Homework 4

Zachariah Galdston

9/19/2024

Problem 1

Problem Statement: Suppose you have A_0 dollars to invest in a savings account earning an annual interest rate of r percent compounded continuously. Furthermore, suppose that you make annual deposits of d dollars to the account. The differential equation governing this situation is:

$$\frac{dA}{dt} = rA + d, \quad A(0) = A_0$$

(a)

Problem Statement: Find an equation for the future value $A(t)$ of the account by solving the initial value problem.

Solution: We start with the differential equation:

$$\frac{dA}{dt} = rA + d$$

Using the integrating factor method with $\mu = e^{rt}$:

$$\begin{aligned} \frac{dA}{dt} - rA &= d \\ e^{rt}\frac{dA}{dt} - re^{rt}A &= de^{rt} \\ \frac{d}{dt}(Ae^{rt}) &= de^{rt} \end{aligned}$$

Now integrate both sides:

$$Ae^{rt} = \int de^{rt}dt = \frac{d}{r}e^{rt} + C$$

Thus, the general solution is:

$$A(t) = \frac{d}{r} + Ce^{-rt}$$

Using the initial condition $A(0) = A_0$, we solve for C:

$$A(0) = \frac{d}{r} + C = A_0 \quad \Rightarrow \quad C = A_0 - \frac{d}{r}$$

Therefore, the solution is:

$$A(t) = \frac{d}{r} + \left(A_0 - \frac{d}{r}\right) e^{-rt}$$

(b)

Problem Statement: Upon graduating from college and starting your career, you have no money. You deposit $d = 1000$ into an account that pays interest at a rate of 8% compounded continuously. Find the future value of the account after 40 years and the interest earned.

Solution: We use the formula from part (a) with $r = 0.08$, $d = 1000$, and $A_0 = 0$. The solution becomes:

$$A(t) = \frac{1000}{0.08} + \left(0 - \frac{1000}{0.08}\right) e^{-0.08t}$$

Simplifying:

$$A(t) = 12500(1 - e^{-0.08t})$$

Substituting $t = 40$ years:

$$A(40) = 12500(1 - e^{-0.08\times40}) \approx 294156.63$$

The interest earned is the difference between the final balance and the total deposits:

$$\text{Interest} = 294156.63 - 1000 \times 40 = 254156.63$$

(c)

Problem Statement: Find the value of d that would produce a balance of one million dollars after 40 years.

Solution: We need to solve for d in the equation:

$$1000000 = \frac{d}{0.08}(1 - e^{-0.08\times40})$$

Solving for d:

$$d = \frac{1000000 \times 0.08}{1 - e^{-0.08\times40}} \approx 3395.95$$

(d)

Problem Statement: If the annual deposit is $d = 2500$, find the value of r that would produce a balance of one million dollars after 40 years.

Solution: We need to solve the equation:

$$1000000 = \frac{2500}{r}(1 - e^{-r\times40})$$

This requires numerical methods, and solving yields $r \approx 0.0904$ or 9.04%.

Problem 2

Problem Statement: You must be anesthetized with a minimum concentration of 50 milligrams per kilogram. Your weight is 100 kg, and the half-life of the anesthetic is 10 hours. Find the dose needed to stay anesthetized for 3 hours.

Solution: The elimination of the anesthetic follows an exponential decay model:

$$A(t) = A_0 e^{-kt}$$

We will solve for k using the half-life of 10 hours:

$$\begin{aligned} \frac{1}{2}A_0 &= A_0 e^{-10k} \\ \frac{1}{2} &= e^{-10k} \\ \ln\frac{1}{2} &= -10k \\ k &= \frac{\ln 2}{10} \end{aligned}$$

At $t = 3$ hours, we need $A(3) = 50 \times 100 = 5000$ mg. Solving for A_0:

$$5000 = A_0 e^{-3\frac{\ln 2}{10}} \quad \Rightarrow \quad A_0 \approx 6160 \text{ mg or } 6.16 \text{ grams}$$

Problem 3

Problem Statement: A tank starts with 300 gallons of pure water, and a salt solution flows in at 3 gallons per minute while solution drains at 1 gallon per minute. Find the salt content when the tank reaches 600 gallons.

Solution: The differential equation governing the salt content $S(t)$ is:

$$\begin{aligned} \frac{dS}{dt} &= (RateIn)(ConcentrationIn) - (RateOut)(ConcentrationOut) \\ &= (3 \cdot 1) - (\frac{S}{300+2t} \cdot 1) \end{aligned}$$

Solving the differential equation:

$$\begin{aligned} \frac{dS}{dt} &= 3 - \frac{S}{300+2t} \\ \frac{dS}{dt} + \frac{S}{300+2t} &= 3 \end{aligned}$$

We use the integrating factor method with $\mu = e^{\int \frac{1}{300+2t}dt} = e^{\frac{1}{2}\ln(300+2t)} = (300+2t)^{\frac{1}{2}}$:

$$\begin{aligned} (300+2t)^{\frac{1}{2}}\frac{dS}{dt} + (300+2t)^{-\frac{1}{2}}S &= 3 \\ \frac{d}{dt}((300+2t)^{\frac{1}{2}}S) &= 3(300+2t)^{\frac{1}{2}} \end{aligned}$$

Integrating both sides:

$$(300+2t)^{\frac{1}{2}}S = 3\int (300+2t)^{\frac{1}{2}}dt$$

Solving the right side integral:

$$\begin{aligned} 3\int (300+2t)^{\frac{1}{2}}dt &= 3\left(\frac{2}{3}(300+2t)^{\frac{3}{2}}\right)+C \\ &= (300+2t)^{\frac{3}{2}}+C \end{aligned}$$

Substitute back into the differential equation:

$$\begin{aligned} (300+2t)^{\frac{1}{2}}S &= (300+2t)^{\frac{3}{2}}+C \\ S &= (300+2t)+\frac{C}{(300+2t)^{-\frac{1}{2}}} \end{aligned}$$

Solving for C using the initial condition $S(0)=0$:

$$0 = 300+\frac{C}{300^{-\frac{1}{2}}}$$
$$C = -300\cdot 300^{\frac{1}{2}}$$

Therefore, the solution is:

$$S(t) = 300+2t-\frac{300(300^{\frac{1}{2}})}{(300+2t)^{-\frac{1}{2}}}$$

To solve for the time when the tank reaches 600 gallons, we set must solve for t. We know the tank is increaseing at rate $RateIN - RateOut = 3-1$ gallons per minute, so we solve for t in the equation $600 = 300+2t$ to find $t=150$ minutes. Substituting $t=150$ into the solution:

$$\begin{aligned} S(t) &= 300+2(t)-\frac{300(300^{\frac{1}{2}})}{(300+2t)^{-\frac{1}{2}}} \\ S(150) &= 300+2(150)-\frac{300(300^{\frac{1}{2}})}{(300+2(150))^{-\frac{1}{2}}} \\ &= 600-\frac{300(300^{\frac{1}{2}})}{(600)^{-\frac{1}{2}}} \\ &= 600-\frac{300(300^{\frac{1}{2}})}{\sqrt{600}} \\ &\approx 387.87 lbs \end{aligned}$$

Problem 4

Problem Statement: A small single-stage rocket of mass $m(t)$ is launched vertically. The air resistance is linear, and the rocket consumes fuel at a constant rate. The velocity of the rocket is modeled by the differential equation:

$$\frac{dv}{dt} + \frac{k-\lambda}{m_0 - \lambda t} v = -g + \frac{R}{m_0 - \lambda t}$$

where $m_0 = 200$ kg, $R = 2000$ N, $\lambda = 1$ kg/s, $g = 9.8$ m/s^2, $k = 3$ kg/s, and $v(0) = 0$.

(a)

Problem Statement: Find the velocity $v(t)$ of the rocket.

Solution: Solve using the integrating factor method with $\mu = (200-t)^{-2}$:

$$\begin{aligned} \frac{dv}{dt} + \frac{3-1}{200-t} v &= -9.8 + \frac{2000}{200-t} \\ (200-t)^{-2}\frac{dv}{dt} + (200-t)^{-3} v &= -9.8(200-t)^{-2} + 2000(200-t)^{-3} \\ \frac{d}{dt}\left((200-t)^{-2} v\right) &= -9.8(200-t)^{-2} + 2000(200-t)^{-3} \end{aligned}$$

Integrating both sides:

$$\begin{aligned} (200-t)^{-2} v &= 9.8(200-t)^{-1} - 1000(200-t)^{-2} + C \\ v &= 9.8(200-t) - 1000 + C(200-t)^2 \end{aligned}$$

Solving for C using the initial condition $v(0) = 0$:

$$\begin{aligned} 0 &= 9.8(200) - 1000 + C(200)^2 \\ C &= \frac{9.8(200) - 1000}{200^2} = \frac{1960 - 1000}{40000} = 0.024 \end{aligned}$$

So the final solution is:

$$v(t) = 9.8(200-t) - 1000 + 0.024(200-t)^2$$

(b)

Problem Statement: Find the height $s(t)$ of the rocket.

Solution: The height is found by integrating $v(t)$:

$$s(t) = \int v(t)dt$$

Substitute $v(t)$ into the integral and C = 0:

$$\begin{aligned} s(t) &= \int -9.8(200-t) + 1000 + 0.024(200-t)^2 dt \\ &= \int 0.024t^2 + 0.2t dt \\ &= 0.008t^3 + 0.1t^2 + C \qquad\qquad = 0.008t^3 + 0.1t^2 \end{aligned}$$

(c)

Problem Statement: Find the burnout time t_b when all the fuel is consumed.

Solution: The burnout time is when $m(t) = 200 - \lambda t = 200 - 50$. Solving for t:

$$\begin{aligned} 200 - 50 &= 200 - t \\ t &= 50 \text{ seconds} \end{aligned}$$

(d)

Problem Statement: Find the velocity at burnout.

Solution: Substitute $t_b = 50$ s into the velocity equation $v(t)$:

$$v(50) = 9.8(200-50) - 1000 + 0.024(200-50)^2$$

After simplifying, the velocity at burnout is approximately $v(50) \approx 1010$ m/s.

(e)

Problem Statement: Find the height at burnout.

Solution: Substitute $t_b = 50$ s into $s(t)$ to find the height:

$$s(50) = 0.008(50)^3 + 0.1(50)^2 = 1250 \text{ m}$$

The height at burnout is approximately $s(50) \approx 1250$ m.

Problem 5

Problem Statement: Your air conditioner breaks down at noon. The temperature inside the house is 75°F, and outside it is 95°F. The time constant for the house is 4 hours. Find the temperature inside the house after 2 hours.

(a)

Solution: The temperature follows Newton's Law of Cooling:

$$T(t) = T_{\text{out}} + (T_0 - T_{\text{out}})e^{-kt}$$

where $k = \frac{1}{4}$ and $T_0 = 75$°F. Substituting $t = 2$ hours:

$$T(2) = 95 + (75-95)e^{-\frac{2}{4}} \approx 82.87°F$$

(b)

Problem Statement: Find the time when the temperature reaches 80°F.

Solution: We solve for t in the equation:

$$80 = 95 + (75 - 95)e^{-\frac{t}{4}}$$

Solving this gives:

$$t \approx 1 \text{ hour and } 9 \text{ minutes}$$

Problem 6

Problem Statement: Consider the differential equation for population growth:

$$\frac{dP}{dt} = kP^{1+c}$$

where $c = 0.01$. Find the solution given the initial population and doubling rate.

(a)

Solution: We separate variables and integrate:

$$\int P^{-1.01} dP = \int k dt$$
$$\frac{-100}{P^{\frac{1}{100}}} = kt + C$$

Solving for C with the initial condition $P(0) = 10$:

$$\frac{-100}{10^{\frac{1}{100}}} = 0 + C$$
$$C = -97.7237$$

Solving for k with the doubling rate condition $P(5) = 20$:

$$\frac{-100}{20^{\frac{1}{100}}} = 5k - 97.7237$$
$$\frac{-100}{(20)^{\frac{1}{100}}} + 97.7237 = 5k$$
$$k \approx 0.135$$

Therefore, the solution is:

$$P(t) = \left(\frac{0.135t - 97.7237}{-100}\right)^{-100}$$

(b)

Problem Statement: Find the population after 50 and 100 months.

Solution: Substitute $t = 50$ and $t = 100$ into the solution from part (a):

$$P(50) = \left(\frac{0.135(50) - 97.7237}{-100}\right)^{-100} \approx 12,835 P(100) \quad = \left(\frac{0.135(100) - 97.7237}{-100}\right)^{-100} \approx 28,613,327$$

(c)

Problem Statement: Find the doomsday time t_0 when the population becomes infinite.

Solution: The population becomes infinite when there is a 0 in the demoninator of the solution from part (a). Solving for t:

$$\frac{-100}{P^{\frac{1}{100}}} = kt + C \frac{-100}{kt + c} = P^{\frac{1}{100}} \frac{-100}{0.135t - 97.7237} = P^{\frac{1}{100}}$$

Sett8ng the denominator to 0 gives:

$$0.135t - 97.7237 = 0$$
$$t = \frac{97.7237}{0.135} \approx 724.28 \text{ months}$$