Week 11: LU decomposition

1.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

The matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -5 & 0 & 1 \end{bmatrix}$$

represents an elementary row operation. What is the determinant of T?

(a) $\det(T) = 1$ (100%)
(b) $\det(T) = 0$
(c) $\det(T) = -1$
(d) $\det(T) = -5$

One can check, e.g., by a Laplace expansion, that $\det(T) = 1$. *Also,* T *adds a multiple of one row to another, and we already now this does not change the determinant (so* $\det(T)$ *had to be equal to* 1*).*

2.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

A matrix A admits an LU decomposition if it can be written as $A = LU$, with L lower triangular, U upper triangular. A matrix A admits an LUP decomposition if it can be written as $PA = LU$, with L lower triangular, U upper triangular, and P a matrix that reorders rows. Which of the following is true?

(a) Any square matrix admits an LUP decomposition. (100%)
(b) Any square matrix admits an LU decomposition.
(c) Any Hermitian matrix admits an LU decomposition.
(d) A matrix admits an LU decomposition if and only if it is invertible.

Reordering the rows might be necessary as discussed in class, so any matrix admits an LUP *decomposition, while only some admit an* LU *decomposition.*

3.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the Hermitian matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Is

$$A = LL^*$$

with

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

a valid Cholesky decomposition?

(a) No. (100%)
(b) Yes.

It is not, since $LL^* \neq A$.

4.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

A matrix A admits an LU decomposition if it can be written as $A = LU$, with L lower triangular, U upper triangular. A matrix A admits an LUP decomposition if it can be written as $PA = LU$, with L lower triangular, U upper triangular, and P a matrix that reorders rows. Which of the following is true for the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}?$$

(a) A admits an LUP decomposition, but not an LU decomposition. (100%)
(b) A admits an LU decomposition, but not an LUP decomposition.
(c) A admits neither an LUP decomposition, nor an LU decomposition.
(d) A admits both an LUP decomposition, and an LU decomposition.

One cannot do Gaussian elimination on A to remove the A_{41} entry of A. Hence, there is no LU decomposition. But if one reorders the rows, A can be brought into upper triangular form, and hence an LUP decomposition exists (as it always does for square matrices).

5.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Does the matrix

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 6 & 3 & 2 \\ 0 & 3 & 8 & 5 \\ 0 & 2 & 5 & 2 \end{bmatrix}?$$

have a Cholesky decomposition?

(a) No. (100%)
(b) Yes.

A is not positive definite, e.g., since $\langle x, Ax \rangle = -2$ for $x = (1, 0, 0, 0)^T$. Hence it cannot have a Cholesky decomposition.

6.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrices

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Check whether $S = LU$ is a valid LU decomposition. If the decomposition is valid, then use L and U to compute $\det(S)$.

(a) $S = LU$ and $\det(S) = 1$ (100%)
(b) $S \neq LU$
(c) $S = LU$ and $\det(S) = 2$
(d) $S = LU$ and $\det(S) = -1$

Since L and U are lower and upper triangular respectively, we only need to check whether $LU = S$. Computing LU we see that this is indeed the case. Since $\det(L) = \det(U) = 1$, $\det(S) = \det(L)\det(U) = 1$.

7.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Compute the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

such that all diagonal entries of L are one. What are the diagonal entries of U? What is the entry below the diagonal in L?

(a) U has diagonal entries $1, -2$ and the entry below the diagonal in L is 3. (100%)
(b) U has diagonal entries $-1, -2$ and the entry below the diagonal in L is 2.
(c) U has diagonal entries $-1, 2$ and the entry below the diagonal in L is 1.
(d) LU decomposition is not possible.

By performing an elementary row operation, we can write

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = LU.$$

Hence the diagonal entries of U are $1, -2$ and the entry below the diagonal of L is 3.

8.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Compute the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

such that the diagonal entries of L are 3 and 5. What are the diagonal entries of U? What is the entry below the diagonal in L?

(a) U has diagonal entries $\frac{1}{3}, -\frac{2}{5}$ and the entry below the diagonal in L is 9. (100%)
(b) U has diagonal entries $\frac{1}{5}, -\frac{2}{3}$ and the entry below the diagonal in L is -9.
(c) U has diagonal entries $\frac{1}{5}, \frac{2}{3}$ and the entry below the diagonal in L is 9.
(d) LU decomposition is not possible.

We find

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{2}{5} \end{bmatrix} = LU.$$

9.

MULTI 1.0 point 0.10 penalty Single Shuffle

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ -4 & 0 & 9 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

Perform an LU decomposition on A to obtain lower and upper triangular matrices L and U such that $A = LU$ and L has ones on the diagonal. What are the diagonal elements of U? What are the entries in the diagonal below the main diagonal of L?

(a) The diagonal of U has $2, 1, 3, -3$ and the entries below the diagonal of L are $0, 0, 1$. (100%)
(b) The diagonal of U has $2, 1, 3, 3$ and the entries below the diagonal of L are $0, 0, 0$.
(c) The diagonal of U has $-2, -1, -3, -3$ and the entries below the diagonal of L are $-2, -1, 0$.
(d) LU decomposition is not possible.

Using elementary row operations, we can write

$$A = \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ -4 & 0 & 9 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix} = U$$

Hence the elementary transformations are given by the composition

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

10.

MULTI 1.0 point 0.10 penalty Single Shuffle

Consider the system of equations

$$\begin{aligned} 6x + 18y + 3z &= 3 \\ 2x + 12y + z &= 19 \\ 4x + 15y + 3z &= 0. \end{aligned}$$

Find an LU decomposition of the associated matrix and then solve the system. What is the determinant of the associated matrix? What is the value of xyz?

(a) $\det = 36$, $xyz = 99$ (100%)
(b) $\det = -18$, $xyz = 33$
(c) $\det = 36$, $xyz = -33$
(d) $\det = 18$, $xyz = 99$

Let M be the associated matrix. One possible LU decomposition of M is

$$M = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $\det(M) = 36$ *(product of all diagonal entries of L and U). Now let $X = (x, y, z)$, $V = (3, 19, 0)$ and $W = (u, v, w) = UX$. Solving the equation $LW = V$ we have*

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}.$$

By substitution, we see that $u = 1, v = 3$ and $w = -11$. Then let $W_0 = (1, 3, -11)$. Finally solving $UX = W_0$, we get

$$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -11 \end{bmatrix}.$$

Back-substituting, we obtain $z = -11$, $y = 3$, $x = -3$. Hence $xyz = 99$.

Total of marks: 10