Week 6: The determinant

1.

MULTI 1.0 point 0 penalty Single Shuffle

A 4×4 invertible matrix A has determinant $\det(A) = \frac{1}{2}$. Find $\det(2A), \det(-A), \det(A^2)$, and $\det(A^{-1})$.

(a) $\det(2A) = 8, \det(-A) = \frac{1}{2}, \det(A^2) = \frac{1}{4}, \det(A^{-1}) = 2$ (100%)
(b) $\det(2A) = 1, \det(-A) = -\frac{1}{2}, \det(A^2) = \frac{1}{4}, \det(A^{-1}) = 2$
(c) $\det(2A) = 2, \det(-A) = \frac{1}{2}, \det(A^2) = \frac{1}{2}, \det(A^{-1}) = \frac{1}{2}$
(d) $\det(2A) = 1, \det(-A) = \frac{1}{2}, \det(A^2) = \frac{1}{2}, \det(A^{-1}) = \frac{1}{16}$

Note that for any two matrices M, N we have $\det(MN) = \det(M)\det(N)$. Hence

$$\begin{aligned}
\det(2A) &= \det(2I \cdot A) = \det(2I)\det(A) = 2^4 \det(A) = 8,\\
\det(-A) &= \det(-I \cdot A) = \det(-I)\det(A) = (-1)^4 \det(A) = \frac{1}{2},\\
\det(A^2) &= \det(A \cdot A) = \det(A)\det(A) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4},\\
A \cdot A^{-1} = I &\implies \det(A)\cdot\det(A^{-1}) = 1 \implies \det(A^{-1}) = \frac{1}{\det(A)} = 2.
\end{aligned}$$

2.

MULTI 1.0 point 0 penalty Single Shuffle

A rotation about the y-axis by an angle θ in $\mathbb{R}^3$ is described by the matrix

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}.$$

What is $\det(R_y(\theta))$?

(a) 1 (100%)
(b) 0
(c) -1
(d) $\cos^2(\theta) - \sin^2(\theta)$

$$\begin{aligned}
\det(R_y(\theta)) &= \det \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}\\
&= \cos(\theta)\cdot\det\begin{bmatrix} 1 & 0 \\ 0 & \cos(\theta) \end{bmatrix} - 0\cdot\det\begin{bmatrix} 0 & 0 \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} + \sin(\theta)\cdot\det\begin{bmatrix} \cos(\theta) & 0 \\ -\sin(\theta) & 0 \end{bmatrix}\\
&= \cos^2(\theta) + \sin^2(\theta)\\
&= 1.
\end{aligned}$$

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

What is the volume of the parallelopiped spanned by the vectors $v_1 = (3, 2, 1)$, $v_2 = (0, 3, 2)$, $v_3 = (0, 0, 3)$?

(a) 27 (100%)
(b) 0
(c) 12
(d) 9

We just need to compute the determinant of the matrix

$$P = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since the matrix is upper triangular, the volume of the parallelopiped is $\det(P) = 3^3 = 27$.

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

What is the determinant of the $n \times n$ matrix

$$U = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (n-1) & (n-1) \\ 0 & \cdots & 0 & 0 & n \end{bmatrix}.$$

(a) $n!$ (100%)
(b) 0
(c) n
(d) $\dfrac{n(n+1)(2n+1)}{6}$

Since the matrix is upper-triangular, the determinant is just the product of the entries in the diagonal. Therefore $\det(U) = 1 \cdot 2 \cdot \cdots (n-1) \cdot n = n!$.

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the $n \times n$ matrix C_n with entries which simply count from 1 to n^2. For example $C_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. What is the determinant of C_n?

(a) $\det(C_1) = 1, \det(C_2) = -2, \det(C_n) = 0$ for $n > 2$. (100%)
(b) $\det(C_n) = (-1)^{n+1}n$
(c) $\det(C_1) = 1, \det(C_2) = 2, \det(C_n) = n$ for $n > 2$.
(d) $\det(C_1) = 1, \det(C_2) = -2, \det(C_3) = 0, \det(C_n) = -n^2$ for $n > 3$.

For $n = 1$ and $n = 2$ we can compute

$$\det(C_1) = 1,$$
$$\det(C_2) = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2.$$

We now fix n, we can see that the entries of C_n are $c_{i,j} = (i-1)n + j$. Whenever $i > 2$,

$$\begin{aligned} c_{i,j} &= (i-1)n + j \\ &= (i-1)n + ((i-1) - (i-2))j \\ &= (i-1)(n+j) - (i-2)j \\ &= (i-1)c_{2,j} - (i-2)c_{1,j}. \end{aligned}$$

Hence the i-th row can be written as a linear combination of rows 1 and 2 and we have $\det(C_n) = 0$ for $n > 2$.

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $u = (2,3,5), v = (-1,4,-10), w = (1,-2,-8)$ be vectors in $\mathbb{R}^3$. Use facts about the determinant to check whether u, v, w are linearly independent.

(a) The vectors are not linearly independent.
(b) The vectors are linearly independent. (100%)
(c) Cannot be determined from the information given.

We can compute the determinant of the matrix that has u, v, w as rows.

$$\det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & -10 \\ 1 & -2 & -8 \end{bmatrix} = 2C_{1,1} + C_{2,1} + C_{3,1} = 2 \cdot (-52) - 14 - 50 = -168.$$

Hence the vectors are linearly independent.

7.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

First recall that in general $\det(A+B) \neq \det(A) + \det(B)$. Now let $p, q, r, s \in \mathbb{R}$ and consider the matrices

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -r & -s \\ p & q \end{bmatrix}.$$

Compute $\det(A) + \det(B)$ and $\det(A+B)$. If these values are equal what is the common value? If not, what is the difference $\det(A+B) - (\det(A) + \det(B))$?

(a) $\det(A) + \det(B) = \det(A+B) = 2(ps - qr)$. (100%)
(b) $\det(A+B) - \det(A) - \det(B) = -rq$
(c) $\det(A+B) - \det(A) - \det(B) = rq$
(d) $\det(A+B) - \det(A) - \det(B) = -2ps$

We use the formula for the determinant of a 2×2 matrix and compute the values directly.

$$\begin{aligned}\det(A) &= ps - qr \\ \det(B) &= -qr - (-ps) = ps - qr \\ \det(A+B) &= (p-r)(q+s) - (q-s)(r+p) = 2(ps-qr) = \det(A) + \det(B).\end{aligned}$$

In this special example we see that the determinant is indeed additive! Keep in mind, however, that this is not true in general.

8.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the matrix

$$H = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

What are the cofactors $C_{1,1}$ and $C_{1,2}$? What is $\det(H)$?

(a) $C_{1,1} = 5, C_{1,2} = -2, \det(H) = 8$ (100%)
(b) $C_{1,1} = 5, C_{1,2} = 2, \det(H) = 8$
(c) $C_{1,1} = 2, C_{1,2} = 2, \det(H) = 8$
(d) $C_{1,1} = 5, C_{1,2} = -2, \det(H) = 0$

$$C_{1,1} = (-1)^{1+1} \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 5,$$

$$C_{1,2} = (-1)^{1+2} \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = -2,$$

$$\det(H) = 2C_{1,1} + C_{1,2} = 2 \cdot 5 - 2 = 8.$$

9.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the matrix

$$H = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{bmatrix}$$

What are the cofactors $C_{3,4}$ and $C_{4,4}$?

(a) $C_{3,4} = -27, C_{4,4} = -1$ (100%)
(b) $C_{3,4} = 27, C_{4,4} = -1$
(c) $C_{3,4} = 27, C_{4,4} = 1$
(d) $C_{3,4} = -27, C_{4,4} = 1$

$$C_{3,4} = (-1)^{3+4} \det \begin{bmatrix} 2 & 5 & -3 \\ -2 & -3 & 2 \\ -1 & 6 & 4 \end{bmatrix} = -(-48 + 76 - 1) = -27,$$

$$C_{4,4} = (-1)^{4+4} \det \begin{bmatrix} 2 & 5 & -3 \\ -2 & -3 & 2 \\ 1 & 3 & -2 \end{bmatrix} = 2 \cdot 0 + 2 \cdot (-1) + 1 \cdot 1 = -1.$$

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the matrix

$$H = \begin{bmatrix} -\lambda & 2 & 7 & 12 \\ 3 & 1-\lambda & 2 & -4 \\ 0 & 1 & -\lambda & 7 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$$

where λ is an unknown. Find the $C_{4,4}$ cofactor and compute the determinant of the matrix.

(a) $\det(H) = (2-\lambda)C_{4,4} = \lambda^4 - 3\lambda^2 - 6\lambda^2 - 5\lambda + 42$ (100%)
(b) $\det(H) = \lambda^4 + \lambda^3 + 6\lambda^2 + 4$
(c) $\det(H) = \lambda^4 + \lambda^3 + \lambda^2 - 5\lambda + 42$
(d) $\det(H) = \lambda^4 + 8\lambda^3 + 3\lambda + 5$

$$C_{4,4} = \det \begin{bmatrix} -\lambda & 2 & 7 \\ 3 & 1-\lambda & 2 \\ 0 & 1 & -\lambda \end{bmatrix} \stackrel{*}{=} \det \begin{bmatrix} -\lambda & 2 & 7+2\lambda \\ 3 & 1-\lambda & 2+\lambda(1-\lambda) \\ 0 & 1 & 0 \end{bmatrix} = -\lambda^3 + \lambda^2 + 8\lambda + 21,$$

$\det(H) = (2-\lambda)C_{4,4} = \lambda^4 - 3\lambda^2 - 6\lambda^2 - 5\lambda + 42.$

Above at $(*)$ we replace the last column by its sum with λ times the second column.

Total of marks: 10