Constructor University
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Elements of Linear Algebra

Homework 5 (covering Weeks 9 and 10)

Problem 1 [4 points]

A *graph* is a mathematical structure consisting of a set $V = \{v_1, \dots, v_n\}$ of vertices and a set E of edges between these vertices. The *adjacency matrix* of a graph is a matrix A such that

$$A_{i,j} = \begin{cases} 1 & \text{there is an edge connecting } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}.$$

A *walk* of length k on a graph is a sequence of vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ such that any two successive vertices have an edge connecting them. More precisely, for any j, v_{i_j} and $v_{i_{j+1}}$ are connected. The total number of walks from vertex v_i to vertex v_j is given by the (i, j) entry of A^k. Consider a graph with three vertices $V = \{v_1, v_2, v_3\}$ such that there are edges between v_1 and v_2, v_2 and v_3, and v_1 and v_3. Find the number of walks of length $k = 10$ starting at v_1 and ending at v_2.

Solution: The adjacency matrix of the given graph is seen to be

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We first compute the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = -(\lambda - 2)(\lambda + 1)^2.$$

Hence the eigenvalues are $-1, -1, 2$. We now compute the eigenvectors

$$\lambda = 2 : 0 = (A - 2I)v = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} v \implies v \in \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right),$$

$$\lambda = -1 : 0 = (A + I)v = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} v \implies v \in \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

Hence A is diagonalizable with $A = V \Lambda V^{-1}$ where

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Hence in our case,

$$A^{10} = V \cdot \begin{bmatrix} (-1)^{10} & 0 & 0 \\ 0 & (-1)^{10} & 0 \\ 0 & 0 & 2^{10} \end{bmatrix} \cdot V^{-1} = \begin{bmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{bmatrix}.$$

Now, the number of walks from v_1 to v_2 is just given by the $(1, 2)$ entry in A^{10} and is therefore 341.

Problem 2 [**4 points**]

In class, we briefly discussed the Jordan normal form. Just to get an idea how it can be useful, let us consider the matrix

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

with some $\lambda \in \mathbb{R}$. Find a general formula for J^k for any $k \in \mathbb{N}$. *Hint: Start by computing J^2, J^3, and then find the general pattern. It is also a good exercise to write down a nice clear proof of the formula using induction.*

Solution:

We're going to prove the following formula by induction:

$$J^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k. \end{pmatrix}$$

The base case is obvious. Now, assume that the formula holds for $k = n$, and we need to prove it for $k = n + 1$. We get

$$\begin{aligned} J^{n+1} &= J^n J \\ &= \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{n+1} & \lambda^n + n\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix} \end{aligned}$$

So the formula holds for $k = n + 1$, and, according to the method of mathematical induction, for all $k \in \mathbb{N}$.

Problem 3 [4 points]
Show that the matrix

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

is unitary, and compute all the eigenvalues and eigenvectors.

Solution:
To show that U is unitary, we compute $UU^\dagger$:

$$\begin{aligned} UU^\dagger &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \\ &= I. \end{aligned}$$

So U is indeed unitary. To find the eigenvalues, we compute the characteristic polynomial:

$$P(\lambda) = \lambda^2 - \frac{1+i}{\sqrt{2}}\lambda + i.$$

The discriminant is $D = \frac{(1+i)^2}{2} - 4i = -3i = 3e^{-i\frac{\pi}{2}}$. So the principal square root of D is $\sqrt{D} = \sqrt{3}e^{-i\frac{\pi}{4}} = \frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}i$. The eigenvalues are then given by the following formulas:

$$\begin{aligned} \lambda_1 &= \frac{1+i+\sqrt{2}\sqrt{D}}{2\sqrt{2}} \\ &= \frac{1+\sqrt{3}+(1-\sqrt{3})i}{2\sqrt{2}}, \\ \lambda_2 &= \frac{1+i-\sqrt{2}\sqrt{D}}{2\sqrt{2}} \\ &= \frac{1-\sqrt{3}+(1+\sqrt{3})i}{2\sqrt{2}}. \end{aligned}$$

To find the eigenvector corresponding to λ_1, we solve the equation $(U - \lambda_1 I)\vec{v} = 0$:

$$\begin{pmatrix} \frac{1-\sqrt{3}-(1-\sqrt{3})i}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{-(1+\sqrt{3})+(1+\sqrt{3})i}{2\sqrt{2}} \end{pmatrix} \vec{v} = 0.$$

Since the rows in this system are linearly dependent, the equation above is equivalent to

$$\begin{pmatrix} \frac{1-\sqrt{3}-(1-\sqrt{3})i}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \vec{v} = 0.$$

It is easy to see that vector $\vec{v}_1 = \left(1, -\frac{1-\sqrt{3}-(1-\sqrt{3})i}{2}\right)^T$ is a solution of this equation, and that $\vec{v}_1$ spans the corresponding eigenspace.

To find the eigenvector corresponding to λ_2, we solve the equation $(U - \lambda_2 I)\vec{v} = 0$:

$$\begin{pmatrix} \frac{1+\sqrt{3}-(1+\sqrt{3})i}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{(\sqrt{3}-1)-(\sqrt{3}-1)i}{2\sqrt{2}} \end{pmatrix} \vec{v} = 0.$$

Since the rows in this system are linearly dependent, the equation above is equivalent to

$$\begin{pmatrix} \frac{1+\sqrt{3}-(1+\sqrt{3})i}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \vec{v} = 0.$$

It is easy to see that vector $\vec{v}_2 = \left(1, -\frac{1+\sqrt{3}-(1+\sqrt{3})i}{\sqrt{2}}\right)^T$ is a solution of this equation, and that $\vec{v}_2$ spans the corresponding eigenspace.

Problem 4 [4 points]

A general unitary 2×2 matrix can be written in the form

$$U = \begin{pmatrix} a & b \\ -e^{i\varphi}\overline{b} & e^{i\varphi}\overline{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$, and $\varphi \in [0, 2\pi)$. For matrices of this form, check explicitly the following properties of unitary matrices that we stated in class: that the columns are orthonormal, that the rows are orthonormal, and that $U^\dagger = U^{-1}$. Also compute the determinant of U, and verify that it has absolute value one. (The computation of eigenvalues and eigenvectors is a bit lengthy, so let us skip it here.)

Solution:
a) Denote the column vectors by $\vec{c}_1, \vec{c}_2$. To check the orthogonality of columns, we compute the scalar product $\langle \vec{c}_1, \vec{c}_2 \rangle$. We get:

$$\begin{aligned}\langle \vec{c}_2, \vec{c}_1 \rangle &= a\bar{b} - e^{i\phi}\bar{b}e^{-i\phi}a \\ &= a\bar{b} - \bar{b}a \\ &= 0.\end{aligned}$$

So the columns are indeed orthogonal. Now we compute the norms of the columns:

$$\begin{aligned}||\vec{c}_1|| &= \sqrt{a\bar{a} + e^{i\phi - i\phi}\bar{b}b} \\ &= \sqrt{a\bar{a} + b\bar{b}} \\ &= \sqrt{|a|^2 + |b|^2} \\ &= 1.\end{aligned}$$

Similarly, we can show that $||\vec{c}_2|| = 1$. Therefore, the columns are indeed orthonormal.
b) Denote the row vectors by $\vec{r}_1, \vec{r}_2$. To check the orthogonality of rows, we compute the dot product $\langle \vec{r}_1, \vec{r}_2 \rangle$. We get:

$$\begin{aligned}\langle \vec{r}_2, \vec{r}_1 \rangle &= -abe^{-i\phi} + abe^{-i\phi} \\ &= 0.\end{aligned}$$

So the rows are indeed orthogonal. Row norms are computed similarly to the column norms.
c) First, we compute $U^\dagger$:

$$U^\dagger = \begin{pmatrix} \bar{a} & -e^{-i\phi}b \\ \bar{b} & e^{-i\phi}a \end{pmatrix}.$$

Now, we compute the product $UU^\dagger$:

$$\begin{aligned}UU^\dagger &= \begin{pmatrix} a & b \\ -e^{i\phi}\bar{b} & e^{i\phi}\bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} & -e^{-i\phi}b \\ \bar{b} & e^{-i\phi}a \end{pmatrix} \\ &= \begin{pmatrix} a\bar{a} + b\bar{b} & 0 \\ 0 & a\bar{a} + b\bar{b} \end{pmatrix} \\ &= I.\end{aligned}$$

The last equality holds because $a\bar{a} + b\bar{b} = |a|^2 + |b|^2 = 1$. Since the inverse matrix is unique, $UU^\dagger = I$ proves that $U^{-1} = U^\dagger$.
d)

$$\begin{aligned}\det(U) &= e^{i\phi}a\bar{a} + e^{i\phi}b\bar{b} \\ &= e^{i\phi}(a\bar{a} + b\bar{b})\end{aligned}$$

Now we compute its absolute value:

$$\begin{aligned}|\det(U)| &= |e^{\phi}||a\bar{a} + b\bar{b}| \\ &= 1 \cdot 1 \\ &= 1.\end{aligned}$$

Problem 5 [4 points]
Show that for anti-Hermitian matrices all eigenvalues are purely imaginary or zero, i.e., that they can be written as $\lambda = i\alpha$ with $\alpha \in \mathbb{R}$.

Solution:
By definition, matrix A is anti-Hermitian if $A^\dagger = -A$. This condition implies that

$$\begin{aligned} A^\dagger A &= (-A)A \\ &= -A^2 \\ &= A(-A) \\ &= AA^\dagger \end{aligned}$$

so A is normal. Then we know that if A has an eigenvalue λ, $A^\dagger$ has an eigenvalue $\overline{\lambda}$. Combined with the property $A^\dagger = -A$, this implies that $\overline{\lambda} = -\lambda$ for every eigenvalue of A. In turn, this means that $\mathrm{Re}(\lambda) = 0$, so λ must be purely imaginary or 0, which proves the claim.

Bonus Problem [8 points]
Let us consider the Hermitian matrix

$$H = \frac{\pi}{4}\left(\begin{array}{cc} 1 & i \\ -i & 1 \end{array}\right).$$

Compute e^{iH} in two different ways: first, by diagonalizing H, and second, by directly computing the power series (think about what H^2 is, and deduce from that what H^k is). Verify that the resulting matrix is unitary.

Solution:
First, we compute e^{iH} by diagonalizing H. The characteristic polynomial is

$$\begin{aligned} P(\lambda) &= \det(H - \lambda I) \\ &= \lambda^2 - \frac{\pi}{2}\lambda \\ &= \lambda\left(\lambda - \frac{\pi}{2}\right). \end{aligned}$$

This means that the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \frac{\pi}{2}$. The corresponding normalized eigenvectors are $\vec{v}_1 = \dfrac{1}{\sqrt{2}}(-i,\ 1)^T$ and $\vec{v}_2 = \dfrac{1}{\sqrt{2}}(i,\ 1)^T$. So the matrix that diagonalizes H is

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$

Since $\vec{v}_1$ and $\vec{v}_2$ form an orthonormal system, P is unitary and $P^{-1} = P^\dagger$. So H can be diagonalized as follows:

$$\begin{aligned} H &= PDP^{-1} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \frac{\pi}{2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}. \end{aligned}$$

This allows us to compute the matrix exponent:

$$\begin{aligned} e^{iH} &= Pe^{iD}P^{-1} \\ &= \frac{1}{2}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^0 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1+i & 1+i \end{pmatrix}. \end{aligned}$$

Now we compute e^{iH} as a sum of the power series. Using mathematical induction, we can show that for $n \geq 1$ the following holds:

$$H^n = \frac{\pi^n}{2^{n+1}}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

To compute e^{iH}, we write the formal power series:

$$\begin{aligned} e^{iH} &= \sum_{k=0}^{\infty} \frac{(iH)^k}{k!} \\ &= I + \sum_{k=1}^{\infty} \left(\frac{(i\pi)^n}{n!2^{n+1}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \right) \\ &= I + \left(\sum_{k=1}^{\infty} \frac{(i\pi)^n}{n!2^{n+1}} \right) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \end{aligned}$$

By considering the exponential series of $i = e^{i\frac{\pi}{2}}$ we can see that $\sum_{k=1}^{\infty} \frac{(i\pi)^n}{n!2^{n+1}} = \frac{e^{i\frac{\pi}{2}} - 1}{2} = \frac{i-1}{2}$. This gives us a simple formula for the expression above:

$$\begin{aligned} e^{iH} &= I + \frac{i-1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{i-1}{2} & i\frac{(i-1)}{2} \\ -i\frac{(i-1)}{2} & 1 + \frac{i-1}{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1+i & -1-i \\ 1+i & 1+i \end{pmatrix}. \end{aligned}$$

So the results of both computations coincide. Finally, we check that e^{iH} is unitary by computing the product $e^{iH}(e^{iH})^\dagger$:

$$\begin{aligned} e^{iH}(e^{iH})^\dagger &= \frac{1}{4} \begin{pmatrix} 1+i & -1-i \\ 1+i & 1+i \end{pmatrix} \begin{pmatrix} 1-i & 1-i \\ -1+i & 1-i \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}. \\ &= I \end{aligned}$$

This implies that $(e^{iH})^\dagger = (e^{iH})^{-1}$, so e^{iH} is indeed unitary.