Week 7: The determinant and eigenvalues

1.

MULTI 1.0 point 0 penalty Single Shuffle

Consider the linear transformation given by the matrix.

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

Is T invertible? Why? (Hint: Pay attention to the first three rows.)

(a) T is not invertible because $\det(T) = 0$ (100%)
(b) T is not invertible because $\det(T) \neq 0$
(c) T is invertible because $\det(T) = 0$
(d) T is invertible because $\det(T) \neq 0$

Note that the third row of the matris is the sum of the first two rows. Hence $\det(T) = 0$ *and therefore it is not invertible.*

2.

MULTI 1.0 point 0 penalty Single Shuffle

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that has the following values on the basis elements:

$$T : (1, 0, 0) \mapsto (-1, 0, 1)$$
$$T : (0, 1, 0) \mapsto (3, -2, -1)$$
$$T : (0, 0, 1) \mapsto (1, 1, 1)$$

Is T invertible? Why?

(a) T is not invertible because $\det(T) = 0$
(b) T is not invertible because $\det(T) \neq 0$
(c) T is invertible because $\det(T) = 0$
(d) T is invertible because $\det(T) \neq 0$ (100%)

The matrix of T in the standard basis is given by

$$T = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Computing the determinant of T ,

$$\det(T) = -C_{1,1} + C_{3,1} = 1 + 5 = 6.$$

Since the determinant is non-zero, the matrix has an inverse and the linear transformation is invertible.

3.

MULTI 1.0 point 0.10 penalty Single Shuffle

Use Cramer's rule to solve for y in

$$A\vec{x} = \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{b}.$$

(a) -1 (100%)
(b) 1
(c) 2
(d) -2

We first note that using Laplace expansion along the top row, we get

$$\det(A) = 2 \cdot (-18) - 6 \cdot (-10) + 2 \cdot (-11) = 2.$$

Let $B_j = A_{j \to \vec{b}}$. *Then we have*

$$B_2 = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{bmatrix} \implies \det(B_2) = -2.$$

Hence, $y = \det(B_2)/\det A = -1$.

4.

MULTI 1.0 point 0.10 penalty Single Shuffle

Use Cramer's rule to solve for y in

$$\begin{aligned} ax + by + cz &= 1 \\ dx + ey + fz &= 0 \\ gx + hy + iz &= 0. \end{aligned}$$

You can assume that the relevant 3×3 matrix has non-zero determinant D.

(a) $\frac{fg - di}{D}$ (100%)
(b) $\frac{di - fg}{D}$
(c) $\frac{ei - fh}{D}$
(d) $\frac{fh - ei}{D}$

The 3×3 matrix corresponding to the system of equations is given by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Since we are interested in the value of y, we compute

$$y = \frac{\det(A_{2\to\vec{b}})}{\det(A)} = \frac{1}{D} \begin{vmatrix} a & 1 & c \\ d & 0 & f \\ g & 0 & i \end{vmatrix} = \frac{fg - di}{D}.$$

5.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Use Cramer's rule to solve for x_1 in

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

(a) $\frac{3}{4}$ (100%)
(b) 3
(c) $\frac{1}{2}$
(d) $\frac{1}{4}$

The 3×3 matrix corresponding to the system above is given by

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

For this matrix

$$\det(A) = 2 \cdot 3 - 1 \cdot 2 = 4.$$

Hence

$$x_1 = \frac{\det(A_{1\to\vec{b}})}{\det(A)} = \frac{1}{4} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \frac{3}{4}.$$

6.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Compute the inverse of the matrix (try using the classical adjoint)

$$H = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

where a, b, c are arbitrary real numbers.

(a) $\begin{bmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$ (100%)
(b) $\begin{bmatrix} 1 & a & c-ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c-ab & b & 1 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{bmatrix}$

Since H is upper-triangular, $\det(H) = 1$. *The matrix of cofactors is given by*

$$C = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{bmatrix}$$

Hence the inverse is given by

$$A^{-1} = \frac{C^\top}{\det(A)} = \begin{bmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

7.

MULTI 1.0 point 0.10 penalty Single Shuffle

Compute the inverse of the matrix (try using the classical adjoint)

$$R_\theta = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

where θ is an arbitrary real number.

(a) $R_{-\theta}$ (100%)
(b) $-R_\theta$
(c) $-R_\theta$
(d) $R_{\theta^{-1}}$

Doing Laplace expansion along the second row, the determinant $\det(R_\theta) = 1$. *The matrix of cofactors is given by*

$$C = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}.$$

Hence the inverse is given by

$$R_\theta^{-1} = \frac{C^\top}{\det(R_\theta)} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & 0 & \sin(-\theta) \\ 0 & 1 & 0 \\ -\sin(-\theta) & 0 & \cos(-\theta) \end{bmatrix} = R_{-\theta}.$$

8.

MULTI 1.0 point 0.10 penalty Single Shuffle

Find the eigenvalues and the associated eigenvectors of the matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(a) $\lambda_1 = 1, v_1 = (1, \sqrt{2} - 1)$
$\lambda_2 = -1, v_2 = (1, -\sqrt{2} - 1)$ (100%)
(b) $\lambda_1 = \sqrt{2}, v_1 = (1, \sqrt{2} - 1)$
$\lambda_2 = -\sqrt{2}, v_2 = (1, -\sqrt{2} - 1)$
(c) $\lambda_1 = 1, v_1 = (\sqrt{2} - 1, 1)$
$\lambda_2 = -1, v_2 = (\sqrt{2} + 1, 1)$
(d) $\lambda_1 = \sqrt{2}, v_1 = (1, 1)$
$\lambda_2 = -\sqrt{2}, v_1 = (1, -1)$

We first compute the characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} (1\sqrt{2}) - \lambda & (1/\sqrt{2}) \\ (1\sqrt{2}) & -(1/\sqrt{2}) - \lambda \end{vmatrix} = -\left(\frac{1}{\sqrt{2}} - \lambda\right)\left(\frac{1}{\sqrt{2}} + \lambda\right) - \frac{1}{2} = \lambda^2 - 1.$$

Hence the eigenvalues are $\lambda_+ = 1$ *and* $\lambda_- = -1$.

$$\lambda_+ = 1 : (A - I)v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 - \sqrt{2} & 1 \\ 1 & -1 - \sqrt{2} \end{bmatrix} v = 0 \implies v_+ = \begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix}$$

$$\lambda_- = -1 : (A + I)v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + \sqrt{2} & 1 \\ 1 & 1 - \sqrt{2} \end{bmatrix} v = 0 \implies v_- = \begin{bmatrix} 1 \\ -\sqrt{2} - 1 \end{bmatrix}.$$

9.

MULTI 1.0 point 0.10 penalty Single Shuffle

Find the eigenvalues and the associated eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

(a) $\lambda_1 = 0, v_1 = (2, -1)$
$\lambda_2 = 5, v_2 = (1, 2)$ (100%)
(b) $\lambda_1 = 1, v_1 = (2, 1)$
$\lambda_2 = 5, v_2 = (1, 2)$
(c) $\lambda_1 = 0, v_1 = (2, 1)$
$\lambda_2 = 3, v_2 = (1, 3)$
(d) $\lambda_1 = 0, v_1 = (-2, -1)$
$\lambda_2 = 5, v_2 = (-1, 2)$

We first compute the characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5).$$

Hence the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$. To compute the eigenvectors we need to solve:

$$\lambda_1 = 0 : (A - 0I)v = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} v = 0 \implies v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\lambda_2 = 5 : (A - 5I)v = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} v = 0 \implies v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

10.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Find all eigenvalues and associated eigenvectors of the matrix

$$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}.$$

(a) $\lambda_1 = \lambda_2 = -2, v = (t, s, 3t)$ for $s, t \in \mathbb{R}$ and $\lambda_3 = 1, v_3 = (1, -1, 2)$ (100%)
(b) $\lambda_1 = \lambda_2 = -2, v = (1, 0, 3)$ and $\lambda_3 = 1, v_3 = (1, -1, 2)$
(c) $\lambda_1 = \lambda_2 = 1, v = (1, 0, 3)$ and $\lambda_3 = 2, v_3 = (1, -1, 2)$
(d) All eigenvalues are zero.

We first compute the characteristic polynomial (using cofactor expansion along the second column)

$$\det(A-\lambda I) = \begin{vmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{vmatrix} = -(\lambda+2)\begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix}$$
$$= -(\lambda+2)(\lambda^2+\lambda-2) = -(\lambda+2)^2(\lambda-1)$$

Hence the eigenvalues are $\lambda_1 = \lambda_2 = -2$ *and* $\lambda_3 = 1$.

$$\lambda_1 = \lambda_2 = -2 : (A+2I)v = \begin{bmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{bmatrix} = 0 \implies v = \begin{bmatrix} t \\ s \\ 3t \end{bmatrix} \text{ where } t, s \in \mathbb{R},$$
$$\lambda_3 = 1 : (A-I)v = \begin{bmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{bmatrix} v = 0 \implies v = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Total of marks: 10