C241 HW10

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1 Problem 1

2) **Claim**: For all $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$, then $a|(b \cdot c)$.

Proof. Choose some $a, b, c \in \mathbb{Z}$ with $a|b$ and $a|c$.
Since $a|b$, there exists some $k_1 \in \mathbb{Z}$ with $b = k_1 a$.
Similarly, since $a|c$ there exists some $k_2 \in \mathbb{Z}$ with $c = k_2 a$.
Therefore $b \cdot c = k_1 a \cdot k_2 a = (k_1 k_2 a) \cdot a$.
Since $k_1, k_2, a \in \mathbb{Z},\ k_1 k_1 a \in \mathbb{Z}$.
Since $b \cdot c = (k_1 k_2 a) \cdot a$ and $k_1 k_2 a \in \mathbb{Z},\ a|(b \cdot c)$. □

4) **Claim**: For all $a, b, c \in \mathbb{Z}$, if $a|b$ and $b|c$, then $a|c$.

Proof. Choose some $a, b, c \in \mathbb{Z}$ with $a|b$ and $b|c$.
Since $a|b$, there exists some $k_1 \in \mathbb{Z}$ with $b = k_1 a$.
Similarly, since $b|c$ there exists some $k_2 \in \mathbb{Z}$ with $c = k_2 b$.
Since $c = k_2 b$ and $b = k_1 a$, $c = k_2(k_1 a) = k_1 k_2 \cdot a$.
Since $k_1, k_2 \in \mathbb{Z},\ k_1 k_2 \in \mathbb{Z}$.
Therefore since $c = k_1 k_2 \cdot a$ and $k_1 k_2 \in \mathbb{Z},\ a|c$. □

6) **Claim**: For all $n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

Proof. Choose some $n \in \mathbb{Z}$ with n^2 odd.
> Suppose towards a contradiction that n is even.
> Since n is even, for some $k \in \mathbb{Z},\ n = 2k$.
> Therefore $n^2 = 2k \cdot 2k = 2 \cdot (2k^2)$.
> Since $2, k \in \mathbb{Z},\ 2k^2 \in \mathbb{Z}$.
> Since $n^2 = 2 \cdot (2k^2)$ and $2k^2 \in \mathbb{Z},\ n^2$ is even, which contradicts our earlier claim.

Therefore n is not even.
Since n is not even and $n \in \mathbb{Z}$, n is odd. □

10) **Claim**: For all $n_1, n_2 \in \mathbb{Z}$, if n_1, n_2 are odd, then $n_1 + n_2$ is even.

Proof. Choose some $n_1, n_2 \in \mathbb{Z}$ with n_1 and n_2 both odd.
Since n_1 is odd, for some $k_1 \in \mathbb{Z},\ n_1 = 2k_1 + 1$.
Similarly, for some $k_2 \in \mathbb{Z},\ n_2 = 2k_2 + 1$.
Therefore $n_1 + n_2 = 2k_1 + 1 + 2k_2 + 1 = 2k_1 + 2k_2 + 2 = 2(k_1 + k_2 + 1)$.
Since $1, k_1, k_2 \in \mathbb{Z},\ k_1 + k_2 + 1 \in \mathbb{Z}$.
Since $k_1 + k_2 + 1 \in \mathbb{Z}$ and $n_1 + n_2 = 2(k_1 + k_2 + 1),\ n_1 + n_2$ is even. □

14) **Def**: An integer n is *sane* if $3|(n^2 + 2n)$.

(a) 5 is an odd integer. $5^2 + 2 \cdot 5 = 35$. $\dfrac{35}{3} = 11 + \frac{1}{3}$, which is not an integer, so $\neg(3|(5^2 + 2 \cdot 5))$, so 5 is not sane.

(b) **Claim**: If $3|n$, then n is sane.

Proof. Choose some $n \in \mathbb{Z}$ with $3|n$.
Since $3|n$, there exists some $k \in \mathbb{Z}$ with $n = k \cdot 3 = 3k$.
Thus $n^2 + 2n = (3k)^2 + 2 \cdot 3k = 9k^2 + 6k$.
$\dfrac{9k^2 + 6k}{3} = 3k^2 + 2k$.
Since $2, 3, k \in \mathbb{Z},\ 3k^2 + 2k \in \mathbb{Z}$.
Since $3k^2 + 2k \in \mathbb{Z}$ and $\dfrac{9k^2 + 6k}{3} = 3k^2 + 2k$ and $9k^2 + 6k = n^2 + 2n,\ 3|(n^2 + 2n)$.
Therefore n is sane. □

(c) **Claim**: If for some $n, j \in \mathbb{Z},\ n = 3j + 2$, then n is not sane.

Proof. Choose some $n, j \in \mathbb{Z}$ with $n = 3j + 2$.
Suppose towards a contradiction that n is sane.
Since n is sane, $3|(n^2 + 2n)$.
Since $n = 3j + 2$ and $3|(n^2 + 2n),\ 3|((3j + 2)^2 + 2 \cdot (3j + 2))$.
Therefore $3|(9j^2 + 12j + 4 + 6j + 4)$.
Therefore $3|(9j^2 + 18j + 8)$.
Since $3|(9j^2 + 18j + 8),\ 9j^2 + 18j + 8 = 3k$ for some $k \in \mathbb{Z}$.
Since $9j^2 + 18j + 8 = 3k,\ k = 3j^2 + 6j + \frac{8}{3}$.
Since $\frac{8}{3} \notin \mathbb{Z},\ 3j^2 + 6j + \frac{8}{3} \notin \mathbb{Z}$.
Since $k = 3j^2 + 6j + \frac{8}{3}$ and $3j^2 + 6j + \frac{8}{3} \notin \mathbb{Z},\ k \notin \mathbb{Z}$, which contradicts our earlier claim.

Therefore $n = 3j + 2$ is not sane. □

18) **Claim**: Rational numbers are closed under addition, i.e. for any $r_1, r_2 \in \mathbb{Q},\ r_1 + r_2 \in \mathbb{Q}$ as well.

Proof. Choose some $r_1, r_2 \in \mathbb{Q}$.
Since $r_1 \in \mathbb{Q}$, there exist $p_1, q_1 \in \mathbb{Z}$ with $r_1 = \frac{p_1}{q_1}$ and $q_1 \neq 0$.
Similarly, there exist $p_2, q_2 \in \mathbb{Z}$ with $r_2 = \frac{p_2}{q_2}$ and $q_2 \neq 0$.
Therefore
$$\begin{aligned} r_1 + r_2 &= \tfrac{p_1}{q_1} + \tfrac{p_2}{q_2} \\ &= \tfrac{p_1 q_2}{q_1 q_2} + \tfrac{p_2 q_1}{q_1 q_2} \\ &= \tfrac{p_1 q_2 + p_2 q_1}{q_1 q_2}. \end{aligned}$$

Since $p_1, p_2, q_1, q_2 \in \mathbb{Z},\ p_1q_2 + p_2q_2 \in \mathbb{Z}$.
Since $q_1, q_2 \in \mathbb{Z},\ q_1q_2 \in \mathbb{Z}$.
Since $q_1 \neq 0$ and $q_2 \neq 0,\ q_1q_2 \neq 0$.
Since $p_1q_2 + p_2q_1 \in \mathbb{Z}, q_1q_2 \in \mathbb{Z}$, and $q_1q_2 \neq 0,\ \frac{p_1q_2+p_2q_1}{q_1q_2} \in \mathbb{Q}$.
Since $r_1 + r_2 = \frac{p_1q_2+p_2q_1}{q_1q_2}$ and $\frac{p_1q_2+p_2q_1}{q_1q_2} \in \mathbb{Q},\ r_1 + r_2 \in \mathbb{Q}$. □

20) The universe is $\mathbb{R}$. **Claim**: For any $x \in \mathbb{Q},\ y \notin \mathbb{Q},\ x + y \notin \mathbb{Q}$.

Proof. Choose some $x \in \mathbb{Q}$ and some $y \notin \mathbb{Q}$.
Since $x \in \mathbb{Q},\ x = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$.

> Suppose towards a contradiction that $x + y \in \mathbb{Q}$.
> Since $x + y \in \mathbb{Q},\ x + y = \dfrac{r}{s}$ for some $r, s \in \mathbb{Z}$ with $s \neq 0$.
> Since $x = \frac{p}{q}$ and $x + y = \frac{r}{s},\ \frac{p}{q} + y = \frac{r}{s}$.
> Therefore $y = \frac{r}{s} - \frac{p}{q}$
> $= \frac{rq}{qs} - \frac{ps}{qs}$
> $= \frac{rq-ps}{qs}$.
> Since $p, q, r, s \in \mathbb{Z},\ rq - ps \in \mathbb{Z}$.
> Since $q, s \in \mathbb{Z},\ qs \in \mathbb{Z}$.
> Since $q \neq 0$ and $s \neq 0,\ qs \neq 0$.
> Since $rq - ps \in \mathbb{Z},\ qs \in \mathbb{Z}$, and $qs \neq 0,\ \frac{rq-ps}{qs} \in \mathbb{Q}$.
> Since $y = \frac{rq-ps}{qs}$ and $\frac{rq-ps}{qs} \in \mathbb{Q},\ y \in \mathbb{Q}$, which contradicts our previous claim.

Therefore $x + y \notin \mathbb{Q}$. □

2 Bonus

7) What's unusual about this claim is this: if x is odd, so is x^2. 1 is also odd. The sum of any two odd numbers is even, so $x + 1$ is even. The sum of an even number and an odd number is odd, so $x^2 + x + 1$ is odd. But the claim's premise is that this quantity is even!
When x is even, x^2 is even. 1 is still odd. The sum of any two even numbers is even, so $x^2 + x$ is even. The sum of an even number and an odd number is odd, so $x^2 + x + 1$ is odd. But, again, the premise of the claim is that $x^2 + x + 1$ is even!
So, for any $x \in \mathbb{Z},\ x^2 + x + 1$ is odd, so the original claim holds true, but only trivially so.