Week 9: Diagonalization and Normal Matrices

1.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Is the matrix

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

diagonalizable?

(a) A is diagonalizable (100%)
(b) A is not diagonalizable
(c) Information given is insufficient

We first compute the characteristic polynomial of A.

$$\det(A - \lambda I) = (6 - \lambda)(3 - \lambda) + 2 = \lambda^2 - 9\lambda + 20 = (\lambda - 5)(\lambda - 4).$$

Since the eigenvalues of A are distinct, A is diagonalizable.

2.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Is the matrix

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

diagonalizable?

(a) A is diagonalizable (100%)
(b) A is not diagonalizable
(c) Information given is insufficient

We first compute the characteristic polynomial of A:

$$\begin{aligned}\det(A - \lambda I) &= -(2 + \lambda)[(1 - \lambda)(5 - \lambda) - 4] + 2[-4(5 - \lambda) - 4] + 4[-8 - 2(1 - \lambda)] \\ &= (\lambda - 6)(\lambda - 3)(\lambda + 5).\end{aligned}$$

Since the eigenvalues of A are distinct, A is diagonalizable.

3.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let $0 < \mu < 1$ and suppose an $n \times n$ matrix B has characteristic equation

$$\lambda(\lambda - \mu)(\lambda - \mu^2) \cdots (\lambda - \mu^{n-1}).$$

Is B diagonalizable?

(a) B is diagonalizable (100%)
(b) B is not diagonalizable
(c) Information given is insufficient

The condition $0 < \mu < 1$ ensures that all μ^k are distinct and non-zero. Hence all eigenvalues of B are distinct and B is diagonalizable.

4.

MULTI 1.0 point 0.10 penalty Single Shuffle

Consider the vector space $\mathcal{P}_n$ of polynomials of degree n and consider the linear map $D : \mathcal{P}_n \to \mathcal{P}_n$ that maps each polynomial to its derivative. First compute the matrix of D in the standard basis $\{1, x, x^2, \ldots, x^n\}$ (you can refer to a previous exercise here). If $n > 0$, is D diagonalizable?

(a) D is diagonalizable
(b) D is not diagonalizable (100%)
(c) Information given is insufficient

Using the solution from a previous exercise, D has the form

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & n \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Clearly all eigenvalues are 0. Moreover it is easy to see that the eigenspace of 0 is spanned by $(1, 0, \ldots, 0)$. Hence for $n > 0$, D is not diagonalizable.

5.

MULTI 1.0 point 0.10 penalty Single Shuffle

Is the matrix

$$B = \begin{bmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{bmatrix}$$

diagonalizable?

(a) B is diagonalizable (100%)
(b) B is not diagonalizable
(c) Information given is insufficient

Observe that $B = B^\dagger$. Hence, $BB^\dagger = B^\dagger B$ and B is normal. Since every normal matrix is diagonalizable, B is diagonalizable.

6.

MULTI 1.0 point 0.10 penalty Single Shuffle

Find all values of k that make the matrix

$$A = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & k \\ 0 & 0 & 2 \end{bmatrix}$$

diagonalizable.

(a) A is diagonalizable only when $k = 0$. (100%)
(b) A is diagonalizable only when $k \neq 0$.
(c) A is diagonalizable only when $|k| \neq 1$.
(d) A is diagonalizable only when $|k| = 1$.

Since the matrix is upper triangular, we can easily compute its characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)^2(2-\lambda).$$

Since 1 is the repeated eigenvalue, we need to ensure that the corresponding eigenspace is 2 dimensional. That is we want two linearly independent vectors $v = (x, y, z)$ satisfying

$$0 = (A - I)v = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ky \\ kz \\ z \end{bmatrix}.$$

If $k = 0$, then we see that, e.g., $(1, 1, 0)$ and $(1, 0, 0)$ are two linearly independent eigenvectors and the eigenspace of 1 is two dimensional. However if $k \neq 0$ then the equation above implies that $z = 0$ and $ky = 0 \implies y = 0$ so the eigenspace of 1 is just one dimensional. Hence A is diagonalizable only when $k = 0$.

7.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the 2×2 matrix

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}.$$

Find a formula for A^k.

(a) $\begin{bmatrix} 2 \cdot 5^k - 4^k & -5^k + 4^k \\ 2 \cdot 5^k - 2 \cdot 4^k & -5^k + 2 \cdot 4^k \end{bmatrix}$. (100%)
(b) $\begin{bmatrix} 2 \cdot 5^k - 4^k & 5^k + 4^k \\ 2 \cdot 5^k - 4^k & -5^k + 2 \cdot 4^k \end{bmatrix}$.
(c) $\begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We first find the eigenvalues of A.

$$\det(A-\lambda I) = (6-\lambda)(3-\lambda)+2 = \lambda^2 - 9\lambda + 20 = (\lambda-5)(\lambda-4).$$

Hence A is diagonalizable so set

$$\Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$

Now the eigenvectors of A are

$$\lambda = 5 : (A-5I)v = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} v \implies v = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\lambda = 4 : (A-4I)v = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} v \implies v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence setting

$$V = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

We see that $A = V\Lambda V^{-1}$ and therefore

$$A^k = V\Lambda^k V^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 5^k - 4^k & -5^k + 4^k \\ 2\cdot 5^k - 2\cdot 4^k & -5^k + 2\cdot 4^k \end{bmatrix}.$$

8.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Suppose A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that for all k, $|\lambda_k| < 1$. Compute the limits $\lim_{p\to\infty} A^p$ and $\lim_{p\to\infty} \exp(A^p)$.

(a) $\lim_{p\to\infty} A^p = 0$, $\lim_{p\to\infty} \exp(A^p) = I$ (100%)
(b) $\lim_{p\to\infty} A^p = I$, $\lim_{p\to\infty} \exp(A^p) = 0$
(c) $\lim_{p\to\infty} A^p = I$, $\lim_{p\to\infty} \exp(A^p) = I$
(d) $\lim_{p\to\infty} A^p = 0$, $\lim_{p\to\infty} \exp(A^p) = 0$

Since A has distinct eigenvalues there exists a matrix V such that $A = V\Lambda V^{-1}$ where Λ is the diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

Since $|\lambda_k| < 1$, $\lim_{p\to\infty} \lambda_k^p = 0$. Therefore

$$\lim_{p\to\infty} A^p = \lim_{p\to\infty} V\Lambda^p V^{-1} = V \begin{bmatrix} \lim_{p\to\infty} \lambda_1^p & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lim_{p\to\infty} \lambda_n^p \end{bmatrix} V^{-1} = V0V^{-1} = 0,$$

$$\lim_{p\to\infty} \exp(A^p) = V \begin{bmatrix} \lim_{p\to\infty} \exp(\lambda_1^p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lim_{p\to\infty} \exp(\lambda_n^p) \end{bmatrix} V^{-1} = VIV^{-1} = I.$$

9.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Recall that when x, y are real numbers, we have the identity $\exp(x)\exp(y) = \exp(x+y)$. In this exercise we will investigate whether this holds for matrices. Consider the matrices

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and } Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Compute $\exp(X)\exp(Y)$ and $\exp(X+Y)$.

(a) $\exp(X)\exp(Y) = \begin{bmatrix} e & e \\ 0 & e^2 \end{bmatrix}, \exp(X+Y) = \begin{bmatrix} e & e^2 - e \\ 0 & e^2 \end{bmatrix}$ (100%)

(b) $\exp(X)\exp(Y) = \exp(X+Y) = \begin{bmatrix} e & e^2 - e \\ 0 & e^2 \end{bmatrix}$

(c) $\exp(X)\exp(Y)\exp(X+Y) = \begin{bmatrix} e & e \\ 0 & e^2 \end{bmatrix}$

(d) The matrix is exponential is undefined for $\exp(Y)$.

The exponential of X can be computed directly as it is in diagonal form. So

$$\exp(X) = \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}.$$

Even though Y is not diagonalizable, we have $Y^k = 0$ for all $k > 2$. Therefore we can compute $\exp(Y)$ using the series formula for the exponential. Hence we can compute

$$\exp(Y) = \sum \frac{Y^k}{k!} = Y^0 + Y^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

to get

$$\exp(X) \cdot \exp(Y) = \begin{bmatrix} e & e \\ 0 & e^2 \end{bmatrix}.$$

We now compute the eigenvalues and eigenvectors of $X + Y$.

$$\det(X + Y - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ 0 & 2 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda)$$

so $X + Y$ is diagonalizable. We also see that the eigenvectors corresponding to 2 and 1 are $(1, 1)$ and $(1, 0)$ respectively and $X + Y = V\Lambda V^{-1}$ where

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Hence

$$\exp(X + Y) = V \cdot \begin{bmatrix} e^2 & 0 \\ 0 & e \end{bmatrix} \cdot V^{-1} = \begin{bmatrix} e & e^2 - e \\ 0 & e^2 \end{bmatrix}.$$

Note. *In case of matrices, $\exp(X)\cdot\exp(Y)$ is given by the Baker-Campbell-Hausdorff formula. You can read more about it in the relevant Wikipedia page.*

10.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 1 & -2 - i & 4\exp(i\frac{\pi}{4}) \\ 1 + i & i & 2 - 7i \end{bmatrix}$$

What is $A^\dagger$?

(a) $\begin{bmatrix} 1 & 1 - i \\ -2 + i & -i \\ 4\exp(-i\frac{\pi}{4}) & 2 + 7i \end{bmatrix}$ (100%)

(b) $\begin{bmatrix} 1 & -2 + i & 4\exp(-i\frac{\pi}{4}) \\ 1 - i & -i & 2 + 7i \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 + i \\ -2 - i & i \\ 4\exp(i\frac{\pi}{4}) & 2 - 7i \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2-i & 4\exp(i\frac{\pi}{4}) \\ 1+i & i & 2-7i \end{bmatrix}$

$$\boxed{A^\dagger = \overline{A}^\top = \begin{bmatrix} 1 & -2+i & 4\exp(-i\frac{\pi}{4}) \\ 1-i & -i & 2+7i \end{bmatrix}^\top = \begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 4\exp(-i\frac{\pi}{4}) & 2+7i \end{bmatrix}}$$

Total of marks: 10