C241 HW13

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1. (a) $\sum_{i=1}^{4} 2i = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 = 2 + 4 + 6 + 8 = 20$

   (b) $$\begin{aligned} \sum_{i=0}^{3}(2i + i^2) &= (2 \cdot 0 + 0^2) + (2 \cdot 1 + 1^2) + (2 \cdot 2 + 2^2) + (2 \cdot 3 + 3^2) \\ &= 0 + 3 + 8 + 15 = 26 \end{aligned}$$

   (c) $\sum_{i=1}^{3} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{6}{6} + \frac{3}{6} + \frac{2}{6} = \frac{6+3+2}{6} = \frac{11}{6x}$

2. (a) **Claim**: For all $n \in \mathbb{N}$, if $n \geq 2$, $3^n > n^2$.

   Proof. Choose some $n \in \mathbb{N}$ with $n \geq 2$.

   Base case ($n = 2$):

   $3^2 = 9$

   $2^2 = 4$

   $9 > 4$ so $3^2 > 2^2$.

   Induction case ($n > 2$):

   Choose some $k \in \mathbb{N}$ such that $k \geq 2$ and $3^k > k^2$.

   $3^{k+1} = 3 \cdot 3^k$

   $3 \cdot 3^k >^{IH} 3 \cdot k^2$ so $3^k + 3^k + 3^k > k^2 + k^2 + k^2$

   $(k+1)^2 = k^2 + k + k + 1 = k^2 + 2k + 1$

   $k^2 + k^2 \geq 2 \cdot k + 4$ because $k \geq 2$, so $k^2 + k^2 + k^2 \geq k^2 + 2k + 4$

   $4 > 1$ so $k^2 + 2k + 4 > k^2 + 2k + 1$

   $$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &= 3^k + 3^k + 3^k \\ &> k^2 + k^2 + k^2 \\ &\geq k^2 + 2k + 4 \\ &> k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

   So $3^{k+1} > (k+1)^2$

   $\square$

(b) **Claim**: For all $n \in \mathbb{N}$, if $n \geq 7$, $3^n < n!$.

Proof. Choose some $n \in \mathbb{N}$ with $n \geq 7$.

Base case ($n = 7$):

$3^7 = 2187$

$7! = 5040$

So $3^7 < 7!$.

Induction case ($n > 7$):

Suppose $3^k < k!$ for some $k \in \mathbb{N}$ with $k \geq 7$.

$3^{k+1} = 3 \cdot 3^k$

$3 \cdot 3^k <^{IH} 3 \cdot k!$

$(k+1)! = (k+1) \cdot k!$

$k + 1 \geq 8$ because $k \geq 7$

So $(k+1) \cdot k! \geq 8 \cdot k!$

Clearly, $8 > 3$, so $8 \cdot k! > 3 \cdot k!$

$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&< 3 \cdot k! \\
&< 8 \cdot k! \\
&\leq (k+1) \cdot k! \\
&= (k+1)!
\end{aligned}
$$

So $3^{k+1} < (k+1)!$

$\square$

(c) **Claim**: For all $n \in \mathbb{N}$, if $n \geq 1$, $\sum_{i=1}^{n} 2i = n(n+1)$.

Proof. Choose some $n \in \mathbb{N}$ with $n \geq 1$.

Base case ($n = 1$):

$\sum_{i=1}^{1} 2i = 2 \cdot 1 = 2$

$1(1+1) = 2$

So $\sum_{i=1}^{1} 2i = 1(1+1)$

Induction case ($n > 1$):

Choose some $k \in \mathbb{N}$ with $k \geq 1$ and $\sum_{i=1}^{k} 2i = k(k+1)$.

$$
\begin{aligned}
\sum_{i=1}^{k+1} 2i &= \sum_{i=1}^{k} 2i + 2(k+1) \\
&=^{IH} k(k+1) + 2(k+1) \\
&= (k+2)(k+1) \\
&= (k+1)(k+2) \\
&= (k+1)(k+1+1)
\end{aligned}
$$

$\square$

(d) **Claim**: For all $n \in \mathbb{N}$, if $n \geq 1$, $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.

Proof. Choose some $n \in \mathbb{N}$ with $n \geq 1$.

Base case ($n = 1$):

$\sum_{i=1}^{1} 2^{i-1} = 2^{1-1} = 1$

$2^1 - 1 = 2 - 1 = 1$

So $\sum_{i=1}^{1} 2^{i-1} = 2^1 - 1$.

Induction case ($n > 1$):

Suppose $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$ for some $k \in \mathbb{N}$ with $k \geq 1$.

$$
\begin{aligned}
\sum_{i=1}^{k+1} 2^{i-1} &= \sum_{i=1}^{k} 2^{i-1} + 2^{k+1-1} \\
&=^{IH} 2^k - 1 + 2^{k+1-1} \\
&= 2^k - 1 + 2^k \\
&= 2 \cdot 2^k - 1 \\
&= 2^{k+1}
\end{aligned}
$$

□

(e) **Claim**: For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i! \cdot i = (n+1)! - 1$.

Proof. Choose some $n \in \mathbb{N}$.

Base case ($n = 0$):

$\sum_{i=0}^{0} i! \cdot i = 0! \cdot 0 = 0$

$(0+1)! - 1 = 1! - 1 = 0$

So $\sum_{i=0}^{0} i! \cdot i = (0+1)! - 1$.

Induction case ($n > 0$):

Suppose $\sum_{i=0}^{k} i! \cdot i = (k+1)! - 1$ for some $k \in \mathbb{N}$.

$\sum_{i=0}^{k+1} i! \cdot i = (k+1)! \cdot (k+1) + \sum_{i=0}^{k} i! \cdot i$

$=^{IH} (k+1)! \cdot (k+1) + (k+1)! - 1$

$= (k+1)! \cdot (k+1+1) - 1$

$= (k+2)! - 1$

$= (k+1+1)! - 1$

□

(f) **Claim**: For all $n \in \mathbb{N}$, $n^2 - 3n$ is even.

Proof. Choose some $n \in \mathbb{N}$.

<u>Base case</u> ($n = 0$):

$0^2 - 3 \cdot 0 = 0 - 0 = 0$

$0 = 2 \cdot 0$ and $0 \in \mathbb{Z}$, so 0 is even.

So $0^2 - 3 \cdot 0$ is even.

<u>Induction case</u> ($n > 0$):

Choose some $k \in \mathbb{N}$ such that $k^2 - 3k$ is even.

Since $k^2 - 3k$ is even (by IH), there exists some $l \in \mathbb{Z}$ such that $k^2 - 3k = 2l$.

$$
\begin{aligned}
(k+1)^2 - 3(k+1) &= (\mathrm{k}^2 + 2k + 1) - (3k + 3) \\
&= k^2 + 2k + 1 - 3k - 3 \\
&= k^2 - 3k + 2k - 2 \\
&= 2l + 2k - 2 \\
&= 2(l + k - 1)
\end{aligned}
$$

Since $l, k, 1 \in \mathbb{Z}$, $l + k + 1 \in \mathbb{Z}$.

Since $(k+1)^2 = 2(l + k - 1)$ and $l + k + 1 \in \mathbb{Z}$, $(k+1)^2 - 3(k+1)$ is even.

$\square$