MA1521 cheatsheet 20/21

by Zachary Chua

Functions

- $(f \pm g)(x) = f(x) \pm g(x)$
- $fg(x) = f(x)g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$
- Let $f : D \to R$ and $g : D1 \to R$
  $(f \circ g)(x) = f(g(x))$ for D1 $\subseteq$ D
- $f \circ g \neq g \circ f$

Limits
Suppose $lim_{x\to a}f(x) = L$ and $lim_{x\to a}g(x) = L'$

- $\lim_{x\to a}(f \pm g)(x) = L \pm L'$
- $\lim_{x\to a}(fg)(x) = LL'$
- $\lim_{x\to a} \frac{f}{g}(x) = \frac{L}{L'}$, provided $L' \neq 0$
- $\lim_{x\to a} kf(x) = kL$, any real number k

Differentiation

$f'(a) = lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

- Linearity
  - $(kf)'(x) = kf'(x)$
  - $(f \pm g)'(x) = f'(x) \pm g'(x)$
- Product rule
  - $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- Quotient rule
  - $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$
- Chain Rule
  - $(f \circ g)'(x) = f'(g(x))g'(x) \cong (f' \circ g)(x)g'(x)$
- Power
  - $\frac{d}{dx}x^n = nx^{n-1}$
- Trigonometry
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\cos x) = -\sin x$
  - $\frac{d}{dx}(\tan x) = \sec^2 x$
  - $\frac{d}{dx}(\sec x) = \sec x \tan x$
  - $\frac{d}{dx}(\cot x) = -\csc^2 x$
  - $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- Exponent and Logarithms
  - $\frac{d}{dx}e^x = e^x$
  - $\frac{d}{dx}a^x = a^x \ln a$
  - $\frac{d}{dx}\ln x = \frac{1}{x}$
  - $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- Inverse Trigonometry
  - $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
  - $\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
  - $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
  - $\frac{d}{dx}(\csc^{-1} x) = 1\frac{1}{|x|\sqrt{x^2-1}}$

Parametric Differentiation

- $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Implicit Differentiation
Implementation of Chain Rule on y
eg. $\frac{d}{dx}y^2 = 2y(\frac{dy}{dx})$

Maxima and Minima
Points where f can have an extreme value are

- Interior points where $f'(x) = 0$
- Interior points where $f'(x)$ does not exist
- End points of the domain of f

First Derivative Test
Suppose that $c \in (a.b)$ is a critical point of f, if

- $f'(x) > 0$ for $x \in (a, c)$, and $f'(x) < 0$ for $x \in (c, b)$, then $f(c)$ is a local maximum.
- $f'(x) < 0$ for $x \in (a, c)$, and $f'(x) > 0$ for $x \in (c, b)$, then $f(c)$ is a local minimum

Second Derivative test

- if $f'(c)$ and $f''(c) < 0$, then f has a local maximum at $x = c$
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$

L'Hospital's Rule
Suppose

- f and $g$are differentiable in a neighbourhood of x_0
- $f(x_0) = g(x_0) = 0/\infty$
- $g'(x) \neq 0$ except possibly at x_0

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ (Can chain this rule multiple times)

Integration

A differentiable function $F(x)$ is an antiderivative of a function $f(x)$ if,
$F'(x) = f(x)$, for all x in domain of f

Rules of definite Integral

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, for any constant k
  $\Big($n paritcular, $\int_a^b -f(x)dx = -\int_a^b f(x)dx\Big)$
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- If $f(x) \geq g(x)$on$[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If f is continuous on $(a, b)and(b, c)$, then
  $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Fundamental Theorem of Calculus
$F(x) = \int_a^x f(t)dt$ has a derivative at every point of [a, b], and

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

eg.

- $\frac{d}{dx}\int_{-\pi}^{\pi} \cos t dt - \cos x$
- $\frac{d}{dx}\int_0^x \frac{dt}{1-t^2} = \frac{1}{1-x^2}$
- $\frac{d}{dx}\int_1^{x^2} \cos t dt = \left(\frac{d}{dx^2}\int_1^{x^2} \cos t dt\right)\frac{d}{dx}x^2 = (\cos x^2)2x = 2x\cos(x^2)$

If f is continuous at every point on [a, b] and F is any antiderivative of f on [a. b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Integration by Substitution

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)} g(b)f(u)du$$

Integration by Parts

$$\int \frac{d}{dx}(uv) = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$$
$$uv = \int udv + \int vdu$$
$$\int udv = uv - \int vdu$$

Area between two curves

$$\text{Area} = \int_a^b [f_2(x) - f_1(x)]dx$$

Sometimes we may like to view the curve as $x = g(y)$ instead of $y = g(x)$ when evaluating area.

Volume of solids of revolution

$$\text{Volume} = \int \pi y^2 dx$$

Series

Geometric Series
Geomeric series converges to the sum $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$

Rules on Series

If $\sum a_n = A$ and $\sum b_n = B$, then

- $\sum(a_n \pm b_n) = A \pm B$
- $\sum(ka_n) = kA$

Ratio Test

Can be used on other series as well, not just geometric series.
For geometric series, check any two consecutive terms, $\frac{a_{n+1}}{a_n}$
For other series, check $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| < 1$

- $|r| > 1$ diverges
- $|r| = 1$ diverges for Geometric series, otherwise not conclusive
- $|r| < 1$ converges

Power series about x = 0

In the form of

$$\sum_{n=0}^{\infty} c_n x^n$$

Power series can be considered a function of x when it is convergent

Power series about x = a

In the form of

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

where a is the centre of the series, something like shifting of the origin in coordinate geoometry.

Convergence of Power Series

Power Series is always convergent at its centre, ie $x = a$
3 possibilities

- converges only at the centre
- converges in a region, $(a-h, a+h)$, where h is known as the radius of convergence
- converges for all values of x

Use ratio test, and put in the form of $|x-a| < h$

Differentiating Power Series

If a power serieshas radius of convergence h, then it defines the function f

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n,\ a-h < x < a+h$$

then f has derivatives of all orders within $(a-h, a+h)$,

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$$

and similarly for higher order derivatives. The differentiated series also converges within $(a-h, a+h)$.

Integrating Power Series

The power series would have anti-derivatives in $(a-h, a+h)$

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

The integrated series also converges for $(a-h, a+h)$.

Taylor Series

Taylor series of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Used to approximate and represent complex functions at a value of x (a).

Taylor Polynomials

The nth order Taylor Polynomial of f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It provides the best polynomial approximation of degree n.
At degree 1 it would be the tangent line.

3D Space

Vectors

Let P and Q be points in the xyz-space with coordinates (x, y, z) and (x_1, y_1, z_1). Then the vector $\overrightarrow{PQ} = (x_1 - x, y_1 - y, z_1 - z)$

Magnitude/ Norm of Vectors

Magnitude of vector $v_1 = (x_1, y_1, z_1)$ is $||v_1|| = \sqrt{x_1^2 + y_1^2 + z_1^2}$, $||cv_1|| = |c|||v_1||$

Angle Between Two Vectors

Using Cosine Rule,

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{||v_1||||v_2||}$$

Or can be written as

$$\cos\theta = \frac{v_1 \cdot v_2}{||v_1||||v_2||}$$

Dot / Scalar Product

Let $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$, then the dot product is given by

$$v_1 \cdot v_2 = x_1x_2 + y_1y_2 + z_1z_2$$

Or can be written as,

$$v_1 \cdot v_2 = ||v_1||||v_2||\cos\theta$$

Properties of Dot Product

- $v \cdot v = ||v||^2$, $v \cdot v = 0$ if and only if $v = 0$
- Commutative $v_1 \cdot v_2 = v_2 \cdot v_1$
- Distributive
- Scalars can be "pulled" out $(cv_1 \cdot v_2) = (v_1 \cdot cv_2) = c(v_1 \cdot v_2)$

Unit Vector

Vector with magnitude or length 1. Can normalize any vector to get a unit vector by $\frac{1}{||w||}w$

Projection

The projection of a vector $\overrightarrow{b}$ onto a vector $\overrightarrow{a}$, is denoted by $proj_a b$ is given by

$$proj_a b = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{||\overrightarrow{a}||^2}\overrightarrow{a}$$

Vector Product

Vector product returns a vector that is perpendicular to the plane including both input vectors. And is given by

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Properties of Vector Product

- Non-commutative
- Distributive
- Scalar Multiple can be put anywhere like dot product
- $v_1 \times v_1 = 0$

Magnitude of Cross Product,

$$||v_1 \times v_2|| = ||v_1||||v_2||\sin\theta$$

Lines in 3D space

Vector equation of a line:

$$\overrightarrow{OP} = \overrightarrow{r} = \overrightarrow{r_0} + t\overrightarrow{v}$$

where $\overrightarrow{r_0}$ is a fixed point on the line and $\overrightarrow{v}$ is a vector parallel to the line.

Symmetric Form of the Equation: no parameter

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

Shortest Distance from point to Line:
Find vector connecting point to line, find length of projection of that vector onto the parallel, and use Pythagoras' Theorem.

Planes in 3D space

Equation of a Plane:
let vector perpendicular to the plane be $\overrightarrow{n}$, and $\overrightarrow{r_0}$ be a known point on the plane, then any point, $\overrightarrow{r}$ on the plane is given by,

$$\overrightarrow{r} \cdot \overrightarrow{n} = \overrightarrow{r_0} \cdot \overrightarrow{n}$$

or,

$$ax + by + cz = d,$$

where $\overrightarrow{n} = (a, b, c)$, $\overrightarrow{r} = (x, y, z)$ and $d = \overrightarrow{r_0} \cdot \overrightarrow{n}$

Distance from a Point to a Plane

$$h = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Partial Differentiation

First-Order Partial Differentiation

Let $f(x, y)$ be a function of two variables. Then the first order partial derivative of f with respect to x at the point (a, b) is

$$\frac{d}{dx}f(x,b)\Big|_{x=a} = \lim_{h\to 0}\frac{f(a+h,b) - f(a,b)}{h}$$

When the above partial derivative exists, it is denoted by,

$$\frac{\partial f}{\partial x}\Big|_{a,b} \text{ or } f_x(a, b)$$

Geometric Interpretation
The gradient of the line y = b or x = a translated upwards to cut the surface of the graph.

Higher Order Partial Derivatives

- $f_{xx} = \frac{\partial^2 f}{\partial x^2}$
- $f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, for functions in this course.
- $f_{yy} = \frac{\partial^2 f}{\partial y^2}$

Can use f_{xy} to find f_{yx} and vice-versa if one is difficult to differentiate.

Chain Rule

Chain Rule for 2 dependent variables and 1 independent variable.
eg $z = f(x, y)$ and $x = x(t)$, $y = y(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Chain rule for 2 independent variables on $f(x, y)$
eg. $z = f(x, y)$ and $x = x(s, t)$, $y = y(s, t)$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Chain Rule for 3 dependent variables and 1 independent variable
eg. $w = f(x, y, z)$ and $z = z(t)$, $y = y(t)$, $x = x(t)$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Gradient Vector

The Vector which shows the direction (and magnitude) of greatest rate of change of f.

$$\nabla f(x, y) = f_x(x, y)\overrightarrow{i} + f_y(x, y)\overrightarrow{j}$$

Directional Derivatives

Measures the gradient with respect to $\overrightarrow{u}$ or gradient in the direction of $\overrightarrow{u}$
Note that $\overrightarrow{u}$ must be a unit vector.

$$D_{\overrightarrow{u}} f(a, b) = \nabla f(a, b) \cdot \overrightarrow{u} = f_x(a, b)u_1 + f_y(a, b)u_2$$

Another way of writing,

$$D_{\overrightarrow{u}} f(a, b) = ||\nabla f(a, b)||||\overrightarrow{u}|| \cos\theta = ||\nabla f(a, b)|| \cos\theta$$

Therefore,

$$-||\nabla f(a, b)|| \leq D_{\overrightarrow{u}} f(a, b) \leq ||\nabla f(a, b)||$$

when $\theta = \pi$ and $\theta = 0$ respectively
Formula for functions of 3 variables and above are the same:
Gradient Vector · unit direction vector.

Maximum and Minimum

The critical point where $\nabla f(a, b) = 0$. ie.

$$f_x(a.b) = 0 \text{ and } f_y(a, b) = 0$$

Or where $f_x(a, b)$ and $f_y(a, b)$ do not exist.

How to tell min/ max/ saddle
Second Derivative Test, using determinant

$$D = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$$

Local min: $D > 0$, $f_{xx}(a, b) > 0$
Local max: $D > 0$, $f_{xx}(a, b) < 0$
Saddle: $D < 0$
No conclusion: $D = 0$

Double Integration

Definition

Let ΔA_i be the area of R_i and (x_i, y_i) be a point in R_i
let $f(x, y)$ be a function of two variables. Then the double integral of f over R is

$$\int\int_R f(x, y)dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

Geometrically, it is the volume of under the surface, over the area R.

Properties of Double Integrals

- $\int\int_R (f(x, y) + g(x, y))dA = \int\int_R f(x, y)dA + \int\int_R g(x, y)dA$
- $\int\int_R cf(x, y)dA = c\int\int_R f(x, y)dA$, where $c \in \mathbb{R}$
- if $f(x, y) \geq g(x, y)$ for all $(x, y) \in R$, then $\int\int_R f(x, y)dA \geq \int\int_R g(x, y)dA$
- $\int\int_R dA = A(R)$, the area of R
- $\int\int_R f(x, y)dA = \int\int_{R_1} f(x, y)dA + \int\int_{R_2} f(x, y)dA$, where $R = R_1 \cup R_2$, and R_1, R_2 do not overlap except on their boundary.

Calculating Rectangular Region

The region can be expressed in terms of inequalities

$$a \leq x \leq b \text{ and } c \leq y \leq d$$

Then the double integral is given by

$$\int\int_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

In general if $f(x, y) = g(x)h(y)$, then

$$\int\int_R g(x)h(y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

Non-Rectangular

Type A: set the left and right extremes to be parallel to the y-axis, ie $x = a$, $x = b$, and top and bottom boundaries to be functions of x, ie $y = g(x)$, $y = h(x)$ then the inequalities become

$$h(x) \leq y \leq g(x) \text{ and } a \leq x \leq b$$

And the double integral becomes

$$\int_a^b \int_{h(x)}^{g(x)} f(x, y)dydx$$

Type B: set the top and bottom extremes, and left and right boundaries become functions of y.
Inequalities

$$g(y) \leq x \leq h(y) \text{ and } c \leq y \leq d$$

Double Integral:

$$\int_c^d \int_{g(y)}^{h(y)} f(x, y)dxdy$$

Polar Coordinates

Used when R is circular.
Converting cartesian coordinates to polar coordinates

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

where r is the radius of the circle and θ is the angle in radians from the x-axis.
Double integral becomes, eg
$\int\int_R (x + y)dA = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r\cos\theta + r\sin\theta r d\theta dr$
$= \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r\cos\theta + r\sin\theta r dr d\theta$

Surface Area

If f has continuous firs partial derivatives on a closed region R of the xy-plane, then the area S of that portion of the surface z = f(x, y) that projects onto R is

$$S = \int\int_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right) + 1}dA$$

Differential Equations

Ordinary Differential Equations

Differential equations of only one variable.
Order of equation: Highest order derivative.
Linearity of equation: power of derivative eg $(f'')^2$

Separable equations

Can be written in the form $M(x)dx - N(y)y' = 0$ or $M(x)dx = N(y)dy$. Separated because everything on the left is x and everything on the right is y.

Radioactive substances equation:

$$Y = Y_0 e^{-\frac{\ln 2}{T}t}$$

where Y is the amount of radioactive substance left, Y_0 is the initial amount, T is the half-life, and t is the time elapsed.

Non-Separable Equations

Use a substitution to convert it into separable form.
Can try to use the substitution $y = vx$ which will result in $\frac{dy}{dx} = \frac{dy}{dx}x + v$

Linear First Order ODEs

A differential equation which can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are funtions of x. Note that the coefficient of $\frac{dy}{dx}$ must be 1.

How to solve:
1. Let $R = e^{\int Pdx}$, no constant
2. $y = \frac{1}{R}\int RQdx$

Reduction to Linear Form

Certain non-linear differential equations can be reduced to linear form. The most important class of such differential equations are called Bernoulli equations which have the form $y' + p(x)y = q(x)y^n$, where $n \in \mathbb{R}$

How to solve:
1. $y' + Py = Qy^n$
2. $y^{-n}y' + Py^{1-n} = Q$
3. let $z = y^{1-n}$, where $n \neq 1$ or 0
4. Solve the Linear First Order ODE in terms of z

Application to Population Growth

let B be the birth rate per capita, and D be the death rate per capita, N be the population, $\hat{N}$ be the original population

Malthusian Model:

$$\frac{dN}{dt} = (B - D)N = kN$$

or equivalently,

$$N = \hat{N}e^{kt}$$

Logistic Model:

$$N = \frac{\hat{N}N_\infty}{\hat{N} + (N_\infty - \hat{N})e^{-Bt}}$$

or equivalently,

$$N = \frac{N_\infty}{1 + (\frac{N_\infty}{\hat{N}} - 1)e^{-Bt}}$$

where N_∞ is the carrying capacity, or $\lim_{t\to\infty} N = N_\infty$, $N_\infty = \frac{B}{S}$

$\frac{temp - min(temp)}{max(temp) - min(temp)}$