Constructor University
Spring 2025

April 8, 2025

# Elements of Calculus
# Mock Midterm Exam

**This is a mock midterm only. It covers the material from Weeks 1-8. The results of this test exam can be used to replace your worst homework sheet in the bonus point calculation.**

**Instructions:**

- The exam has 12 multiple choice questions (several answers can be correct!) and 1 longer question.
- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.) There are no negative points, but of course there are fewer points if wrong answers are selected, or if right answers are not selected.
- For the longer exercise 13, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

**Name:** ________________________________________

**Matric./Student No.:** ___________________________

1. **(6 points)** Consider the function

$$f_\lambda(x) = x^2 + 2\lambda x - \lambda,$$

with parameter $\lambda \in \mathbb{R}$. Which of the following is true?

A. * For $\lambda < -1$, the equation $f_\lambda(x) = 0$ has two real solutions.

B. For $\lambda > 0$, the equation $f_\lambda(x) = 0$ has no real solution.

C. For $\lambda = 2$, the equation $f_\lambda(x) = 0$ has exactly one real solution.

D. * The range of $f_\lambda(x)$ is the interval $[-2\lambda^2 - \lambda, \infty)$.

E. The range of $f_\lambda(x)$ is the interval $[0, \infty)$.

F. The domain of $f_\lambda(x)$ is the interval $[-1, \infty)$.

2. **(6 points)** Which of the following statements about limits are true?

A. $\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = \sqrt{5}$.

B. * $\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = 1$.

C. * $\lim_{x\to\infty} \frac{x^6}{e^x} = 0$.

D. * $\lim_{h\to 0} \frac{h^2-h}{h} = -1$.

E. For any differentiable function $f$, we have that $\lim_{h\to 0} \frac{f(x+h)}{h} = f'(x)$, where $f'(x)$ is the derivative of $f$.

F. $\lim_{x\to\infty} \frac{x^6}{\ln(x)} = 0$.

3. **(6 points)** Which of the following statements are true?

A. * The exponential function $e^x$ is continuous.

B. The exponential function $e^x$ is NOT continuous.

C. * If the function $f : [a, b] \to \mathbb{R}$ is continuous, then it assumes its minimum and maximum.

D. * A function $f : (a, b) \to \mathbb{R}$ is continuous at $x_0 \in (a, b)$ if $\lim_{x\to x_0} f(x) = f(x_0)$.

E. If the function $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $y \in (a, b)$ such that $f(y) = 0$.

F. A function $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ if $\lim_{x\to x_0} f(x) = f(x_0)$.

4. **(4 points)** Compute the derivative $f'(x)$ of the function

$$f(x) = x^3 e^x.$$

A. $f'(x) = 3x^2 e^x$.
B. $f'(x) = 3x^2 e^x + x^3 \ln(x)$.
C. $f'(x) = \left(\frac{x^4}{4} + x^3\right) e^x$.
D. $f'(x) = 3x^2 e^x + x^3 e^{-x}$.
E. $f'(x) = (x^3 + 3x) e^x$.
F. * $f'(x) = (x^3 + 3x^2) e^x$.

5. **(6 points)** Consider the function

$$f(x) = \ln(1 + e^x).$$

Which of the following is true?

A. The derivative of $f(x)$ is $f'(x) = e^x \ln(1 + e^x)$.
B. * The derivative of $f(x)$ is $f'(x) = \frac{e^x}{1+e^x}$.
C. The derivative of $f(x)$ is $f'(x) = \frac{1}{(1+e^x)^2}$.
D. * The domain of $f$ is $\mathbb{R}$.
E. The domain of $f$ is $(0, \infty)$.
F. The domain of $f$ is $(-1, 1)$.

6. **(4 points)** Compute the (infinite) Taylor series of $e^x$ around $x = 0$.

A. $\sum_{k=0}^{\infty} \frac{kx^k}{k!}$.
B. $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!}$.
C. * $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.
D. $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k-1)!}$.
E. $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k-1)!}$.
F. $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k-1)!}$.

7. **(4 points)** Evaluate the indefinite integral

$$\int \frac{\cos(\pi/x)}{x^2}\,dx.$$

*Hint*: Substitute $\frac{\pi}{x}$.

A. $\frac{1}{\pi}\sin\frac{1}{x} + C$.

B. $\sin\frac{\pi}{x} + C$.

C. $\pi$.

D. $\cos(\pi/x) + C$.

E. * $-\frac{1}{\pi}\sin\frac{\pi}{x} + C$

F. $-\frac{1}{\pi}\sin \pi x + C$.

8. **(4 points)** Evaluate the indefinite integral

$$\int (\sin x)\ln(\cos x)\,dx.$$

*Hint*: Integration by parts.

A. $(\cos x)(1 + \ln\cos x) + C$.

B. $(\cos x)(-1 - \ln\cos x) + C$.

C. * $(\cos x)(1 - \ln\cos x) + C$.

D. $(\cos x)(-1 + \ln\cos x) + C$.

E. $(\sin x)\ln(\cos x) + C$.

F. $\frac{\pi}{2}$.

9. **(4 points)** Calculate

$$\int_0^{1/2} \frac{2x^2+2}{x^2-1}\,dx.$$

A. $2\ln(3)$.

B. $-1$.

C. $2\ln(2) - 1$.

D. $2\ln(2) - 2$.

E. $x^2 + 1$.

F. * $1 - 2\ln(3)$.

10. **(4 points)** Compute the area between the curves of

$$f(x) = 6x - 2x^2$$

and

$$g(x) = x^2.$$

A. 1.
B. 2.
C. 3.
D. * 4.
E. 5.
F. 6.

11. **(6 points)** Consider the improper integral

$$\int_1^\infty \frac{1}{x^\alpha} \, \mathrm{d}x$$

for different parameters $\alpha \in \mathbb{R}$.

A. * For $\alpha = 1$ the improper integral is infinite.
B. For $\alpha = 1$ the improper integral is finite and its value is 1.
C. For $\alpha = 1$ the improper integral is finite and its value is $\ln(1 + \alpha)$.
D. For $\alpha > 1$ the improper integral is infinite.
E. * For $\alpha > 1$ the improper integral is finite and its value is $\frac{1}{\alpha - 1}$.
F. For $\alpha > 1$ the improper integral is finite and its value is $\ln \alpha$.

12. **(4 points)** Solve the ordinary differential equation $\frac{dy}{dt} = -2yt^2$ with initial condition $y(0) = 2$.

A. $y(t) = 2e^{-t^3}$.
B. $y(t) = 3e^{-t^3} - 1$.
C. $y(t) = e^{-\frac{t^3}{3}} + 1$.
D. * $y(t) = 2e^{-\frac{2t^3}{3}}$.
E. $y(t) = e^{-\frac{2t^3}{3}} + 1$.
F. $y(t) = 2e^{-2t^3}$.

13. **(25 points)**

We consider the function
$$f(x) = \frac{\ln(x)}{x}.$$

(a) (2 points) What is the domain of the function?

(b) (3 points) What are the intercepts with the $x$-axis and with the $y$-axis?

(c) (2 points) What are the horizontal asymptotes?

(d) (2 points) What are the vertical asymptotes?

(e) (6 points) Compute and analyze the first derivative. In which intervals is the function increasing or decreasing, what are the local minima or maxima?

(f) (5 points) Compute and analyze the second derivative. In which intervals is the function concave up or concave down, what are the points of inflection?

(g) (5 points) Sketch the function. Your drawing needs to include all the qualitative features of the graph discussed in the questions above.