ME303 - ADVANCED ENGINEERING MATHEMATICS

Chapter 6: Analytical solution to PDEs (Lecture 34)

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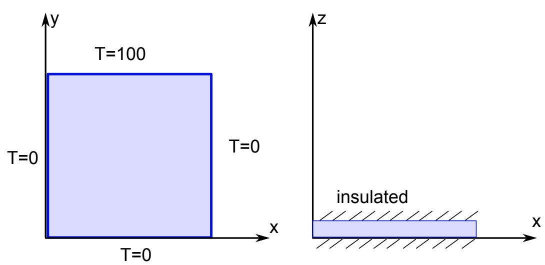
SUMMARY

- 6.2 Solution of diffusion equation with non-homogeneous BC

6.3 ANALYTICAL SOLUTION TO LAPLACE EQUATION

Let's consider the following elliptical, steady-state problem:



- The objective is to find $T(x, y)$
- The problem is governed by Laplace equation: $\nabla^2 T = 0$ or:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (1)$$

NOTES

- The boundary conditions are **homogeneous** in the x-direction, but **non-homogeneous** in the y-direction:

$$\begin{aligned} T(0, y) &= 0 & \\ T(a, y) &= 0 & \quad (homogeneous) \\ T(x, 0) &= 0 & \\ T(x, b) &= 100 & \quad (non-homogeneous) \end{aligned}$$

- We can use the separation of variable as the PDE and the BC are homogeneous in the x-direction. We use BCs in y-direction in a similar way to ICs in heat equation.

$$T(x, y) = \Phi(x)\eta(y)$$

- Step 1: Separate the PDE:

$$\eta \frac{d^2\Phi}{dx^2} + \Phi \frac{d^2\eta}{dy^2} = 0$$

We rearrange:

$$\underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{dx^2}}_{\text{function of x}} = -\underbrace{\frac{1}{\eta}\frac{d^2\eta}{dy^2}}_{\text{function of y}} = \underbrace{k}_{\text{separation constant}}$$

Our equation becomes:

$$\frac{d^2\Phi}{dx^2} - k\Phi = 0$$
$$\frac{d^2\eta}{dy^2} + k\eta = 0$$

- Step 2: Separate the homogeneous pair of BCs (x-direction):

$$T(0,y) = 0 \quad \rightarrow \quad \Phi(0)\eta(y) = 0 \quad \rightarrow \quad \Phi(0) = 0$$
$$T(a,y) = 0 \quad \rightarrow \quad \Phi(a)\eta(y) = 0 \quad \rightarrow \quad \Phi(a) = 0$$

We cannot separate the second pair on BCs (y-direction) as they are non-homogenous!

- Step 3: Solve for $\Phi(x)$.
  - i) $k = \lambda^2$ (trivial solution)
  - ii) $k = 0$ (trivial solution)
  - iii) $k = -\lambda^2$

$$\frac{d^2\Phi}{dx^2} + \lambda^2\Phi = 0 \quad \rightarrow \quad \Phi_n(x) = B_n \sin(\lambda_n x)$$

where $\lambda_n = n\pi/a$ with $n = 1, 2, 3, \ldots$

- Step 4: Solve for $\eta(y)$.

$$\frac{d^2\eta}{dy^2} - \lambda_n^2\eta = 0$$
$$r^2 - \lambda_n^2 = 0 \quad \rightarrow \quad r = \pm\lambda_n$$

The solution is:

$$\eta_n = a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}$$

The above equation can be rewritten as (trigonometric transformations)

$$\eta_n = c_n \cosh(\lambda_n y) + d_n \sinh(\lambda_n x)$$

Recall that $\sinh(z) = \frac{e^z - e^{-z}}{2}$

- Step 5: Apply superposition

$$T_n(x,y) = \Phi(x)\eta(y)$$

The general solution is:

$$T(x,y) = \sum_{n=1}^{\infty} \underbrace{B_n \sin(\lambda_n x)}_{\Phi_n(x)} \underbrace{[c_n \cosh(\lambda_n y) + d_n \sinh(\lambda_n x)]}_{\eta_n(y)}$$

or

$$\boxed{T(x,y) = \sum_{n=1}^{\infty} \sin(\lambda_n x)\left[C_n \cosh(\lambda_n y) + D_n \sinh(\lambda_n x)\right]}$$

where $\lambda_n = \frac{n\pi}{a}$, $C_n = c_n B_n$ and $D_n = d_n B_n$

- Step 6: Apply other BCs to find C_n and D_n

$$T(x,0)=0 \qquad T(x,b)=100$$

We look at the first boundary:

$$T(x,0)=0 \rightarrow 0=\sum_{n=1}^{\infty} C_n \sin(\lambda_n x) \qquad \rightarrow \qquad \text{then } C_n=0$$

$\rightarrow$ we can formally prove that C_n must be zero if we use the orthogonality of sine.

Apply other BC

$$T(x,b)=100 \rightarrow 100=\sum_{n=1}^{\infty} C_n \sin(\lambda_n x) \qquad \underbrace{D_n \sinh(\lambda_n b)}_{\text{nothing more than a constant } E_n}$$

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Use orthogonality:

$$E_n=\frac{\int_0^a 100\sin(\lambda_n x)dx}{\int_0^a 100\sin^2(\lambda_n x)dx}=\frac{100\frac{a}{n\pi}\left[-cos\left(\frac{n\pi}{a}a\right)-cos\left(\frac{n\pi}{a}0\right)\right]}{a/2}$$

or:

$$E_n=\frac{200}{n\pi}\left[1-(-1)^n\right]$$

We recall that:

$$D_n=\frac{E_n}{\sinh(\lambda_n b)}=\frac{200}{n\pi\sinh(\lambda_n b)}\left[1-(-1)^n\right]$$

We note that if n is even, then $D_n=0$. If it is odd, then: $D_n=\frac{400}{n\pi}\frac{1}{\sinh(\lambda_n b)}$

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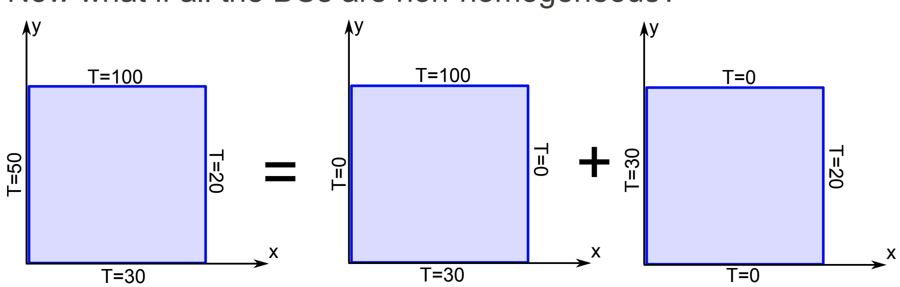
The solution takes the form:

$$\boxed{T(x,y)=\frac{400}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\frac{\sinh\left(\frac{n\pi}{a}y\right)}{\sinh\left(\frac{n\pi}{a}b\right)}\sin\left(\frac{n\pi x}{a}\right)}$$

Now what if all the BCs are non-homogeneous?



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