Homework 2

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Problem 1

Problem Statement: Consider the autonomous ODE $y' = 2y - y^2$

Part (a)

Problem Statement: Find the equilibria.

Solution: To find the equilibrium solutions, we set the derivative equal to zero:

$$2y - y^2 = 0$$

Factorizing the equation:

$$y(2 - y) = 0$$

Thus, the equilibrium solutions are:

$$y = 0 \quad \text{or} \quad y = 2$$

Problem Statement: Determine the stability at each of equilibrium point

Solution: Solve for the sign of the derivative $y' = 2y - y^2$.

1. For $y = 0$:

$$\frac{dy}{dt} = 2y - y^2 = 2(0) - (0)^2 = 0$$

Consider $\frac{dy}{dt}$ near $y = 0$. For $y > 0$, $2y - y^2 > 0$, meaning the solution increases, and for $y < 0$, $2y - y^2 < 0$, meaning the solution decreases. Hence, $y = 0$ is unstable.

2. For $y = 2$:

$$\frac{dy}{dt} = 2(2) - (2)^2 = 4 - 4 = 0$$

Consider $\frac{dy}{dt}$ near $y = 2$. For $y > 2$, $2y - y^2 < 0$, meaning the solution decreases, and for $y < 2$, $2y - y^2 > 0$, meaning the solution increases. Hence, $y = 2$ is stable.

Part (b)

Problem Statement: Sketch the direction field and solution through $(0, 1)$.

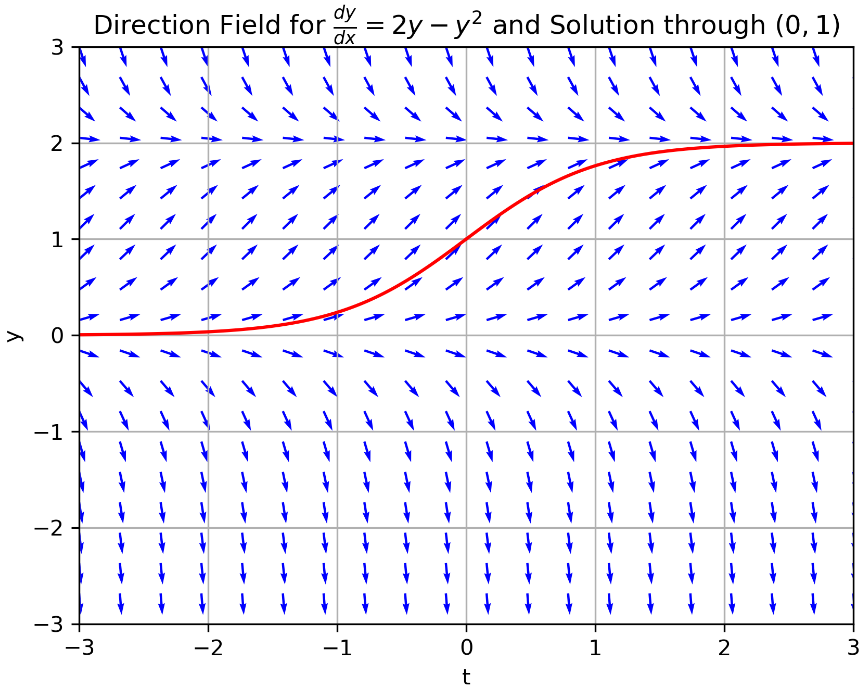

Figure 1: Direction Field and Solution passing through $(0, 1)$

Part (c)

Problem Statement: Describe the long term behavior.

(i) For the initial point $(-1, 3)$, the solution approaches $y = 2$ as $t \to \infty$, since $y = 2$ is a stable equilibrium and the initial value of $y = 3$ is greater than 2.

(ii) For the initial point $(0, -1)$, since $y = 0$ is an unstable equilibrium, the solution decreases indefinitely as $t \to \infty$.

Part (d)

Problem Statement: Find the general solution.

Solution: The differential equation can be rewritten as:

$$\frac{dy}{dt} = 2y - y^2$$

This is separable. We rewrite it as:

$$\frac{1}{y(2-y)}dy = dt$$

Using partial fraction decomposition:

$$\frac{1}{y(2-y)} = \frac{A}{y} + \frac{B}{2-y}$$

Multiplying both sides by $y(2-y)$, we get:

$$1 = A(2-y) + By$$

Equating coefficients:

$$A = \frac{1}{2}, \quad B = \frac{1}{2}$$

Thus, the equation becomes:

$$\frac{1}{2}\left(\frac{1}{y} + \frac{1}{2-y}\right) dy = dt$$

Integrating both sides:

$$\frac{1}{2}\left(\ln|y| - \ln|2-y|\right) = t + C$$

Simplifying:

$$\ln\left(\frac{y}{2-y}\right) = 2t + C$$

Exponentiating both sides:

$$\frac{y}{2-y} = e^{2t+C}$$

Let $C = e^C$, then:

$$\frac{y}{2-y} = Ce^{2t}$$

Solving for y:

$$y = \frac{2Ce^{2t}}{1 + Ce^{2t}}$$

Part (e)

Problem Statement: Find particular solution for IVP

Solution: We are given $y(\ln 2) = 1$. Using the general solution:

$$y = \frac{2e^{2t}}{1 + Ce^{2t}}$$

Substitute $t = \ln 2$ and $y = 1$:

$$1 = \frac{2Ce^{2\ln 2}}{1 + Ce^{2\ln 2}}$$

Simplifying $e^{2\ln 2} = 4$:

$$1 = \frac{8C}{1+4}$$

Solving for C:

$$1 + 4C = 8C$$
$$1 = 4C$$
$$C = \frac{1}{4}$$

Thus, the particular solution is:

$$y(t) = \frac{2 \times \frac{1}{4}e^{2t}}{1 + \frac{1}{4}e^{2t}} = \frac{\frac{1}{2}e^{2t}}{1 + \frac{1}{4}e^{2t}} = \frac{2}{1 + 4e^{-2t}}$$

Thus, the solution to the initial value problem is:

$$y(t) = \frac{2}{1 + 4e^{-2t}}$$

Problem 2

Problem Statement: Graph the isoclines of the ODE $y' = y + t^2$ corresponding to the slopes -1, 0, and 1.

Solution: Isoclines are curves where the slope of the solution is constant. For the ODE $y' = y + t^2$, set $y' = m$ (where m is the constant slope).

$$y + t^2 = m \implies y = m - t^2$$

Thus, the isoclines corresponding to the slopes are:

- For $m = -1$, the isocline is $y = -1 - t^2$.
- For $m = 0$, the isocline is $y = -t^2$.
- For $m = 1$, the isocline is $y = 1 - t^2$.

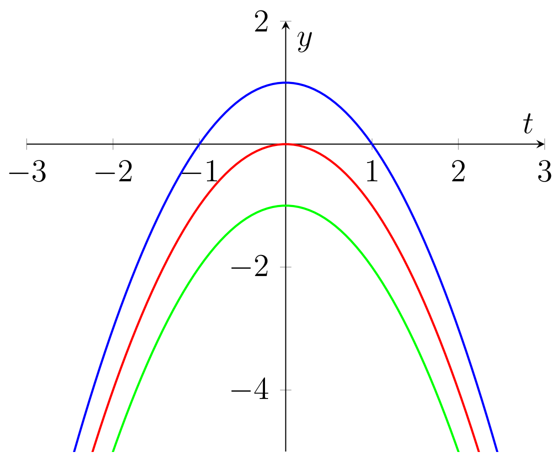

Figure 2: Isoclines for slopes $m = -1$, $m = 0$, and $m = 1$

Problem 3

Part (a)

Problem Statement: Solve the IVP $y' = \frac{t^2+7}{y^4-4y^3}$, $y(1) = 2$, and leave the answer in implicit form.

Solution: This equation can be separated:

$$(y^4 - 4y^3)\,dy = (t^2 + 7)\,dt$$

Integrating both sides:

$$\int (y^4 - 4y^3)\,dy = \int (t^2 + 7)\,dt$$

The integrals are:

$$\frac{y^5}{5} - y^4 = \frac{t^3}{3} + 7t + C$$

Given $y(1) = 2$, substitute to solve for C:

$$\frac{(2)^5}{5} - (2)^4 = \frac{(1)^3}{3} + 7(1) + C$$

Simplifying:

$$\frac{32}{5} - 16 = \frac{1}{3} + 7 + C$$

Solve for C:

$$C = -\frac{254}{15}$$

Thus, the implicit solution is:

$$\frac{y^5}{5} - y^4 = \frac{t^3}{3} + 7t - \frac{254}{15}$$

Part (b)

Problem Statement: Solve $y' = \cos^2(y) \ln|t|$.

Solution: Separate the variables:

$$\frac{1}{\cos^2(y)} dy = \ln|t|\, dt$$

Integrating both sides:

$$\int \sec^2(y)\, dy = \int \ln|t|\, dt$$

$$\tan(y) = t\ln|t| - t$$

Thus, the explicit solution is:

$$y = \tan^{-1}(t \ln t - t + C)$$

Part (c)

Problem Statement: Solve $(t^2 + t)y' + y^2 = ty^2$, with $y(1) = -1$.

Solution: Rewriting the equation:

$$(t^2 + t)y' = ty^2 - y^2 = y^2(t - 1)$$

Separate variables:

$$\frac{1}{y^2} dy = \frac{(t-1)}{(t^2+t)}\, dt$$

Integrating both sides:

$$\int \frac{1}{y^2} dy = \int \frac{(t-1)}{t^2+t}$$

Integration of the right side by partial fractions

$$\int \frac{(t-1)}{t^2+t}$$

$$\frac{(t-1)}{t^2+t} = \frac{(t-1)}{t(t+1)}$$

$$\frac{(t-1)}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

Solve for A and B:

$$\begin{aligned} t-1 &= A(t+1)+Bt \\ &= t(A+B)+A \\ A &= -1 \\ B &= 2 \end{aligned}$$

Substitute:

$$\begin{aligned} \int \frac{(t-1)}{t^2+t} &= \int -\frac{1}{t} + \frac{2}{t+1} \\ &= -\ln|t| + 2\ln|t+1| + C \end{aligned}$$

Plugging the integral back into the original equation:

$$\begin{aligned} \int \frac{1}{y^2}dy &= \int \frac{(t-1)}{t^2+t} \\ -\frac{1}{y} &= -\ln|t| + 2\ln|t+1| + C \end{aligned}$$

Solve for C with y(1) = -1

$$\begin{aligned} -\frac{1}{y} &= -\ln|t| + 2\ln|t+1| + C \\ -\frac{1}{(-1)} &= -\ln|(1)| + 2\ln|(1)+1| + C \\ 1 &= 0 + 2\ln|2| + C \\ C &= 1 - 2\ln|2| \end{aligned}$$

Thus, the explicit solution is:

$$y = \frac{1}{\ln|t| - 2\ln|t+1| - 1 + 2\ln|2|}$$

Problem 4

Problem Statement: Solve given the differential equation: $\frac{dy}{dt} = \frac{2y^4+t^4}{ty^3}$

Solution: We use the substitution $v = \frac{y}{t}$, which gives $y = vt$. Differentiating both sides with respect to t, we obtain:

$$\frac{dy}{dt} = v + t\frac{dv}{dt}$$

Substituting this into the original differential equation:

$$v + t\frac{dv}{dt} = \frac{2(vt)^4 + t^4}{t(vt)^3} = \frac{2v^4t^4 + t^4}{t^4v^3} = \frac{2v^4+1}{v^3}$$

Thus, the equation becomes:

$$v + t\frac{dv}{dt} = \frac{2v^4 + 1}{v^3}$$

Rearranging to isolate $\frac{dv}{dt}$, we get:

$$\frac{dv}{dt} = \frac{v^4 + 1}{tv^3}$$

Finally, separating the variables gives:

$$\frac{v^3}{v^4 + 1}dv = \frac{dt}{t}$$

Integrating both sides:

$$\int \frac{v^3}{v^4 + 1}dv = \int \frac{dt}{t}$$

Integration of left side by u-sub with $u = v^4 + 1$ and $dx = 1/v^3$:

$$\int \frac{v^3}{4vu} = \int \frac{1}{4u}$$
$$\frac{1}{4}\int \frac{1}{u} = \frac{1}{4}\ln|u| = \frac{1}{4}\ln|v^4 + 1|$$
$$\int \frac{v^3}{4vu} = \frac{1}{4}\ln|v^4 + 1|$$

Plugging the integral back into the original equation:

$$\int \frac{v^3}{v^4 + 1}dv = \int \frac{dt}{t}$$

Recalling that $v = \frac{y}{t}$, we get the implicit solution:

$$\frac{1}{4}\ln|(\frac{y}{t})^4 + 1| = \ln|t| + C$$

Problem 5

Problem Statement: Solve the ODE $y' = (y + t)^2$ using $u = y + t$.

Solution: Substitute $u = y + t$, so $y' = u^2$. The equation becomes:

$$\frac{du}{dt} = u^2$$

Because we use the substitution $u = y + t$ we can rearange to give us $y = u - t$. Differentiating both sides with respect to t:

$$\frac{dy}{dt} = \frac{du}{dt} - 1$$

Substituting into the equation $y' = (y+t)^2$:

$$\frac{du}{dt} - 1 = u^2$$
$$\frac{du}{dt} = u^2 + 1$$

This is now a separable differential equation. We separate the variables and integrate:

$$\int \frac{du}{u^2+1} = \int dt$$

Integration of left side by:

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} tan^{-1}(x/a) + C$$

$$\int \frac{du}{u^2+1} = tan^-1(u)$$

Plugging the integral back into the equation:

$$\int \frac{du}{u^2+1} du = \int dt$$

$$tan^{-1}(u) = t + C$$

Solving for u:

$$u = \tan(t+C)$$

Recalling that $u = y + t$, we substitute back:

$$y + t = \tan(t+C)$$

Finally, solving for y:

$$y = \tan(t+C) - t$$

Thus, the general solution to the original differential equation is:

$$y = \tan(t+C) - t$$

Problem 6

Part (a)

Problem Statement: Express the radius of the raindrop as a function of time.

Solution: The rate of decrease of the volume is proportional to the surface area. For a sphere, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. The differential equation is:

$$\frac{dV}{dt} = -kS$$

Substitute $S = 4\pi r^3$ and $V = \frac{4}{3}\pi r^3$

$$\frac{d(\frac{4}{3}\pi r^3)}{dt} = -4k\pi r^3$$

Take the derivative of r with respect to t

$$\begin{aligned} 4\pi r^3 \frac{dr}{dt} &= -4k\pi r^3 \\ r &= -kt + C \end{aligned}$$

Thus r is a linear fucntion with respct to t. We can solve for the unkowns k and c by substituting the intitial conditions $r(0) = 1$ and $r(2) = 1/2$

$$\begin{aligned} r &= -kt + C \\ 1 &= -k(0) + C \\ 1 &= C \end{aligned}$$

$$\begin{aligned} \frac{1}{2} &= -k(2) + 1 \\ \frac{1}{4} &= k \end{aligned}$$

Thus our final function of r with respect to time is:

$$r(t) = -\frac{1}{4}t + 1$$

Part (b)

Problem Statement: When will the raindrop evaporate completely?

Solution: Solve for t when $r(t) = 0$.

$$\begin{aligned} r(t) &= -\frac{1}{4}t + 1 \\ 0 &= -\frac{1}{4}t + 1 \\ -1 &= -\frac{1}{4}t \\ 4 &= t \end{aligned}$$

Problem 7:

Problem Statement: Consider the IVP $y' = t\sqrt{t}, y(1) = 4$. Use Euler's Method to determine an estimate to the value of $y(1.5)$ using step sizes of $h_1 = 0.1$ and $h_2 = 0.05$.

Solution: Yes. The smaller step size results in a more accurate approximation.

t	Approximate y h = 0.1	Approximate y h=0.05	Actual y	Error h=0.1	Error h=0.05
1.0	4.0	4.0	4.0	0.0	0.0
1.1	4.2	4.206304	4.212756	0.012756	0.006452
1.2	4.425433	4.438605	4.4521	0.026667	0.013495
1.3	4.677873	4.698549	4.719756	0.041883	0.021207
1.4	4.959043	4.987936	5.0176	0.058557	0.029664
1.5	5.270807	5.308709	5.347656	0.076849	0.038948

Table 1: Euler's Method Approximations