538 Riddler Express

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Question: You and your team of cyclists are climbing a mountain with a constant gradient in the Tour de FiveThirtyEight. You make the climb single file, with one rider in front of the other. The first rider sets the tempo for the rest to follow — that is, until that first rider runs out of energy, or "cracks." Upon cracking, a rider can no longer maintain the tempo and falls well behind any riders who still have some remaining energy.

When you begin the climb, there are eight riders on your team, and you start at the very back. The seven riders in front of you have exactly enough energy to make it up the entire climb, as long as they are not the first rider. Setting the tempo is hard work, and riding first in line is twice as exhausting as being one of the other riders.

At some point up the mountain, the first rider cracks. Then the next rider cracks, then the next. Eventually, the last rider in front of you cracks, leaving you to contend with the remaining portion of the mountain all on your own. What fraction of the mountain do you climb alone?

Solution: Begin by noting that as it requires exactly twice as exhausting as being one of the other riders, the rider who begins in the front will "crack" exactly halfway up the mountain. Now, the rider who started in second will be in front. They will have expended half of their energy while riding behind the first rider. Now, with half of the mountain to go, they will expend twice as much energy, so they will crack $\frac{3}{4}$ of the way up the mountain. We can quickly see a pattern: the rider who begins in the k-th position will spend $\frac{1}{2^{k-1}} - \frac{1}{2^k} = 12^k$ time in front before cracking. From this pattern, we can see that the eighth and final rider will ride $\frac{1}{2^7} - \frac{1}{2^8} = \frac{1}{256}$th of the way up before cracking. In particular, they will ride from $\frac{254}{256}$th of the way up to $\frac{255}{256}$th of the way up.

We will prove by induction that the k-th of n riders will ride $\frac{1}{2^k}$ distance in front before cracking.

Base Case: The first rider must expend twice as much energy while riding in front. They have exactly enough energy to reach the top of the mountain. They are in front from the beginning of the ride until they reach the top or crack. Clearly, they will crack halfway up the mountain.

Induction Hypothesis: Suppose that for all riders $1 \leq i \leq k$, the i-th rider will ride $\frac{1}{2^i}$ distance in front before cracking.

Induction Step: Now, we must prove that the $k+1$-th rider will ride $\frac{1}{2^{k+1}}$ distance in front before cracking.

We know that the $k+1$-th rider will take over the front after the k riders in front of them have cracked. From the induction hypothesis, we know that this will occur at the distance:

$\sum_{i=1}^{k} \frac{1}{2^i}$ up the mountain.

$\sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k} = \frac{2^k-1}{2^k}$

As each rider begins with exactly enough energy to ride up the mountain (assuming they do not ride in front), and the $k+1$-th rider has not yet spent any time in front, we know that they have expended $\frac{2^k-1}{2^k}$ of their energy so far, and therefore they have $1 - \frac{2^k-1}{2^k} = \frac{1}{2^k}$ of their energy remaining. As they are now riding in front, they must expend energy at twice the rate they were before. So, they will be able to ride $\frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}$ distance in front before cracking.

This completes the proof.