CSCI-C 241 HW #9 Assigned Thursday, October 18th, 2018
Instructor: Wennstrom Due Tuesday, October 23rd, 2018

As always, when writing informal proofs (otherwise known as "proofs"), follow the guidelines I laid out in the lecture (and that were repeated in last week's homework instructions). Below are a few more guidelines for writing proofs involving existential claims.

When writing proofs about numbers, you may use most basic facts about numbers, arithmetic, and algebra (with some exceptions listed below). For example, you may use basic facts of arithmetic (addition and multiplication are commutative and associative, $1 + 1 = 2$, $2x = x + x$, $x^2 = x \cdot x$, $2 > 1$, etc.) You may also use algebraic manipulations to simplify or rewrite algebraic expressions or equations (e.g., factoring, distribution, collecting like terms, simplifying fractions, doing the same thing to both sides of an equation, adding the same thing to both sides of an inequality, etc.) You may also use general facts about integers (e.g., 0, 1, -1, etc. are integers; the integers are closed under addition, multiplication, and subtraction; etc.) These are only a few examples.

Any time you have to prove an existential claim (such as proving that a number is even), you must give an example (existential introduction). And any time you have to *use* an existential claim (such as the assumption that a particular number is odd), you must use existential instantiation.

The purpose of the proofs about even and odd numbers, divisibility, and rational numbers is to teach you how to use existential instantiation and existential introduction. So do *not* use more general rules about arithmetic with even and odd numbers, how divisibility interacts with arithmetic, or closure properties for rational numbers. So for example, while it's true that an even number plus another even number is always even, you may not use this fact. Similarly, you may not assume that the rational numbers are closed under addition or multiplication.

The only rules you can use about divisibility, rational numbers, and even and odd numbers are the basic definitions:

Definition. n is **divisible** by m ($m \mid n$) if and only if there exists an integer k such that $n = m \cdot k$.

Definition. n is **even** if and only if there exists an integer k such that $n = 2 \cdot k$.

Definition. n is **odd** if and only if there exists an integer k such that $n = 2 \cdot k + 1$.

Definition. A number r is rational if it can be written in the form $n = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $q \neq 0$.

1. For this problem, you'll be creating toy models to meet the given requirements.

   This time, you'll write down a collection of letters to serve as a universe for each problem. Some of the letters will be vowels (A, a, E, e, I, i, ...) and some will be consonants (B, b, D, d, F, f, G, g, ...) Some will be in uppercase (A, B, D, E, ...), and some will be in lowercase (a, b, d, e, ...).

   To avoid confusion, avoid letters where it's hard to tell the difference between upper and lowercase (e.g., don't use C or O or X), and for this assignment, we will treat Y as a vowel.

In addition, you will be drawing arrows between the letters. Arrows are allowed to point from a letter to any other letter or to itself.

Use the following interpretations for the predicate symbols. $V(x)$: "x is a vowel." $L(x)$: "x is lowercase." $P(x, y)$: "There is an arrow pointing from x to y."

There must be at least three letters for each model.

If you think that there is no such model, you must explain why not.

(a) Satisfying $\forall x \exists y \big(V(x) \to P(x, y)\big)$ and $\exists x V(x)$, but not $\forall x \exists y P(x, y)$.

(b) Satisfying $\forall x \forall y \big(V(y) \to P(x, y)\big)$ and $\exists x V(x)$, but not $\forall x \forall y P(x, y)$.

(c) Satisfying $\forall x \exists y \big(V(y) \land P(x, y)\big)$ but not $\forall x \forall y \big(V(y) \to P(x, y)\big)$.

(d) Satisfying $\forall x \exists y \neg P(x, y)$ and $\forall x \exists y P(x, y)$.

(e) Satisfying $\exists x \forall y \neg P(x, y)$ and $\exists x \forall y P(x, y)$.

(f) Satisfying $\forall x \forall y \big(L(x) \to \neg P(x, y)\big)$, $\exists x \forall y P(x, y)$, and $\exists x L(x)$.

(g) Satisfying $\exists x \forall y \big(\neg V(x) \land P(x, y)\big)$.

For problems 2 through 3, let the universe be the set of functions usable in some program, and use the following definitions. $R(x)$: "x is recursive." $S(x)$: "x returns a string." $D(x, y)$: "x is dependent on y." $M(x, y)$: "x has more arguments than y."

Warning: $\neg M(x, y)$ means "x does not have more arguments than y." It does *not* mean "x has fewer arguments than y," because that would be ignoring the possibility that they have the *same* number of arguments. If you want to express "x has fewer arguments than y," you should write $M(y, x)$ instead.

2. Translate the following English sentences into predicate logic.

   (a) Every function has at least one function that it is not dependent on.

   (b) Every function has at least one function that is not dependent on it.

   (c) There is a function that doesn't have more arguments than any function.

   (d) There is a function that isn't dependent on every function.

   (e) Some function that returns a string isn't dependent on every function.

   (f) Not every function has a function that it is dependent on.

   (g) Not every function is dependent on all functions.

   (h) Not every recursive function is dependent on all functions.

   (i) No function is dependent on all functions.

   (j) No function that returns a string is dependent on all functions.

   (k) There are no functions that are dependent on all recursive functions.

   (l) No functions are dependent any recursive functions.

3. Translate the following predicate logic formulas into English sentences. These should all be natural-sounding English sentences, with no variables.

(a) $\forall x \forall y \neg D(x, y)$

(b) $\forall x \neg \forall y M(x, y)$

(c) $\neg \forall x \forall y D(x, y)$

(d) $\exists x \forall y \neg D(x, y)$

(e) $\forall x \big(R(x) \to \neg \forall y D(x, y)\big)$

(f) $\forall x \big(R(x) \to \exists y \neg D(x, y)\big)$

(g) $\forall x \neg \forall y \big(S(y) \to M(x, y)\big)$

(h) $\neg \exists x \forall y \big(R(x) \wedge D(x, y)\big)$