Week 2: Elementary Analytical Geometry

1.

MULTI 1.0 point 0 penalty Single Shuffle

What is the angle (in radian, i.e., where 360° corresponds to 2π) between the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 \\ 7 \\ 17 \end{bmatrix}?$$

(a) 0
(b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (100%)
(d) π

Recall the formula for the scalar product: $u \cdot v = |u||v|\cos(\theta)$, with θ the angle between the vectors u and v. Here, we find

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \\ 17 \end{bmatrix} = 0,$$

i.e., the angle is $\frac{\pi}{2}$ (or 90°).

2.

MULTI 1.0 point 0 penalty Single Shuffle

If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, then

(a) $|\vec{u} - \vec{v}| = \sqrt{2}$ (100%)
(b) $|\vec{u} - \vec{v}|$ cannot be computed without further information on $\vec{u}$ and $\vec{v}$
(c) $|\vec{u} - \vec{v}| = 1$
(d) $|\vec{u} - \vec{v}| = 0$

We find

$$|\vec{u} - \vec{v}|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 = 1 + 0 + 1.$$

3.

MULTI 1.0 point 0 penalty Single Shuffle

How long is the vector $(1, 1, \ldots, 1)$ in 16 dimensions?

(a) Length $= 4$ (100%)
(b) Length $= 16$

(c) Length $= 1$
(d) Length $= 32$

We find

$$|(1,1,\dots,1)| = \sqrt{\sum_{j=1}^{16} 1^2} = \sqrt{16} = 4.$$

4.

MULTI 1.0 point 0 penalty Single Shuffle

Which of the following formulas is not true (for $\vec{u}, \vec{v} \in \mathbb{R}^n$)?

(a) $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta) + |\vec{v}|^2$
(b) $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$
(c) $|\vec{u} \times \vec{v}| \leq |\vec{u}||\vec{v}|$
(d) $\cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2|\vec{v}|^2}$ (100%)

We have $\cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$.

5.

MULTI 1.0 point 0 penalty Single Shuffle

Let x, y, z be such that $x + y + z = 0$. Define $u = (x, y, z)$ and $v = (z, x, y)$. What is the value of $\dfrac{u \cdot v}{|u||v|}$?

(a) $-\dfrac{1}{2}$ (100%)
(b) 1
(c) $\dfrac{-x^2 + yz}{x^2 + y^2 + (x - y)^2}$
(d) 0

A direct computation using the constraint $x + y + z = 0$ *yields the result.*

6.

MULTI 1.0 point 0 penalty Single Shuffle

A line is given by $\vec{r} = \lambda\vec{a} + \vec{b}$, with $\vec{a} = (1, -1, 4)$ and $\vec{b} = (4, 5, 6)$, while the equation of a plane is given by $-2x + 2y + z = 17$. What are the coordinates of the point P where the line and plane intersect?

(a) The line and the plane do not intersect (100%)
(b) $P = (-1, 4, 7)$
(c) $P = (3, 3, 17)$
(d) The line and the plane intersect infinitely many times

Consider the vector normal to the plane $\vec{n} = (-2, 2, 1)$.
One can observe $\vec{a} \cdot \vec{n} = 1 \cdot (-2) + (-1) \cdot 2 + 4 \cdot 1 = 0$. *Thus, the line and the plane either do not intersect or intersect infinitely many times.*
Analyzing $\lambda = 0$ *and plugging in the coordinates of the line into the plane equation:* $-2 \cdot 4 + 2 \cdot 5 + 1 \cdot 6 = 8 \neq 17$.
Thus, the line and plane do not intersect.

7.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

What is the equation of the hyperplane, given by $\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \vec{p}_0 + \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ with

$\vec{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\alpha, \beta, \gamma \in \mathbb{R}$

(a) $-t - x - y + z + 1 = 0$ (100%)
(b) $-t - x - y - z + 1 = 0$
(c) $t + x - y + z - 1 = 0$
(d) $t + x - y - z - 1 = 0$

Note that $\vec{d} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ +1 \end{bmatrix}$ *is perpendicular to* $\vec{a}, \vec{b}, \vec{c}$.

This defines the following equation (expanded version of $\vec{d} \cdot \left(\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} - \vec{p}_0 \right) = 0$*):*

$$-(t-1) - x - y + z = 0 \Leftrightarrow -t - x - y + z + 1 = 0$$

8.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find the cross product $\vec{u} \times \vec{v}$ of $\vec{u} = \langle 3, 2, -1 \rangle$, $\vec{v} = \langle 1, 1, 0 \rangle$

(a) $\langle 1, -1, 1 \rangle$ (100%)
(b) $\langle -1, -1, 5 \rangle$
(c) $\langle -6, -4, 2 \rangle$
(d) $\langle 6, -4, 2 \rangle$

Direct computation.

9.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find the unit vector along the direction of the cross product $\vec{u}\times\vec{v}$ of $\vec{u} = \langle 7, -1, 3\rangle$, $\vec{v} = \langle 2, 0, -2\rangle$.

(a) $\frac{1}{\sqrt{408}}\langle 2, 20, 2\rangle$ (100%)
(b) $\frac{1}{108}\langle -2, -10, 2\rangle$
(c) $\frac{1}{408}\langle 2, 20, 2\rangle$
(d) $\frac{1}{\sqrt{108}}\langle -2, -10, 2\rangle$

Direct computation yields:

$$\vec{u}\times\vec{v} = \begin{bmatrix} 2 \\ 20 \\ 2 \end{bmatrix}$$

Then the norm squared is simply $2^2 + 20^2 + 2^2 = 408$. *Thus, we take the square root to find the norm and divide by it to normalize.*

10.

MULTI 1.0 point 0 penalty Single Shuffle

Let $\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i\,j\,k) = (1\,2\,3),\ (2\,3\,1),\ \text{or } (3\,1\,2) \\ -1 & \text{if } (i\,j\,k) = (1\,3\,2),\ (3\,2\,1),\ \text{or } (2\,1\,3) \\ 0 & \text{else} \end{cases}$

Consider $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$. Which of the following is equivalent to the kth component of $\vec{u}\times\vec{v}$

(a) $[\vec{u}\times\vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} u_i\, v_j$ (100%)
(b) $[\vec{u}\times\vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} v_i\, u_j$
(c) $[\vec{u}\times\vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} (u_i\, v_j - v_i\, u_j)$
(d) $[\vec{u}\times\vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} (u_i\, v_j + v_i\, u_j)$

Direct computation yields:

$$\vec{u}\times\vec{v} = \begin{bmatrix} u_2\,v_3 - u_3\,v_2 \\ u_3\,v_1 - u_1\,v_3 \\ u_1\,v_2 - u_2\,v_1 \end{bmatrix} = \begin{bmatrix} \epsilon_{231}u_2\,v_3 + \epsilon_{321}u_3\,v_2 + 0 \\ \epsilon_{312}u_3\,v_1 + \epsilon_{132}u_1\,v_3 + 0 \\ \epsilon_{123}u_1\,v_2 + \epsilon_{213}u_2\,v_1 + 0 \end{bmatrix} = \sum_{i,j=1}^{3} \begin{bmatrix} \epsilon_{ij1}u_i\,v_j \\ \epsilon_{ij2}u_i\,v_j \\ \epsilon_{ij3}u_i\,v_j \end{bmatrix}$$

Where the $+0$ *represents all the other* ϵ_{ijk} *terms.*

Total of marks: 10