Week 1: Basic Calculus Review

1.

MULTI 1.0 point 0 penalty Single Shuffle

Find the (complex) roots of the polynomial

$$p(x) = x^2 + 4\,x + 13$$

(a) $x_1 = -2 - 3i$, $x_2 = -2 + 3i$ (100%)
(b) $x_1 = -3 + 2i$, $x_2 = -3 - 2i$
(c) $x_1 = +2 + 3i$, $x_2 = +2 - 3i$
(d) $x_1 = +3 - 2i$, $x_2 = +3 + 2i$

Quadratic formula:

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i$$

2.

MULTI 1.0 point 0 penalty Single Shuffle

Let $p(x)$ be a polynomial of degree n with **arbitrary complex coefficients**. Which of the following is true?

(a) $p(x)$ has exactly n roots (considering multiplicities) (100%)
(b) If z is a root, then its complex conjugate $\overline{z}$ is also a root
(c) If $p(x) = c(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$ with $\alpha_1, ..., \alpha_n \in \mathbb{R}$, then the roots of $p(x)$ can be real and also imaginary.
(d) $p(x)$ can have no roots

Consider the Fundamental Theorem of Algebra: "Any polynomial of degree n with complex coefficients is the product of n linear factors." (These factors being the roots.) And roots come in complex conjugate pairs only if the coefficients are real.

3.

MULTI 1.0 point 0 penalty Single Shuffle

Find all the values of the parameter λ for which the equation

$$2\,x^2 - \lambda\,x + \lambda = 0$$

has no real solutions.

(a) $\lambda \in (0, 8)$ (100%)
(b) $\lambda \in (-\infty, 0) \cup (8, \infty)$
(c) $\lambda \in \{0, 8\}$
(d) $\lambda \in (-8, 0)$

Check the so-called discriminant: $\Delta = \lambda^2 - 4 \cdot 2 \cdot \lambda = \lambda(\lambda - 8)$

- $\Delta = 0$ *for* $\lambda = 0$ *or* $\lambda = 8$ *(1 real solution)*
- $\Delta > 0$ *for* $\lambda > 8$ *or* $\lambda < 0$ *(2 real solutions)*
- $\Delta < 0$ *for* $\lambda \in (0, 8)$ *(pair of complex-conjugate roots)*

4.

MULTI | 1.0 point | 0 penalty | Multiple | Shuffle

The number $5.21\overline{37}$ is:

(a) a rational number (50%)
(b) a natural number (−50%)
(c) an integer (−50%)
(d) a real number (50%)

Let $x = 0.3737... \implies 99x = 100x - x = 37.37... - 0.37... = 37 \implies x = \frac{37}{99}$
Now note that:
$5.21\overline{37} = 5 + 0.2 + 0.01 + 0.00\overline{37} = 5 + \frac{2}{10} + \frac{1}{100} + \frac{1}{100}\frac{37}{99} = 5\frac{529}{2475}$
which is indeed a fraction.

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Assuming that $z = a + bi$ is a complex number, compute real and imaginary part of $\frac{1}{z^2}$

(a) $\mathrm{Re}\left(\frac{1}{z^2}\right) = \frac{a^2 - b^2}{(a^2+b^2)^2}, \mathrm{Im}\left(\frac{1}{z^2}\right) = \frac{-2ab}{(a^2+b^2)^2}$ (100%)
(b) $\mathrm{Re}\left(\frac{1}{z^2}\right) = \frac{a^2 + b^2}{(a^2+b^2)^2}, \mathrm{Im}\left(\frac{1}{z^2}\right) = \frac{-2ab}{(a^2+b^2)^2}$
(c) $\mathrm{Re}\left(\frac{1}{z^2}\right) = \frac{a^2 - b^2}{(a^2-b^2)^2}, \mathrm{Im}\left(\frac{1}{z^2}\right) = \frac{2ab}{(a^2-b^2)^2}$
(d) $\mathrm{Re}\left(\frac{1}{z^2}\right) = \frac{a^2 + b^2}{(a^2+b^2)^2}, \mathrm{Im}\left(\frac{1}{z^2}\right) = \frac{2ab}{(a^2+b^2)^2}$

$$\frac{1}{z^2} = \frac{1}{a^2 + 2abi - b^2} = \frac{a^2 - 2abi - b^2}{(a^2 + 2abi - b^2)(a^2 - 2abi - b^2)} = \frac{a^2 - b^2}{(a^2+b^2)^2} + i\frac{(-2ab)}{(a^2+b^2)^2}$$

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $p(x)$ be a polynomial of degree n with **real** coefficients. Which of the following is true?

(a) If z is a root, then its complex conjugate is z^* is also a root (100%)
(b) $p(x)$ has n distinct real roots

(c) If $p(x)$ is odd, it can have no roots
(d) $p(x)$ can have less than n complex roots

As discussed in class.
It can be easily verified that $p(z^*) = (p(z))^*$ if the coefficients are real. Then $p(z) = 0 \implies (p(z))^* = 0 \implies p(z^*) = 0$,

7.

MULTI 1.0 point 0 penalty Single Shuffle

Compute $\left|\frac{1+i}{2-i}\right|$.

(a) $\left|\frac{1+i}{2-i}\right| = \sqrt{\frac{2}{5}}$ (100%)
(b) $\left|\frac{1+i}{2-i}\right| = \sqrt{\frac{2}{3}}$
(c) $\left|\frac{1+i}{2-i}\right| = \frac{2}{5}$
(d) $\left|\frac{1+i}{2-i}\right| = \frac{2}{3}$

$$\left|\frac{1+i}{2-i}\right|^2 = \frac{(1+i)(1-i)}{(2-i)(2+i)} = \frac{1+1}{4+1} = \frac{2}{5} \Rightarrow \left|\frac{1+i}{2-i}\right| = \sqrt{\frac{2}{5}}$$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Which of the following does not describe the rational numbers $\mathbb{Q}$?

(a) $\mathbb{Q} = \left\{\frac{n}{m} \mid n, m \in \mathbb{N}\right\}$ (100%)
(b) $\mathbb{Q} = \left\{\frac{n}{m} \mid n, m \in \mathbb{Z} \text{ and } m \neq 0\right\}$
(c) $\mathbb{Q} = \left\{\frac{n}{m} \mid n, m \in \mathbb{N}\right\} \cup \left\{\frac{-n}{m} \mid n, m \in \mathbb{N}\right\} \cup \{0\}$
(d) $\mathbb{Q} = \left\{\frac{n}{m} \mid n \in \mathbb{Z} \text{ and } m \in \mathbb{N}\right\}$

This set would only describe the positive rational numbers.

9.

MULTI 1.0 point 0 penalty Single Shuffle

Let $g(x) = x^2 + 1$. Determine the domain and range of $g(x)$.

(a) $Domain(g) = (-\infty, \infty)$, $Range(g) = [1, \infty)$ (100%)
(b) $Domain(g) = [0, \infty)$ $Range(g) = (-\infty, \infty)$
(c) $Domain(g) = (-\infty, \infty)$, $Range(g) = [0, \infty)$
(d) $Domain(g) = [0, \infty)$, $Range(g) = (-\infty, \infty)$

x^2+1 is well defined for all real numbers. It has a minimum at $x = 0$, with $g(0) = 1$, thus its range is $[1, \infty)$.

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $f(x) = 2^{-9x+3}$. Determine the domain and range of $f(x)$ and its inverse $f^{-1}(x)$.

(a) $Dom(f) = (-\infty, \infty)$, $Ran(f) = (0, \infty)$,
$Dom(f^{-1}) = (0, \infty)$, $Ran(f^{-1}) = (-\infty, \infty)$ (100%)
(b) $Dom(f) = (-\infty, \infty)$, $Ran(f) = [0, \infty)$,
$Dom(f^{-1}) = [0, \infty)$, $Ran(f^{-1}) = (-\infty, \infty)$
(c) $Dom(f) = (0, \infty)$, $Range(f) = (-\infty, \infty)$,
$Dom(f^{-1}) = (-\infty, \infty)$, $Ran(f^{-1}) = (0, \infty)$
(d) $Dom(f) = [0, \infty)$, $Ran(f) = [0, \infty)$,
$Dom(f^{-1}) = [0, \infty)$, $Ran(f^{-1}) = [0, \infty)$

2^x is defined for all real values, and $0 < 2^x \leq 1$ for $2 \in (-\infty, 0]$, and $1 < 2^x < \infty$ for $x \in (0, \infty)$. Since 2^x is invertible, the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Total of marks: 10