Week 13: Singular Value Decomposition (SVD)

1.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Find the singular values of the matrix

$$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

(a) The singular values are $\sqrt{2}$ and 1. (100%)
(b) The singular values are 2 and 1.
(c) The singular values don't exist because the matrix is not square.
(d) The singular values are 4 and 1.

We first compute

$$M^T M = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

The matrix has characteristic polynomial $(\lambda-2)(\lambda-1)$ *and therefore has eigenvalues* 1 *and* 2. *The singular values are thus* 1 *and* $\sqrt{2}$.

2.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Find the singular values of the matrix

$$M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

(a) The singular values are 5 and 3. (100%)
(b) The singular values are $\sqrt{5}$ and $\sqrt{3}$.
(c) The singular values are 25 and 9.
(d) The singular values don't exist because the matrix is not square.

We first compute

$$MM^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.$$

This matrix has characteristic polynomial $(\lambda - 25)(\lambda - 9)$ *and therefore has the eigenvalues* 25 *and* 9. *The singular values are thus* 5 *and* 3.

3.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

and the matrices

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Is the decomposition $A = U\Sigma V^T$ a valid singular value decomposition? If not, why not?

(a) The SVD decomposition is valid. (100%)
(b) The decomposition is not valid as the singular values are incorrect.
(c) The decomposition is not valid because U is not orthogonal.
(d) The decomposition is not valid because V is not orthogonal.

We check that

$$AA^T = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$$

has eigenvalues 45 *and* 5. *Hence the singular values are* $\sqrt{5}$ *and* $3\sqrt{5}$. *We also check that* U *and* V *satisfy* $UU^T = U^TU = I$ *and* $VV^T = V^TV = I$ *respectively and hence are orthogonal. Finally, we check that the product* $U\Sigma V^T$ *is indeed equal to* A. *Hence* $A = U\Sigma V^T$ *is a valid singular value decomposition.*

4.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 17/10 & 1/10 & -17/10 & -1/10 \\ 3/5 & 9/5 & -3/5 & -9/5 \end{bmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad V^* = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Let

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Find all solutions x to the linear equations $Ax = b$.

(a) $x = \frac{1}{8}(1, 1, 1, 1)^T + \lambda(1, -1, 1, -1)$ for any $\lambda \in \mathbb{R}$. (100%)
(b) $x = \frac{1}{8}(1, 1, 1, 1)^T + \lambda(1, -1, -1, -1)$ for any $\lambda \in \mathbb{R}$.
(c) $x = \frac{1}{8}(1, 1, 1, 1)^T$.
(d) There are no solutions to $Ax = b$.

Following the lecture notes we compute

$$\hat{x} = V\widetilde{\Sigma}U^*b = \frac{1}{8}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

Then the general solution is $x = \hat{x} + \lambda v_4$, with v_4 the forth column of V, i.e., the fourth row of V^*.

5.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 17/10 & 1/10 & -17/10 & -1/10 \\ 3/5 & 9/5 & -3/5 & -9/5 \end{bmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad V^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

What are the rank and the nullity of A, i.e., what are the dimensions of the range and kernel of A?

(a) $\text{rank}(A) = 4$, $\text{nullity}(A) = 3$.
(b) $\text{rank}(A) = 3$, $\text{nullity}(A) = 4$.
(c) $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$. (100%)
(d) $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.

A has three nonzero singular values, hence the rank is 3. There is one zero column in Σ, hence the nullity is 1.

6.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Consider the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & -\sqrt{3} & \sqrt{2} \\ 2 & 0 & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let furthermore

$$b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve the least square problem for $Ax = b$ using the singular value decomposition of A, i.e., find the vector x that makes $\|Ax - b\|$ minimal.

(a) $x_1 = -\frac{4}{3}, x_2 = -\frac{7}{3}$.
(b) $x_1 = -4, x_2 = -7$.
(c) $x_1 = -\sqrt{3}, x_2 = 1$.
(d) $x_1 = -1, x_2 = 1$. (100%)

According to class, the least square problem is solved by

$$x = V\widetilde{\Sigma}U^*b.$$

Direct computation yields $x_1 = -1, x_2 = 1$.

7.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let A be an $m \times n$ matrix with $m \neq n$. Which of the following statements is false?

(a) The matrix AA^* is an $n \times n$ matrix. (100%)
(b) The matrix AA^* is Hermitian.
(c) The matrix AA^* is positive semidefinite.
(d) The matrix A^*A is Hermitian.

The matrix AA^* *is an* $m \times m$ *matrix, not an* $n \times n$ *matrix.*

8.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let A be an $m \times n$ matrix with $m > n$. Which of the following statements is false?

(a) A might have m non-zero singular values. (100%)
(b) A might have n non-zero singular values.
(c) Some of the singular values of A might be zero.
(d) None of the singular values of A are negative.

A has $\min(m, n)$ *singular values, all of them are nonnegative, and some of them might be zero. Here,* $\min(m, n) = n$*, so A can have at most n non-zero singular values.*

9.

MULTI | 1.0 point | 0.10 penalty | Single | Shuffle

Let A be an $m \times n$ matrix with rank r and singular value decomposition $A = U\Sigma V^*$ (with singular values in descending order). Let $u_1, \ldots, u_m$ be the columns of U, and $v_1, \ldots, v_n$ the columns of V. Which of the following statements is false?

(a) $\mathrm{span}(v_1, \ldots, v_r) = \mathrm{Ran}(A)$ (100%)
(b) $\mathrm{span}(u_1, \ldots, u_r) = \mathrm{Ran}(A)$.
(c) $\mathrm{span}(v_{r+1}, \ldots, v_n) = \mathrm{Ker}(A)$.
(d) $\{v_{r+1}, \ldots, v_n\}$ is an orthonormal basis for $\mathrm{Ker}(A)$.

As discussed in the lecture notes, $\operatorname{span}(v_1, \ldots, v_r) = \operatorname{Ran}(A^*)$*, not* $\operatorname{Ran}(A)$.

10.

MULTI 1.0 point 0.10 penalty Single Shuffle

An experiment has collected $N = 10$ two-dimensional data points, which were collected in the 2×10 matrix

$$A = \begin{bmatrix} 3 & -2 & -3 & -1 & -4 & 3 & 4 & -3 & 2 & 1 \\ 6 & -8 & -7 & -9 & -5 & 5 & 9 & -6 & 7 & 8. \end{bmatrix}.$$

Note that the averages in both rows are already zero. Use a suitable online calculator (e.g., Wolfram Alpha) to compute the singular value decomposition of A and identify the first principal component. Visualize the data for yourself in a two-dimensional coordinate system to make sure the result makes sense.

(a) The first principal component is approximately $(0.334, 0.943)$. (100%)
(b) The first principal component is approximately $(-0.943, 0.334)$.
(c) The first principal component is approximately $(-0.323, 0.843)$.
(d) The first principal component is approximately 23.916.

The first principal component is the first column u_1 *of the matrix* U *in the SVD* $A = U\Sigma V^*$*. A numerical evaluation yields* $u_1 \approx (0.334, 0.943)$*, which makes sense looking at a plot of the data.*

Total of marks: 10