Week 10: Hermitian/real symmetric and unitary/orthogonal matrices

1.

MULTI 1.0 point 0.10 penalty Single Shuffle

A matrix H is called *Hermitian* if $H = H^\dagger$. A matrix A is called *anti-Hermitian* if $A = -A^\dagger$. It is possible to write any matrix as a sum of a Hermitian and an anti-Hermitian matrix. Let

$$C = \begin{bmatrix} 1 & -i \\ 2i & 3 \end{bmatrix}$$

and find a Hermitian matrix H and an anti-Hermitian matrix A such that $C = H + A$.

(a) $H = \begin{bmatrix} 1 & -\frac{3}{2}i \\ \frac{3}{2}i & 3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}$ (100%)

(b) $H = \begin{bmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -\frac{3}{2}i \\ \frac{3}{2}i & 3 \end{bmatrix}$

(c) $H = \begin{bmatrix} 1 & \frac{3}{2}i \\ -\frac{3}{2}i & 3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix}$

(d) $H = \begin{bmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & \frac{3}{2}i \\ -\frac{3}{2}i & 3 \end{bmatrix}$

Note that for any matrix C the matrix $H = \frac{1}{2}(C + C^\dagger)$ is Hermitian.

$$H^\dagger = \frac{1}{2}(C + C^\dagger)^\dagger = \frac{1}{2}(C^\dagger + (C^\dagger)^\dagger) = \frac{1}{2}(C^\dagger + C) = H.$$

Moreover the matrix $A = \frac{1}{2}(C - C^\dagger)$ is always anti-Hermitian.

$$A^\dagger = \frac{1}{2}(C - C^\dagger)^\dagger = \frac{1}{2}(C^\dagger - (C^\dagger)^\dagger) = \frac{1}{2}(C^\dagger - C) = -A.$$

If we choose H and A as above, then

$$A + H = \frac{1}{2}C - \frac{1}{2}C^\dagger + \frac{1}{2}C + \frac{1}{2}C^\dagger = C.$$

Hence the required matrices are

$$H = \begin{bmatrix} 1 & -\frac{3}{2}i \\ \frac{3}{2}i & 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}.$$

2.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Find the characteristic polynomial of the matrix

$$H = \begin{bmatrix} -1 & 1-2i & 0 \\ 1+2i & 0 & -i \\ 0 & i & 1 \end{bmatrix}.$$

Find this polynomial explicitly and determine the number of real roots.

(a) The characteristic polynomial $-\lambda^3 + 7\lambda - 4$ has three real roots (100%)
(b) The characteristic polynomial $-\lambda^3 - 4$ has three real roots
(c) The characteristic polynomial $-\lambda^3 + 7\lambda - 4$ has two real roots
(d) The characteristic polynomial $-\lambda^3 + 7\lambda - 4$ has only one real root

$$\begin{aligned} \det(H - \lambda I) &= \begin{vmatrix} -1-\lambda & 1-2i & 0 \\ 1+2i & -\lambda & -i \\ 0 & i & 1-\lambda \end{vmatrix} \\ &= -(\lambda+1)(\lambda(\lambda-1)-1) - (1+2i)((1-2i)(1-\lambda)) \\ &= -\lambda^3 + 7\lambda - 4. \end{aligned}$$

Since H is Hermitian, all its three roots must be real.

3.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

The matrix

$$A = \begin{bmatrix} i & 2+i & 3 \\ -2+i & 2i & -1 \\ -3 & 1 & 3i \end{bmatrix}$$

is

(a) Skew-Hermitian (100%)
(b) Hermitian
(c) Unitary
(d) Orthogonal

Since $A^* = -A$ the matrix is skew-Hermitian.

4.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let U be unitary, and define the matrix

$$A = U^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} U.$$

Is A positive definite?

(a) Yes. (100%)

(b) No.
(c) For some U it is, for others not.

A is a Hermitian matrix with positive eigenvalues for any choice of unitary U, hence it is positive definite.

5.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

normal and is it unitary?

(a) U is not normal and U is not unitary
(b) U is normal but U is not unitary
(c) U is not normal but U is unitary
(d) U is normal and U is unitary (100%)

$$UU^\dagger = \frac{1}{4}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$U^\dagger U = \frac{1}{4}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence U is normal and unitary.

6.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

normal and is it unitary?

(a) U is not normal and U is not unitary
(b) U is normal but U is not unitary (100%)
(c) U is not normal but U is unitary
(d) U is normal and U is unitary

$$UU^\dagger = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$U^\dagger U = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Hence U is normal but not unitary.

7.

MULTI 1.0 point 0.10 penalty Single Shuffle

A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

normal and is it unitary?

(a) U is not normal and U is not unitary
(b) U is normal but U is not unitary
(c) U is not normal but U is unitary
(d) U is normal and U is unitary (100%)

$$UU^\dagger = \frac{1}{4}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U^\dagger U = \frac{1}{4}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence U is normal and unitary.

8.

MULTI 1.0 point 0.10 penalty Single Shuffle

Consider the matrix

$$U = \begin{bmatrix} i & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

and the vector

$$x = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Compute the length of Ux, i.e., compute $|Ux|$.

(a) $|Ux| = \sqrt{17}$. (100%)
(b) $|Ux| = 1$.
(c) $|Ux| = \sqrt{28}$.
(d) $|Ux| = \sqrt{39}$.

Since U is unitary (its columns are orthonormal), we have $|Ux| = |x|$ and

$$|x| = \sqrt{2^2 + 2^2 + 3^3} = \sqrt{17}.$$

9.

MULTI 1.0 point 0.10 penalty Single Shuffle

Let U_1 and U_2 both be unitary $n \times n$ matrices. Then the product U_1U_2 is

(a) unitary (100%)
(b) Hermitian
(c) orthogonal
(d) real symmetric

We have

$$(U_1U_2)^* = U_2^*U_1^* = U_2^{-1}U_1^{-1} = (U_1U_2)^{-1},$$

so U_1U_2 is unitary.

10.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let Q be an orthogonal 5×5 matrix. Then

(a) Q must have an eigenvalue $+1$ or -1. (100%)
(b) All eigenvalues of Q are either $+1$ or -1.
(c) All eigenvalues of Q are real.
(d) At least one eigenvalue must have non-zero imaginary part.

Since Q is orthogonal, $\det Q = \pm 1$. Also, all eigenvalues of Q come in complex conjugate pairs, and have absolute value 1. Hence, there can be at most two complex conjugate pairs of eigenvalues, and either 1, 3, or 5 eigenvalues are real with absolute value 1, i.e., are ± 1.

Total of marks: 10