Constructor University
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Elements of Linear Algebra

Homework 6 (covering Weeks 11, 12, 13)

Problem 1 [16 points]
Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}.$$

Compute

(a) if possible, the diagonalization,

(b) the LU decomposition with the convention that L has 1's on the diagonal,

(c) if possible, a Cholesky decomposition,

(d) a QR decomposition in two ways: once with Gram-Schmidt, once using Householder reflections,

(e) a singular value decomposition.

Then, use each of these decompositions to compute A^{-1} and $\det(A)$. Use the LU, QR, and singular value decompositions to find the solutions to $Ax = b$ with $b = (1, 2)^T$.

(Note: Here, we are just considering a simple 2×2 matrix example to illustrate all these methods. Of course, inverses, determinants and solutions to linear equations can be easily found in other ways here.)

a) Diagonalization

To diagonalize the matrix A, we first compute the characteristic polynomial:

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0.$$

Calculating the determinant:

$$(2-\lambda)(1-\lambda) - (-1)(2) = 0.$$

Expanding this gives:

$$2 - 2\lambda - \lambda + \lambda^2 + 2 = 0,$$

which simplifies to:

$$\lambda^2 - 3\lambda + 4 = 0.$$

Using the quadratic formula, we find the eigenvalues:

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}.$$

Thus, the eigenvalues are:

$$\lambda_+ = \frac{1}{2}(3 + i\sqrt{7}), \quad \lambda_- = \frac{1}{2}(3 - i\sqrt{7}).$$

Next, we find the eigenvector corresponding to λ_+:

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_+ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This leads to the system of equations:

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_+ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This leads to the system of equations:

$$2x_1 + 2x_2 = \frac{1}{2}(3 + i\sqrt{7})x_1,$$
$$-x_1 + x_2 = \frac{1}{2}(3 + i\sqrt{7})x_2.$$

From the above, we find the eigenvector:

$$\mathbf{v}_+ = \begin{pmatrix} \frac{1}{2}(-1 - i\sqrt{7}) \\ 1 \end{pmatrix}.$$

Now, for the eigenvalue λ_-:

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_- \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This leads to the system of equations:

$$2x_1 + 2x_2 = \frac{1}{2}(3 - i\sqrt{7})x_1,$$
$$-x_1 + x_2 = \frac{1}{2}(3 - i\sqrt{7})x_2.$$

From the above, we find the eigenvector:

$$\mathbf{v}_- = \begin{pmatrix} \frac{1}{2}(-1 + i\sqrt{7}) \\ 1 \end{pmatrix}.$$

Now, we can construct the matrix V and the diagonal matrix Λ:

$$V = \frac{1}{2} \begin{pmatrix} -1 - i\sqrt{7} & -1 + i\sqrt{7} \\ 1 & 1 \end{pmatrix},$$

$$\Lambda = \frac{1}{2} \begin{pmatrix} 3 + i\sqrt{7} & 0 \\ 0 & 3 - i\sqrt{7} \end{pmatrix}.$$

Finally, we can express the diagonalization of the matrix A as:

$$A = V\Lambda V^{-1}.$$

b) LU decomposition of $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

Method 1: Gaussian elimination:

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}R1 + R2 \to R2} \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

In matrix notation: $\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}}_{U}$$

Method 2: Direct comparison

We want $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ L_{21} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ L_{21} U_{11} & L_{21} U_{12} + U_{22} \end{pmatrix}$

$$\Rightarrow U_{11} = 2, \quad U_{12} = 2,$$

$$L_{21} U_{11} = L_{21} 2 = -1 \Rightarrow L_{21} = -\frac{1}{2}$$

$$L_{21} U_{12} + U_{22} = -\frac{1}{2} \cdot 2 + U_{22} = 1 \Rightarrow U_{22} = 2$$

$$\Rightarrow A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

c) A Cholesky decomposition does not exist since A is not Hermitian.

(d)

QR Decomposition of the Matrix

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}.$$

QR Decomposition using Gram-Schmidt

1. **Define the columns of A**: Let

$$\mathbf{a_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{a_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

2. **Normalize $\mathbf{a_1}$** to get $\mathbf{q_1}$:

$$\|\mathbf{a_1}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

Thus,

$$\mathbf{q_1} = \frac{\mathbf{a_1}}{\|\mathbf{a_1}\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}.$$

3. **Project $\mathbf{a_2}$ onto $\mathbf{q_1}$**:

$$\text{proj}_{\mathbf{q_1}} \mathbf{a_2} = \left(\frac{\mathbf{a_2} \cdot \mathbf{q_1}}{\mathbf{q_1} \cdot \mathbf{q_1}} \right) \mathbf{q_1}.$$

First, compute $\mathbf{a_2} \cdot \mathbf{q_1}$:

$$\mathbf{a_2} \cdot \mathbf{q_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} = \frac{4-1}{\sqrt{5}} = \frac{3}{\sqrt{5}}.$$

Thus,

$$\text{proj}_{\mathbf{q_1}} \mathbf{a_2} = \frac{3}{\sqrt{5}} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix}.$$

4. **Compute $\mathbf{q_2}$**:

$$\mathbf{q_2} = \mathbf{a_2} - \text{proj}_{\mathbf{q_1}} \mathbf{a_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{8}{5} \end{pmatrix}.$$

5. **Normalize $\mathbf{q_2}$**:

$$\|\mathbf{q_2}\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = \sqrt{\frac{80}{25}} = \frac{4}{\sqrt{5}}.$$

Thus,

$$\mathbf{q_2} = \frac{1}{\|\mathbf{q_2}\|}\mathbf{q_2} = \frac{\sqrt{5}}{4} \begin{pmatrix} \frac{4}{5} \\ \frac{8}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{5}}$$

Form the Matrix Q and R

The matrix Q is formed by the normalized vectors $\mathbf{q_1}$ and $\mathbf{q_2}$:

$$Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

The matrix R is formed by the coefficients from the projections:

$$R = \begin{pmatrix} \sqrt{5} & \frac{3}{\sqrt{5}} \\ 0 & \frac{4}{\sqrt{5}} \end{pmatrix}.$$

QR Decomposition using Householder Reflections

1. **Define the vector $\mathbf{x}$**:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

2. **Compute the norm of $\mathbf{x}$**:

$$\|\mathbf{x}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

3. **Define the Householder vector**: Let $\mathbf{u} = \mathbf{x} - \|\mathbf{x}\|\mathbf{e_1}$, where $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sqrt{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{5} \\ -1 \end{pmatrix}.$$

4. **Normalize $\mathbf{u}$**:

$$\|\mathbf{u}\| = \sqrt{(2 - \sqrt{5})^2 + (-1)^2}.$$

The Householder reflection matrix H is given by:

$$H = I - \frac{2\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}}.$$

5. **Apply the Householder reflection**: The resulting matrix R will be upper triangular, and Q will be the product of the Householder matrices.

Final Result

Thus, we can express the QR decomposition of the matrix A as:

$$A = QR,$$

where

$$Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{5} & \frac{3}{\sqrt{5}} \\ 0 & \frac{4}{\sqrt{5}} \end{pmatrix}.$$

(e) The Singular Value Decomposition of a matrix A is given by:

$$A = U\Sigma V^T$$

where U and V are orthogonal matrices, and Σ is a diagonal matrix containing the singular values of A. 1. Compute A^TA and AA^T:

$$A^TA = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

2. Compute the eigenvalues and eigenvectors of A^TA and AA^T: - For A^TA:

$$\text{Eigenvalues: } \lambda_1 = 8, \quad \lambda_2 = 2$$

$$\text{Eigenvectors: } \mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- For AA^T:

$$\text{Eigenvalues: } \lambda_1 = 8, \quad \lambda_2 = 2$$

$$\text{Eigenvectors: } \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

3. Form the matrices U, Σ, and V:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Thus, the Singular Value Decomposition (SVD) of A is:

$$A = U\Sigma V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Problem 2 [4 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the following slight perturbation of this matrix,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ \frac{1}{60\,000} & 0 & 0 & 0 \end{pmatrix}.$$

Rest of Problem 1: A^{-1}, $\det A$, $Ax=b$

- Diagonalization: $\det A = \det V\Lambda V^{-1} = \det V \det\Lambda \underbrace{\det V^{-1}}_{=\frac{1}{\det V}} = \det\Lambda = \lambda_+\lambda_- = \left|\frac{1}{2}(3+i\sqrt{7})\right|^2 = 4$

  - $A = V\Lambda V^{-1} \Rightarrow A^{-1} = V\Lambda^{-1}V^{-1} = \dots$

- LU: $\det A = \det LU = \det L \det U = \det U = 2\cdot 2 = 4$

  - $A^{-1} = U^{-1}L^{-1} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

  - $Ax = b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Leftrightarrow L\underbrace{Ux}_{=:y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow Ly = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

  $\Rightarrow y_1 = 1, \quad -\frac{1}{2}y_1 + y_2 = 2 \Rightarrow y_2 = 2 + \frac{1}{2} = \frac{5}{2}$

  $Ux = y = \begin{pmatrix} 1 \\ \frac{5}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \end{pmatrix} \Rightarrow 2x_2 = \frac{5}{2} \Rightarrow x_2 = \frac{5}{4},$

  $2x_1 + 2x_2 = 1 \Rightarrow x_1 = \frac{1}{2} - \frac{5}{4} = -\frac{3}{4}$

  $\Rightarrow x = \begin{pmatrix} -\frac{3}{4} \\ \frac{5}{4} \end{pmatrix}$

- QR: $\det A = \det Q \det R = \underbrace{1}_{\det Q} R_{11}\cdot R_{22} = \sqrt{5}\,\frac{4}{\sqrt{5}} = 4$

  - $A^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T = \dots$

  - $Ax = b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow QRx = b \Rightarrow x = R^{-1}Q^Tb$

  $\Rightarrow x = \begin{pmatrix} \sqrt{5} & \frac{3}{\sqrt{5}} \\ 0 & \frac{4}{\sqrt{5}} \end{pmatrix}^{-1} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{4}\underbrace{\begin{pmatrix} \frac{4}{\sqrt{5}} & -\frac{3}{\sqrt{5}} \\ 0 & \sqrt{5} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}}_{\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$

• SVD: $\det A = \det U \det \Sigma \det V = 1 \cdot \Sigma_{11} \Sigma_{22} \cdot 1 = 2\sqrt{2} \cdot \sqrt{2} = 4$

• $A^{-1} = V \Sigma^{-1} U^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

• $Ax = b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow U \Sigma V^T x = b \Rightarrow x = V \Sigma^{-1} U^T b$

$\Rightarrow x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ \frac{5}{4} \end{pmatrix}$

Problem 2 ctd.:

Compute

(a) the eigenvalues and eigenvectors of A and B,

(b) a singular value decomposition of A and B.

What can you conclude?

(a) Eigenvalues and Eigenvectors

Matrix A

To find the eigenvalues, we solve the characteristic equation $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$$

The determinant is:

$$\det(A - \lambda I) = (-\lambda)(-\lambda)(-\lambda)(-\lambda) = \lambda^4$$

Thus, the eigenvalues of A are:

$$\lambda = 0, 0, 0, 0$$

To find the eigenvectors, we solve $(A - \lambda I)\mathbf{v} = 0$. Since all eigenvalues are zero, we solve $A\mathbf{v} = 0$.

$$A\mathbf{v} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us the system:

$$\begin{cases} v_2 = 0 \\ 2v_3 = 0 \\ 3v_4 = 0 \\ 0 = 0 \end{cases}$$

Thus, $v_2 = v_3 = v_4 = 0$ and v_1 can be any value. Therefore, the eigenvectors are:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Matrix B

To find the eigenvalues, we solve the characteristic equation $\det(B - \lambda I) = 0$.

$$B - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ \frac{1}{60000} & 0 & 0 & -\lambda \end{pmatrix}$$

The determinant is:

$$\det(B - \lambda I) = \lambda^4 - \frac{1}{60000} = 0$$

Thus, the eigenvalues of B are approximately:

$$\lambda = \pm\frac{1}{10}, \pm\frac{i}{10}$$

(b) Singular Value Decomposition (SVD)

Matrix A

The singular values of A are the square roots of the eigenvalues of $A^T A$.

$$A^T A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

The eigenvalues of $A^T A$ are $0, 1, 4, 9$. Thus, the singular values of A are:

$$\sigma = 3, 2, 1, 0$$

Matrix B

The singular values of B are the square roots of the eigenvalues of $B^T B$.

$$B^T B = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{60000} \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ \frac{1}{60000} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3600000000} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

The eigenvalues of $B^T B$ are $\frac{1}{3600000000}, 1, 4, 9$. Thus, the singular values of B are:

$$\sigma = 3, 2, 1, \frac{1}{60000}$$

Conclusion

The eigenvalues of A are highly sensitive to perturbations, as seen by the significant change in eigenvalues from $0, 0, 0, 0$ to $\pm\frac{1}{10}, \pm\frac{i}{10}$ with a small perturbation. This indicates instability in the eigenvalues.

In contrast, the singular values of A and B are more stable, changing only slightly from $3, 2, 1, 0$ to $3, 2, 1, \frac{1}{60000}$. This demonstrates the stability of singular values compared to eigenvalues.