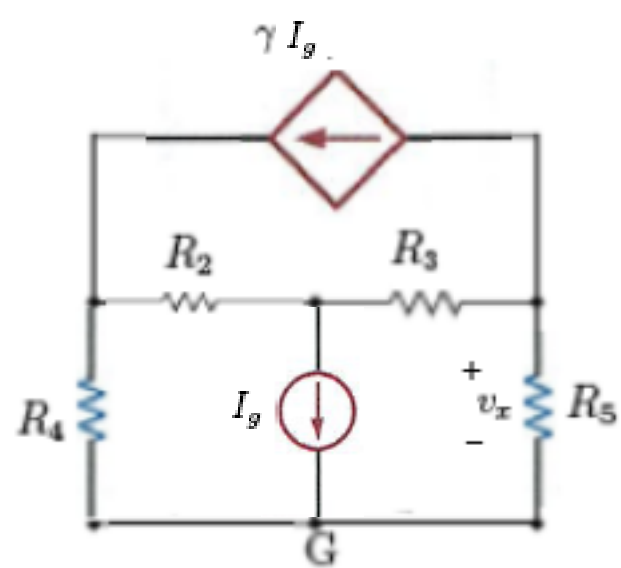


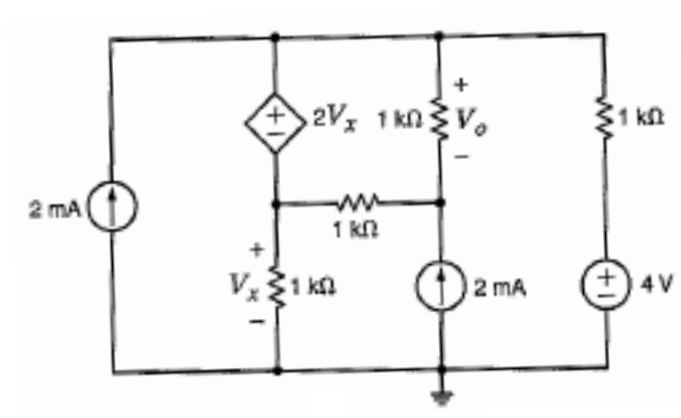


v









There are four nodes, v_a , v_b , v_c , and $v_d = 4\text{ V}$ with respect to ground.

We observe that $v_a = V_x$, $v_b = v_a + 2V_x = 3v_a$. We need two more equations. We can write a node equation at c,

$$(v_c - v_a)10^{-3} + (v_c - v_b)10^{-3} = (2)10^{-3} \Rightarrow v_c - v_a + v_c - v_b = 2 \Rightarrow 2v_c - v_a - v_b = 2.$$

We can rewrite the above as

$$2v_c - v_a - v_b = 2 \Rightarrow 2v_c - 4v_a = 2.$$

The other equation is a super node equation at the nodes a and b,

$$v_a + v_a - v_c + v_b - v_c + v_b - v_d = 2 \Rightarrow 8v_a - 2v_c = 6$$

This implies that $v_a = 2\text{ V}$, $v_b = 3v_a = 6\text{ V}$, $v_c = 5\text{ V}$.

