Constructor University
Fall 2024

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Elements of Linear Algebra

Homework 4 (covering Weeks 7 and 8)

Problem 1 [4 points]

Recall the following theorem from class. For an invertible $n \times n$ matrix A, an explicit formula for the inverse is

$$A^{-1} = \frac{1}{\det A} \mathrm{Adj} A.$$

Let's consider the product $A \cdot \mathrm{Adj}(A)$. For any entry (i, j) in this product:

$$[A \cdot \mathrm{Adj}(A)]_{ij} = \sum_{k=1}^{n} a_{ik} (\mathrm{Adj}(A))_{kj} = \sum_{k=1}^{n} a_{ik} C_{jk}$$

where C_{jk} is the (j, k) cofactor of A.
We need to consider two cases:
Case 1: $i = j$
In this case, the sum $\sum_{k=1}^{n} a_{ik} C_{ik}$ is precisely the Laplace expansion of $\det(A)$ along row i. Therefore:

$$[A \cdot \mathrm{Adj}(A)]_{ii} = \det(A)$$

Case 2: $i \neq j$
In this case, $\sum_{k=1}^{n} a_{ik} C_{jk}$ represents the Laplace expansion of a determinant of a matrix that has two identical rows (the ith and jth rows of A), which equals zero. Therefore:

$$[A \cdot \mathrm{Adj}(A)]_{ij} = 0$$

Combining both cases, we can write:

$$A \cdot \mathrm{Adj}(A) = \det(A) \cdot I_n$$

where I_n is the $n \times n$ identity matrix.
If A is invertible, then $\det(A) \neq 0$, and we can multiply both sides by $\frac{1}{\det(A)}$:

$$\frac{1}{\det(A)} (A \cdot \mathrm{Adj}(A)) = I_n$$

$$A \cdot \left(\frac{1}{\det(A)} \mathrm{Adj}(A) \right) = I_n$$

By the uniqueness of the inverse matrix, we conclude:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A)$$

This completes the proof.

Problem 2 [4 points]
Consider the linear system of equations

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 8, \\ x_1 + 2x_2 + 2x_3 &= 5, \\ 2x_1 + 2x_2 + 3x_3 &= 7. \end{aligned}$$

Use Cramer's rule to determine the solution x_3. (Just use Sarrus' rule to quickly compute the necessary determinants.)

Solution:
If we rewrite the system as $Ax = b$, then, according to Cramers' rule, we get

$$x_3 = \frac{\det(A_3)}{\det(A)},$$

where A_3 is obtained from A by using the vector b in place of the third column. We have:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 3 & 2 & 5 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ &= 3(6-4) - 2(3-4) + 5(2-4) \\ &= -2. \end{aligned}$$

Similarly, we compute $\det(A_3)$:

$$\begin{aligned} \det(A_3) &= \begin{vmatrix} 3 & 2 & 8 \\ 1 & 2 & 5 \\ 2 & 2 & 7 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & 5 \\ 2 & 7 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + 8\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ &= 3(14-10) - 2(7-10) + 8(2-4) \\ &= 2. \end{aligned}$$

Finally, we get:

$$x_3 = \frac{\det(A_3)}{\det(A)} = -1.$$

Problem 3 [4 points]

Recall the definition of the classical adjoint $\text{Adj}(A)$ of an $n \times n$ matrix A from class. We assume that A is an invertible matrix.

(a) Compute the determinant of $\text{Adj}(A)$ in terms of the determinant of A.

(b) Show that the adjoint of the adjoint of A is guaranteed to equal A if $n = 2$, but not necessarily for $n > 2$.

Solution:

a) We know the following equation:

$$A \cdot \text{Adj}(A) = \det(A)I.$$

Since A is invertible, it can be rewritten as

$$\text{Adj}(A) = \det(A)A^{-1}. \tag{1}$$

Also, if $c \in \mathbb{R}$ and B is an $n \times n$ matrix, it is easy to show that $\det(cB) = c^n \det(B)$. This statement holds, because multiplication of a single matrix row by c multiplies the determinant by c, and for cB all n rows are multiplied. After taking the determinant of both sides of (1) and using this formula, we get:

$$\begin{aligned}
\det(\text{Adj}(A)) &= \det(\det(A)A^{-1}) \\
&= \det(A)^n \det(A^{-1}) \\
&= \det(A)^n \det(A)^{-1} \\
&= \det(A)^{n-1}.
\end{aligned}$$

b) After substituting $\text{Adj}(A)$ in place of A into (1), we obtain the following equation:

$$\text{Adj}(\text{Adj}(A)) = \det(\text{Adj}(A))(\text{Adj}(A))^{-1}. \tag{2}$$

To find $(\text{Adj}(A))^{-1}$, we take the inverse of both sides of (1). This yields:

$$(\text{Adj}(A))^{-1} = (\det(A)A^{-1})^{-1} = \det(A)^{-1}A.$$

After plugging this and the result from part a) into (2), we get:

$$\begin{aligned}
\text{Adj}(\text{Adj}(A)) &= \det(A)^{n-1} \det(A)^{-1} A \\
&= \det(A)^{n-2} A.
\end{aligned}$$

If $n = 2$ then, since A is invertible, $\det(A)^{n-2} = \det(A)^0 = 1$, so $\text{Adj}(\text{Adj}(A)) = A$. For other values of n, however, this is not true, unless $\det(A) = 1$ (or ± 1 for even n).

Problem 4 [8 points]

In class, we discussed several properties of eigenvalues. Let us exemplify them for the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

Solution:

a) First, we compute the characteristic polynomial of A:

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= \lambda^2 - (a+d)\lambda + (ad - bc). \end{aligned}$$

Eigenvalues are roots of $p(\lambda)$:

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

If $D = (a+d)^2 - 4(ad-bc) > 0$, then we have two distinct real eigenvalues. If $D = 0$, we have one real eigenvalue of multiplicity 2. If $D < 0$, we have two eigenvalues that are complex conjugates.

b) A is invertible whenever $\det(A)$ is non-zero. Since $\det(A) = ad - bc$, we get:

$$\exists A^{-1} \iff ad \neq bc.$$

To find the inverse, we use the classical adjoint:

$$\begin{aligned} A^{-1} &= \frac{\mathrm{Adj}(A)}{\det(A)} \\ &= \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}. \end{aligned}$$

c) We have:

$$\begin{aligned}\lambda_1 + \lambda_2 &= \frac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2} + \frac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \\ &= \frac{2(a+d)}{2} = (a+d).\end{aligned}$$

d) We have:

$$\begin{aligned}\lambda_1\lambda_2 &= \left(\frac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2}\right)\left(\frac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}\right) \\ &= \frac{(a+d)^2 - ((a+d)^2 - 4(ad-bc))}{4} \\ &= ad - bc \\ &= \det(A).\end{aligned}$$

e) We plug A into the characteristic polynomial from a):

$$\begin{aligned}B &= p(A) \\ &= A^2 - (a+d)A + (ad-bc)I \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}.\end{aligned}$$

Now, we compute B component-wise:

$$\begin{aligned}B_{11} &= a^2 + bc - a^2 - ad + ad - bc \\ &= 0.\end{aligned}$$

Similarly, we can show that the other components are equal to 0 as well. So B is the zero matrix, and the Cayley-Hamilton theorem holds.