Week 5: Kernel, range, rank-nullity theorem, matrix inverse

1.

MULTI 1.0 point 0 penalty Single Shuffle

Find the inverse of $A = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$.

(a) $A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{bmatrix}$ (100%)

(b) $A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ 4 & 4 & 2 \\ 2 & -4 & -2 \end{bmatrix}$

(c) $A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ 4 & 4 & 2 \\ 2 & 4 & -2 \end{bmatrix}$

(d) the inverse does not exist

Augmented matrix:

$$\left[\begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\textit{reorder rows}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right] \xrightarrow{R2-2R1\to R2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right] \xrightarrow{R3-R2\to R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & 0 & \frac{1}{2} & 1 & 2 & -1 \end{array}\right] \xrightarrow[2\,R3\to R3]{2\,R2\to R2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -4 & 2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right] \xrightarrow[R2+2\,R3\to R2]{R1-R3\to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right]$$

2.

MULTI 1.0 point 0 penalty Single Shuffle

Find the inverse of $A = \begin{bmatrix} 3.5 & -1 & 0.5 \\ 10 & -3 & 2 \\ 2.5 & -1 & 1.5 \end{bmatrix}$.

(a) $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$

(c) $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$

(d) the inverse does not exist (100%)

Note that:

$$\begin{bmatrix} 3.5 & -1 & 0.5 \\ 10 & -3 & 2 \\ 2.5 & -1 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Meaning that $\dim(\ker(A)) > 0 \Rightarrow A$ *is singular and doesn't have an inverse*

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) $A^{-1} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$ (100%)

(b) $A^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$

(c) $A^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{bmatrix}$

(d) the inverse does not exist

Augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R2+R1\to R2]{R2+R3\to R3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R2+R1\to R1]{\substack{R3\to R2 \\ 0.5R2\to R2 \\ 0.5R3\to R3}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0.5 & 0.5 \\ 0 & 1 & 1 & 0 & 0.5 & 0.5 \\ 0 & 0 & 1 & 0.5 & 0.5 & 0 \end{array}\right] \xrightarrow[R2-R3\to R2]{R1-2R3\to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -0.5 & 0.5 \\ 0 & 1 & 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 & 0.5 & 0 \end{array}\right]$$

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find the kernel of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) $\ker(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \right\}$ (100%)

(b) $\ker(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) $\ker(A) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right\}$

(d) $\ker(A) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$

Write out the augmented matrix corresponding to $Ax = 0$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow x = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

An $n \times k$ matrix A has the following kernel:

$$\ker(A) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

What is the dimension of its image $\dim(\text{im}(A))$?

(a) $\dim(\text{im}(A)) = k - 1$
(b) $\dim(\text{im}(A)) = k - 2$ (100%)
(c) $\dim(\text{im}(A)) = n - 3$
(d) $\dim(\text{im}(A)) = n$

Using the rank–nullity theorem ($\dim(\ker(A)) + \dim(\text{im}(A)) = \dim(\text{Dom}(A)) = k$):

$$\dim(\ker(A)) = 2 \Rightarrow \dim(\text{im}(A)) = k - 2$$

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Given the matrix $A = B \cdot C \cdot D$ find its inverse A^{-1}.

(a) $A^{-1} = D^{-1}C^{-1}B^{-1}$ (100%)
(b) $A^{-1} = B^{-1}C^{-1}D^{-1}$
(c) $A^{-1} = D^{-1}B^{-1}C^{-1}$
(d) $A^{-1} = C^{-1}D^{-1}B^{-1}$

Using $(B \cdot C)^{-1} = C^{-1}B^{-1}$:

$$(BCD)^{-1} = D^{-1} \cdot (BC)^{-1} = D^{-1}C^{-1}B^{-1}$$

7.

MULTI 1.0 point 0 penalty Single Shuffle

A linear map $D : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the following matrix in the standard basis: $D_{\text{st}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. How is the map represented in the following basis: $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$?

(a) $D_{\text{new}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (100%)

(b) $D_{\text{new}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $D_{\text{new}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(d) $D_{\text{new}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

To act on an element given by the coordinates in the new basis one can:

(a) convert coordinates to the standard ones

(b) act with the map, represented in the standard basis

(c) convert back to the new basis

Converting the above mentioned map composition to the matrix multiplication language:

$$D_{new} = \underbrace{\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1}}_{(c)} \cdot \overbrace{D_{st}}^{(b)} \cdot \underbrace{\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)}_{(a)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Consider the standard basis in $\mathbb{R}^3$: $\{e_x, e_y, e_z\}$.
Which of the following matrices represents a counterclockwise rotation around the z-axis?

(a) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (100%)

(b) $\mathcal{R} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}$

(c) $\mathcal{R} = \begin{bmatrix} 1 & 0 & -\sin\varphi \\ \cos\varphi & 1 & \cos\varphi \\ \sin\varphi & 0 & 1 \end{bmatrix}$

(d) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The rotation matrix should act as follows on the basis vectors:

$$\mathcal{R}e_x = \cos\varphi\, e_x + \sin\varphi\, e_y; \quad \mathcal{R}e_y = -\sin\varphi\, e_x + \cos\varphi\, e_y; \quad \mathcal{R}e_z = e_z$$

Given that $e_x \equiv \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_y \equiv \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_z \equiv \begin{bmatrix}0\\0\\1\end{bmatrix}$, *then*

$$\mathcal{R} = \begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}$$

9.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the space of 2×2 Hermitian Matrices $H_2(\mathbb{C})$ (the space of 2×2 matrices A with complex entries such that $A^\dagger := \overline{A}^T = A$).
Which of the following is true?

(a) $H_2(\mathbb{C})$ is a vector space over the field of real numbers, but not over the complex numbers. (100%)
(b) $H_2(\mathbb{C})$ is a vector space over the field of real numbers and over the complex numbers.
(c) $H_2(\mathbb{C})$ is not a vector space over the field of real numbers or complex numbers.
(d) $H_2(\mathbb{C})$ is a vector space over the field of complex numbers, but not over the real numbers.

Consider two matrices $A, B \in H_2(\mathbb{C})$. *We can see that* $(A+B)^\dagger = A^\dagger + B^\dagger = A+B$, *so the space is closed under addition. Now consider* $(cA)^\dagger = \overline{c}A^\dagger = \overline{c}A$, *with* $\overline{c}$ *being the complex conjugate of* c. *If* $c \in \mathbb{R}$, *then* $\overline{c} = c$, *and thus the space is closed under scalar multiplication. However, if* $c \in \mathbb{C}$, *then* $(cA)^\dagger = \overline{c}A \neq cA$, *and thus the space is not closed under scalar multiplication.*

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the vector space $P_2(\mathbb{R}) = \{p(x) \mid p(x) \text{ is a quadratic polynomial}\}$.
Is the derivative operator $\mathcal{D} : p(x) \mapsto p'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}p(x)$ a linear operator? If it is, how is it represented in the standard basis $\mathfrak{B} = \{1, x, x^2\}$

Hint: You can express a polynomial $ax^2 + bx + c$ *as* $\begin{bmatrix}c\\b\\a\end{bmatrix}$

(a) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}$ (100%)

(b) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

(c) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(d) $\mathcal{D}$ is not a linear operator

The derivative is a linear operator since $\frac{\mathrm{d}}{\mathrm{d}x}(\alpha p(x) + \beta q(x)) = \alpha \frac{\mathrm{d}}{\mathrm{d}x} p(x) + \beta \frac{\mathrm{d}}{\mathrm{d}x} q(x)$

We know: $\mathcal{D}(p(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(ax^2 + bx + c) = 2ax + b \equiv \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$

Therefore, $[\mathcal{D}]_{\mathfrak{B}} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$. *This is only satisfied by* $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Total of marks: 10