Homework 3

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Problem 1

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: Picards theorem sates that if $f(t, y)$ is continuous in some region R defined by

$$\{(t, y) \,|\, a < t < b, c < y < d\}$$

and $(t_0, y_0) \in \mathbb{R}$. Then there exists a postivie number h such that the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

Has a solution in an open interval containing $(t_0 - h, t_0 + h)$. The solution is unique if $\frac{\partial f}{\partial y}$ is continuous in R

(a) $y' = ty^{\frac{4}{3}}, \quad y(0) = 0$

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: We will start by determining the continutiy of the function $f(t) = ty^{\frac{4}{3}}$. We will start by finding the partial derivative of f with respect to y

$f(t) = ty^{\frac{4}{3}}$ is continuous for all t and y in the region R defined by $a \leq t \leq b$ and $|y - y_0| \leq M$ for some constant M. Therefore, Picard's Theorem can be used to show the existence of a solution in an open interval containing $t = 0$.

Next we will determine the uniqueness of the solution. We will start by finding the partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \frac{4}{3}ty^{\frac{1}{3}}$$

Since $\frac{\partial f}{\partial y}$ is continuous for all t and y in the region R defined by $a \leq t \leq b$ and $|y - y_0| \leq M$ for some constant M, Picard's Theorem can be used to show the existence of a unique solution in an open interval containing $t = 0$.

(b) $y' = ty^{1/3}, \quad y(0) = 0$

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: We will start by determining the continutiy of the function $f(t) = ty^{\frac{1}{3}}$

$f(t) = ty^{\frac{1}{3}}$ is continuous for all t and y in the region R Therefore, Picard's Theorem can be used to show the existence of a solution in an open interval containing $t = 0$.

Next we will determine the uniqueness of the solution. We will start by finding the partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \frac{1}{3}ty^{-\frac{2}{3}}$$

Since $\frac{\partial f}{\partial y}$ is not continuous for all t and y in the region R defined by $a \leq t \leq b$ and $|y - y_0| \leq M$ for some constant M, Picard's Theorem cannot be used to show the existence of a unique solution in an open interval containing $t = 0$.

(c) $y' = ty^{\frac{1}{3}}, \quad y(0) = 1$

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: We will start by determining the continutiy of the function $f(t) = ty^{\frac{1}{3}}$

$f(t) = ty^{\frac{1}{3}}$ is continuous for all t and y in the region R defined by $a \leq t \leq b$ and $|y - y_0| \leq M$ for some constant M. Therefore, Picard's Theorem can be used to show the existence of a solution in an open interval containing $t = 0$.
Next we will determine the uniqueness of the solution. We will start by finding the partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \frac{1}{3}ty^{-\frac{2}{3}}$$

Since $\frac{\partial f}{\partial y}$ is continuous for all t and y in the region R defined by $a \leq t \leq b$ and $|y - y_0| \leq M$ for some constant M, Picard's Theorem can be used to show the existence of a unique solution in an open interval containing $t = 0$.

Problem 2

Problem Statement: Find the order, linearity, homogeneity, and the variablity of the coefficients of the following:

Problem	Diff Eq	Order	Linearity	Homogeneity	coefficients
(a)	$y'' + 2y' + y = 0$	2	Linear	Homogeneous	Constant
(b)	$\ddot{x} + 2\dot{x} + tx = \sin(t)$	2	Linear	Non-Homogeneous	Variable
(c)	$\cos(y') + ty = 0$	1	Non-Linear	Homogeneous	Variable
(d)	$y'' + ey' + \pi y = 0$	2	Linear	Homogeneous	Constant
(e)	$y' + \frac{1}{1+t^2}y = 7$	1	Linear	Non-Homogeneous	Variable

Problem 3

Problem Statement: Which of the following operators are linear?

Solution: An operator is linear if it satisfies the following properties:

$$L(k\vec{\mathbf{u}}) = kL(\vec{\mathbf{u}}), \quad k \in \mathbb{R}$$
$$L(\vec{\mathbf{u}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{u}}) + L(\vec{\mathbf{w}})$$

(a) $L(\vec{\mathbf{y}}) = y' + 2ty$

Solution: We will start by checking the first property

$$\begin{aligned} L(k\vec{\mathbf{u}}) &= kL(\vec{\mathbf{u}}) \\ L(ky) &= k(y' + 2ty) \\ ky' + 2kty &= ky' + 2kty \\ ky' + 2kty &= ky' + 2kty \end{aligned}$$

Now we will check the second property

$$\begin{aligned} L(\vec{\mathbf{u}} + \vec{\mathbf{w}}) &= L(\vec{\mathbf{u}}) + L(\vec{\mathbf{w}}) \\ L(y_1 + y_2) &= L(y_1) + L(y_2) \\ y_1' + 2ty_1 + y_2' + 2ty_2 &= y_1' + 2ty_1 + y_2' + 2ty_2 \end{aligned}$$

Since both properties are satisfied, the operator $L(\vec{\mathbf{y}}) = y' + 2ty$ is linear

(b) $L(\vec{\mathbf{y}}) = y'' + (1 - y^2) + y$

Solution: We will start by checking the first property

$$\begin{aligned} L(k\vec{\mathbf{u}}) &= kL(\vec{\mathbf{u}}) \\ L(ky) &= k(y'' + (1 - y^2) + y) \\ ky'' + (1 - (ky)^2) + ky &\neq ky'' + k(1 - y^2) + ky \end{aligned}$$

This does not satisfy the first property, so the operator $L(\vec{\mathbf{y}}) = y'' + (1 - y^2) + y$ is not linear. We will check the second property to be thorough.

$$L(\vec{\mathbf{u}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{u}}) + L(\vec{\mathbf{w}})$$
$$L(y_1 + y_2) = L(y_1) + L(y_2)$$
$$y_1'' + (1 - y_1^2) + y_1 + y_2'' + (1 - y_2^2) + y_2 \neq y_1'' + (1 - y_1^2) + y_1 + y_2'' + (1 - y_2^2) + y_2$$

Since the second property is not satisfied, the operator $L(\vec{\mathbf{y}}) = y'' + (1 - y^2) + y$ is not linear.

Problem 4

Problem Statement: Show that if $y_1(t)$ and $y_2(t)$ are solutions of $y' + p(t)y = 0$, then so are $y_1(t) + y_2(t)$ and $cy_1(t)$ for any constant c.

Solution: We will start by assuming that $y_1(t)$ and $y_2(t)$ are solutions of $y' + p(t)y = 0$ and follow the properties of linear homogenoues ODEs. Thus,

$$y_1' + p(t)y_1 = 0$$
$$y_2' + p(t)y_2 = 0$$

Using the first property, we will show that $y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = 0$

$$(y_1 + y_2)' + p(t)(y_1 + y_2) = 0$$
$$y_1' + y_2' + p(t)y_1 + p(t)y_2 = 0$$
$$y_1' + p(t)y_1 + y_2' + p(t)y_2 = 0 + 0 = 0$$

This proves that $y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = 0$.

Next we will show that $cy_1(t)$ is a solution of $y' + p(t)y = 0$

$$(cy_1)' + p(t)(cy_1) = 0$$
$$cy_1' + cp(t)y_1 = 0$$
$$c(y_1' + p(t)y_1) = c \cdot 0 = 0$$

This proves that $cy_1(t)$ is a solution of $y' + p(t)y = 0$.

Problem 5

Problem Statement: Verify that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then show that $c_1y_1(t) + c_2y_2(t)$ is also a solution for any real numbers c_1 and c_2.

(a) $y'' - y' + 6y = 0; \quad y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}$

Solution: We will start by verifying that $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation.

$$\begin{aligned} y_1(t) &= e^{3t} \\ y_1' &= 3e^{3t} \\ y_1'' &= 9e^{3t} \\ 9e^{3t} - 3e^{3t} - 6e^{3t} &= 0 \end{aligned}$$

Therefore, $y_1(t)$ is a solution of the given differential equation. Next we will verify that $y_2(t)$ is a solution

$$\begin{aligned} y_2(t) &= e^{-2t} \\ y_2' &= -2e^{-2t} \\ y_2'' &= 4e^{-2t} \\ 4e^{-2t} + 2e^{-2t} - 6e^{-2t} &= 0 \end{aligned}$$

Therefore, $y_2(t)$ is a solution of the given differential equation. Next we will show that $c_1y_1(t) + c_2y_2(t)$ is also a solution for any real numbers c_1 and c_2.

$$c_1y_1(t) + c_2y_2(t) = c_1e^{3t} + c_2e^{-2t}$$

Problem 6

Problem Statement: Find the general solution to the non homogenoues ODE $y' + \frac{1}{t+1}y = 2$

Solution: We will solve using the superposition principal. $y_{G,NH} = y_{G,H} + y_{P,NH}$

(a)

Problem Satement: Show that $y_p = \frac{t^2+2t}{t+1}$ is a particular solution to the Non-Homogeneous ODE $y' + \frac{1}{t+1}y = 2$

Solution: We will start by finding the derivative of y_p

$$\begin{aligned} y_p &= \frac{t^2 + 2t}{t+1} \\ y_p' &= \frac{(t+1)(2t+2) - (t^2+2t)}{(t+1)^2} \\ y_p' &= \frac{2t^2 + 2t + 2t + 2 - t^2 - 2t}{(t+1)^2} \\ y_p' &= \frac{t^2 + 4t + 2}{(t+1)^2} \end{aligned}$$

Now we will plug y_p and y_p' into the Non-Homogeneous ODE

$$\begin{aligned}
y_p' + \frac{1}{t+1}y_p &= 2 \\
\frac{t^2+4t+2}{(t+1)^2} + \frac{t^2+2t}{(t+1)^2} &= 2 \\
\frac{t^2+4t+2}{t^2+2t+1} + \frac{t^2+2}{t^2+2t+1} &= 2 \\
\frac{2t^2+4t+2}{t^2+2t+1} &= 2 \\
\frac{2(t^2+2t+1)}{t^2+2t+1} &= 2 \\
2 &= 2
\end{aligned}$$

(b)

Problem Statement: Find the general solution to the Non-Homogeneous ODE $y' + \frac{1}{t+1}y = 2$

Solution: We will use the superposition principal to find the general solution to the Non-Homogeneous ODE

$$y_{G,NH} = y_{G,H} + y_{P,NH}$$

We will start by finding the general solution to the Homogeneous ODE

$$\begin{aligned}
y' + \frac{1}{t+1}y &= 0 \\
\frac{1}{y}dy &= -\frac{1}{t+1}dt \\
\ln|y| &= -\ln|t+1| + c \\
y &= \frac{c}{t+1}
\end{aligned}$$

Now we will find the general solution to the Non-Homogeneous ODE

$$y_{G,NH} = \frac{c}{t+1} + \frac{t^2+2t}{t+1}$$

Problem 7

Problem 8

Problem Statement: Solve the differential equation $(x+1)\frac{dy}{x} + y = \ln(t)$

Solution: We will solve using the integrating factor method We will start by rewriting the equation

$$\frac{dy}{dt} + \frac{y}{t+1} = \frac{lnt}{t+1}$$

Set right side equal to zero

$$\frac{dy}{dt} + \frac{y}{t+1} = 0$$

Solve for y

$$\begin{aligned}\frac{dy}{dt} &= -\frac{y}{t+1}\\ \frac{dy}{y} &= -\frac{dt}{t+1}\end{aligned}$$

Integrate both sides

$$\begin{aligned}\int \frac{1}{y}dy &= -\frac{1}{t+1}dt\\ \ln|y| &= -ln|t+1| + c\\ y &= e^{-\ln|t+1|+c}\\ y &= \frac{c}{t+1}\end{aligned}$$

Solve for integrating factor μ

$$\begin{aligned}\mu &= \frac{1}{y_{G,H}}\\ \mu &= t+1\end{aligned}$$

Multiply orignial equation by μ

$$\begin{aligned}\mu\frac{dy}{dt} + \mu\frac{y}{t+1} &= \mu\frac{lnt}{t+1}\\ D[(t+1)y] &= \frac{\ln(t)}{t+1}(t+1)\\ (t+1)y &= \int \ln(t)\\ (t+1)y &= t\ln(t) - t + c\\ y &= \frac{t\ln(t) - t + c}{t+1}\end{aligned}$$