Points, Vectors, Lines, Spheres and Matrices

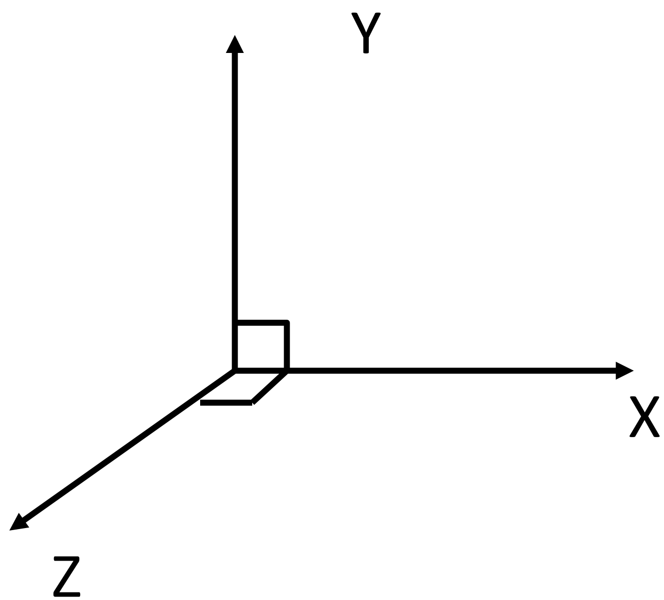
Overview

– Points
– Vectors
– Lines
– Spheres
– Matrices
– 3D transformations as matrices
– Homogenous co-ordinates

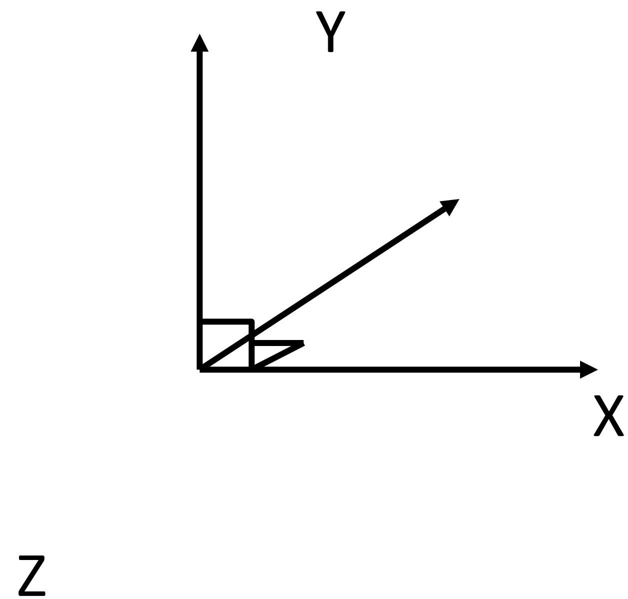
Basic Maths

- In computer graphics we need mathematics both for describing our scenes and also for performing operations on them, such as projection and various transformations.
- Coordinate systems (right- and left-handed), serve as a reference point.
- 3 axes labelled x, y, z at right angles.

Co-ordinate Systems



Right-Handed System
(Z comes out of the screen)



Left-Handed System
(Z goes in to the screen)

Points, P (x, y, z)

- Gives us a position in relation to the origin of our coordinate system

Vectors, V (x, y, z)

- Represent a *direction* (and magnitude) in 3D space

- Points != Vectors

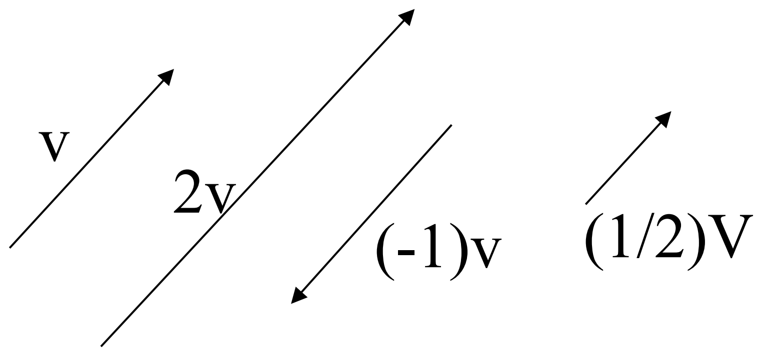
Vector +Vector = Vector

Point – Point = Vector

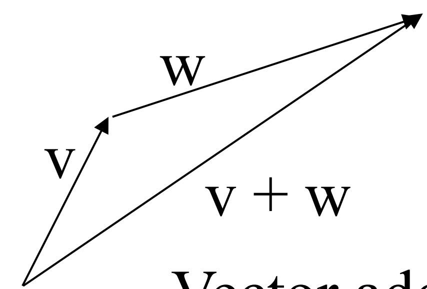
Point + Vector = Point

Point + Point = ?

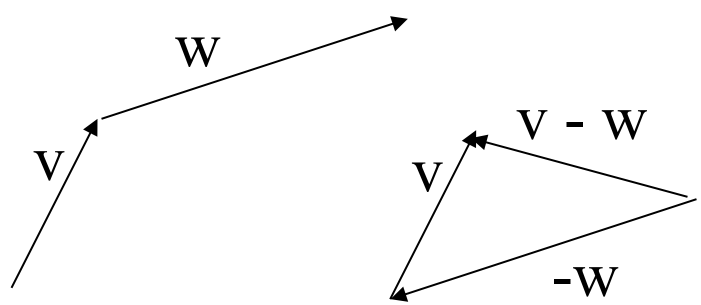
Vectors, V (x, y, z)



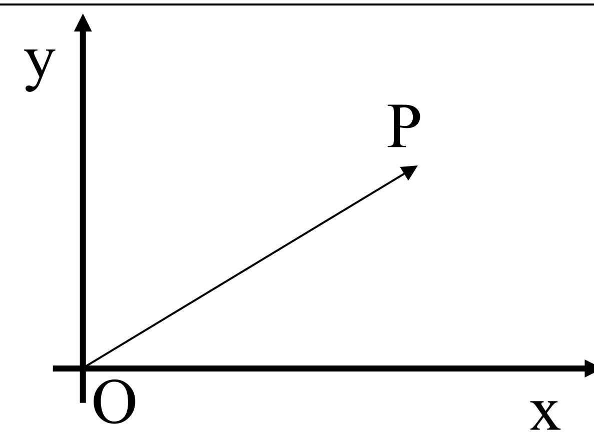
Scalar multiplication of vectors (they remain parallel)



Vector addition
sum v + w



Vector difference
v - w = v + (-w)



Vector $\overrightarrow{OP}$

Vectors $\underline{V}$

- Length (modulus) of a vector $\underline{V}$ (x, y, z)

  $|\underline{V}| = \sqrt{x^2 + y^2 + z^2}$

- A unit vector: a vector can be *normalised* such that it retains its direction, but is scaled to have unit length:

$$\hat{V} = \frac{\text{vector V}}{\text{modulus of V}} = \frac{\underline{V}}{|\underline{V}|}$$

Dot Product

$$u \cdot v = x_u \cdot x_v + y_u \cdot y_v + z_u \cdot z_v$$

$$u \cdot v = |u|\ |v| \cos\theta$$

$$\therefore \cos\theta = u \cdot v / |u||v|$$

- This is purely a scalar number not a vector.
- What happens when the vectors are unit
- What does it mean if dot product == 0 or == 1?

Cross Product

- The result is not a scalar but a vector which is normal to the plane of the other 2
- Can be computed using the determinant of:

$$\begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \qquad \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \qquad \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \qquad \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$u \times v = \mathbf{i}(y_u z_v - z_u y_v), -\mathbf{j}(x_u z_v - z_u x_v), \mathbf{k}(x_u y_v - y_u x_v)$

- Size is $|u \times v| = |u||v|\sin\theta$
- Cross product of vector with itself is null

Parametric equation of a line (ray)

Given two points $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ the line passing through them can be expressed as:

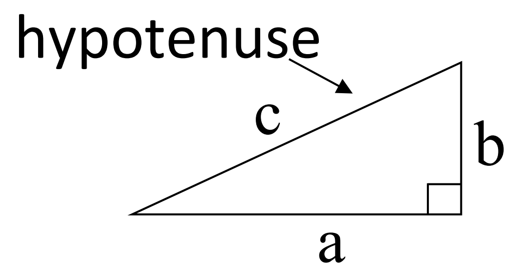
$$P(t) = P_0 + t(P_1 - P_0) \quad = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}$$

With $-\infty < t < \infty$

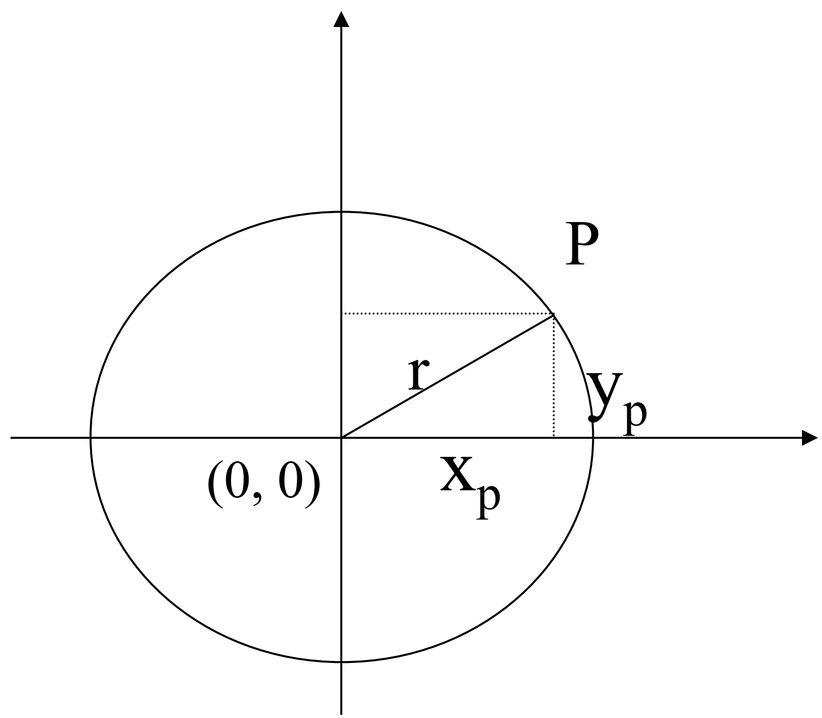
Equation of a sphere

- Pythagoras Theorem:

$$a^2 + b^2 = c^2$$



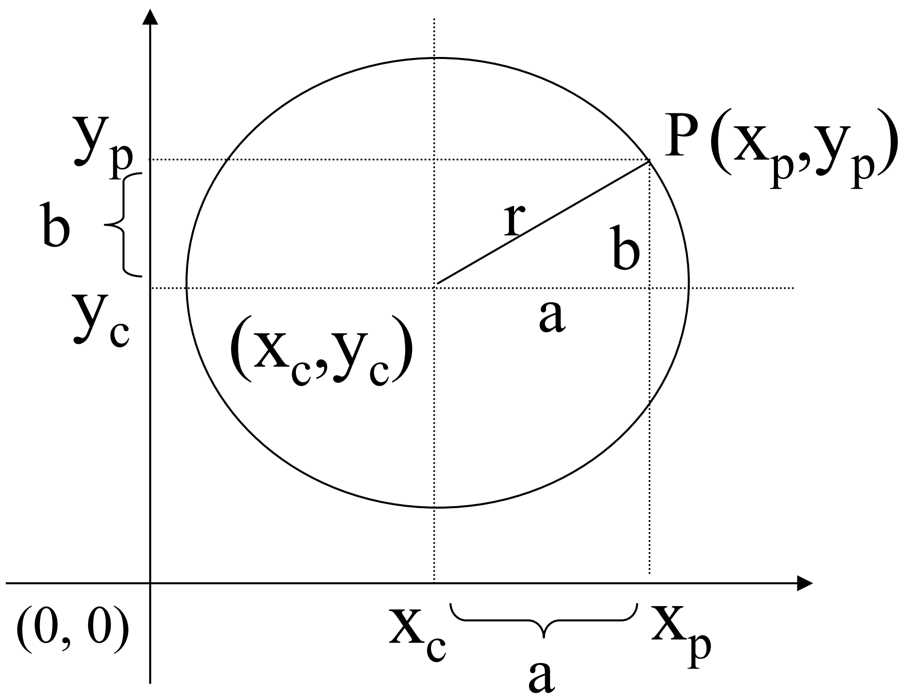
- Given a circle through the origin with radius r, then for any point P on it we have:



$$x^2 + y^2 = r^2$$

Equation of a sphere

If the circle is not centred on the origin, we still have: $a^2 + b^2 = r^2$



where

$a = x_p - x_c$

$b = y_p - y_c$

So for the general case $(x - x_c)^2 + (y - y_c)^2 = r^2$

Equation of a sphere

* Pythagoras theorem generalises to 3D giving

  $a^2 + b^2 + c^2 = d^2$ Based on that we can easily

  prove that the general equation of a sphere is:

$$\boxed{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2}$$

and at origin: $\boxed{x^2 + y^2 + z^2 = r^2}$

Matrix Math

Vectors and Matrices

- Matrix is an array of numbers with dimensions M (rows) by N (columns)
  - 3 by 6 matrix
  - element 2,3 is (3)

$$\begin{pmatrix} 3 & 0 & 0 & -2 & 1 & -2 \\ 1 & 1 & 3 & 4 & 1 & -1 \\ -5 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Vector can be considered a 1 x N matrix

$$v = \begin{pmatrix} x & y & z \end{pmatrix}$$

Types of Matrix

- Identity matrices - I

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Diagonal

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

- Symmetric

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

  - Symmetric matrix is equal to its transpose
  - Diagonal matrices are (of course) symmetric
  - Identity matrices are (of course) diagonal

Operation on Matrices

- Addition
  - Done elementwise

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$$

- Transpose
  - "Flip" (M by N becomes N by M)

$$\begin{pmatrix} 1 & 4 & 9 \\ 5 & 2 & 8 \\ 6 & 7 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & 6 \\ 4 & 2 & 7 \\ 9 & 8 & 3 \end{pmatrix}$$

Operations on Matrices

- Multiplication
  - Only possible to multiply of dimensions
    - m_1 by n_1 and m_2 by n_2 iff $n_1 = m_2$
      - i.e. iff number of columns in first matrix equals number of rows in second matrix
      - resulting matrix is m_1 by n_2
    - e.g. Matrix A is 2 by 3 and Matrix by 3 by 4
      - resulting matrix is 2 by 4
    - Just because A x B is possible doesn't mean B x A is possible!

Matrix Multiplication Order

- A is m by k , B is k by n
- C = A x B defined by

$$c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$

- BxA not necessarily equal to AxB

$$\begin{pmatrix} * & * & * & * & * \\ & & & & \\ & & & & \end{pmatrix}\begin{pmatrix} * & & \\ * & & \\ * & & \\ * & & \\ * & & \end{pmatrix} = \begin{pmatrix} * & & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} * & * & * & * & * \\ & & & & \\ & & & & \end{pmatrix}\begin{pmatrix} . & * & \\ . & * & \\ . & * & \\ . & * & \\ . & * & \end{pmatrix} = \begin{pmatrix} . & * & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} . & . & . & . & . \\ * & * & * & * & * \\ & & & & \end{pmatrix}\begin{pmatrix} . & * & \\ . & * & \\ . & * & \\ . & * & \\ . & * & \end{pmatrix} = \begin{pmatrix} . & . & \\ . & * & \\ & & \end{pmatrix}$$

Example Multiplications

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 3 \\ -3 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix}$$

Inverse

- If $A \times B = I$ and $B \times A = I$ then
  $A = B^{-1}$ and $B = A^{-1}$

3D Transforms

- In 3-space vectors and points are transformed by 3 by 3 matrices

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} xa + yd + zg & xb + ye + zh & xc + yf + zi \end{pmatrix}$$

Scale

- Scale uses a diagonal matrix

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} xa & yb & zc \end{pmatrix}$$

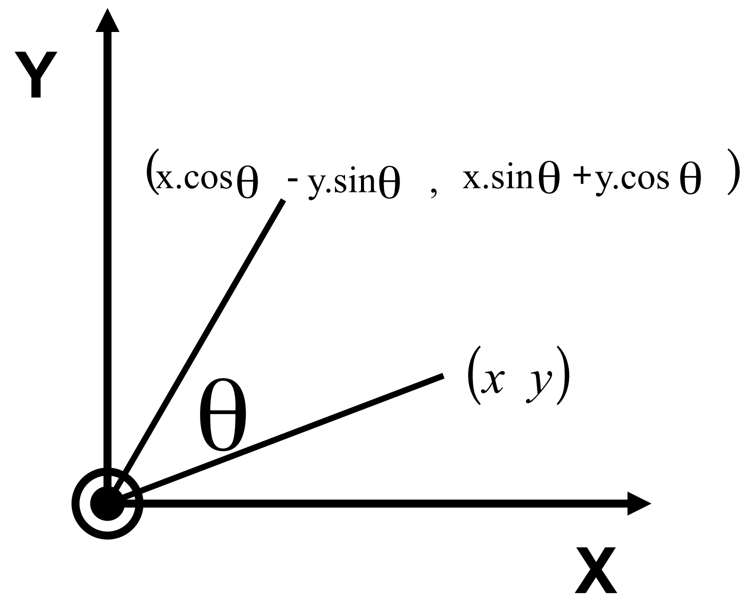
- Scale by 2 along x and -2 along z

$$\begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 4 & -10 \end{pmatrix}$$

Rotation

- Rotation about z axis

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



- Note **z values** remain the same whilst **x and y change**

Rotation

- About X

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

- About Y

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Homogenous Points

- Add 1D, but constrain that to be equal to 1 (x,y,z,1)
- Homogeneity means that any point in 3-space can be represented by an infinite variety of homogenous 4D points
  - (2 3 4 1) = (4 6 8 2) = (3 4.5 6 1.5)
- Why?
  - 4D allows us to include 3D translation in matrix form

Homogenous Vectors

- Vectors != Points
- Remember points can not be added
- If A and B are points A-B is a vector
- Vectors have form (x y z 1)
- Addition makes sense

Translation in Homogenous Form

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} = \begin{pmatrix} x+a & y+b & z+c & 1 \end{pmatrix}$$

- Note that the homogenous component is preserved (* * * 1), and aside from the translation the matrix is I

Putting it Together

$$\begin{pmatrix} R_1 & R_2 & R_3 & 0 \\ R_4 & R_5 & R_6 & 0 \\ R_7 & R_8 & R_9 & 0 \\ T_1 & T_2 & T_3 & 1 \end{pmatrix} = R.T$$

- R is rotation and scale components
- T is translation component

Order Matters

- Composition order of transforms matters
  - Remember that basic vectors change so “direction” of translations changed

$$\begin{pmatrix} X & Y & Z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} X & Z & -Y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} X+2 & Z+3 & -Y+4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} X & Y & Z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X+2 & Y+3 & Z+4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X+2 & Z+4 & -Y-3 & 1 \end{pmatrix}$$

Exercises

- Calculate the following matrix: $\pi/2$ about X then $\pi/2$ about Y then $\pi/2$ about Z (remember "then" means multiply on the right). What is a simpler form of this matrix?

- Compose the following matrix: translate 2 along X, rotate $\pi/2$ about Y, translate -2 along X. Draw a figure with a few points (you will only need 2D) and then its translation under this transformation.

Matrix Summary

- Rotation, Scale, Translation
- Composition of transforms
- The homogenous form