# Week 4: Systems of Linear Equations, Gaussian Elimination

1.

MULTI 1.0 point 0 penalty Single Shuffle

Solve the following system of linear equations:

$$\begin{aligned} x_1 + 3\,x_2 - 5\,x_3 &= 4 \\ x_1 + 4\,x_2 - 8\,x_3 &= 7 \\ -3\,x_1 - 7\,x_2 + 9\,x_3 &= -6 \end{aligned}$$

(a)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$$

(100%)

(b)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 6 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 6 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

*Write out the augmented matrix:*

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right] \xrightarrow[R3+3R1\to R3]{R2-R1\to R2} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right] \xrightarrow[R3-2R2\to R3]{R1-3R2\to R1} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow x = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$$

2.

MULTI 1.0 point 0 penalty Single Shuffle

Find $\alpha \in \mathbb{R}$ such that following system of linear equations has infinitely many solutions:

$$\begin{aligned} 3x_1 + (6+\alpha)x_2 &= 11 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

(a) There exists no such $\alpha$. (100%)
(b) $\alpha = 0$
(c) $\alpha = 1$
(d) $\alpha = 2$

*After elimination we arrive at*

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & \alpha & 2 \end{array}\right).$$

*The choice $\alpha = 0$ makes the last row read $0 = 2$, i.e., no solution exists. Any $\alpha \neq 0$ leads to a unique solution.*

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following is true for homogeneous systems of linear equations?

(a) If $\vec{a}$ and $\vec{b}$ are both solutions, then $\vec{a} + \vec{b}$ is also a solution (100%)
(b) The system might not have a solution
(c) If $\vec{a}$ is a solution, $\exists k \in \mathbb{R}$ such that $k\vec{a}$ is not a solution
(d) We can always find a solution $\vec{a}$ such that all its components $a_i$ are positive

*If we have $\vec{a}$ and $\vec{b}$ be solutions of $A\vec{x} = 0$, then we have*

$$A(\vec{a} + \vec{b}) = A\vec{a} + A\vec{b} = 0 + 0 = 0,$$

*where the first equality comes from the linearity of $A$.*

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $\vec{a}$ and $\vec{b}$ be both solutions to a system of linear equations $A\vec{x} = \vec{v}$. When is $\vec{a} + \vec{b}$ also a solution?

(a) When $\vec{v} = 0$ (100%)
(b) When $\vec{v} \neq 0$
(c) Always
(d) Never

$$A(\vec{a} + \vec{b}) = A\vec{a} + A\vec{b} = 2\vec{v}$$

*And $2\vec{v} = \vec{v} \Rightarrow \vec{v} = 0$*

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Suppose the homogeneous system of linear equations $Av = 0$ has the **unique** solution $v = 0$. Let $b \neq 0$. Then $Ax = b$:

(a) has a unique solution. (100%)
(b) might not have a solution.
(c) might have infinitely many solutions.
(d) might have exactly two solutions.

*If the zero vector* $0$ *is the unique solution to* $Av = 0$*, then* $Ax = b$ *can never have more than two solutions. If it would have two solutions* $x \neq y$*, then* $A(x - y) = b - b = 0$*, i.e.,* $Av = 0$ *would have another non-zero solution. But* $Ax = b$ *definitely has a solution, namely* $0$ *shifted such that* $Ax = b$ *holds.*

*In terms of Gaussian elimination: If* $Av = 0$ *has only* $0$ *as solution, then we can always perform Gaussian elimination until we have a system with only* $1$*'s on the diagonal. But then we can do the same elimination with* $Ax = b$*, and we arrive at the unique solution.*

6.

MULTI | 1.0 point | 0 penalty | Multiple | Shuffle

Consider some $2 \times 5$ matrix $A$, and some vector $b \in \mathbb{R}^2$. Then the system of linear equations $Ax = b$ might have

(a) infinitely many solutions. (50%)
(b) exactly one solution. (−50%)
(c) exactly two solutions. (−50%)
(d) no solution. (50%)

*This system of equations has infinitely many solutions:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

*This system of equations has no solutions:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

*We cannot have exactly one solution, because there are at most* $2$ *equations, but* $5$ *variables. We cannot have exactly two solutions: Once we have two, we could construct infinitely many.*

7.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the standard basis in $\mathbb{R}^3$: $\{e_x, e_y, e_z\}$.
Which of the following matrices represents a counterclockwise rotation with angle $\varphi$ around the $z$-axis?

(a) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (100%)

(b) $\mathcal{R} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}$

(c) $\mathcal{R} = \begin{bmatrix} 1 & 0 & -\sin\varphi \\ \cos\varphi & 1 & \cos\varphi \\ \sin\varphi & 0 & 1 \end{bmatrix}$

(d) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$

*The rotation matrix should act as follows on the basis vectors:*

$$\mathcal{R}e_x = \cos\varphi\, e_x + \sin\varphi\, e_y; \quad \mathcal{R}e_y = -\sin\varphi\, e_x + \cos\varphi\, e_y; \quad \mathcal{R}e_z = e_z$$

*Given that* $e_x \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_y \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_z \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, *then*

$$\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Consider the vector space $P_2(\mathbb{R}) = \{p(x) \mid p(x) \text{ is a quadratic polynomial}\}$.
Is the derivative operator $\mathcal{D} : p(x) \mapsto p'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}p(x)$ a linear operator? If it is, how is it represented in the standard basis $\mathfrak{B} = \{1, x, x^2\}$?

*Hint: You can express a polynomial* $ax^2 + bx + c$ *as* $\begin{bmatrix} c \\ b \\ a \end{bmatrix}$

(a) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (100%)

(b) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(d) $\mathcal{D}$ is not a linear operator

*The derivative is a linear operator since* $\frac{\mathrm{d}}{\mathrm{d}x}(\alpha p(x) + \beta q(x)) = \alpha \frac{\mathrm{d}}{\mathrm{d}x} p(x) + \beta \frac{\mathrm{d}}{\mathrm{d}x} q(x)$
*We know:* $\mathcal{D}(p(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(ax^2 + bx + c) = 2ax + b \equiv \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$
*Therefore,* $[\mathcal{D}]_{\mathfrak{B}} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$. *This is only satisfied by* $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

9.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following is equivalent to $(A \cdot B \cdot C)^T$

(a) $C^T \cdot B^T \cdot A^T$ (100%)
(b) $B^T \cdot C^T \cdot A^T$
(c) $A^T \cdot B^T \cdot C^T$
(d) $C^T \cdot A^T \cdot B^T$

$$(A \cdot B \cdot C)^T = (A \cdot (B \cdot C))^T = (B \cdot C)^T \cdot A^T = C^T \cdot B^T \cdot A^T$$

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $A$ be a $(3 \times 4)$ matrix, and $B$ be a matrix such that $A^T \cdot B$ and $B \cdot A^T$ are both defined. What are the dimensions of $B$

(a) $(3 \times 4)$ (100%)
(b) $(3 \times 3)$
(c) $(4 \times 4)$
(d) $(4 \times 3)$

*If* $A$ *is* $(3 \times 4)$, *then* $A^T$ *is* $(4 \times 3)$. *Then if* $B$ *is* $(3 \times 4)$, $A^T \cdot B$ *and* $B \cdot A^T$ *are well defined.*

*Total of marks: 10*