Constructor University
Fall 2024

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Elements of Linear Algebra

Homework 2 (covering Weeks 3 and 4)

Problem 1 [5 points]

Prove the following statement: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly independent. If a vector $\mathbf{w}$ can be written as

$$\mathbf{w} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k,$$

then the choice of the coefficients $\alpha_1, \ldots, \alpha_n$ is unique.

Suppose that $\mathbf{w}$ can be written as

$$\mathbf{w} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k = \sum_{k=1}^{n} \beta_k \mathbf{v}_k.$$

Then, we have

$$\sum_{k=1}^{n} (\alpha_k - \beta_k) \mathbf{v}_k = \mathbf{0}.$$

Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, this implies that $\alpha_k - \beta_k = 0$ for all $k = 1, \ldots, n$. Therefore, $\alpha_k = \beta_k$ for all k, and the choice of coefficients is unique.

Problem 2 [5 points]

Solve the following system of linear equations using the method taught in class.

$$\begin{aligned} x_1 + 3\,x_2 - 5\,x_3 &= 4 \\ x_1 + 4\,x_2 - 8\,x_3 &= 7 \\ -3\,x_1 - 7\,x_2 + 9\,x_3 &= -6 \end{aligned}$$

Initial augmented matrix:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix}$$

Step 1: $R_2 \to R_2 - R_1$(Eliminate 1 in R)

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ -3 & -7 & 9 & -6 \end{bmatrix}$$

Step 2: $R_3 \to R_3 + 3R_1$(Eliminate -3 in R)

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix}$$

Step 3: $R_3 \to R_3 - 2R_2$(Eliminate 2 in R)

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 4: $R_1 \to R_1 - 3R_2$ (Make zero in R column 2)

$$\begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{x} = (5, 3, 0)^T + \lambda\,(4, -3, -1)^T$

Problem 3

Find conditions on α such that the following system of linear equations has (a) exactly one solution, (b) no solutions, or (c) an infinite number of solutions.

Write out augmented matrix:

$$\begin{pmatrix} 1 & \alpha & 0 & | & 1 \end{pmatrix} \xrightarrow{R2 \to R1} \begin{pmatrix} 1 & -1 & 3 & | & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 3 & | & -1 \\ 2 & -2 & \alpha & | & -2 \end{pmatrix} \xrightarrow{R1 \to R2 \to R3} \begin{pmatrix} 0 & \alpha+1 & -3 & | & 2 \\ 0 & 0 & \alpha-6 & | & 0 \end{pmatrix}$$

This matrix has 3 pivot numbers when $\alpha = -1$ or $\alpha = 6$, thus rank A = 3 and the system has a unique solution.

Let's look at the exceptional cases:

- When $\alpha = -1$, the last two equations are

$$-3x_3 = 2 \text{ and } -7x_3 = 0$$

$$\Rightarrow \text{ the system is inconsistent}$$

- When $\alpha = 6$, we obtain the augmented matrix

$$\begin{pmatrix} 1 & -1 & 3 & | & -1 \\ 0 & 7 & -3 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow \text{ the solution is not unique.}$$

Problem 4

Solve
$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 2 \\ 2x_1 + 6x_2 \qquad - x_4 &= 1 \end{aligned}$$

Augmented matrix:
$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 0 & -1 & 1 \end{array}\right)$$

$$-2R1+R2 \to R2 \quad \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & -2 & -3 & -3 \end{array}\right)$$

$$\begin{array}{r} \frac{1}{2}R2+R1 \to R1 \\ -\frac{1}{2}R2 \to R2 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$$

To read off the general solution we insert 0-rows:

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \overset{-1}{\cancel{0}} & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & \overset{-1}{\cancel{0}} & 0 \end{array}\right)$$

With the (-1) trick the general solution is

$$x = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ -1 \end{pmatrix} \quad \text{for any } \lambda, \mu \in \mathbb{R}$$