Homework 3

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Problem 1

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: Picards theorem sates that if $f(t, y)$ is continuous in some region R defined by

$$\{(t, y) \,|\, a < t < b, c < y < d\}$$

and $(t_0, y_0) \in \mathbb{R}$. Then there exists a postivie number h such that the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

Has a solution in an open interval containing $(t_0 - h, t_0 + h)$. The solution is unique if $\frac{\partial f}{\partial y}$ is continuous in R

(a) $y' = ty^{\frac{4}{3}}, \quad y(0) = 0$

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: We will start by determining the continutiy of the function $f(t) = ty^{\frac{4}{3}}$ by finding the partial derivative of f with respect to y

Existence: $f(t) = ty^{\frac{4}{3}}$ is continuous for all t and y in the region R. Therefore, Picard's Theorem can be used to show the existence of a solution in an open interval containing $t = 0$.

Uniqueness: We will find the partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \frac{4}{3}ty^{\frac{1}{3}}$$

Since $\frac{\partial f}{\partial y}$ is continuous for all t and y in the region RPicard's Theorem can be used to show the existence of a unique solution in an open interval containing $t = 0$.

(b) $y' = ty^{1/3}, \quad y(0) = 0$

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: We will start by determining the continutiy of the function $f(t) = ty^{\frac{1}{3}}$

Existence: $f(t) = ty^{\frac{1}{3}}$ is continuous for all t and y in the region R Therefore, Picard's Theorem can be used to show the existence of a solution in an open interval containing $t = 0$.

Uniqueness: We will find the partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \frac{1}{3}ty^{-\frac{2}{3}}$$

Since $\frac{\partial f}{\partial y}$ is not continuous ($\frac{1}{\sqrt[3]{0}^2} = DNE$) for all t and y in the region R , Picard's Theorem cannot be used to show the existence of a unique solution in an open interval containing $t = 0$.

(c) $y' = ty^{\frac{1}{3}}, \quad y(0) = 1$

Problem Statement: Determine weather Picard's Theorem can be used to show the existence of a unique solution in an open interval containing t = 0

Solution: We will start by determining the continutiy of the function $f(t) = ty^{\frac{1}{3}}$

Existence: $f(t) = ty^{\frac{1}{3}}$ is continuous for all t and y in the region R. Therefore, Picard's Theorem can be used to show the existence of a solution in an open interval containing $t = 0$.

Uniqueness: We will find the partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \frac{1}{3}ty^{-\frac{2}{3}}$$

Since $\frac{\partial f}{\partial y}$ is continuous for all t and y in the region R, Picard's Theorem can be used to show the existence of a unique solution in an open interval containing $t = 0$.

Problem 2

Problem Statement: Find the order, linearity, homogeneity, and the variablity of the coefficients of the following:

Problem	Diff Eq	Order	Linearity	Homogeneity	coefficients
(a)	$y'' + 2y' + y = 0$	2	Linear	Homogeneous	Constant
(b)	$\ddot{x} + 2\dot{x} + tx = \sin(t)$	2	Linear	Non-Homogeneous	Variable
(c)	$\cos(y') + ty = 0$	1	Non-Linear	Homogeneous	Variable
(d)	$y'' + ey' + \pi y = 0$	2	Linear	Homogeneous	Constant
(e)	$y' + \frac{1}{1+t^2}y = 7$	1	Linear	Non-Homogeneous	Variable

Problem 3

Problem Statement: Which of the following operators are linear?

Solution: An operator is linear if it satisfies the following properties:

$$L(k\vec{\mathbf{u}}) = kL(\vec{\mathbf{u}}), \quad k \in \mathbb{R}$$
$$L(\vec{\mathbf{u}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{u}}) + L(\vec{\mathbf{w}})$$

(a) $L(\vec{\mathbf{y}}) = y' + 2ty$

Solution: We will start by checking the first property

$$\begin{aligned} L(k\vec{\mathbf{u}}) &= kL(\vec{\mathbf{u}}) \\ L(ky) &= k(y' + 2ty) \\ ky' + 2kty &= ky' + 2kty \\ ky' + 2kty &= ky' + 2kty \end{aligned}$$

Now we will check the second property

$$\begin{aligned} L(\vec{\mathbf{u}} + \vec{\mathbf{w}}) &= L(\vec{\mathbf{u}}) + L(\vec{\mathbf{w}}) \\ L(y_1 + y_2) &= L(y_1) + L(y_2) \\ y_1' + 2ty_1 + y_2' + 2ty_2 &= y_1' + 2ty_1 + y_2' + 2ty_2 \end{aligned}$$

Since both properties are satisfied, the operator $L(\vec{\mathbf{y}}) = y' + 2ty$ is linear

(b) $L(\vec{\mathbf{y}}) = y'' + (1 - y^2) + y$

Solution: We will start by checking the first property

$$\begin{aligned} L(k\vec{\mathbf{u}}) &= kL(\vec{\mathbf{u}}) \\ L(ky) &= k(y'' + (1 - y^2) + y) \\ ky'' + (1 - (ky)^2) + ky &\neq ky'' + k(1 - y^2) + ky \end{aligned}$$

This does not satisfy the first property, so the operator $L(\vec{\mathbf{y}}) = y'' + (1 - y^2) + y$ is not linear. We will check the second property to be thorough.

$$\begin{aligned} L(\vec{\mathbf{u}} + \vec{\mathbf{w}}) &= L(\vec{\mathbf{u}}) + L(\vec{\mathbf{w}}) \\ L(y_1 + y_2) &= L(y_1) + L(y_2) \\ y_1'' + (1 - y_1^2) + y_1 + y_2'' + (1 - y_2^2) + y_2 &\neq y_1'' + (1 - y_1^2) + y_1 + y_2'' + (1 - y_2^2) + y_2 \end{aligned}$$

Since the second property is not satisfied, the operator $L(\vec{\mathbf{y}}) = y'' + (1 - y^2) + y$ is not linear.

Problem 4

Problem Statement: Show that if $y_1(t)$ and $y_2(t)$ are solutions of $y' + p(t)y = 0$, then so are $y_1(t) + y_2(t)$ and $cy_1(t)$ for any constant c.

Solution: Assuming that $y_1(t)$ and $y_2(t)$ are solutions of $y' + p(t)y = 0$ and follow the properties of linear homogenoues ODEs. Thus,

$$\begin{aligned} y_1' + p(t)y_1 &= 0 \\ y_2' + p(t)y_2 &= 0 \end{aligned}$$

Using the first property, we will show that $y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = 0$

$$\begin{aligned} (y_1 + y_2)' + p(t)(y_1 + y_2) &= 0 \\ y_1' + y_2' + p(t)y_1 + p(t)y_2 &= 0 \\ y_1' + p(t)y_1 + y_2' + p(t)y_2 &= 0 + 0 = 0 \end{aligned}$$

This proves that $y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = 0$.

Next we will show that $cy_1(t)$ is a solution of $y' + p(t)y = 0$

$$\begin{aligned} (cy_1)' + p(t)(cy_1) &= 0 \\ cy_1' + cp(t)y_1 &= 0 \\ c(y_1' + p(t)y_1) &= c \cdot 0 = 0 \end{aligned}$$

This proves that $cy_1(t)$ is a solution of $y' + p(t)y = 0$.

Problem 5

Problem Statement: Verify that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then show that $c_1y_1(t) + c_2y_2(t)$ is also a solution for any real numbers c_1 and c_2.

(a) $y'' - y' + 6y = 0; \quad y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}$

Solution: We will start by verifying that $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation.

$$\begin{aligned} y_1(t) &= e^{3t} \\ y_1' &= 3e^{3t} \\ y_1'' &= 9e^{3t} \\ 9e^{3t} - 3e^{3t} - 6e^{3t} &= 0 \end{aligned}$$

Therefore, $y_1(t)$ is a solution of the given differential equation. Next we will verify that $y_2(t)$ is a solution

$$\begin{aligned} y_2(t) &= e^{-2t} \\ y_2' &= -2e^{-2t} \\ y_2'' &= 4e^{-2t} \\ 4e^{-2t} + 2e^{-2t} - 6e^{-2t} &= 0 \end{aligned}$$

Therefore, $y_2(t)$ is a solution of the given differential equation. Next we will show that $c_1y_1(t) + c_2y_2(t)$ is also a solution for any real numbers c_1 and c_2.

$$c_1y_1(t) + c_2y_2(t) = c_1e^{3t} + c_2e^{-2t}$$

Now we will find the first and second derivatives of $c_1y_1(t) + c_2y_2(t)$

$$\begin{aligned}(c_1y_1 + c_2y_2)' &= 3c_1e^{3t} - 2c_2e^{-2t}\\ (c_1y_1 + c_2y_2)'' &= 9c_1e^{3t} + 4c_2e^{-2t}\end{aligned}$$

Pluging $c_1y_1 + c_2y_2$ into the differential equation,

$$\begin{aligned}9c_1e^{3t} + 4c_2e^{-2t} - 3c_1e^{3t} + 2c_2e^{-2t} - 6c_1e^{3t} - 6c_2e^{-2t} &= 0\\ (9c_1 - 3c_1 - 6c_1)e^{3t} + (4c_2 + 2c_2 - 6c_2)e^{-2t} &= 0\\ 0e^{3t} + 0e^{-2t} &= 0\\ 0 &= 0\end{aligned}$$

Therefore, $c_1y_1(t) + c_2y_2(t)$ is a solution of the given differential equation.

(b) $y'' - 25y = 0; \quad y_1 = cosh(5t); y_2 = sinh(5t)$

Solution: We will start by verifying that $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation.

$$\begin{aligned}y_1(t) &= cosh(5t)\\ y_1' &= 5sinh(5t)\\ y_1'' &= 25cosh(5t)\\ 25cosh(5t) - 25cosh(5t) &= 0\end{aligned}$$

Therefore, $y_1(t)$ is a solution of the given differential equation. Next we will verify that $y_2(t)$ is a solution

$$\begin{aligned}y_2(t) &= sinh(5t)\\ y_2' &= 5cosh(5t)\\ y_2'' &= 25sinh(5t)\\ 25sinh(5t) - 25sinh(5t) &= 0\end{aligned}$$

Therefore, $y_2(t)$ is a solution of the given differential equation. Next we will show that $c_1y_1(t) + c_2y_2(t)$ is also a solution for any real numbers c_1 and c_2.

$$c_1y_1(t) + c_2y_2(t) = c_1cosh(5t) + c_2sinh(5t)$$

Now we will find the first and second derivatives of $c_1y_1(t) + c_2y_2(t)$

$$(c_1y_1 + c_2y_2)' = 5c_1sinh(5t) + 5c_2cosh(5t)$$
$$(c_1y_1 + c_2y_2)'' = 25c_1cosh(5t) + 25c_2sinh(5t)$$

Pluging $c_1y_1 + c_2y_2$ into the differential equation,

$$25c_1cosh(5t) + 25c_2sinh(5t) - 25c_1cosh(5t) - 25c_2sinh(5t) = 0$$

Therefore, $c_1y_1(t) + c_2y_2(t)$ is a solution of the given differential equation.

Problem 6

Problem Statement: Find the general solution to the non homogenoues ODE $y' + \frac{1}{t+1}y = 2$

Solution: We will solve using the superposition principal. $y_{G,NH} = y_{G,H} + y_{P,NH}$

(a)

Problem Satement: Show that $y_p = \frac{t^2+2t}{t+1}$ is a particular solution to the Non-Homogeneous ODE $y' + \frac{1}{t+1}y = 2$

Solution: We will start by finding the derivative of y_p

$$y_p = \frac{t^2 + 2t}{t + 1}$$
$$y_p' = \frac{(t + 1)(2t + 2) - (t^2 + 2t)}{(t + 1)^2}$$
$$y_p' = \frac{2t^2 + 2t + 2t + 2 - t^2 - 2t}{(t + 1)^2}$$
$$y_p' = \frac{t^2 + 4t + 2}{(t + 1)^2}$$

Now we will plug y_p and y_p' into the Non-Homogeneous ODE

$$\begin{aligned} y_p' + \frac{1}{t+1}y_p &= 2 \\ \frac{t^2+4t+2}{(t+1)^2} + \frac{t^2+2t}{(t+1)^2} &= 2 \\ \frac{t^2+4t+2}{t^2+2t+1} + \frac{t^2+2}{t^2+2t+1} &= 2 \\ \frac{2t^2+4t+2}{t^2+2t+1} &= 2 \\ \frac{2(t^2+2t+1)}{t^2+2t+1} &= 2 \\ 2 &= 2 \end{aligned}$$

(b)

Problem Statement: Find the general solution to the Non-Homogeneous ODE $y' + \frac{1}{t+1}y = 2$

Solution: We will use the superposition principal to find the general solution to the Non-Homogeneous ODE

$$y_{G,NH} = y_{G,H} + y_{P,NH}$$

Now we will the general solution to the Homogeneous ODE

$$\begin{aligned} y' + \frac{1}{t+1}y &= 0 \\ \frac{1}{y}dy &= -\frac{1}{t+1}dt \\ \ln|y| &= -\ln|t+1| + c \\ y &= \frac{c}{t+1} \end{aligned}$$

Now we will find the general solution to the Non-Homogeneous ODE

$$y_{G,NH} = \frac{c}{t+1} + \frac{t^2+2t}{t+1}$$

Problem 7

Problem Statement: Cosider the differential equation governing the current of an RL circuit $L\frac{dI}{dt} + RI = E(t)$ Suppose that $R = 100$ ohms, $L = 2.5$ henries, and the constant impressed voltage is $E(t) = E_0 = 110$ volts. Time t is measured in seconds.

(a) If the initial curret is zero, find the current at any time

Solution: We will start by finding the general solution to the Non-Homogeneous ODE

$$L\frac{dI}{dt} + RI = E(t)$$
$$2.5\frac{dI}{dt} + 100I = 110$$
$$\frac{dI}{dt} + 40I = 44$$

We will use the integrating factor method.

$$\frac{dI}{dt} + 40I = 44$$
$$\mu = e^{\int 40dt}$$
$$\mu = e^{40t}$$

Multiply the original equation by μ

$$e^{40t}\frac{dI}{dt} + 40e^{40t}I = 44e^{40t}$$
$$D[\mu I] = 44\mu$$
$$e^{40t}I = \int 44e^{40t}$$
$$e^{40t}I = \frac{44}{40} + ce^{40t}$$
$$I = 1.1 + ce^{-40t}$$

Solve the IVP $I(0) = 0$

$$0 = 1.1 + c$$
$$c = -1.1$$
$$I = 1.1 - 1.1e^{-40t}$$

(b) Find the transient and steady state currents)

Solution: We will start by finding the transient current

$$I_{transient} = -1.1e^{-40t}$$
$$\lim_{t\to\infty} -1.1e^{-40t} = 0$$
$$I_{transient} = 0$$

Next we will find the steady state current

$$\lim_{t\to\infty} 1.1 + ce^{-40t} = 1.1$$
$$I_{steady} = 1.1$$

(c) Find the time required for the current to reach 0.6 amps

Solution: Set I = 0.6 and solve for t

$$0.6 = 1.1(1 - e^{-40t})$$
$$\frac{0.6}{111} = 1 - e^{-40t}$$
$$e^{-40t} = \frac{0.5}{1.1}$$
$$-40t = \ln(\frac{0.5}{1.1})$$
$$t = \frac{\ln(\frac{0.5}{1.1})}{-40}$$

Problem 8

Problem Statement: Solve the differential equation $(x+1)\frac{dy}{x} + y = \ln(t)$

Solution: We will solve using the integrating factor method Rewriting the equation:

$$\frac{dy}{dt} + \frac{y}{t+1} = \frac{lnt}{t+1}$$

Set right side equal to zero

$$\frac{dy}{dt} + \frac{y}{t+1} = 0$$

Solve for y

$$\frac{dy}{dt} = -\frac{y}{t+1}$$
$$\frac{dy}{y} = -\frac{dt}{t+1}$$

Integrate both sides

$$\int \frac{1}{y}dy = -\frac{1}{t+1}dt$$
$$\ln|y| = -ln|t+1| + c$$
$$y = e^{-\ln|t+1|+c}$$
$$y = \frac{c}{t+1}$$

Solve for integrating factor μ

$$\mu = \frac{1}{y_{G,H}}$$
$$\mu = t + 1$$

Multiply orignial equation by μ

$$\mu\frac{dy}{dt} + \mu\frac{y}{t+1} = \mu\frac{lnt}{t+1}$$
$$D[(t+1)y] = \frac{\ln(t)}{t+1}(t+1)$$
$$(t+1)y = \int \ln(t)$$
$$(t+1)y = t\ln(t) - t + c$$
$$y = \frac{t\ln(t) - t + c}{t+1}$$

Problem 9

Problem Statement: Use the Euler-Lagrange Two-Stage Method to sole the IVP $\frac{dy}{dt}+2ty = t^3, \quad y(1) = 1$

Solution: We will start by finding the general solution to the Homogeneous ODE

$$\frac{dy}{dt} + 2ty = 0$$
$$\frac{dy}{y} = -2tdt$$
$$\ln|y| = -t^2 + c$$
$$y = ce^{-t^2}$$

We will now substitute a fucntion $v(t)$ for the constant c:

$$y = v(t)e^{-t^2}$$

We will now determine the function $v(t)$ by substituting y into the oringinal Non-Homogeneous ODE

$$\begin{aligned}
\frac{dy}{dt} + 2ty &= t^3 \\
\frac{d(v(t)e^{-t^2})}{dt} + 2t(v(t)e^{-t^2}) &= t^3 \\
\frac{dv}{dt}e^{-t^2} - 2te^{-t^2}v + 2te^{-t^2}v &= t^3 \\
\frac{dv}{dt}e^{-t^2} &= t^3 \\
\frac{dv}{dt} &= t^3e^{t^2}
\end{aligned}$$

Integrating both sides using $u = t^2$,

$$\begin{aligned}
v &= \frac{1}{2}\int ue^u du \\
v &= \frac{1}{2}(t^2 - 1) + Ce^{-t^2}
\end{aligned}$$

Now we will substitute v back into the equation for y

$$\begin{aligned}
y &= (\frac{1}{2}(t^2 - 1) + Ce^{-t^2})e^{-t^2} \\
y &= \frac{1}{2}(t^2 - 1)e^{-t^2} + Ce^{-2t^2}
\end{aligned}$$

Using the superposition principal, $y_{G,NH} = y_{G,H} + y_{P,NH}$, we will solve for the general solution to the Non-Homogeneous ODE

$$y_G, NH = \frac{1}{2}(t^2 - 1) + ce^{-2t^2} + ce^{-t^2} y_G, NH \qquad = \frac{1}{2}(t^2 - 1) + ce^{-2t^2}$$

Solve the IVP $y(1) = 1$

$$\begin{aligned}
1 &= \frac{1}{2}(1 - 1) + ce^{-1^2} \\
1 &= c \cdot e^1 \\
c &= e^1
\end{aligned}$$

Therefore, the solution to the IVP is

$$y = \frac{1}{2}(t^2 - 1) + e^1e^{-2t^2} y \qquad = \frac{1}{2}(t^2 - 1) + e^{1-2t^2}$$

Problem 10

Problem Statement: Solve the bernoulli equation $t^2y' + 2ty = 3y^4$

Solution: We will use the substitution $v = y^{1-4}$

$$\begin{aligned} v &= y^{1-4} \\ v &= y^{-3} \\ y^3 &= v^{-1} \\ y &= v^{-1/3} \end{aligned}$$

Now we will find the derivative of y

$$\begin{aligned} y &= v^{-1/3} \\ \frac{dy}{dt} &= -\frac{1}{3}v^{-4/3}\frac{dv}{dt} \end{aligned}$$

Now we will substitute y and $\frac{dy}{dt}$ into the Bernoulli equation

$$\begin{aligned} t^2y' + 2ty = 3y^4 y' + \frac{2y}{t} &= \frac{3}{t^2}y^4 \\ -\frac{1}{3}v^{-4/3}\frac{dv}{dt} + \frac{2}{t}v^{-1/3} &= \frac{3}{t^2}v^{-4/3} \\ -\frac{1}{3}\frac{dv}{dt} + \frac{2}{t}v &= \frac{3}{t^2}v \\ \frac{dv}{dt} - \frac{6}{t}v &= -\frac{9}{t^2}v \end{aligned}$$

Now we will solve the ODE using the integrating factor method.

$$\begin{aligned} \mu &= e^{\int -\frac{6}{t}dt} \\ \mu &= e^{-6\ln|t|} \\ \mu &= e^{\ln|t^{-6}|} \\ \mu &= t^{-6} \end{aligned}$$

Multiplying the ODE by μ

$$\begin{aligned}
\mu\frac{dv}{dt} - \frac{d\mu}{dt}v &= \frac{-9}{t^2}\mu \\
D[\mu v] &= \frac{-9}{t^2}\mu \\
t^{-6}v &= \int -9t^{-8}v \\
t^{-6}v &= -\frac{9}{-7}t^{-7}v + c \\
v &= \frac{9}{7}t^{-1} + ct^6
\end{aligned}$$

Now we will substitute y back into the equation for v

$$\begin{aligned}
v &= y^3 \\
y^3 &= \frac{9}{7}t^{-1} + ct^6 \\
y &= (\frac{9}{7}t^{-1} + ct^6)^{-1/3}
\end{aligned}$$