Constructor University
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Elements of Linear Algebra

Homework 3 (covering Weeks 5 and 6)

Problem 1 [4 points]

Using your knowledge about the rank of a matrix, prove that if $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are both bases of a vector space V, then $n = m$.

Proof. Let V be a vector space with two bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$. Since both are bases, we can express each w_j as a linear combination of the v_i's:

$$w_j = \sum_{i=1}^{n} a_{ij} v_i \quad \text{for } j = 1, \ldots, m$$

Let $A = (a_{ij})$ be the $n \times m$ matrix of coefficients. Similarly, we can express each v_i as a linear combination of the w_j's:

$$v_i = \sum_{j=1}^{m} b_{ij} w_j \quad \text{for } i = 1, \ldots, n$$

Let $B = (b_{ij})$ be the $m \times n$ matrix of these coefficients.

Substituting the expression for each w_j into the equation for v_i:

$$v_i = \sum_{j=1}^{m} b_{ij} \left(\sum_{k=1}^{n} a_{kj} v_k \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{m} b_{ij} a_{kj} \right) v_k$$

This means that $BA = I_n$, where I_n is the $n \times n$ identity matrix. Similarly, substituting the expression for each v_i into the equation for w_j yields $AB = I_m$.

Therefore:

- $\text{rank}(A) = n$ (since $BA = I_n$ implies A has full row rank)
- $\text{rank}(A) = m$ (since $AB = I_m$ implies A has full column rank)

Thus, $n = \text{rank}(A) = m$.

The key insight is that the matrix A must be both full row rank and full column rank, which is only possible when the number of rows equals the number of columns, proving that $n = m$. □

Problem 2 [4 points]

Consider two vectors in $\mathbb{R}^3$,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In the standard basis

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we can write the vectors as $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$ and $\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2 + y_3\vec{e}_3$. Now compute the determinant of the matrix

$$\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

(Note that our notation is a bit symbolic here, since we have put vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as matrix entries; but that should not bother us.) The result should be familiar to you. Where have you encountered the resulting expression before?

Solution:

$$\begin{aligned}
\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} &= \vec{e}_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \vec{e}_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \vec{e}_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \\
&= \vec{e}_1(x_2y_3 - x_3y_2) - \vec{e}_2(x_1y_3 - x_3y_1) + \vec{e}_3(x_1y_2 - x_2y_1) \\
&= \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}
\end{aligned}$$

This expression corresponds to the cross product $\vec{x} \times \vec{y}$.

Problem 3 [4 points]

Consider the vector space $\mathcal{P}_n$ of polynomials of degree at most n and consider the linear map $D : \mathcal{P}_n \to \mathcal{P}_n$ that maps each polynomial to its derivative. Compute the matrix of D in the basis $\{1, x, x^2, \ldots, x^n\}$ and find its determinant.

Solution: It suffices to check the action of D on the basis elements. The power rule of differentiation tells us that $D(x^k) = kx^{k-1}$, hence as a matrix

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & n \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

Since the matrix is upper triangular and all diagonal entries are zero, $\det(D) = 0$.

Problem 4 [4 points]

Let A be an invertible $n \times n$ matrix. If we know that both A and A^{-1} only have integer entries, what are possible values of $\det(A)$? *Hint: Consider how* $\det(A)$ *can be written as a function of entries of* A.

Solution: The determinant is a polynomial in the entries of the matrix. Hence as A and A^{-1} have integer entries $\det(A), \det(A^{-1}) \in \mathbb{Z}$. Since we must have

$$\det(A) \cdot \det(A^{-1}) = \det(I) = 1,$$

we conclude that $\det(A) = \det(A^{-1}) = \pm 1$.

Problem 5 [4 points]

Let A be an $n \times n$ matrix with entries $a_{i,j}$ and consider the matrix B with entries $b_{i,j} = \frac{ia_{i,j}}{j}$. What is $\det(B)$ in terms of $\det(A)$?

Solution: Note that B looks like

$$B = \begin{bmatrix} a_{1,1} & \frac{1}{2}a_{1,2} & \cdots & \frac{1}{n}a_{1,n} \\ 2a_{2,1} & a_{2,2} & \cdots & \frac{2}{n}a_{2,n} \\ \vdots & \cdots & \cdots & \vdots \\ na_{n,1} & \frac{n}{2}a_{1,2} & \cdots & a_{n,n} \end{bmatrix}.$$

Now using linearity of the determinant we can first multiply the j-th column by j and then multiply the i-th row by $(1/i)$. Repeatedly doing this gives us

$$\det(B) = \begin{vmatrix} a_{1,1} & \frac{1}{2}a_{1,2} & \cdots & \frac{1}{n}a_{1,n} \\ 2a_{2,1} & a_{2,2} & \cdots & \frac{2}{n}a_{2,n} \\ \vdots & \cdots & \cdots & \vdots \\ na_{n,1} & \frac{n}{2}a_{1,2} & \cdots & a_{n,n} \end{vmatrix} = 1 \cdot 2 \cdots n \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 2a_{2,1} & 2a_{2,2} & \cdots & 2a_{2,n} \\ \vdots & \cdots & \cdots & \vdots \\ na_{n,1} & na_{1,2} & \cdots & na_{n,n} \end{vmatrix}$$

$$= n! \cdot \frac{1}{1 \cdot 2 \cdots n} \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n,1} & a_{1,2} & \cdots & a_{n,n} \end{vmatrix} = \frac{n!}{n!} \det(A) = \det(A).$$