Elements of Linear Algebra
Homework 1 Solutions

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Problem 1

(a) The BAC-CAB identity

Let $\mathbf{w} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Then:

1. $\mathbf{w} \perp \mathbf{a} \implies \mathbf{w} \cdot \mathbf{a} = 0$
2. $\mathbf{w} \perp \mathbf{b} \times \mathbf{c} \implies \mathbf{w} = \lambda \mathbf{b} + \mu \mathbf{c}$ for some $\lambda, \mu \in \mathbb{R}$

Combining (i) and (ii), we see that $\lambda \mathbf{b} \cdot \mathbf{a} + \mu \mathbf{c} \cdot \mathbf{a} = 0$, so

$$\frac{\lambda}{\mu} = -\frac{\mathbf{c} \cdot \mathbf{a}}{\mathbf{b} \cdot \mathbf{a}}$$

Moreover, $\mathbf{w}$ must be linear in each of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (scaling each of the vectors must scale the triple product by the same factor).

So the only possibility is $\lambda = \mathbf{c} \cdot \mathbf{a}$ and $\mu = -\mathbf{b} \cdot \mathbf{a}$, or the opposite sign. Testing, e.g., taking $\mathbf{a} = \mathbf{e}_1$ and $\mathbf{b} = \mathbf{e}_2$, using the right-hand rule, confirms the choice of sign as stated.

Note: This solution is elegant, but hard to find. You can always check such identities by computing both sides and comparing.

(b) The Jacobi identity

Use (a):

$$\begin{aligned} &\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \\ &= 0 \end{aligned}$$

Problem 2

(a) The Cauchy-Binet formula

We use the identity $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \mathbf{c} \cdot (\mathbf{d} \times (\mathbf{a} \times \mathbf{b})) \\ &= \mathbf{c} \cdot [\mathbf{a}(\mathbf{d} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{d} \cdot \mathbf{a})] \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \end{aligned}$$

(b) The squared norm identity

Solution 1: Use (a) with $\mathbf{c} = \mathbf{a}$, $\mathbf{d} = \mathbf{b}$:

$$\|\mathbf{a} \times \mathbf{b}\|^2 = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{b}) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

Solution 2: $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 \sin^2\theta$, where θ is angle between $\mathbf{a}$ and $\mathbf{b}$
$(\mathbf{a} \cdot \mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 \cos^2\theta$
So identity follows from $\sin^2\theta = 1 - \cos^2\theta$.

Problem 3

We need one more direction vector for the plane, e.g.:

$$\mathbf{w} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

So a parallel parametric representation of the plane is:

$$\mathbf{x} = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

(Others are possible!)

Problem 4

Line: $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$
Square distance to $\mathbf{p}$: $g(\lambda) = \|\mathbf{p} - (\mathbf{a} + \lambda\mathbf{v})\|^2$

$$\begin{aligned} &= \|(\mathbf{p} - \mathbf{a}) - \lambda\mathbf{v}\|^2 \\ &= \|\mathbf{p} - \mathbf{a}\|^2 - 2\lambda\mathbf{v} \cdot (\mathbf{p} - \mathbf{a}) + \lambda^2\|\mathbf{v}\|^2 \end{aligned}$$

To find the minimum, we differentiate:

$$g'(\lambda) = -2\mathbf{v} \cdot (\mathbf{p} - \mathbf{a}) + 2\lambda\|\mathbf{v}\|^2 = 0$$

$$\implies \lambda = \frac{\mathbf{v} \cdot (\mathbf{p} - \mathbf{a})}{\|\mathbf{v}\|^2}$$

Where $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \implies \|\mathbf{v}\|^2 = 2$
$\mathbf{p} - \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$
$\mathbf{v} \cdot (\mathbf{p} - \mathbf{a}) = 2 \cdot 1 - 1 \cdot 0 = 1$

$$\begin{aligned} \implies g_{min} &= \left\| \mathbf{p} - \mathbf{a} - \frac{\mathbf{v} \cdot (\mathbf{p} - \mathbf{a})}{\|\mathbf{v}\|^2}\mathbf{v} \right\|^2 \\ &= \left\| (2, 1, -3) - \frac{1}{2}(1, -1, 0) \right\|^2 \\ &= \left\| (\frac{3}{2}, \frac{3}{2}, -3) \right\|^2 \end{aligned}$$

$$= \frac{9}{4} + \frac{9}{4} + 9 = \frac{54}{4}$$

$$\therefore d = \frac{\sqrt{54}}{2}$$

Note: This formula is the projection of $\mathbf{p} - \mathbf{a}$ *on the direction* $\mathbf{v}$*, which will still be discussed in class.*