Position-Location Solutions by Taylor-Series Estimation

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Abstract

Taylor-series estimation gives a least-sum-squared-error solution to a set of simultaneous linearized algebraic equations. This method is useful in solving multimeasurement mixed-mode position-location problems typical of many navigational applications. While convergence is not proved, examples show that most problems do converge to the correct solution from reasonable initial guesses. The method also provides the statistical spread of the solution errors.

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Introduction

The classical techniques of navigation have been based on position estimation by finding the intersection of two lines of position, but there has developed in recent years a trend toward the use of multiple readings and multiple measurement devices with statistical combination of the readings to compute position. The general result, when rate data are available or constraints across time can be invoked, is the time series of position estimates produced by some form of Kalman-Bucy filter.

There still exist a large number of navigational functions for which the full-blown filter is not warranted, and an operator can be content with a single position estimate from a set of essentially simultaneous observations. Examples include cases where the vehicle attitude is too erratic (e.g., police cars), where the vehicle is usually stationary (e.g., appliance repairmen), where motion is comparatively slow and subject to unknown biases (e.g., ships and lighter-than-air craft), and where the rate data or measurement statistics are such that the Kalman-Bucy filter provides no significant improvement in accuracy (e.g., many aircraft problems). The purpose of this paper is to point out that a large class of these position-location problems can be solved effectively by a common mathematical technique. The method called Taylor-series estimation (also Gauss or Gauss-Newton interpolation) is an iterative scheme for solution of the simultaneous set of algebraic position equations (generally nonlinear), starting with a rough initial guess and improving the guess at each step by determining the local linear least-sum-squared-error correction. The disadvantages are:

1. The method is iterative, requiring an initial guess.
2. It is computationally complex compared to simple plotting of lines of position (LOP).
3. Being a local correction, its convergence is not assured.

The advantages are:

1. Multiple independent measurements to a single station are averaged naturally.
2. Multiple measurements and mixed-mode measurements are combined properly, i.e., with the correct geometric factors, and can be weighted according to their a priori accuracies.
3. The statistical spread of the solution can be found easily and naturally.
4. Experience indicates that the initial position guess can be quite far off without preventing good convergence.
5. Failure to converge is easy to detect.
6. Simulation is easy, so convergence can be readily tested.
7. Computational complexity is less than that of a Kalman-Bucy filter.

We present here no new theoretical results. Our intent is to emphasize the utility of a known mathematical method. The salient points are (1) the Taylor-series method works for a variety of problems including mixed-mode, (2) convergence almost always happens and the solution then is independent of the initial guess, and (3) the computations can be accomplished in a rather simple computing device.

Mathematical Procedure

For notational convenience and ease of exposition in presenting the method, we restrict attention to two-dimensional position location in a plane. Extension of the ideas to more complicated spaces is straightforward but cumbersome and not particularly enlightening.

Adopt a rectangular set of coordinate axes in the plane, and on these axes let

x, y = true position of the vehicle

x_k, y_k = true (known) position of the kth station, $k = 1, 2, \cdots, n_s$

n_s = number of fixed navigational stations or reference points.

Write m_{ki} for the ith navigational measurement (range, angle of bearing, etc., to station k) and we have an algebraic relation of the form

$$f_i(x, y, x_k, y_k) = u_i = m_{ki} - e_i, \qquad i = 1, 2, \cdots, n$$

where

u_i = correct value of the measured quantity

e_i = error in the m_{ki} measurement.

The mathematical problem is to estimate x, y given the n measurements m_{ki}, the functional forms f_i, and the station positions. The errors e_i are statistically distributed. We take the errors to have zero-mean values $\langle e_i \rangle = 0$. We write $r_{ij} = \langle e_i e_j \rangle$ for the i-jth term in the error covariance matrix[1]

$$R = [r_{ij}].$$

If x_v, y_v are guesses of the true vehicle position, write

$$x = x_v + \delta_x \qquad y = y_v + \delta_y,$$

and expand f_i in Taylor's series keeping only terms below second order:

$$f_{iv} + a_{i1}\delta_x + a_{i2}\delta_y \simeq m_{ki} - e_i \tag{1}$$

where

$$f_{iv} = f_i(x_v, y_v, x_k, y_k)$$

$$a_{i1} = \partial f_i/\partial x|_{x_v, y_v} \qquad a_{i2} = \partial f_i/\partial y|_{x_v, y_v}.$$

[1]Capital letters A, B, R, or square brackets [ ] will usually denote matrixes. The identity matrix is denoted I. We write A^T or $[\ ]^T$ for the transpose, A^{-1} or $[\ ]^{-1}$ for the inverse, and det$A$ or det [ ] for the determinant. $\langle\cdot\rangle$ denotes the expected value of the argument over the appropriate probability distribution.

Define the matrixes and vectors

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix} \quad \delta = \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \quad z = \begin{bmatrix} m_{k1} - f_{1v} \\ m_{k2} - f_{2v} \\ \vdots \\ m_{kn} - f_{nv} \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

and the approximate relations of (1) can be written as

$$A\delta \cong z - e. \tag{2}$$

The choice of δ that gives the least-sum-squared error in these relations with the terms in the sum weighted according to the covariances of the measurement errors is

$$\delta = [A^T R^{-1} A]^{-1} A^T R^{-1} z. \tag{3}$$

Thus, to estimate the solution point, compute δ_x, δ_y with (3), replace

$$x_v \leftarrow x_v + \delta_x \qquad y_v \leftarrow y_v + \delta_y$$

in (2), and repeat the computation. The iterations will have converged when δ_x and δ_y are essentially zero. The covariance matrix of the errors in the corresponding position estimate x_v, y_v will be

$$Q_0 = [A^T R^{-1} A]^{-1}. \tag{4}$$

Considerable insight into the way this works can be gained by considering the special case where the noise covariance matrix $R = \sigma^2 I$ (i.e., the noise terms e_i are independent with the same variance) for which the correction vector at each step is

$$\delta = [A^T A]^{-1} A^T z. \tag{5}$$

The matrix $[A^T A]^{-1}$ is sensitive to the geometry of the situation. For instance, if $n = 1$ or if all the measurements are the same type made on the same station we have det $[A^T A] = 0$, and $[A^T A]^{-1}$ does not exist; no solution is possible. If several independent measurements of a given quantity are made to a single station, and are included as separate rows in the A matrix and z vector, the operator $[A^T A]^{-1} A^T$ will average them together and resolve the result properly into δ_x, δ_y components. In the special diagonal-covariance-matrix equal-variances case $Q_o = [A^T A]^{-1}\sigma^2$. The det $[A^T A]^{-1}$ is directly proportional to the amount of geometric dilution in the fix.

For all cases, the covariance matrix Q_o at the final step gives the statistics of the final position estimate. That is, write

$$Q_0 = \begin{bmatrix} \sigma_x^2 & \rho_{xy} \\ \rho_{xy} & \sigma_y^2 \end{bmatrix},$$

and with the assumption that the error distribution is Gaussian, the parameters of the error ellipse with a = semimajor axis and b = semiminor axis are determined by

$$a^2 = 2\{\sigma_x^2\sigma_y^2 - \rho_{xy}^2\} / \{\sigma_x^2 + \sigma_y^2 - [(\sigma_x^2 - \sigma_y^2)^2 + 4\rho_{xy}^2]^{1/2}\}$$

$$b^2 = 2\{\sigma_x^2\sigma_y^2 - \rho_{xy}^2\} / \{\sigma_x^2 + \sigma_y^2 + [(\sigma_x^2 - \sigma_y^2)^2 + 4\rho_{xy}^2]^{1/2}\}.$$

The circular error probability (CEP) is a complicated function of the Q_0 terms, but it can be approximated with no more than an 11 percent error by

$$\text{CEP} \cong (3/4)\sqrt{a^2 + b^2}$$

which gives an easily calculated measure of the spread of the distribution of final position estimates.

Background

The least-sum-squared-error solution (5) to the linear system of (2) is given in [1], and an early application to simultaneous nonlinear equations is found in [2]. For alternative methods of solution see [3]; we chose the Taylor-series method because it seemed the most straightforward. There have been many applications of the method to specific navigational problems; [4] and [5] give examples of particular interest. A useful example showing the role of the position-estimate covariance matrix is found in [6]. Another good example and the approximation for CEP are shown in [7].

Reviews, applications, and discussions of the Kalman-Bucy filtering techniques have been presented in many places; [8] and [9] are useful for navigational problems. The problem formulations shown in [8] and [9] can be compared to ours. Position estimates of comparable dimension with a Kalman-Bucy filter require the integration of two simultaneous nonlinear equations for position and of three simultaneous equations for covariance terms; they would use more data and provide continuous position estimates in time. The Taylor-series method requires inversion of a succession of 2 X 2 matrixes and iterates toward a position estimate at a given time. Convergence of the integration steps or of the iterations is sometimes in question.

Error Statistics

The most important question in applying the method is not mathematical but practical: how well do the statistical assumptions correspond to the properties of actual navigational measurements? In the mathematical development it was supposed that each measurement error had zero-mean value, had variance known on the basis of a priori information, and had known covariances with the other measurements in the set. That is, we imagine that the given measurement can be taken many times, keeping the geometric relationship between the station and the vehicle fixed; the average of these readings should be the true measurement, the average of their squares will be the variance, and so on. These suppositions have practical significance, and the most important is the assumption of zero mean.

There are many cases in which the measurement error usually does not have zero-average value such as HF direction finder bearings, radar ranges subject to multipath, and Loran readings. The following conditions concern bias error.

A. Bias value is known. Any known bias or average error should be subtracted from the measurement. In the DF example, this may correspond to using a set of calibration curves with angle and frequency arguments.

B. Functional form of bias is known, but value is not. If the functional form is comparatively simple it may be possible to solve for the bias value as well as the vehicle position with redundant measurements. For example, [10] uses bearing-angle measurements to several stations to show a solution for both position and a bias that is constant for all measurements. This process is somewhat risky because the solution may be highly sensitive to small random errors in the measurements.

C. Bias is known to be present, but its value is not known. With unknown bias, there seems to be little choice but to assume that the mean value is zero. This may occur in the DF example with the polarization error component or with the ionospheric component. The total error variance will then be the variance due to random error plus the square of the bias error spread.

D. Bias is not known, and may not be present. With essentially no information about bias available, the best choice is to assume that only random zero-mean errors are present.

Thus, with proper handling of the information available about fixed or bias errors, it appears that the measurements should be effectively reduced to the cases for which the zero-mean assumption is appropriate. In any case, values for the bias and covariances of a measurement must be established by detailed examination and/or test of the measurement process itself.

Multiple Lines of Position

As a first illustration of the application of the Taylor-series method, suppose that we have n estimated lines of position in the x-y plane. The ith line can be defined conveniently by the measured distance d_i from the origin and direction angle α_i of its perpendicular as shown in Fig. 1(a), and

$$d_i = d_{oi} + e_{di}$$

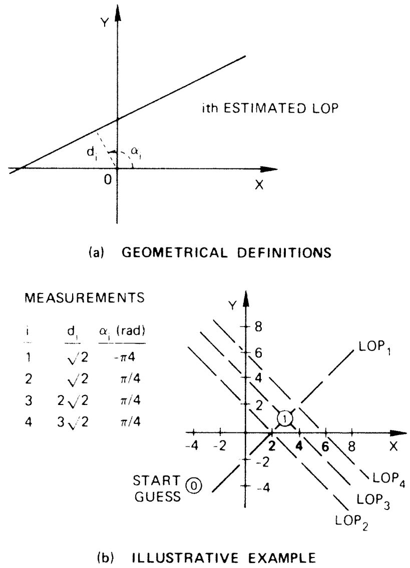

Fig. 1. Multiple lines of position.

$$\alpha_i = \alpha_{oi} + e_{\alpha i}, \qquad i = 1, 2, \cdots, n$$

with d_{oi} and α_{oi} the true values, and e's the measurement errors. The points x, y located on the ith LOP must satisfy

$$x \cos \alpha_{oi} + y \sin \alpha_{oi} = d_{oi}. \tag{6}$$

With $x = x_v + \delta_x$, $y = y_v + \delta_y$, the first-order Taylor-series expansion in terms of δ_x, δ_y, and the errors is

$$\begin{aligned}(\cos \alpha_i)\delta_x + (\sin \alpha_i)\delta_y \simeq d_i &- (x_v \cos \alpha_i + y_v \sin \alpha_i) \\ &- e_{di} + (y_v \cos \alpha_i - x_v \sin \alpha_i) \\ &e_{\alpha i}.\end{aligned} \tag{7}$$

Thus, the terms in the matrix A and $\mathbf{z}$ vector of (2) are

$$a_{i1} = \cos \alpha_i \qquad a_{i2} = \sin \alpha_i$$

$$z_i = d_i - (x_v \cos \alpha_i + y_v \sin \alpha_i).$$

Suppose that the measurement errors are independent and have the same variances for distance and the same for angle measurements

$$\langle e_{di}^2 \rangle = \sigma_d^2 \qquad \langle e_{\alpha i}^2 \rangle = \sigma_\alpha^2.$$

Fig. 1(b) will serve to demonstrate the computations. From the α_i measurements we have

$$A = (1/\sqrt{2}) \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[A^T A]^{-1} = (1/3) \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

We can start the first iteration with the guess $x_v = -4, y_v = -4$ and calculate

$$\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} = [A^T A]^{-1} A^T \mathbf{z} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

Thus, the next iteration is started with the new position guess $x_v + \delta_x = 3$, $y_v + \delta_y = 1$, and we get $A^T \mathbf{z} = \mathbf{0}$. The iterations have converged, and the point (3, 1) is the best position estimate, as is obvious from the plotted LOP in Fig. 1(b). In this case, the basic relations are linear, so the corrections are computed exactly by the linear form in (2). At the convergence point, the noise-variance terms are

$$r_{11} = \sigma_d^2 + 8\sigma_\alpha^2$$
$$r_{22} = r_{33} = r_{44} = \sigma_d^2 + 2\sigma_\alpha^2,$$

and the covariance matrix of the position estimate in x and y is

$$Q_o = (1/3) \begin{bmatrix} 2\sigma_d^2 + 13\sigma_\alpha^2 & -\sigma_d^2 - 11\sigma_\alpha^2 \\ -\sigma_d^2 - 11\sigma_\alpha^2 & 2\sigma_d^2 - 13\sigma_\alpha^2 \end{bmatrix},$$

from which the semiaxes of the error ellipse can be computed and an approximate CEP can be easily found.

Standard Position-Location Techniques

The classical triangulation method involves measuring the bearing angles to a pair of stations of known location and estimating one's position as the intersection of the two lines of bearing. It is easy to generalize this idea to the use of n angle measurements to n known stations (or reference points); the position estimate would then be found from the n lines of bearing. Fig. 2 shows the definition of the angles. For the ith station, the true bearing angle satisfies the relation

$$\arcsin \{(y_i - y)/[(x_i - x)^2 + (y_i - y)^2]^{1/2}\} = \theta_{oi} \tag{8}$$

and we get the first-order expansion for the error and noise terms

$$\begin{aligned}[(y_i - y_v)/\rho_{iv}^2]\,\delta_x - [(x_i - x_v)/\rho_{iv}^2]\,\delta_y \cong \theta_i &- \arctan \\ &[(y_i - y_v)/(x_i - x_v)] \\ &- e_{\theta i}\end{aligned} \tag{9}$$

where

$\rho_{iv} = [(x_i - x_v)^2 + (y_i - y_v)^2]^{1/2}$

and θ_i is the measured bearing angle.

Another method also indicated in Fig. 2 involves measurement of the ranges to two or more known points or stations (multilateration). The corresponding position loci are arcs of circles centered on the stations. The true range to the ith station is given by

$$[(x - x_i)^2 + (y - y_i)^2]^{1/2} = \rho_{oi} \tag{10}$$

and the first-order expansion yields

$$[(x_v - x_i)/\rho_{iv}]\,\delta_x + [(y_v - y_i)/\rho_{iv}]\,\delta_y \cong \rho_i - \rho_{iv} - e_{\rho i} \tag{11}$$

with ρ_i the measured range.

Consider the case in Fig. 2 where bearing-angle measurements to two stations are used to estimate position. If the angle between the two lines of bearing is small, then even small measurement errors will lead to large positon estimation errors roughly along a line bisecting the angle between the lines of bearing, while the estimation errors along a line perpendicular to the bisector will be nominal. This phenomenon is known as "geometric dilution" of position accuracy and will occur no matter how many lines of bearing we can measure that lie between the ith and jth lines (the extremes) shown. An exactly analogous phenomenon will occur when using range measurements if the angle between the tangents to the extreme arcs of position at their intersection is small. Indeed, geometric dilution of position estimation accuracy must be considered in every method of location. It will appear as a large difference between the semimajor and the semiminor axes of the statistical error ellipse.

Instead of measuring range or bearing angle directly, one can estimate a position locus by measuring the difference between the ranges to a pair of known points or stations or by measuring the angle between them. The measurement of range difference is often made by measuring the difference between the times of arrival of electromagnetic waves (e.g., Loran, Omega) or sound waves from the two stations; any error in the assumed speed of propagation will of course appear in the range-difference error. If a range-difference measurement is made between stations i and j, the estimated locus of position will be an hyperbola (see Fig. 2). The relation for the true range difference is simply $\rho_{oi} - \rho_{oj}$ using (10), and the corresponding first-order Taylor-series expansion for the position and measurement errors is

$$\begin{aligned}[(x_v - x_i)/\rho_{iv} - (x_v - x_j)/\rho_{jv}]\,\delta_x &+ [(y_v - y_i)/\rho_{iv} \\ &- (y_v - y_j)/\rho_{jv}]\,\delta_y \\ &\cong \Delta\rho_{ij} - (\rho_{iv} - \rho_{jv}) \\ &- e_{\Delta\rho i}, \end{aligned} \tag{12}$$

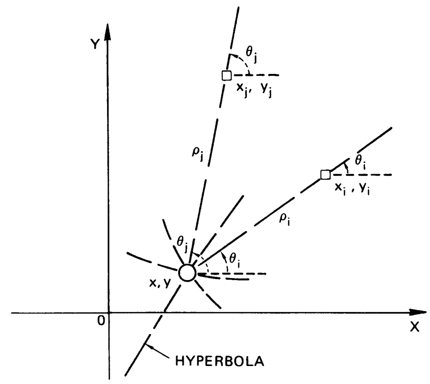

Fig. 2. Angle, range, and range-difference measurements.

where $\Delta\rho_{ij}$ is the measured range difference between stations i and j. Another method, mentioned above, is to measure the angle intercepted by the two stations (i.e., the difference between the bearing angles); the position locus is then an arc of a circle with the stations on its circumference. The linearized error relation for this intercepted-angle measurement method is, by a similar calculation,

$$\begin{aligned}[(y_j - y_v)/\rho_{jv}^2 - (y_i - y_v)/\rho_{iv}^2]\,\delta_x &- [(x_j - x_v)/\rho_{jv}^2 \\ &- (x_i - x_v)/\rho_{iv}^2]\,\delta_y \\ &\cong \Delta\theta_{ji} - (\mathrm{at}_j - \mathrm{at}_i) \\ &- e_{\Delta\theta j}, \end{aligned} \tag{13}$$

where

$\Delta\theta_{ji}$ = measured angle between stations j and i

$\mathrm{at}_i = \arctan[(y_i - y_v)/(x_i - x_v)]$.

All the basic relations for these different techniques lead to correction equations in the form of (1). The solution procedure described above is therefore directly applicable to any problem where enough measurements have been made.

Example: Location of Target by DF

The terms *stations* and *vehicle* have been used so far in the context of a navigator taking measurements to determine his position with respect to known fixed stations. It should be clear, however, that the basic structure of the problem consists of a set of reference points of known positions to which measurements are made in order to find an unknown position. Thus, for instance, we can treat the problem of locating a target (assumed stationary) by radio-direction-finder (DF) readings taken at several points along the known track of a moving vehicle. Each measurement in this case is the horizontal angle of arrival of the radio wave at the measurement point. Fig. 3 shows an example that was set up on a computer. The simulated vehicle moving along the track shown made angle measurements to a target at the points labeled a, b, c, and d.

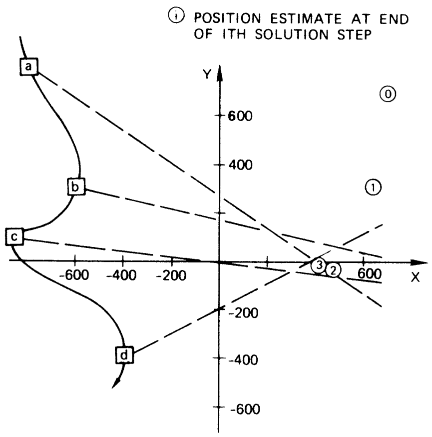

Fig. 3. Target location by DF.

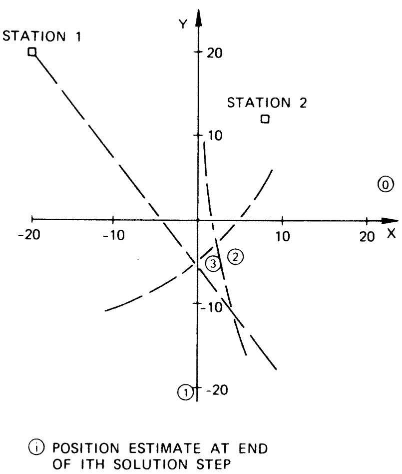

Fig. 4. Bearing, range, range-difference solution.

The angles were supposed to be subject to a random error having zero mean and a standard deviation of 3°; the resulting estimated LOPs are plotted in the figure. The solution was run using (9) and the initial guess indicated. The length of each step of the iterative procedure was limited to 400, and the successive position estimates at the end of each step are plotted. After four steps, the iterations had converged to the final estimate shown, which is quite reasonable for the data. The initial guess, of course, is a long way from the correct solution. In 67 repetitions of the problem, with different 3°-sigma random errors introduced in each run but with the same initial guess, the process always converged to a reasonable estimate. This illustrates the power of the method of solution.

Mixed-Mode Techniques

A major advantage of the Taylor-series estimation method is that it can handle in a straightforward way position-location problems in which different types of measurements are combined. The key step in the general mathematical development is the formulation of the set of simultaneous equations (2). It is important to recognize that there is no restriction to measurements of one type. For instance, one row may relate to a bearing measurement and thus have the form of (9); the next may relate to a range-difference measurement and look like (12); the next may be determined by a plotted line of position and take the form of (7); etc. The iterative correction process is capable of finding the "best," in a least-sum-squared-error sense, position estimate determined by the position loci for the complete collection of measurements.

This point can be best illustrated by a numerical example. Consider the problem shown in Fig. 4 with two known stations.

$$x_1 = -20 \qquad x_2 = 8$$

$$y_1 = 20 \qquad y_2 = 12$$

Suppose we make a bearing-angle measurement θ_1 and range measurement ρ_1 to station 1 and a difference measurement $\Delta\rho_{12}$ between the station 1 range and the station 2 range, observing

$$\theta_1 = 128^\circ = 2.234 \text{ rad}$$

$$\rho_1 = 32 \qquad \Delta\rho_{12} = 16.$$

The estimated position loci determined by these measurements are plotted in Fig. 4. The angle-measurement first-order expansion is shown in (9); but in order to weight angle errors properly in proportion to the range and range-difference errors, we multiply by the distance between the guess position and station 1 ρ_{1v} and get

$$[(y_1 - y_v)/\rho_{1v}]\,\delta_x - [(x_1 - x_v)/\rho_{1v}]\,\delta_y \cong \rho_{1v} \{\theta_1 - \arctan[(y_1 - y_v)/(x_1 - x_v)]\} \quad (14)$$

which gives the functions for a_{11}, a_{12}, and z_1. The second and third rows of (2) are given by (11) for the range and by (12) for the range difference, respectively. Assume that the bearing-angle measurement has a standard deviation of 3° = 0.0524 rad, the range has $\sigma_\rho = 2$, and the range difference has $\sigma_\Delta = 1$. Assume also that the errors are independent. Then the covariance matrix of the errors is

$$R = \begin{bmatrix} (0.0524)^2\rho_{1v}^2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us start with the position guess $x_v = 22$, $y_v = 4$. (This is probably a difficult spot.) Equation (3) gives $\delta_x = -23.5$, $\delta_y = -24.4$, so at the end of the first iteration the new position guess is $x_v = -1.5$, $y_v = -20.4$. The new guess at the end of each iteration is plotted in Fig. 4 to demonstrate the way in which the iterative Taylor-series method steps toward a solution point. At the end of iteration 3, the position is $x_v = 2, y_v = -5.1$ for which the next step calculation gives $\delta_x = 0.05$, $\delta_y = -0.05$; these are within our plotting accuracy and so it is clear that the iterations have converged. As the LOPs show, this is a reasonable solution to the given position-location problem. At the solution point $x_v = 2, y_v = -5.1$, we compute

$$A = \begin{bmatrix} 0.752 & 0.659 \\ 0.659 & -0.752 \\ 0.990 & 0.192 \end{bmatrix}$$

$$Q_0 = [A^T R^{-1} A]^{-1} = \begin{bmatrix} 0.899 & -0.640 \\ -0.640 & 3.578 \end{bmatrix}$$

from which the semimajor and semiminor axes of the error ellipse are

$$a = 1.929 \qquad b = 0.868,$$

and we get an approximate CEP of 1.587, all in the x, y distance units.

Other start guesses we have tried still converge to this same solution. If different measurement variances are assumed, the solution point shifts toward the LOPs that are presumably more accurate; this also, of course, changes the error ellipse.

A significant advantage is that the method leads naturally to measures of the accuracy of the position-location solution obtained at convergence. If information about error variance values is available, the terms in the matrix $Q_0 = [A^T R^{-1} A]^{-1}$ determine the semimajor and semiminor axes of the error ellipse and an approximate CEP can be easily calculated as we have noted. Whether error variances are known or not, the length of the **z** vector in (2) will constitute an overall measure of the differences between the navigational measurements used as input data and the corresponding quantities obtained from the position solution. This vector length will be proportional to the inaccuracy of the solution fix. Further, it can indicate the occurrence of a measurement blunder and thus give a useful check on the validity of the solution.

The major idea here is that the navigator should use all the available measurements to fix his position. This is in contrast to techniques that have been suggested based on the best cut or the best two LOPs. Of course, proper use of multiple measurements requires knowledge of error statistics. It appears that the navigational literature is somewhat lacking in this subject area, and not many detailed treatments of practical measurement errors are avilable. The Taylor-series method places many different measurements in a common mathematical framework. It is hoped that this will encourage the consistent evaluation of the different measurement processes.

Conclusion

The iterative Taylor-series method is a navigation computation scheme of general utility. Some of the advantages and disadvantages have been mentioned. The need for an initial position guess to start the solution process and the reliance on iterative linearized corrections are theoretical weaknesses. However, a reasonable initial guess is often easy to make, and it turns out that the process will usually converge even with an initial guess that is way off. A validity test at each step is easy to implement. We simply compute $\det[A^T R^{-1} A]$ if error variance information is available, or $\det[A^T A]$ if not, and reject the input data or the position guess if this number is too small. Failure of the iterations to converge is also easy to detect. We compute the length $(\delta_x^2 + \delta_y^2)^{1/2}$ of the correction vector at the end of each iteration and, after some 3 steps, start to compare it with that of the previous step. If the ratio is not much less than unity, the process is not converging; we should try a new start guess or take more measurements.

It is important to remark that convergence of the iterations is dependent on having (1) in good form. Sometimes it is tempting, for reasons of computational convenience, to transform mathematically the basic measurment relation, e.g., to write the equation for a bearing angle in terms of the sine or the tangent of the angle rather than the angle itself. Our experience has been that these transformations tend to restrict the region of convergence and that the temptation should be resisted. An exception, of course, is writing the angle relation in terms of displacements by multiplying by the distance as we did in (14).

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