CS2040S CheatSheet

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Searching and Sorting

Binary Search

Can be used for any monotonic function, not necessarily just for searching for number in array

```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  key <= A[mid] then
            end = mid
        else begin = mid+1
    return (A[begin]==key) ? begin : -1
```

Peak Finding

For finding local peak - Binary Search + recurse on larger side

Sorts

- BubbleSort: $\Omega(n)$ already sorted, $O(n^2)$, stable
- SelectionSort: $\Omega(n^2)$, $O(n^2)$, not stable
- InsertionSort: $\Omega(n)$ already sorted, $O(n^2)$ reverse order, stable
- MergeSort: $O(nlogn)$, $T(n) = 2T(\frac{n}{2} + O(n)$, stable if merge step is stable.

QuickSort

Deterministic: $O(n^2)$ for worst case (sorted), $O(nlogn)$ best case.
Random: $O(nlogn)$ for worst and best case.
Paranoid random: Expected 2 tries.
Can make stable with extra $O(n)$ space.

```
partition(A[1..n], n, pIndex)      // Assume no duplicates, n>1
    pivot = A[pIndex];             // pIndex is the index of pivot
    swap(A[1], A[pIndex]);         // store pivot in A[1]
    low = 2;                       // start after pivot in A[1]
    high = n+1;                    // Define: A[n+1] = ∞
    while (low < high)
        while (A[low] < pivot) and (low < high) do low++;
        while (A[high] > pivot) and (low < high) do high– – ;
        if (low < high) then swap(A[low], A[high]);
    swap(A[1], A[low–1]);
    return low–1;
```

```
// option 1: two pass, pack duplicates // option 2: one pass, four regions
void 3wayPartition(A, n, pIdx)
  pivot = A[pIdx];
  low = 1; curr = 1; high = n;
  while (curr < high)
    if (A[curr] < pivot) low++; swap(A[curr], A[low]); curr++;
    else if (A[curr] == pivot) curr++;
    else swap(A[curr], A[high]); high--;
```

QuickSelect

Select the kth element, runtime = $O(n)$
1. Partition then recurse into the correct side.
2. When recursing into right half need to take note of index, $(k - i)$ where i is the index of the partition.

Trees

Tree Operations

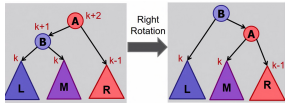
- insert: $O(h)$, only insert at leaves
- delete: $O(h)$, no child just delete, 1 child join parent and child, 2 children swap with successor and delete
- successor (predecessor): $O(h)$, minimal element on right subtree or walk up tree until it is a left child.

AVL trees

Augmentation: store $v.height = max(v.left.height, v.right.height) + 1$
Balance Condition: $|v.left.height - v.right.height| \leq 1$
Balanced tree with n nodes has $h < 2logn$
and any balanced tree with height h has $n > 2^{\frac{h}{2}}$

Rotation

1. Left heavy
1.1 Balanced or Left-heavy: right-otation
1.2 Right-heavy: Left-rotate left child then right-rotation



Insertions: $\leq$ 2 rotations, Deletions: $O(logn)$ rotations(walk up to root)

Trie - Search: $O(L)$ Insert: $O(L + Overhead)$
Useful for prefix, longest prefix, wildcard queries

Augmented Tree

Order Statistics Tree - weights $v.weight = v.left.weight + v.right.weight + 1$ and parent pointers

```
select(k)
    rank = left.weight + 1;
    if (k == rank) then
        return v;
    else if (k < rank) then
        return left.select(k);
    else if (k > rank) then
        return right.select(k-rank);
```

```
rank(node)
    rank = node.left.weight + 1;
    while (node != null) do
        if node is left child then
            do nothing
        else if node is right child then
            rank += node.parent.left.weight + 1;
        node = node.parent;
    return rank;
```

Interval Tree - Keyed by left edge, hold max endpoint in subtree in node

```
interval-search(x)
    c = root;
    while (c != null and x is not in c.interval) do
        if (c.left == null) then
            c = c.right;
        else if (x > c.left.max) then
            c = c.right;
        else c = c.left;
    return c.interval;
```

Orthogonal Range Finding - nodes are max value in left subtree, values are stored at leaves $O(k + logn)$, where k is the number of points found.
- Find split node, LeftTraversal on node.left, RightTraversal on node.right

```
RightTraversal(v, low, high)
    if (v.key <= high) {
        all-leaf-traversal(v.left);
        RightTraversal(v.right, low, high);
    }
    else {
        RightTraversal(v.left, low, high);
    }
}
```

```
LeftTraversal(v, low, high)
    if (low <= v.key) {
        all-leaf-traversal(v.right);
        LeftTraversal(v.left, low, high);
    }
    else {
        LeftTraversal(v.right, low, high);
    }
}
```

(a, b) **trees** - B-trees are (B, 2B) trees.
1. Root $[2, b]$ children, $[1, b - 1]$ keys; Internal $[a, b]$ children, $[a - 1, b - 1]$ keys; Leaf 0 children, $[a - 1, b - 1]$ keys
2. Non-leaf node must have one more child than number of keys (ensures all ranges covered by children)
3. All leaf nodes must be at same depth
Proactive Split - when node's keylist size = b - 1, push median key up (before parent key) and split keylist. Left keylist becomes median node's child, median node's child becomes left keylist rightmost child.
Proactive Merge / Share - when node's keylist = $a - 1$, delete parent key and put between two neighbour keylists, delete the resulting empty keylist (from merging one into other). If keylist is now $>= b - 1$, split (Share).

Hashing

Implement symbol table - key, value pairs without ordering.
Search and Insert - $O(1)$

Collision resolution

Collision if $h(k_1) = h(k_2), k_1 \neq k_2$

Chaining - bucket contains list, add to list if collide
Insertion: $O(1 + cost(h))$, insert at front of LL
Search: $O(n + cost(h))$
Assuming SUHA (every key has equal chance of hashing to any bucket),
Expected Search = $1 + \frac{n}{m}$ (expected no. of items per bucket), 1 for hashing and accessing hashtable.

Open Addressing - finding another bucket
For an item k, in a hashtable with m elements, on the nth try,
- Linear probing: $h(k, n) = h(k) + n \mod m$
- Double Hashing: $h(k, n) = h(k) + n * g(k) \mod m$, if g(k) is coprime to m, then $h(k, n)$ will hit all buckets
Good Hash Function:
- enumerate all possible buckets (permutation of buckets)
- UHA: every key is equally likely to be mapped to every permutation, indep of other keys (NOT linear probing)
Operations' cost: $\leq \frac{1}{1-\alpha}$, assuming UHA.

Table Resizing

Increasing Table Size:
1. Increment by 1: Each insert is $O(n + 1)$, overall cost of inserting n items: $O(n^2)$.
2. Double Table Size: Each time resize ($O(n)$), would have inserted $O(n)$ items, amortized cost of n inserts: $O(n)$.
3. Square Table Size: resizing becomes $O(n^2)$, avg cost of insert $O(n)$, inefficient space usage.
Decreasing Table Size:
1. Halve, $n < m/2$: Consecutive insert, delete on table size n, with n items $\rightarrow$ each operation $O(n)$
2. Halve, $n < m/4$: $O(n)$ operations between resizes, Amortized cost $O(1)$
Amortized Cost - Operation has amortized cost $T(n)$ if for every integer k, the cost of k operations is $\leq kT(n)$.

Sets - FHT, Bloom Filters - Space and Speed critical
FingerprintHash Table - Only stores 0/1 in HT instead of keys (save space)
Collision (Insertion)- just leave table entry as 1
Look up - might false positive; 2 items hash to same bucket, 1 in table, 1 not, both reflect present as bucket == 1
Bloom Filter - > 1 hash functions, need more space: k functions, $k * n$ bits. Requires all k buckets = 1 for item present, reduces collisions and false +ve

Graphs

Adjacency List (store outgoing edges in directed graphs) - sparse graph less space, fast find any nbr/ enum all nbr
Adjacency Matrix - fast query if two nodes are connected

Searching Graph - DFS, BFS explores all nodes, edges but not all paths.
BFS - $O(V + E)$, can be used for unweighted, graphs w same weight, trees

Returns shortest path graph from source to all nodes.
DFS - $O(V+E)$, can be used for unweighted, graphs w same weight, trees
Implemented implicitly by call stack, doesn't return shortest path graph

Shortest Path in Weighted Graphs

Bellman-Ford - $O(VE)$, doesn't work on graphs w -ve weight cycles (NWC), can terminate early if sequence of $|E|$ relax have no effect.

```
n = V.length;

for (i=0; i<n; i++)

    for (Edge e : graph)

            relax(e)
```

Key property:
If P is the shortest path from S to D, and if P goes through X, then P is also the shortest path from S to X (and from X to D).

After k times, k hop estimate for nodes on shortest path is correct.
Run n times to detect NWC (change on nth time)
Dijkstra's - Relax in the right order (each edge relaxed once), works on graphs without -ve weight edges, $O(ElogV)$

```
public Dijkstra{
    private Graph G;
    private IPriorityQueue pq = new PriQueue();
    private double[] distTo;

    searchPath(int start) {
        pq.insert(start, 0.0);
        distTo = new double[G.size()];
        Arrays.fill(distTo, INFTY);
        distTo[start] = 0;
        while (!pq.isEmpty()) {
            int w = pq.deleteMin();
            for (Edge e : G[w].nbrList)
                relax(e);
        }
    }
```

```
relax(Edge e) {
    int v = e.from();
    int w = e.to();
    double weight = e.weight();
    if (distTo[w] > distTo[v] + weight) {
        distTo[w] = distTo[v] + weight;
        parent[w] = v;
        if (pq.contains(w))
            pq.decreaseKey(w, distTo[w]);
        else
            pq.insert(w, distTo[w]);
    }
}
```

Main Requirement: Extending a path does not decrease its estimate.
Cannot reweight graphs w -ve weight edges as paths with multiple edges will add more weight.

DAG, Topo Sort - Only DAGs have topological orderings, converse is true
1. Sequential total ordering of all nodes.
2. edges only point forward.

Constructing Topo Ordering - not unique, $O(V+E)$
1. Post Order DFS - Process (prepend to list) each node when it is last visited

```
DFS-visit(Node[] nodeList, boolean[] visited, int startId){

  for (Integer v : nodeList[startId].nbrList) {

    if (!visited[v]){

        visited[v] = true;

        DFS-visit(nodeList, visited, v);

        ProcessNode(v);

    }

  }

}
```

2. Kahn's Algorithm, runtime here $O(ElogV)$ but can be done in $O(V+E)$
Add nodes with no incoming edges to topo order, remove outgoing edges from that node, repeat. Implement using a PQ, nodes keyed by incoming edges.

Shortest Path in DAG - relax in topo sort order
Works because if relax outgoing edge from node V, all incoming edges have been relaxed, so estimate is correct.
Longest path - negate edges (can because no cycles) / modify relax function.

PQ, Heaps, Union Find Data Structure

Heap - max height: $O(floor(logn))$, operations: $O(logn)$
1. Heap Ordering: priority[parent] $\geq$ priority[child]
2. Complete Binary Tree: Every level full except last, nodes far left

Insert: insert at bottom left, bubble up until in position (swap w parent)
IncreaseKey: bubble up, DecreaseKey: bubble down the larger child
Delete: swap with last node, bubble up/ down as necessary
ExtractMax: delete root
Can be implemented with array: level order, left to right; left child $= 2x+1$, right child $= 2x+2$, parent $= floor((x-1)/2)$

HeapSort: unsorted list to heap to sorted
1. unsorted list to heap: $O(n)$, Leaves are proper heaps, recursively correct node's whos children are proper heaps by bubbling down. Base case: root
2. Heap to sorted: $O(logn)$, ExtractMax and put at the end n times

UFDS - QuickFind: Flat Trees; Find: $O(1)$, Union: $O(n)$
All nodes that are in the same set are connected to same node.
Find - compare root, Union - change all nodes in one set to new root

UFDS - QuickUnion: not Flat Tree; Find $O(n)$, Union: $O(n)$
Find - walk up tree of both nodes to compare roots, Union - walk up tree to roots, join one of the roots to the other

Weighted Union - Join the smaller tree's root to the larger tree's root
Max Height $= O(logn)$, tree's height only increases if joining to another tree $\geq$ itself, can happen $O(logn)$ times.

Path Compression - When walking up a tree, set parent of each node to the root, make it flatter. Find, Union- $O(logn)$

Weighted Union w Path Compression - Find, Union - $O(\alpha(m,n))$, m operations n objects

Minimum Spanning Trees - Prim's. Kruskals, Boruvka's

Properties:
1. Cannot be used to find Shortest Path
2. If cut an edge of MST, get 2 MSTs (converse is not true)
3. Cycle property: for every cycle, max weight edge of cycle not in MST
4. False edge property: min weight edge of cycle may or may not be in MST
5. Cut property: min weight edge across a cut is in MST
6. Min weight adj edge of every node is in MST (6), converse not true

Can reweight edges, so can MST on graph w -ve weights, MaxST: run Kruskal's in reverse.

Prim's - $O(ElogV)$

```
// Initialize priority queue
PriorityQueue pq = new PriorityQueue();
for (Node v : G.V()) {
        pq.insert(v, INFTY);
}
pq.decreaseKey(start, 0);

// Initialize set S
HashSet<Node> S = new HashSet<Node>();
S.put(start);

// Initialize parent hash table
HashMap<Node,Node> parent = new HashMap<Node,Node>();
parent.put(start, null);

while (!pq.isEmpty()){
    Node v = pq.deleteMin();
    S.put(v);
    for each (Edge e : v.edgeList()){
        Node w = e.otherNode(v);
        if (!S.get(w)) {
            pq.decreaseKey(w, e.getWeight());
            parent.put(w, v);
        }
    }
}
```

Assume for today (only): decreaseKey does nothing if new weight is larger than old weight

Have a set S of nodes, initially start node, add min weight edge across cut $\{S, V-S\}$, MST, node to S.

Kruskal's: $O(ElogV)$
Sort weights in ascending order. For each edge, if both endpoints in same tree (using UFDS), then discard, if not add to MST

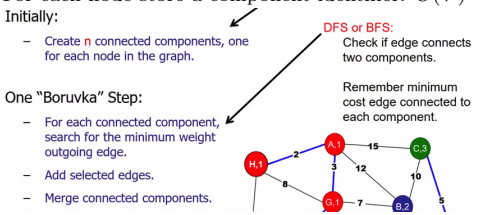
```
// Sort edges and initialize
Edge[] sortedEdges = sort(G.E());
ArrayList<Edge> mstEdges = new ArrayList<Edge>();
UnionFind uf = new UnionFind(G.V());

// Iterate through all the edges, in order
for (int i=0; i<sortedEdges.length; i++) {
        Edge e = sortedEdges[i]; // get edge
        Node v = e.one(); // get node endpoints
        Node w = e.two();

        if (!uf.find(v,w)) { // in the same tree?
            mstEdges.add(e); // save edge
            uf.union(v,w); // combine trees
        }
}
```

Sorting: $O(ElogE) = O(ElogV)$; Find and Union: $O(logV)$ E times

Boruvka's: $O(ElogV)$
For all nodes, add min adj weight edge, adds $\geq n/2$ edges each step, n is the number of connected components
For each node store a component identifier: $O(V)$



DFS/ BFS: $O(V+E)$, Merge CC: $O(V)$, update component IDs
Each Boruvka Step: $O(V+E)$, $O(logV)$ steps.

Steiner Tree 2 approximation
For every pair of required nodes, calculate shortest path $O(ElogV)$, Form new graph with just required nodes, run MST on new graph, Map edges back to original graph.

Dynamic Programming

Used on problems with Optimal Substructure and Overlapping Subproblems. Solve problems with no dependencies first, memoize, use to solve bigger problems

Useful Stuff

Sum of AP formula: $\frac{n}{2}(2a+(n-1)d)$
Sum of GP formula: $S_n = \frac{a(1-r^n)}{1-r}$
$n^{log_n k} = k$

Common Reccurences:
$T(n) = kT(n/k) + O(n) \rightarrow O(nlogn)$
$T(n) = T(n/k) + O(n) \rightarrow O(n)$