Week 12: QR decomposition

1.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the basis

$$\left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

of $\mathbb{R}^3$. Find an orthonormal basis of $\mathbb{R}^3$ containing $u_1/\|u_1\|$ by applying Gram-Schmidt to the basis above.

(a) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$ (100%)

(b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \right\}$

(c) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

Step 1: We start with

$$v_1 = \frac{u_1}{\|u_1\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Step 2: we compute

$$\tilde{v}_2 = u_2 - (v_1 \cdot u_2)v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which is already normalized. Thus

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Step 3: we compute

$$\tilde{v}_3 = u_3 - (v_1 \cdot u_3)v_1 - (v_2 \cdot u_3)v_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - 2\sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Normalizing yields

$$v_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

2.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

This exercise assumes basic knowledge of integration. Consider the vector space of polynomials of degree at most 2 spanned by $\{1, x, x^2\}$. On this space, consider the inner product

$$\langle p, q \rangle := \int_0^1 p(x)q(x)dx$$

and the corresponding norm (length)

$$\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{\int_0^1 p(x)^2 dx}.$$

Use the Gram-Schmidt procedure on the set of vectors $\{1, x, x^2\}$ to obtain an orthonormal basis that contains the vector 1.

(a) $1, \sqrt{12}\Big(x-\frac{1}{2}\Big), \sqrt{180}\Big(x^2-x+\frac{1}{6}\Big)$ (100%)

(b) $1, x, x^2$

(c) $1, x-1, x^2+x-\frac{1}{3}$

(d) The Gram-Schmidt procedure cannot be used

Let $u_1 = 1, u_2 = x, u_3 = x^2$. We will find the orthonormal basis $\{v_1, v_2, v_3\}$ starting with $v_1 = u_1 = 1$. Using the Gram-Schmidt procedure,

$$\tilde{v}_2 = u_2 - \langle v_1, u_2\rangle v_1 = x - \int_0^1 x dx = x - \frac{1}{2}.$$

s Its norm square is

$$\|\tilde{v}_2\|^2 = \int_0^1 \left(x-\frac{1}{2}\right)^2 dx = \frac{1}{12},$$

hence

$$v_2 = \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \sqrt{12}\Big(x-\frac{1}{2}\Big).$$

In the next step:

$$\begin{aligned}
\tilde{v}_3 &= u_3 - \langle v_1, u_3\rangle v_1 - \langle v_2, u_3\rangle v_2 \\
&= x^2 - \int_0^1 x^2 dx - 12\int_0^1 x^2\Big(x-\frac{1}{2}\Big)dx\left(x-\frac{1}{2}\right) \\
&= x^2 - \frac{1}{3} - \left(x-\frac{1}{2}\right) \\
&= x^2 - x + \frac{1}{6}.
\end{aligned}$$

Its norm squared is

$$\|\tilde{v}_3\|^2 = \int_0^1 \left(x^2-x+\frac{1}{6}\right)^2 dx = \frac{1}{180},$$

hence

$$v_3 = \frac{\tilde{v}_3}{\|\tilde{v}_3\|} = \sqrt{180}\Big(x^2-x+\frac{1}{6}\Big).$$

Hence the orthogonal basis is

$$\left\{1, \sqrt{12}\Big(x-\frac{1}{2}\Big), \sqrt{180}\Big(xs^2-x+\frac{1}{6}\Big)\right\}.$$

3.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the decomposition

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{7}{5} & -\frac{1}{6} \\ 1 & \frac{14}{5} & \frac{1}{3} \\ 0 & 1 & -\frac{7}{6} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix}.$$

Is this a valid QR decomposition? If not, why not?

(a) The QR decomposition is not valid because the columns of Q are not normalized. (100%)
(b) The QR decomposition is not valid because R is not lower-triangular.
(c) The QR decomposition is not valid because columns of Q are not orthogonal.
(d) The QR decomposition is valid.

Note that R is upper triangular and the columns of Q are indeed mutually orthogonal. However since they the are not normalized, the decomposition is not a valid QR decomposition.

4.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Use the Gram-Schmidt procedure to find an orthogonal matrix Q and upper triangular matrix R to obtain a QR-decomposition $A = QR$.

(a) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad R = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$. (100%)

(b) $Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$.

(c) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ 0 & 0 & \frac{2\sqrt{2}}{3} \end{bmatrix}$.

(d) QR decomposition is not possible.

Let $u_1 = (1,1,0)$, $u_2 = (1,0,1)$ and $u_3 = (0,1,1)$. Then we can use the Gram-Schmidt process to obtain orthonormal vectors v_1, v_2, v_3.

$$\tilde{v}_1 = u_1 = (1,1,0) \implies v_1 = \frac{\tilde{v}_1}{\|\tilde{v}_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right),$$
$$\tilde{v}_2 = u_2 - (v_1 \cdot u_2)v_1 = \left(\frac{1}{2}, -\frac{1}{2}, 1\right) \implies v_2 = \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right),$$
$$\tilde{v}_3 = u_3 - (v_1 \cdot u_3)v_1 - (v_2 \cdot u_3)v_2 = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \implies v_3 = \frac{\tilde{v}_3}{\|\tilde{v}_3\|} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

Now Q is just the matrix whose columns are given by the different v_i and R is the matrix with entries $v_i \cdot v_j$ in the upper triangle. Hence

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad R = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}.$$

5.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Use the Gram-Schmidt procedure to find an orthogonal matrix Q and upper triangular matrix R to obtain a QR-decomposition $A = QR$.

(a) $Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}, \quad R = \frac{2}{\sqrt{10}} \begin{bmatrix} 5 & 7 \\ 0 & 1 \end{bmatrix}$, (100%)
(b) $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \frac{2}{\sqrt{10}} \begin{bmatrix} 5 & 7 \\ 0 & 1 \end{bmatrix}$,
(c) $Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}, \quad R = \frac{2}{\sqrt{10}} \begin{bmatrix} 7 & 5 \\ 0 & 1 \end{bmatrix}$,
(d) QR decomposition is not possible.

This example has been computed in the Week 12 Example Session, so take a look at the solution there.

6.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Which of the following statements is wrong?

(a) Only invertible real square matrices have QR decompositions. (100%)
(b) Every real square matrix has a QR decomposition.
(c) Every invertible real square matrix has a QR decomposition.
(d) Every real $m \times n$ matrix with $m > n$ has a QR decomposition.

As discussed in the lecture notes, every real square matrix has a QR decomposition, not just invertible ones.

7.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 2 \\ 1 & 3 & 8 \\ 1 & 2 & 4 \end{bmatrix}.$$

Which of the following is a valid QR-decomposition (for 4×3 matrices)?

(a) $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 5 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (100%)

(b) $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 5 & 10 \\ 0 & 1 & 2 \end{bmatrix}$

(c) $Q = \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 5 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(d) None of the options are valid QR decompositions.

This can be checked by direct computation.

8.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix} 2 & 5 & 7 & 0 \\ 2 & 1 & -1 & 2 \\ 2 & 5 & 11 & 8 \\ 2 & 1 & 3 & -2 \end{bmatrix}.$$

Note that A has a QR decomposition with

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \qquad R = \begin{bmatrix} 2 & 3 & 5 & 2 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

with Q an orthogonal matrix with $\det(Q) = 1$. Use the QR decomposition to compute the determinant of A.

(a) $\det(A) = 24$ (100%)
(b) $\det(A) = 12$
(c) $\det(A) = 6$
(d) $\det(A) = 32$

We have $\det(A) = \det(QR) = \det(Q)\det(R) = \det(R)$*, since* $\det(Q) = 1$*. Since* R *is upper triangular, the determinant is simply the product of its diagonal entries, hence* $\det(A) = \det(R) = 2 \cdot 2 \cdot 2 \cdot 3 = 24$*.*

9.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the plane with normal vector

$$n = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Find the orthogonal matrix Q that describes reflection of a vector on the plane with normal vector n. (Such reflections are used in the Householder construction.)

(a) $Q = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 8 & 1 & -4 \\ 4 & -4 & 7 \end{bmatrix}$ (100%)

(b) $Q = \frac{1}{9}\begin{bmatrix} 1 & 10 & 4 \\ 8 & 10 & -3 \\ 4 & -1 & 7 \end{bmatrix}$

(c) $Q = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 8 & 2 & -4 \\ 4 & -6 & 7 \end{bmatrix}$

(d) $Q = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 3 & 1 & -4 \\ 4 & -4 & 8 \end{bmatrix}$

The reflection is described by $Q = 1 - 2nn^T$*. We find*

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{2}{3}, & \frac{2}{3}, & \frac{1}{3} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 8 & 1 & -4 \\ 4 & -4 & 7 \end{bmatrix}.$$

10.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Solve the least-square problem for

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}.$$

In other words: Consider the system of linear equations $Ax = b$, and compute the x with the smallest $\|Ax - b\|$.

(a) $x = \left(-\frac{4}{7}, \frac{5}{7}\right)$. (100%)

(b) $x = \left(4, -2, 3\right)$.

(c) $x = \left(-4, 5\right)$.

(d) $x = \left(\frac{1}{3}, \frac{2}{3}\right)$.

The columns of A are already orthogonal, so they just have to be normalized to yield the orthogonal matrix in the QR decomposition. This yields a QR decomposition with

$$Q = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{5}{\sqrt{42}} & * \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{42}} & * \\ -\frac{2}{\sqrt{14}} & \frac{4}{\sqrt{42}} & * \end{bmatrix},$$

where $*$ stands for some entries we do not need to compute. Then we can directly see that

$$R = \begin{bmatrix} \sqrt{14} & 0 \\ 0 & \sqrt{42} \\ 0 & 0 \end{bmatrix}.$$

We now follow the approach from the lecture notes. We find

$$Q^T b = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{14}} \\ \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ * & * & * \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{8}{\sqrt{14}} \\ \frac{30}{\sqrt{42}} \\ * \end{bmatrix}.$$

Hence the minimal x is

$$x = R_1^{-1} c_1 = \begin{bmatrix} \frac{1}{\sqrt{14}} & 0 \\ 0 & \frac{1}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} -\frac{8}{\sqrt{14}} \\ \frac{30}{\sqrt{42}} \end{bmatrix} = \begin{bmatrix} -\frac{4}{7} \\ \frac{5}{7} \end{bmatrix}.$$

Total of marks: 10