Week 8: Eigenvalues and eigenspaces

1.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

A 5×5 matrix has eigenvalues $\lambda_1, \ldots, \lambda_5$. If $\lambda_1 = 2+i$, $\lambda_2 = \frac{i}{\sqrt{2}}$, $\lambda_3 = 2$ then what are the values of λ_4 and λ_5?

(a) $\lambda_4 = 2 - i, \lambda_5 = -\frac{i}{\sqrt{2}}$ (100%)
(b) $\lambda_4 = 2 + i, \lambda_5 = \frac{i}{\sqrt{2}}$
(c) $\lambda_4 = \lambda_5 = 0$
(d) The information given is insufficient to determine λ_4 and λ_5

Recall that complex eigenvalues come in pairs, hence we must have

$$\lambda_4 = \overline{\lambda_1} = (2-i),$$
$$\lambda_5 = \overline{\lambda_2} = -\frac{i}{\sqrt{2}}.$$

2.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let A be an $n \times n$ matrix that has eigenvalues $1, 3, 5, \ldots, 2n-1$. Compute $\operatorname{tr}(A)$ and $\det(A)$. (Recall that $n! := 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.)

(a) $\operatorname{tr}(A) = n^2, \det(A) = \frac{(2n)!}{2^n(n!)}$ (100%)
(b) $\operatorname{tr}(A) = (n+1)^2, \det(A) = \frac{(2n+2)!}{2^n((n+1)!)}$
(c) $\operatorname{tr}(A) = n^2, \det(A) = \frac{(2n)!}{2(n!)}$
(d) $\operatorname{tr}(A) = n, \det(A) = \frac{(2n)!}{n!}$

Recall that the trace is given by the sum of the eigenvalues and the determinant by the product of the eigenvalues. Hence

$$\operatorname{tr}(A) = 1 + 2 + \cdots + (2n-1) = n^2$$
$$\det(A) = 1 \cdot 2 \cdots (2n-1) = \frac{1 \cdot 2 \cdots 2n}{2 \cdot 4 \cdots 2(n-1)} = \frac{(2n)!}{2^n(n!)}$$

3.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let A be a matrix such that

$$\det(A - \lambda I) = -\lambda^3(\lambda - 1)(2\lambda + 1)^2.$$

Compute $\operatorname{tr}(A)$ and $\det(A)$.

(a) $\mathrm{tr}(A) = 0, \det(A) = 0$ (100%)
(b) $\mathrm{tr}(A) = \frac{1}{2}, \det(A) = 0$
(c) $\mathrm{tr}(A) = 0, \det(A) = -1$
(d) $\mathrm{tr}(A) = \frac{1}{2}, \det(A) = -\frac{1}{2}$

Clearly, the eigenvalues of the matrix are $0, 1, -\frac{1}{2}$. *These eigenvalues respectively occur with multiplicity* $3, 1, 2$. *Hence*

$$\mathrm{tr}(A) = \sum m_i \cdot \lambda_i = 3 \cdot 0 + 1 \cdot 1 + 2 \cdot \left(-\frac{1}{2}\right) = 0$$
$$\det(A) = \prod \lambda_i^{m_i} = 0^3 \cdot 1^1 \cdot \left(-\frac{1}{2}\right)^2 = 0.$$

4.

MULTI 1.0 point 0.10 penalty Single Shuffle

Let A be a 7×7 matrix such that

$$\det(A - \lambda I) = (\lambda - 2 + i)(\lambda - i)(\lambda - \sqrt{2})^2(\lambda - 1)q(\lambda)$$

where $q(\lambda)$ is some polynomial. Compute $\mathrm{tr}(A)\ \det(A)$.

(a) $\mathrm{tr}(A) = 5 + 2\sqrt{2}, \det(A) = 10.$ (100%)
(b) $\mathrm{tr}(A) = 2 + \sqrt{2}, \det(A) = \sqrt{2} + 2\sqrt{2}i.$
(c) $\mathrm{tr}(A) = 2 + 2\sqrt{2}, \det(A) = 2 + 4i.$
(d) Information given in insufficient to determine $\mathrm{tr}(A)$ and $\det(A)$.

From the given expression, we can easily see that five of the eigenvalues are $(2 - i), i, \sqrt{2}, 1$ *(here* $\sqrt{2}$ *appears with multiplicity 2). Since the non-real eigenvalues always appear in pairs, the remaining two eigenvalues are* $(2 + i)$ *and* $-i$. *Hence*

$$\mathrm{tr}(A) = \sum m_i \cdot \lambda_i = (2 - i) + (2 + i) + i - i + 1 + 2 \cdot \sqrt{2} = 5 + 2\sqrt{2}$$
$$\det(A) = \prod \lambda_i^{m_i} = (2 - i) \cdot (2 + i) \cdot (i) \cdot (-i) \cdot (\sqrt{2})^2 \cdot 1 = 5 \cdot 1 \cdot 2 \cdot 1 = 10.$$

5.

MULTI 1.0 point 0.10 penalty Single Shuffle

Consider the eigenvalues of the matrix

$$S = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 1 \\ 2 & 4 & 3 & 5 & 5 & 7 \\ 2 & 3 & 9 & 1 & 1 & 0 \\ 3 & 5 & 1 & 0 & 1 & 2 \\ 4 & 5 & 1 & 1 & 3 & 1 \\ 1 & 7 & 0 & 2 & 1 & 8 \end{bmatrix}.$$

How many eigenvalues of the matrix are **not** real?

(a) 0 (100%)
(b) 1
(c) 7
(d) 3

Since we have $S = S^\top$ all eigenvalues are real.

6.

MULTI 1.0 point 0.10 penalty Single Shuffle

Let A be a matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)(\lambda - 5).$$

What is the dimension of $\ker(A)$?

(a) 0
(b) 1 (100%)
(c) 2
(d) 3

Note that the kernel is the eigenspace corresponding to the eigenvalue 0*. All eigenvalues occur with algebraic multiplicity* 1*, so all eigenspaces are one-dimensional.*

7.

MULTI 1.0 point 0.10 penalty Single Shuffle

Consider the matrix

$$A = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}.$$

Compute the set of tuples $(\lambda, a_\lambda, g_\lambda)$ where λ is an eigenvalue, a_λ its algebraic multiplicity and g_λ is its geometric multiplicity.

(a) $(\lambda_1 = 1, a_1 = 2, g_1 = 1)$ (100%)
(b) $(\lambda_1 = 1, a_1 = 2, g_1 = 2)$
(c) $(\lambda_1 = 1, a_1 = 1, g_1 = 2)$
(d) $(\lambda_1 = 0, a_1 = 2, g_1 = 2)$

We first compute the characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 8 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2.$$

Hence the eigenvalue $\lambda_1 = 1$ occurs with algebraic multiplicity 2.

$$\lambda_1 = 1 : (A - \lambda I)v = \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix} v = 0 \implies v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since the eigenspace corresponding to the eigenvalue 1 *is one dimensional, the geometric multiplicity is* 1*. Hence the required tuple is $(\lambda_1 = 1, a_1 = 2, g_1 = 1)$.*

8.

MULTI 1.0 point 0.10 penalty Single Shuffle

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{bmatrix}.$$

Compute the set of tuples $(\lambda, a_\lambda, g_\lambda)$ where λ is an eigenvalue, a_λ its algebraic multiplicity and g_λ is its geometric multiplicity.

(a) $(\lambda_1 = 1, a_1 = 1, g_1 = 1), (\lambda_2 = 2, a_2 = 1, g_2 = 1), (\lambda_3 = 3, a_3 = 1, g_3 = 1)$ (100%)
(b) $(\lambda_1 = 1, a_1 = 1, g_1 = 0), (\lambda_2 = 2, a_2 = 1, g_2 = 0), (\lambda_3 = 3, a_3 = 1, g_3 = 0)$
(c) $(\lambda_1 = 1, a_1 = 0, g_1 = 1), (\lambda_2 = 2, a_2 = 0, g_2 = 1), (\lambda_3 = 3, a_3 = 0, g_3 = 1)$
(d) $(\lambda_1 = 1, a_1 = 3, g_1 = 2)$

We first compute the characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -3 & 3-\lambda & 0 \\ 3 & 2 & 2-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(2-\lambda).$$

Hence the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ *with algebraic multiplicities* $1, 1,$ *and* 1 *respectively.*

$$\lambda_1 = 1 : (A - I)v = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix} v = 0 \implies v_1 = \mu \begin{bmatrix} -2 \\ -3 \\ 12 \end{bmatrix}$$

$$\lambda_2 = 2 : (A - 2I)v = \begin{bmatrix} -1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} v = 0 \implies v_2 = \mu \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 3 : (A - 3I)v = \begin{bmatrix} -2 & 0 & 0 \\ -3 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix} v = 0 \implies v_3 = \mu \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

The eigenspaces corresponding to the three eigenvalues are one dimensional, hence all geometric multiplicities are 1. *The required tuples are*

$$(\lambda_1 = 1, a_1 = 1, g_1 = 1), (\lambda_2 = 2, a_2 = 1, g_2 = 1), (\lambda_3 = 3, a_3 = 1, g_3 = 1).$$

9.

MULTI 1.0 point 0.10 penalty Single Shuffle

A matrix A has characteristic polynomial

$$p(\lambda) = \lambda(\lambda + 2)^2 + 3\lambda - 1.$$

Use the Cayley-Hamilton theorem to deduce a formula for A^{-1} in terms of A.

(a) $A^{-1} = (A + 2)^2 + 3I$ (100%)

(b) $A^{-1} = A(A+2)^2 + 3A - I$
(c) $A^{-1} = A$
(d) A is not invertible

The Cayley-Hamilton theorem states that a matrix satisfies its own characteristic equation, i.e., $p(A) = 0$. Hence,

$$p(A) = A(A+2)^2 + 3A - 1 = 0.$$

Multiplying by A^{-1} yields

$$(A+2)^2 + 3 - A^{-1} = 0.$$

Note that A is indeed invertible since 0 is not an eigenvalue ($p(0) = -1 \neq 0$).

10.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 1/2 & 10 & 0 \\ 0 & 1/2 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Which of the following statements is false?

(a) A^4 has an eigenvalue $1/4$. (100%)
(b) A^4 has an eigenvalue 1.
(c) A is invertible.
(d) A^{-1} has an eigenvalue 2.

A has eigenvalues $1/2$ and 1. Since 0 is not an eigenvalue, A is invertible, with eigenvalues 1 and 2. The eigenvalues of A^4 are $1/16$ and 1.

Total of marks: 10