C241 HW11

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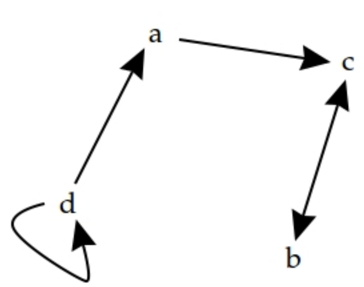
1. (a)
    i. No. $\neg L(0,0)$.
    ii. No. $L(\frac{5}{2}, \frac{5}{2})$.
    iii. No. $L(0,-5)$ but $\neg L(-5,0)$.
    iv. Yes. Consider some $x, y \in \mathbb{R}$ such that $L(x,y)$ and $L(y,x)$. Thus $y+5=3x$ and $x+5=3y$. Solving this out gives one solution pair $(x,y) = (\frac{5}{2}, \frac{5}{2})$, so $x=y$.
    v. No. $L(0,-5)$ and $L(-5,-20)$, but $\neg L(0,-20)$.

   (b)
    i. No. $\neg C("hello", "hello")$.
    ii. Yes. The definition of a proper substring is a substring that is not equal to the entire string, so no string is a proper substring of itself, so for any string s, $\neg C(s,s)$.
    iii. No. $C("hi", "high")$ but $\neg C("high", "hi")$.
    iv. Yes. For some string s to be a proper substring of a string t, s must appear inside t, and $s \neq t$. If $C(s,t)$, then the length of s is less than the length of t. Assuming (towards a contradiction) that $C(t,s)$ implies that the length of t is greater than the length of s, which contradicts our earlier statement, so C is anti-symmetric.
    v. Yes. Consider strings r, s, t with $C(r,s)$ and $C(s,t)$. Since $C(s,t)$, s is contained inside t. Since $C(r,s)$, r is contained inside s. Since r is contained inside s which is contained inside t, r is contained inside t, and thus $C(r,t)$.

   (c)
    i. No. $D(\{1,2,3\},\{1,2,3\}) = \{1,2,3\} \neq \varnothing$.
    ii. No. $D(\varnothing, \varnothing) = \varnothing$.
    iii. Yes. This is because for any sets A and B, $A \cap B = B \cap A$.
    iv. No. $D(\{1,2\},\{3,4\})$ and $D(\{3,4\},\{1,2\})$.
    v. No. $D(\{1,2\},\{3,4\})$ and $D(\{3,4\},\{2,5\})$ but $\neg D(\{1,2\},\{2,5\})$.

2. (a) $\{(a,d),(b,c),(c,b),(d,a)\}$
   (b) $\{(1,1),(2,1),(2,2),(3,3)\}$
   (c) $\{(2,4),(2,-6),(4,-6)\}$
   (d) Here's my relation:



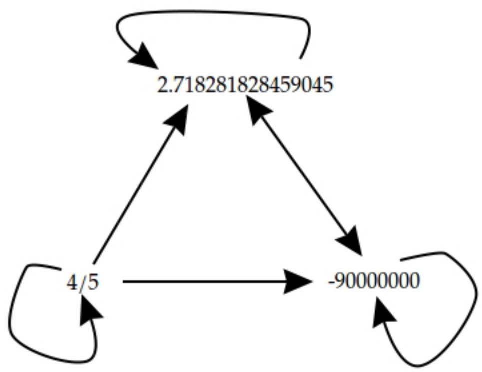
   (e) $\{(w_1, w_2) : w_1 \text{ is spelled the same as } w_2\}$
   (f) $\{(m,n) : m = n^2\}$

(g) $\{(p_1, p_2) : p_1$ and p_2 have the same social security number (not illegally or accidentally) $\}$

(h) $\{(A, B), (B, C), (A, C)\}$

(i) $\{(s, t)$: the student ID # of s is greater than or equal to the student ID # of $t\}$

(j) Here's my relation:



3. (a) **Claim**: D is reflexive.

Proof. Choose some $n \in \mathbb{N}$.
$\frac{n}{n} = 1$, and $1 \in \mathbb{N}$, so $n \mid n$.
Thus $D(n, n)$. $\square$

(b) **Claim**: D is anti-symmetric.

Proof. Choose some $m, n \in \mathbb{N}$ with $D(m, n)$ and $D(n, m)$.
So $m \mid n$ and $n \mid m$.
Since $m \mid n$, there exists some $k \in \mathbb{N}$ with $n = k \cdot m$.
Similarly, since $n \mid m$, there exists some $l \in \mathbb{N}$ with $m = l \cdot n$.
Since $n = k \cdot m$ and $m = l \cdot n$, $n = k \cdot (l \cdot n) = k \cdot l \cdot n$.
Thus $\frac{n}{n} = \frac{k \cdot l \cdot n}{n} \to 1 = k \cdot l$. (Ignoring the case where $n = 0$; since for any $p \in \mathbb{N}$, $p \mid 0$, the case where $n = 0$ isn't as interesting or relevant.)
Since $k \cdot l = 1$, $k = 1$ and $l = 1$.
Since $n = k \cdot m$ and $k = 1$, $n = m$. $\square$

(c) **Claim**: D is transitive.

Proof. Choose some $j, k, l \in \mathbb{N}$ with $D(j, k)$ and $D(k, l)$.
Thus $j \mid k$ and $k \mid l$.
Thus there exists some $m, n \in \mathbb{N}$ with $k = m \cdot j$ and $l = n \cdot k$.
Since $k = m \cdot j$ and $l = n \cdot k$, $l = n \cdot (m \cdot j) = n \cdot m \cdot j$.
Since $n, m \in \mathbb{N}$, $m \cdot m \in \mathbb{N}$.
Since $l = n \cdot m \cdot j$ and $n \cdot m \in \mathbb{N}$, $j \mid l$.
Thus $D(j, l)$. $\square$

Since D is reflexive, anti-symmetric, and transitive, D is a partial order.

4. (a) **Claim**: $\equiv_5$ is reflexive.

Proof. Choose some $n \in \mathbb{Z}$.
$n - n = 0$, and $0 = 5 * 0$, and $0, 5 \in \mathbb{Z}$, so $5 \mid 0$, or, stated differently, $5 \mid (n - n)$.
Thus $n \equiv_5 n$. $\square$

(b) **Claim**: $\equiv_5$ is symmetric.

Proof. Choose some $m, n \in \mathbb{Z}$ with $m \equiv_5 n$.
So $5 \mid (m - n)$.
Thus there exists some $k \in \mathbb{Z}$ with $m - n = k \cdot 5$.
Multiplying both sides of $m - n = k \cdot 5$ by -1, we get $-1 \cdot (m - n) = -1 \cdot (k \cdot 5) \rightarrow n - m = -k \cdot 5$.
Since $-1, k \in \mathbb{Z},\ -k \in \mathbb{Z}$.
Since $n - m = -k \cdot 5$ and $-k \in \mathbb{Z},\ 5 \mid (n - m)$.
Thus $n \equiv_5 m$. □

(c) **Claim**: $\equiv_5$ is transitive.

Proof. Choose some $j, k, l \in \mathbb{Z}$ with $j \equiv_5 k$ and $k \equiv_5 l$.
Thus $5 \mid (j - k)$ and $5 \mid (k - l)$.
So there exist some $m, n \in \mathbb{Z}$ with $j - k = m \cdot 5$ and $k - l = n \cdot 5$.
Thus $k = j - m \cdot 5$ and $k = n \cdot 5 + l$.
So $j - m \cdot 5 = n \cdot 5 + l$.
So $j - l = m \cdot 5 + n \cdot 5 = 5 \cdot (m + n)$.
Since $m, n \in \mathbb{Z},\ m + n \in \mathbb{Z}$.
Since $m + n \in \mathbb{Z}$ and $j - l = 5 \cdot (m + n),\ 5 \mid (j - l)$.
Thus $j \equiv_5 l$. □

Since $\equiv_5$ is reflexive, symmetric, and transitive, $\equiv_5$ is an equivalence relation.

5. (a) R_1 is a function.
   (b) R_2 is not a function; it is not well-defined. $R_2(1,1)$ and $R_2(1,2),\ 1 = 1$.
   (c) S is a function.
   (d) T_1 is not a function because there is no value $\in T_1$ that relates $0 \in \{0,2,4,6\}$ to any member of $\{1,2,3\}$, thus T_1 is not total.

6. (a) L is a function.

Proof. (*well-defined*)
Choose some $x, y_1, y_2 \in \mathbb{R}$ with $L(x, y_1)$ and $L(x, y_2)$.
Thus $y_1 + 5 = 3x$ and $y_2 + 5 = 3x$.
So $y_1 + 5 = y_2 + 5 \rightarrow y_1 = y_2$.
Thus L is well-defined.
(*total*)
Let $y_3 = 3x - 5$.
Since $3, 5, x \in \mathbb{R},\ y_3 \in \mathbb{R}$.
So $y_3 + 5 = 3x - 5 + 5 = 3x$.
So $L(x, y_3)$.
Therefore L is total. □

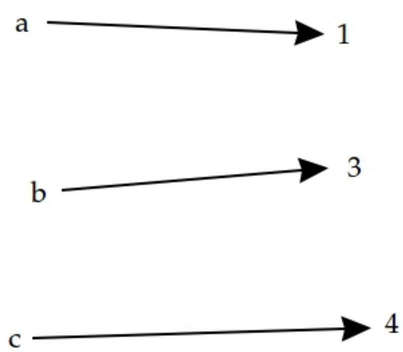
   (b) P_1 is not a function because there is no value $\in \mathbb{Z}$ to which 0 is related via P_1 (so P_1 is not total).
   (c) U is not a function. $U(\{1,2\}, \mathbb{Z})$ and $U(\{1,2\}, (\mathbb{Z} \setminus \{1\})$, and $\mathbb{Z} \neq (\mathbb{Z} \setminus \{1\})$, so U is not well-defined.
   (d) I is a function.

Proof. (*well-defined*)
Choose some sets of integers X, Y_1, Y_2 with $I(X, Y_1)$ and $I(X, Y_2)$.
So $Y_1 = \mathbb{N} \cap X$ and $Y_2 = \mathbb{N} \cap X$.
Therefore $Y_1 = Y_2$.
Thus I is well-defined.
(*total*)
Let $Y_3 = \mathbb{N} \cap X$.

Choose some $n \in \mathbb{N} \cap X$.
So $n \in \mathbb{N}$ and $n \in X$.
Since $X \subseteq \mathbb{Z}$ and $n \in X$, $n \in \mathbb{Z}$.

Thus $Y_3 \subseteq \mathbb{Z}$.
Therefore $I(X, Y_3)$.
Thus I is total. $\square$

7. (a) $\{(a, c), (b, c), (c, c)\}$
   (b) Here's my function:



   (c) $\{(1, c), (2, a), (3, b), (4, a)\}$
   (d) $f(x) = -4x + 9$
   (e) $g(x) = x^4$
   (f) $l = \{(s, n) : s \text{ has length } n\}$