Week 3: Vector Spaces, Linear Maps, Matrices

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Does the set of all positive reals together with the following addition and multiplication by scalar $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ form a vector space over $\mathbb{R}$ (with the scalars $c \in \mathbb{R}$)?

$$v_1 \tilde{+} v_2 \overset{def}{=} v_1 \cdot v_2; \;\; c \tilde{\cdot} v_2 \overset{def}{=} c \cdot v_2$$

(a) $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ is not a vector space over $\mathbb{R}$ (100%)
(b) $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ is a vector space over $\mathbb{R}$

The space is not closed under multiplication by a scalar, note e.g. that:

$$-1 \tilde{\cdot} 1 = -1 \notin \mathbb{R}_+$$

Thus, $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ *is not a vector space over* $\mathbb{R}$

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Is $\mathbb{Z}$, the set of all integers, a field?

(a) No. (100%)
(b) Yes.

$\mathbb{Z}$ *is not a field, since multiplicative inverses do not exist (except for* 1 *and* -1*).*

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find a basis for $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| 7x + 2y - 5z = 0 \right\} \subset \mathbb{R}^3$.

(a) $\begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$ (100%)
(b) $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$
(c) $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$
(d) $\begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \\ 14 \end{bmatrix}$

The plane is spanned by vectors perpendicular to the normal vector $\vec{n} = \begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix}$

For a basis take a vector, perpendicular to $\vec{n}$*, e.g.* $\vec{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$*, and add another one which is not proportional to the first one (to have 2 linearly independent vectors which form a basis of a plane), e.g.* $\vec{v}_2 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find a basis for $\left\{ \begin{bmatrix} 3a \\ -7a \\ 11a \end{bmatrix} \in \mathbb{R}^3 \middle| a \in \mathbb{R} \right\} \subset \mathbb{R}^3$.

(a) $\begin{bmatrix} 15 \\ -35 \\ 55 \end{bmatrix}$ (100%)

(b) $\begin{bmatrix} 51 \\ -118 \\ 187 \end{bmatrix}$

(c) $\begin{bmatrix} -3 \\ -7 \\ 11 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}$

A line is spanned by one vector (such that any point on the line is proportional to the basis vector), thus we can take e.g. $\begin{bmatrix} 3 \\ -7 \\ 11 \end{bmatrix}$ *to be a basis vector, or any other multiple of it, e.g.* $\begin{bmatrix} 3 \\ -7 \\ 11 \end{bmatrix} \cdot 5 = \begin{bmatrix} 15 \\ -35 \\ 55 \end{bmatrix}$

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following is not a basis for the space of all cubic polynomials $P_3(\mathbb{R})$?

(a) $\mathfrak{B} = \{x^3 - x^2,\ x^3 - x,\ x^2 - x,\ x^3 - 1\}$ (100%)
(b) $\mathfrak{B} = \{x^3,\ x^2,\ x,\ 1\}$
(c) $\mathfrak{B} = \{x^3 - x^2,\ x^2 - x,\ x - 1,\ 1\}$
(d) $\mathfrak{B} = \{x^3 + x^2 + x + 1,\ (x - 6)^2,\ x - 10,\ 1\}$

We can see: $(-1)(x^3-x^2)+(1)(x^3-x)+(-1)(x^2-x) = -x^3+x^2+x^3-x-x^2+x = 0$ *This means that three of the basis vectors are linearly dependent, and thus cannot be a basis.*

6.

MULTI 1.0 point 0 penalty Single Shuffle

Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto f(x_1, x_2)$ is linear (in the sense of linear maps as defined in class)?

(a) $f(x_1, x_2) = 5x_1$ (100%)
(b) $f(x_1, x_2) = 7x_1x_2$
(c) $f(x_1, x_2) = \sin(x_1) + \sin(x_2)$
(d) $f(x_1, x_2) = (x_1)^3 + 6(x_2)^4$

For $f(x_1, x_2) = 5x_1$ *we find* $f(\lambda x_1 + \tilde{x}_1, \lambda x_2 + \tilde{x}_2) = \lambda 5x_1 + 5\tilde{x}_1 = \lambda f(x_1, x_2) + f(\tilde{x}_1, \tilde{x}_2)$*, whereas this does not hold for the other functions.*

7.

MULTI 1.0 point 0 penalty Single Shuffle

Calculate the matrix product:

$$\begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = ?$$

(a) $\begin{bmatrix} -6 & 10 & 8 \\ 2 & 6 & 4 \\ 5 & 9 & 7 \end{bmatrix}$ (100%)
(b) $\begin{bmatrix} -6 & 9 & 8 \\ 2 & 6 & 4 \\ 5 & 9 & 8 \end{bmatrix}$
(c) $\begin{bmatrix} -6 & 10 & 8 \\ 2 & 5 & 4 \\ 5 & 9 & 7 \end{bmatrix}$
(d) $\begin{bmatrix} -6 & 9 & 8 \\ 2 & 5 & 4 \\ 5 & 9 & 8 \end{bmatrix}$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Let

$$\mathcal{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Which is the inverse of $\mathcal{R}$? (The inverse of $\mathcal{R}$ is the matrix $\mathcal{R}^{-1}$ such that $\mathcal{R}^{-1}\mathcal{R} = 1$ (the identity matrix with 1's on the diagonal, and 0's everywhere else).

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (100%)

(b) $\begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$
(c) $\begin{bmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$
(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Recall that $A^{-1} \cdot A = \mathrm{id}$ *with* id *being the identity matrix. We have that*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} C = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

Calculate $A \cdot B \cdot C$

(a) $\begin{bmatrix} 12 & 18 \\ 12 & 18 \end{bmatrix}$ (100%)
(b) $\begin{bmatrix} 24 & 24 \\ 2 & 6 \end{bmatrix}$
(c) $\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$
(d) $\begin{bmatrix} 6 & 6 \\ 12 & 18 \end{bmatrix}$

$$A \cdot B \cdot C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 6 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} 12 & 18 \\ 12 & 18 \end{bmatrix}$$

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} B = \begin{bmatrix} 99 & 0 \\ 99 & 99 \\ 99 & 0 \\ 99 & 99 \end{bmatrix} C = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

Which of the following is a valid matrix multiplication?

(a) $B \cdot A^T \cdot C$ (100%)
(b) $A \cdot B \cdot C$
(c) $A^T \cdot B^T \cdot C$
(d) $C^T \cdot B^T \cdot A^T$

We can simply analyse the dimensions of the matrices. Then $B \cdot A^T \cdot C$ reads

$$(5 \times 2) \cdot (3 \times 2)^T \cdot (3 \times 1) = (5 \times 2) \cdot (2 \times 3) \cdot (3 \times 1)$$

And thus the dimensions match and the output is a (5×1) matrix

Total of marks: 10