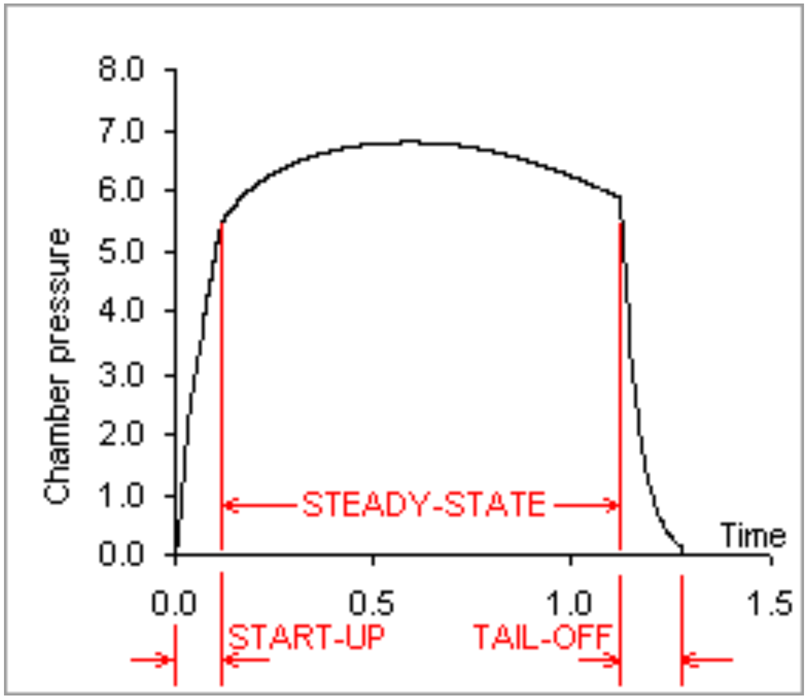
Modeling Transient Rocket Operation (Lecture 6.2: Solid Rockets)



- .. The primary goal of man is survival … food, shelter
  … basic necessities …

- ***A second aim of man is to build things that run very HOT and very LOUD and move really, really FAST ...***

- Sutton and Biblarz: Chapter 11
- Richard Nakka Web Page:

  * *http://members.aol.com/ricnakk/th_pres.html*

Transient Pressure Model

- Combustion Produces High temperature gaseous By-products
- Gases Escape Through Nozzle Throat
- Nozzle Throat Chokes (maximum mass flow)
- Since Gases cannot escape as fast as they are produced
  … Pressure builds up
- As Pressure Builds .. Choking mass flow grows
- Eventually Steady State Condition is reached

Choking Massflow per Unit Area

- maximum Massflow/area Occurs when When M=1
- Effect known as *Choking* in a Duct or Nozzle
- i.e. nozzle will Have a mach 1 throat

$$\left(\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right)_{\max} = \left(\frac{\dot{m}}{A^*}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right) =$$

$$\frac{\sqrt{\gamma}}{\left[1+\frac{(\gamma-1)}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} = \sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow$$

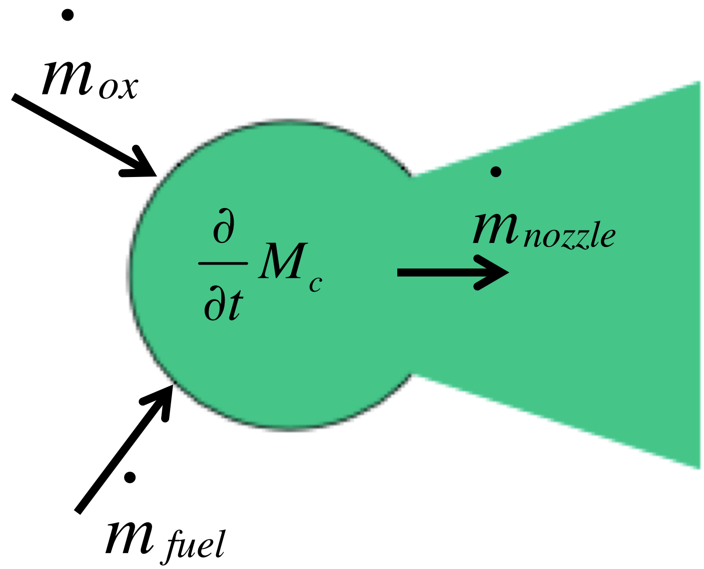
$$\boxed{\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{p_0}{\sqrt{T_0}}}$$

Chamber Pressure Model

- Gaseous Mass Trapped in Chamber

$$\frac{\partial}{\partial t} M_c = \left[\dot{m}_{fuel} + \dot{m}_{ox} \right] - \dot{m}_{nozzle}$$

$$\frac{\partial}{\partial t} M_c = \frac{\partial}{\partial t}\left[\rho_c V_c\right] = \frac{\partial}{\partial t}\left[\rho_c\right] V_c + \rho_c \frac{\partial}{\partial t}\left[V_c\right]$$



- Assuming nozzle chokes immediately

$$\frac{\partial}{\partial t}\left[\rho_c\right] V_c + \rho_c \frac{\partial}{\partial t}\left[V_c\right] = \left[\dot{m}_{fuel} + \dot{m}_{ox} \right] - A^* \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \frac{P_0}{\sqrt{T_0}}$$

Chamber Pressure Model (cont'd)

- Using ideal gas law, *Assuming constant flame temperature*

$$\rho_c = \frac{P_0}{R_g T_0} \rightarrow \frac{\partial}{\partial t}\left[\rho_c\right] \approx \frac{1}{R_g T_0}\frac{\partial}{\partial t}\left[P_0\right]$$

- Subbing into mass flow equation

$$\frac{\partial P_0}{\partial t}\frac{V_c}{R_g T_0} + \frac{P_0}{R_g T_0}\frac{\partial V_c}{\partial t} = \left[\dot{m}_{fuel} + \dot{m}_{ox}\right] - \frac{R_g T_0}{V_c} A^* \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{P_0}{\sqrt{T_0}}$$

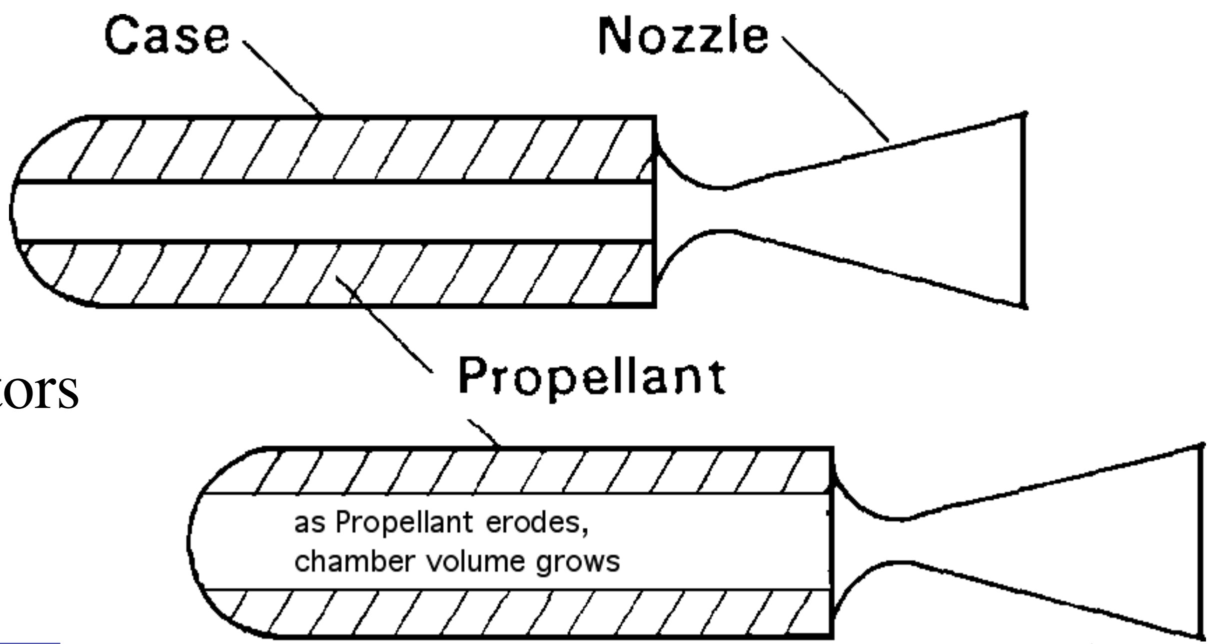
$$\frac{\partial P_0}{\partial t} + P_0 \frac{1}{V_c}\frac{\partial V_c}{\partial t} + \frac{R_g T_0}{V_c} A^* \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{P_0}{\sqrt{T_0}} = \frac{R_g T_0}{V_c}\left[\dot{m}_{fuel} + \dot{m}_{ox}\right]$$

$$\frac{\partial P_0}{\partial t} + P_0\left[\frac{1}{V_c}\frac{\partial V_c}{\partial t} + \frac{A^*}{V_c}\sqrt{\gamma R_g T\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] = \frac{R_g T_0}{V_c}\left[\dot{m}_{fuel} + \dot{m}_{ox}\right]$$

Transient Operation Model For Solid Rockets

- Revisit General Model

$$\frac{\partial P_0}{\partial t} + P_0 \left[\frac{1}{V_c}\frac{\partial V_c}{\partial t} + \frac{A^*}{V_c}\sqrt{\gamma R_g T \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \right] = \frac{R_g T_0}{V_c}\left[\dot{m}_{propellant} \right]$$



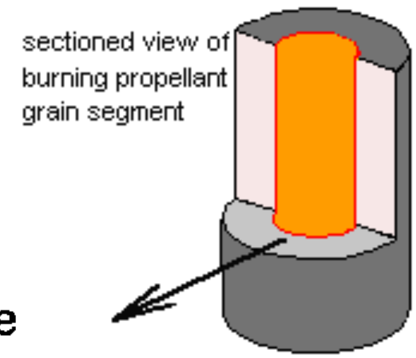
- For Solid Rocket Motors

$$\frac{\partial V_c}{\partial t} \neq 0$$

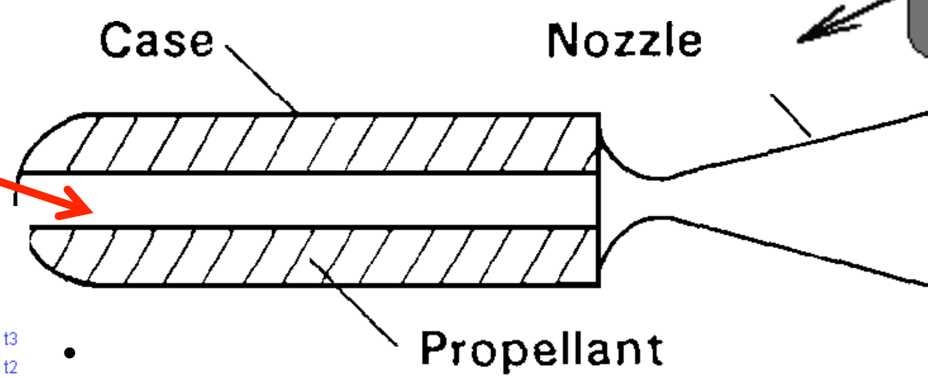
Solid Rocket Example

$$\frac{\partial P_0}{\partial t} + P_0\left[\frac{1}{V_c}\frac{\partial V_c}{\partial t} + \frac{A^*}{V_c}\sqrt{\gamma R_g T\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] = \frac{R_g T_0}{V_c}\left[\dot{m}_{propellant}\right]$$

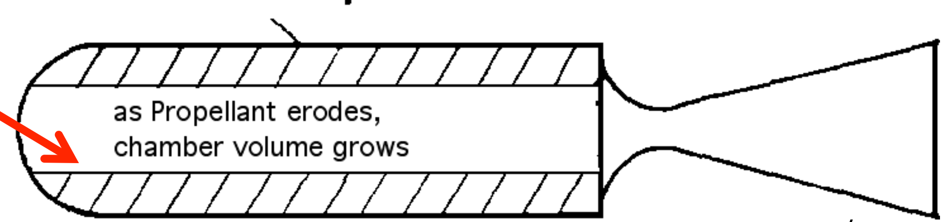
$$\frac{\partial V_c}{\partial t} = A_{burn}\,\dot{r} \rightarrow \left[\begin{array}{l} A_{burn} = \text{Grain Surface Burn Area} \\ \dot{r} = \text{Grain Linear Recession Rate} \end{array}\right]$$

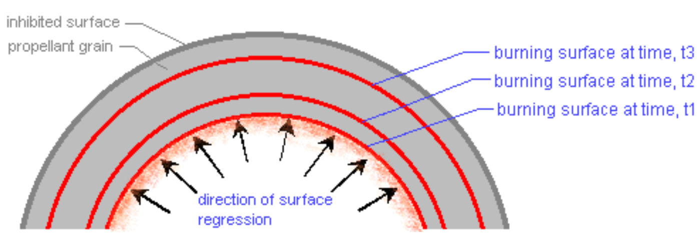


A_{burn}



$\dot{r}$

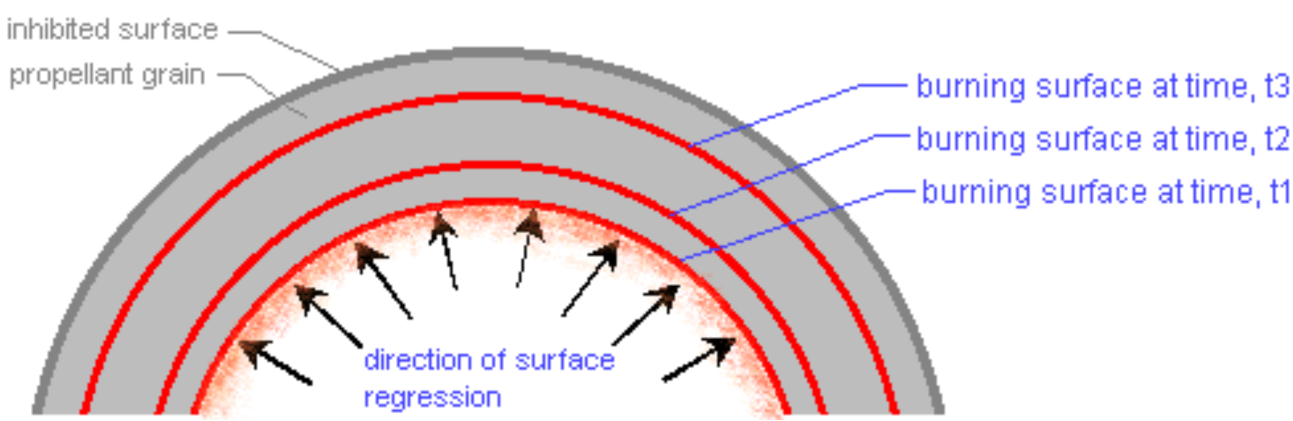




Solid Rocket Example (cont'd)

$$\frac{\partial V_c}{\partial t} = A_{burn}\,\dot{r} \rightarrow \begin{bmatrix} A_{burn} = \text{Grain Surface Burn Area} \\ \dot{r} = \text{Grain Linear Recession Rate} \end{bmatrix}$$

$$\left[\dot{m}_{propellant}\right] = \rho_p \cdot A_{burn} \cdot \dot{r}$$



EFFECT OF PRESSURE ON BURN RATE - Saint-Robert's Law

$$\boxed{\dot{r} = ap^{n}}$$

r = linear burning rate

a = an empirical constant moderately influenced by propellant grain temperature

n = burning rate pressure exponent

$$\dot{r} = aP_o^{\,n} \rightarrow \{a,n\} \rightarrow \text{empirically derived constants}$$

Solid Rocket Example (cont'd)

•Propellant burn rate may be expressed in terms of the chamber pressure by the Saint Robert's law …

$$\dot{r} = aP_o^{\,n} \rightarrow \{a,n\} \rightarrow \text{empirically derived constants}$$

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0 \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

• careful with units on a …

$$a \sim \frac{m}{\sec}\left(\frac{1}{kPa}\right)^n$$

Solid Rocket Example (cont'd)

$$\log\left[\dot{r}\right] = n\log\left[P\right] + y_0$$

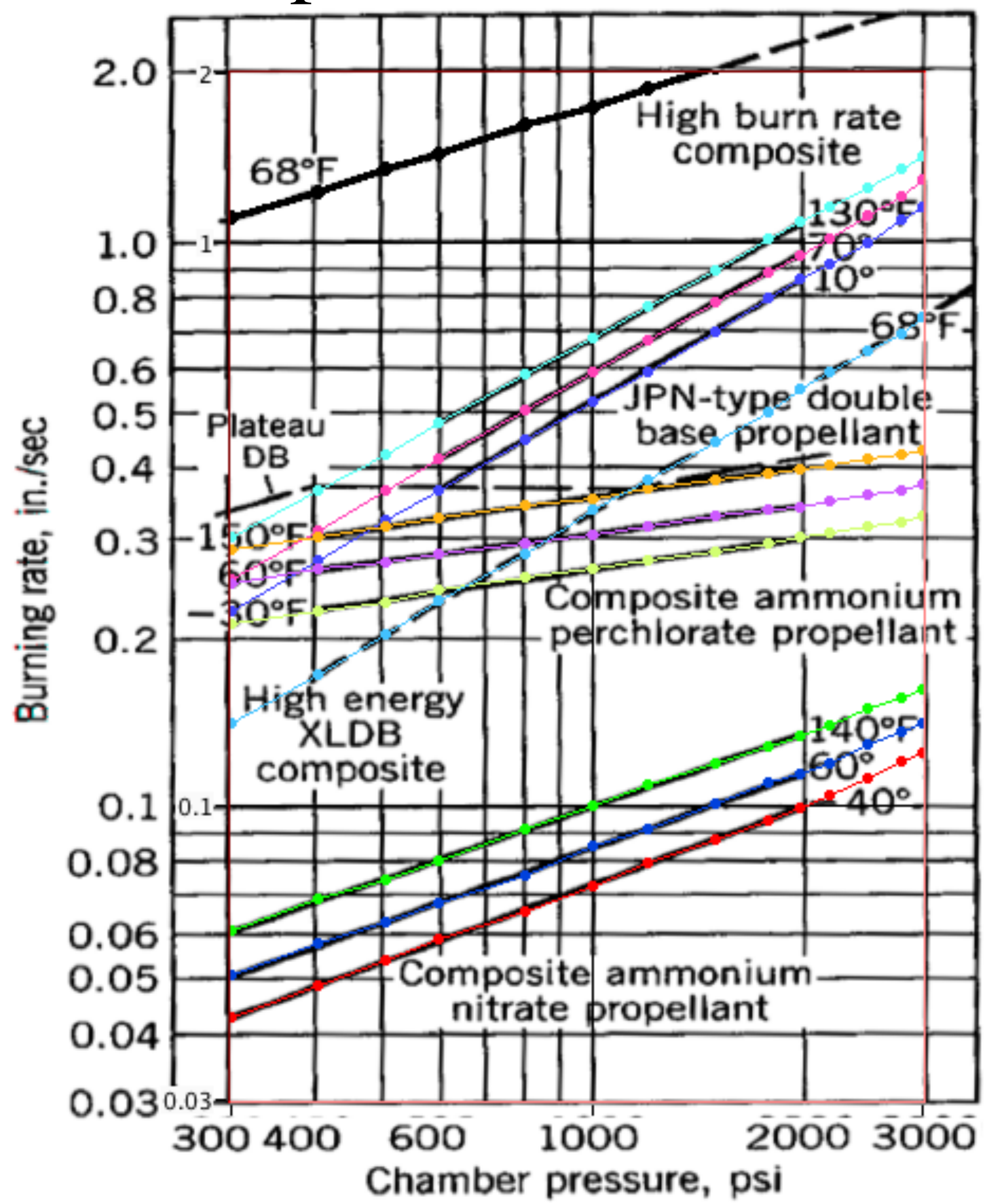
$$\dot{r} = e^{Log\left[P_o^{\,n}\right]+y_0} = e^{y_0}P_o^{\,n} \rightarrow e^{y_0} \equiv a$$

$$\rightarrow$$

$$\dot{r} = aP_o^{\,n}$$

- careful with units on *a* …

Sutton and Biblarz, Chapter 11



Solid Rocket Example (cont'd)

Solid Propellant Saint Robert's Curve Fit (P$_0$-psia, rdot- in/sec)

Propellant Name	*n*	*a (in/sec-psia^n)*
Composite Ammonium Nitrate, -40F	0.463474	0.002965
Composite Ammonium Nitrate, 60F	0.445084	0.003909
Composite Ammonium Nitrate, 140F	0.426803	0.005243
High Energy XLDB Composite	0.720473	0.002293
Composite Ammonium Perchlorate, -30F	0.187867	0.072001
Composite Ammonium Perchlorate, 60F	0.170286	0.094044
Composite Ammonium Perchlorate, 150F	0.172255	0.107348
JPN-type Double Base, 10F	0.712606	0.003818
JPN-type Double Base, 70F	0.701667	0.004624
JPN-type Double Base, 130F	0.678433	0.006260
High Burn Rate Composite @ 68F	0.380710	0.126409

$$\dot{r} = a P_o^{\,n}$$

- Input, Psia
- Output, in/sec

Solid Rocket Example (cont'd)

Propellant, Saint Robert's Curve Fit (P0-kPa, rdot- cm/sec)

propellant name	*n*	*a (cm/sec-kPa^n)*
Composite Ammonium Nitrate, -40F	0.463474	0.003077
Composite Ammonium Nitrate, 60F	0.445084	0.004204
Composite Ammonium Nitrate, 140F	0.426803	0.005841
High Energy XLDB Composite	0.720473	0.001449
Composite Ammonium Perchlorate, -30F	0.187867	0.127245
Composite Ammonium Perchlorate, 60F	0.170286	0.171940
Composite Ammonium Perchlorate, 150F	0.172255	0.195519
JPN-type Double Base, 10F	0.712606	0.002450
JPN-type Double Base, 70F	0.701667	0.003030
JPN-type Double Base, 130F	0.678433	0.004291
High Burn Rate Composite @ 68F	0.380710	0.153949

$$\dot{r} = a P_o^{\,n}$$

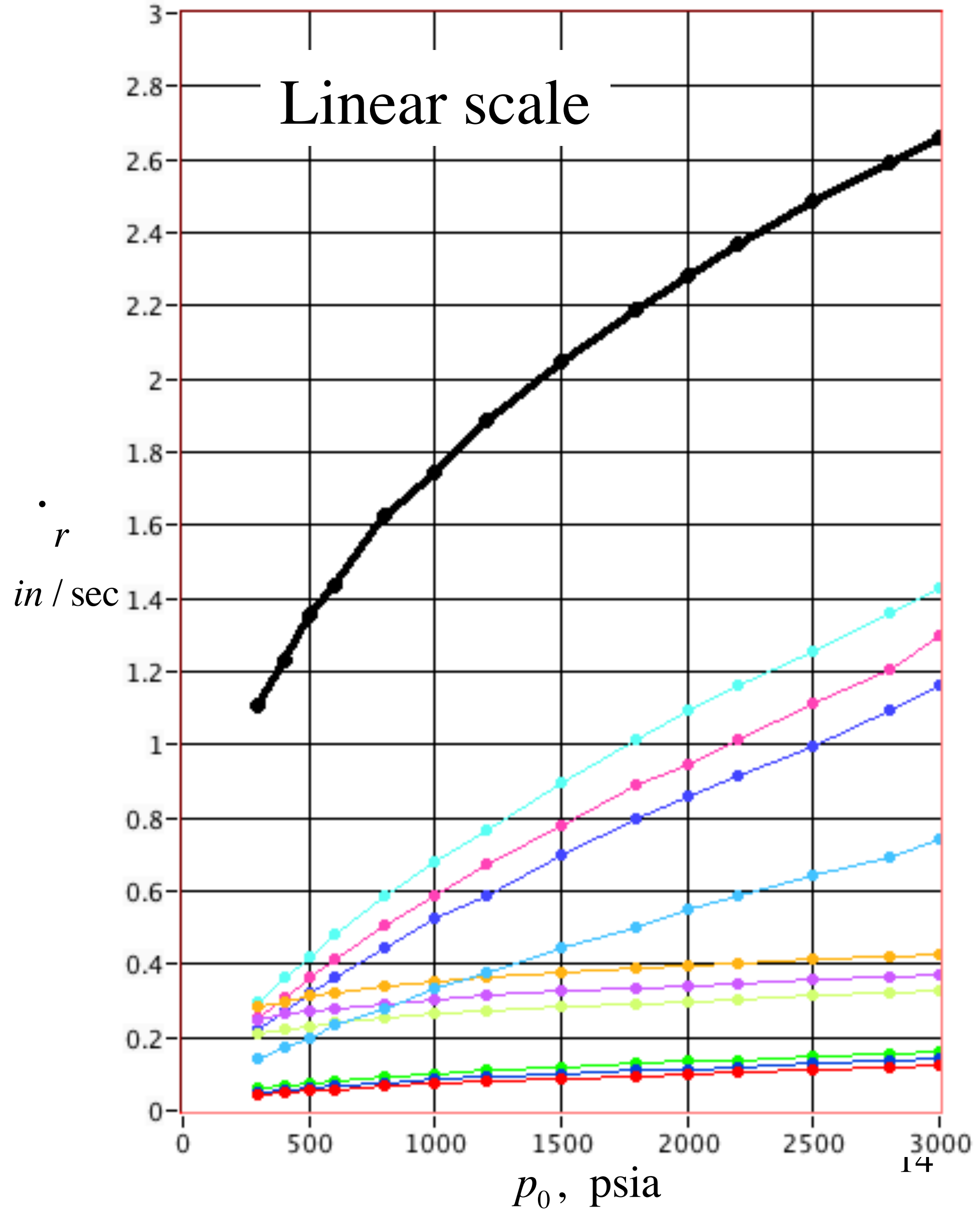
- Input, kPa
- Output, cm/sec

Solid Rocket Example (cont'd)

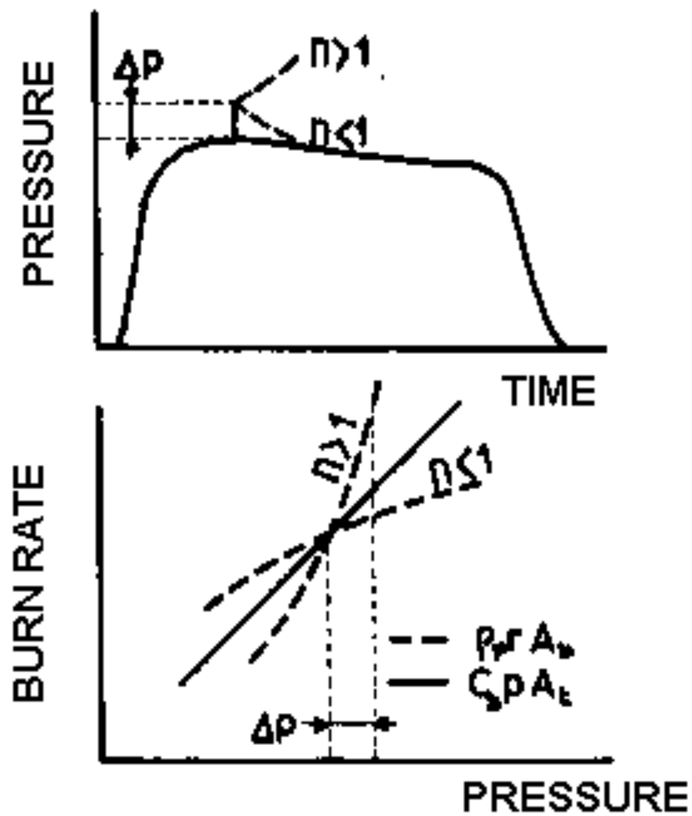
- *Propellant Burn rate is extremely sensitive to exponent, n*

- *Stable operation requires $0.001 < n < 0.990$*

- *High values of n make for a propellant whose burn rate is sensitive to chamber pressure*



Exponent Effect on Burn Rate (Pressure Excursion)



Source: Barrere et al., Raketenantriebe, Fig 5.1 (1961)

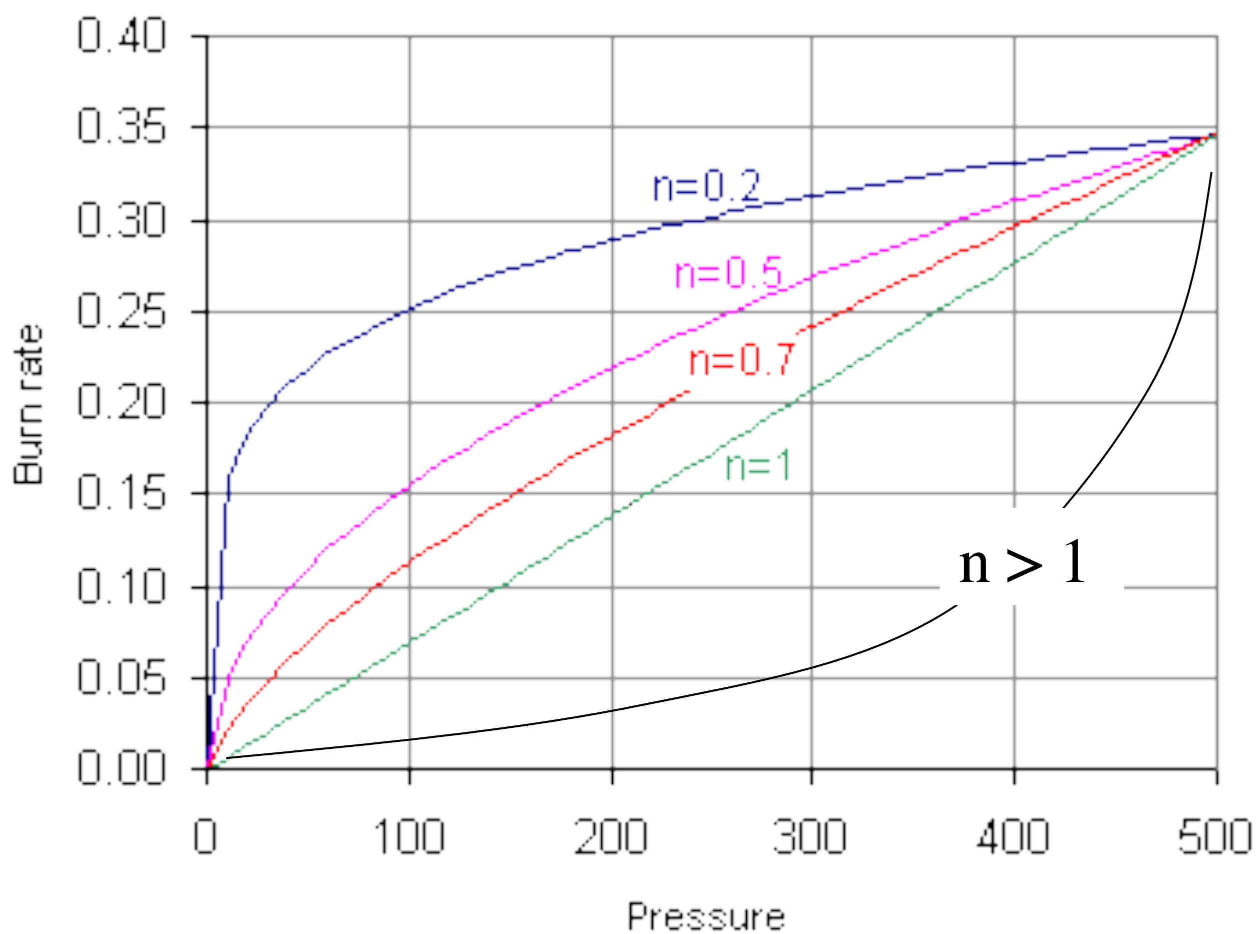
Effect of Burn Exponent

n>1 : Slight positive pressure excursion might lead to explosion of the chamber.

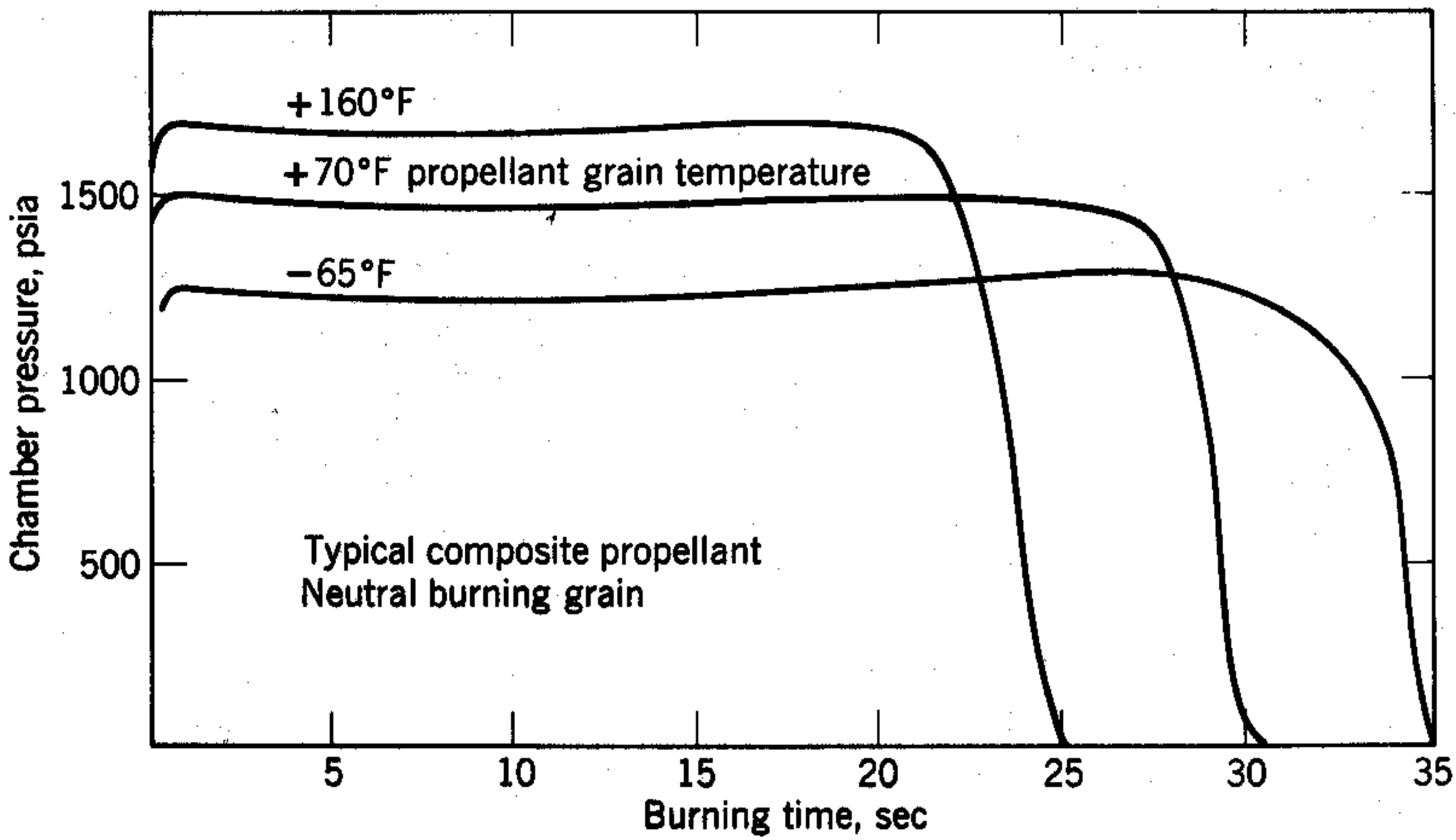
n ~ < 0.8: Maximum pressure exponent tolerated with typical solid rocket propellants.

n < 0: Slight negative pressure excursion might lead to continuing decay of chamber pressure and premature extinguishment of propellant.

Effect of Burn Exponent (2)



Grain Temperature Effect on Burn Rate



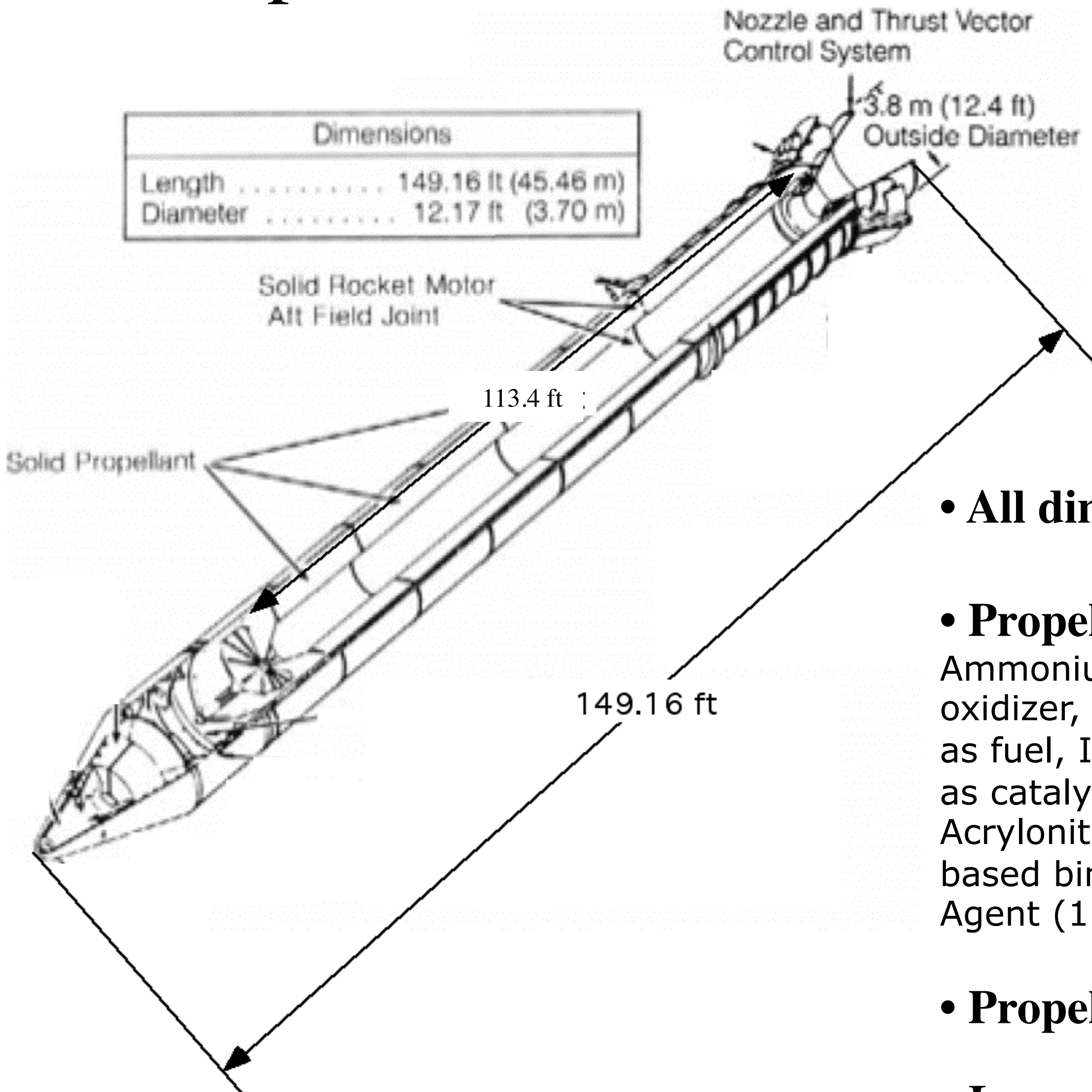
Sutton, & Biblarz p. 268 (1986)

SOLID ROCKET MOTOR
GRAIN DESIGN PROGRAMS

- **Grain Design Program (GDP-Light)**
- **http://home.vianetworks.nl/users/aed/gdp/gdp.htm**

- **Useful Code to test new and unusual grain shapes to achieve certain thrust profiles or minimize slivers and residual burning.**

Space Shuttle SRB Numerical Example



- **All dimensions are approximate**

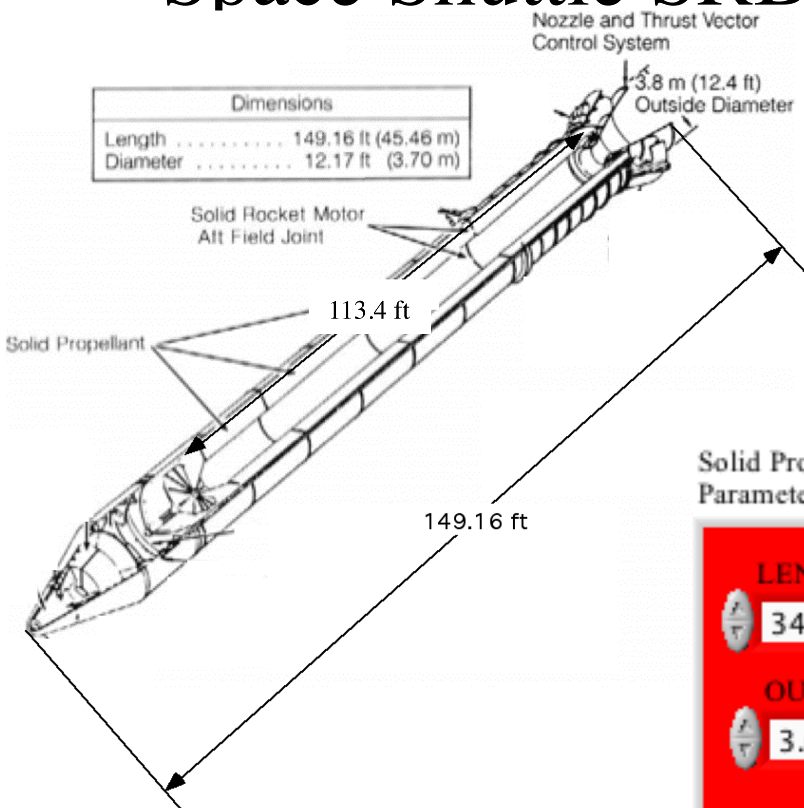
- **Propellant:**

Ammonium Perchlorate (69.6%) as oxidizer, Aluminum Powder (16%) as fuel, Iron Oxidizer Powder (0.4%) as catalyst, Polybutadiene Acrylic Acid Acrylonitrile (12.04%) **(PBAN)** as rubber-based binder, Epoxy Curing Agent (1.96%)

- **Propellant Density .. 1760 kg/m^3**

- **Launch Propellant mass … 502,0 kg**

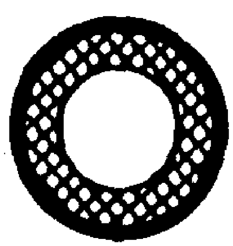
Space Shuttle SRB Numerical Example (cont'd)



- D_{exit} = 3.8 m ---> A_{exit} = 11.341 m^2
- A_{exit}/A^* = 7.78
  --> A^*=1.458 m^2

- **Propellant Density .. 1760 kg/m^3**
- **Launch Propellant mass … 502,000 kg**

Internal Grain Pattern



Solid Propellant Grain Parameters

LENGTH, M
34.57

OUTSIDE DIAMETER, M
3.66

INSIDE DIAMETER, M
1.7

PROPELLANT DENSITY KG/M^3
1760

Solid Propellant Grain Parameters 2

Propellant thickness,M
1.96

PROPELLANT Volume M^3
285.24

PROPELLANT MASS, KG
502022

PROPELLANT MASS, Mton
502.022

Chamber Cross Section Area, M2
2.2698

Propellant Cross Section Area, M^2
8.25108

Chamber Volume, M^3
78.467

Propellant Burn area, M^2
184.628

Space Shuttle SRB Numerical Example (cont'd)

Propellant, Saint Robert's Curve Fit

propellant name	***n***	***a (cm/sec-kPa^n)***
Composite Ammonium Perchlorate, 60F	0.172	0.192

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

$$\begin{bmatrix} A_{burn} = 2\pi R_{chamber} L_{prop} \\ V_c = \pi R_{chamber}^{\;2} L_{prop} \end{bmatrix} \rightarrow R_{chamber} = R_{i_{initial}} + \int_0^t \dot{r}\,dt = R_{i_{initial}} + \int_0^t a P_o^{\,n}\,dt$$

Space Shuttle SRB Numerical Example (cont'd)

- Use Trapezoidal rule or Runge-Kutta to integrate

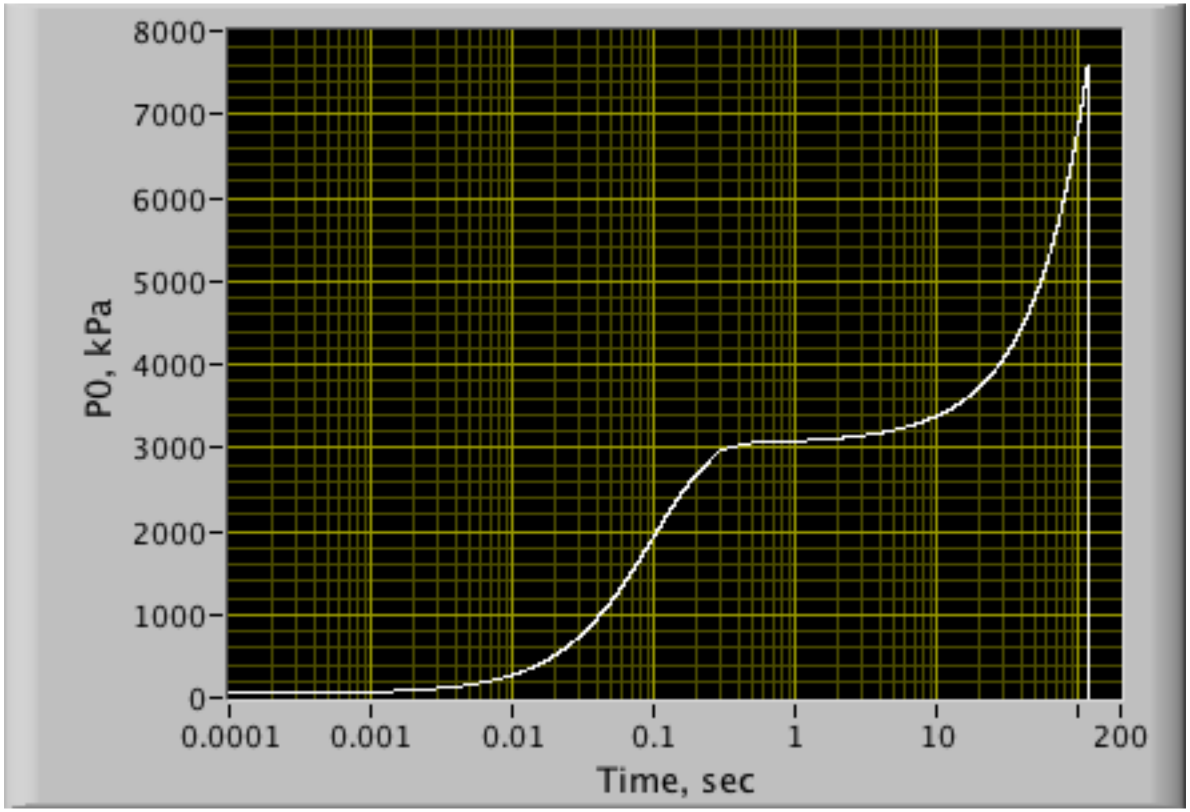
$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\ n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

- Recursive propagation of chamber diameter

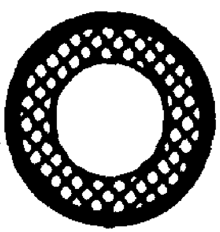
$$R_{burn_{k+1}} = R_{i_{initial}} + \int_0^{(k+1)\Delta t} \dot{r}dt = R_{i_{initial}} + \int_0^{(k)\Delta t} \dot{r}dt + \int_{(k)\Delta t}^{(k+1)\Delta t} \dot{r}dt \rightarrow$$

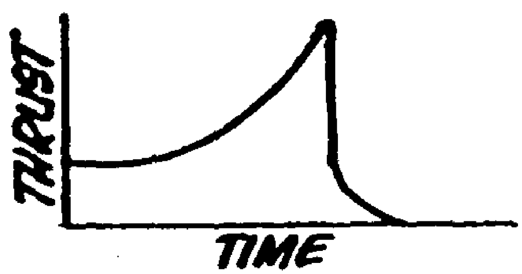
$$R_{burn_{k+1}} = R_{burn_k} + \int_{(k)\Delta t}^{(k+1)\Delta t} \dot{r}dt \approx R_{burn_k} + \dot{r}\,\Delta t = R_{burn_k} + a P_{o\ k}^{\ n}\,\Delta t$$

SRB Burn Time History



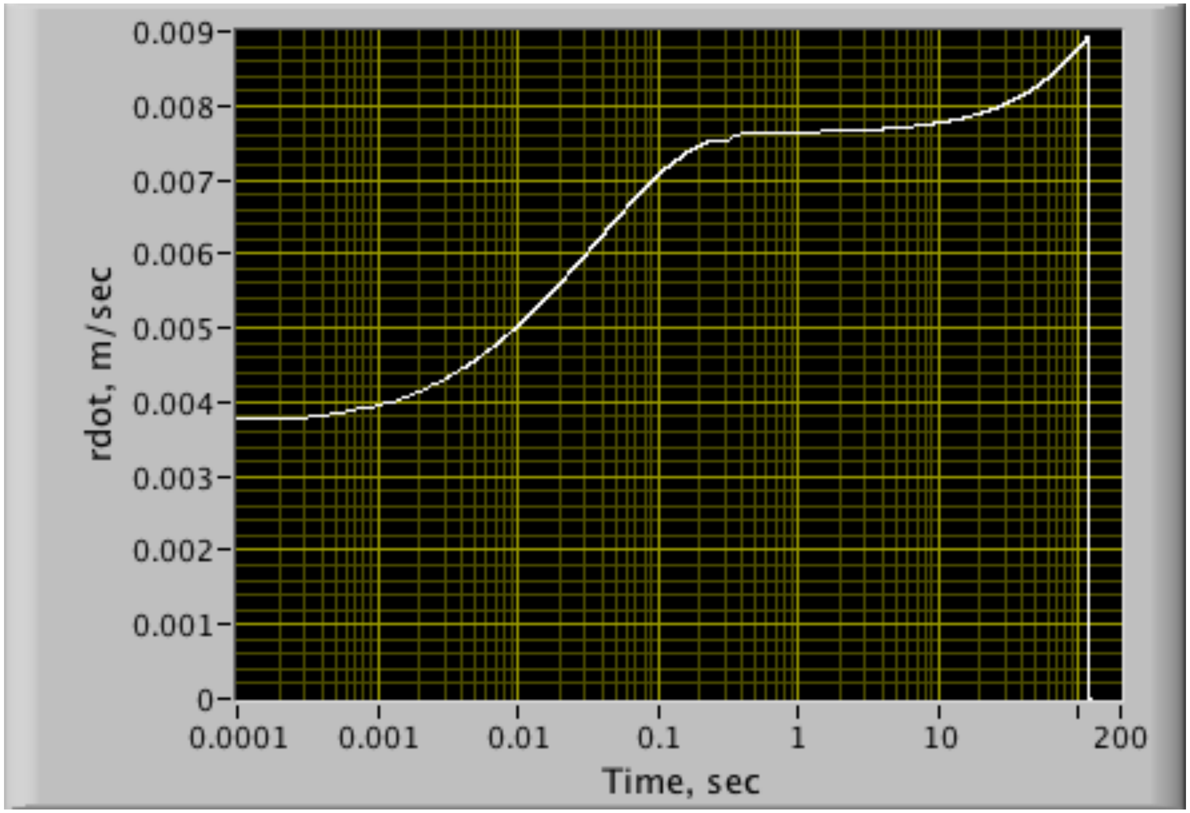
- "Progressive Burn Pattern"



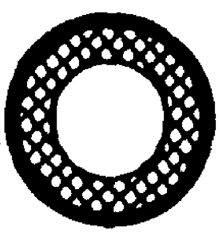


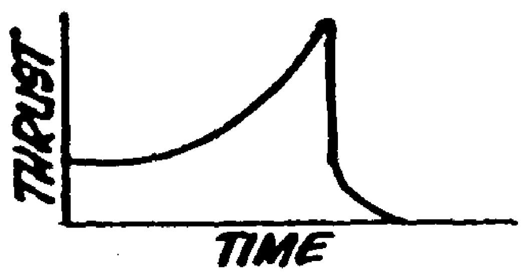
Burn Time: 121 seconds

SRB Burn Time History (cont'd)



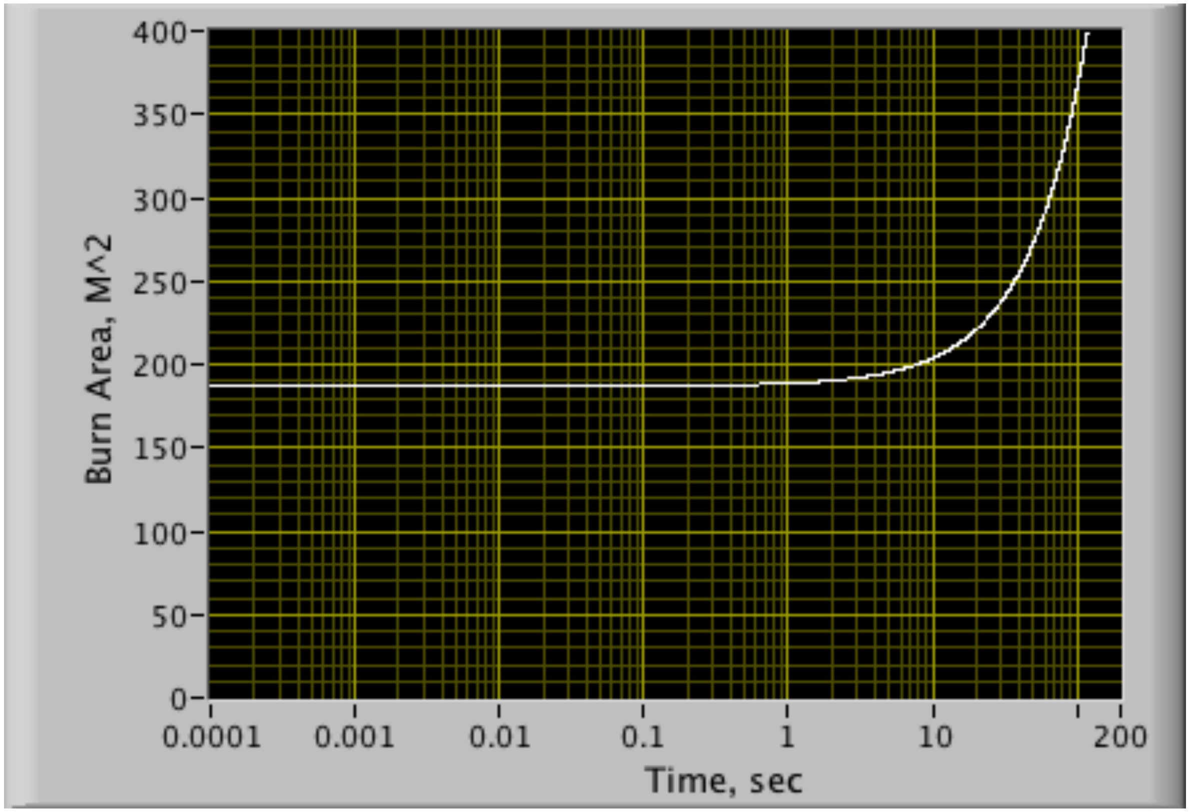
- "Progressive Burn Pattern"



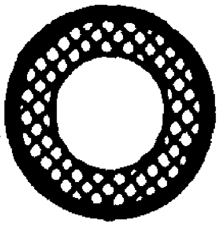


Burn Time: 121 seconds

SRB Burn Time History (cont'd)

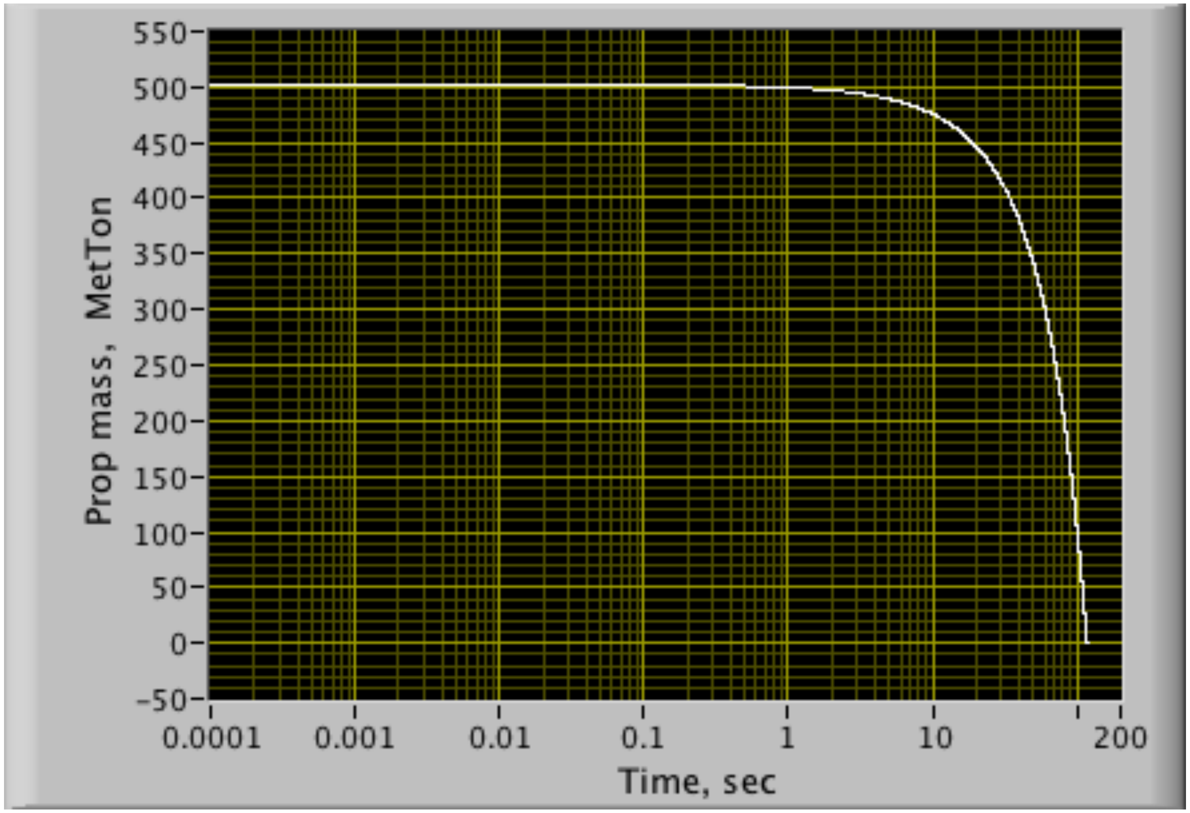


- "Progressive Burn Pattern"



Burn Time: 121 seconds

SRB Burn Time History (cont'd)

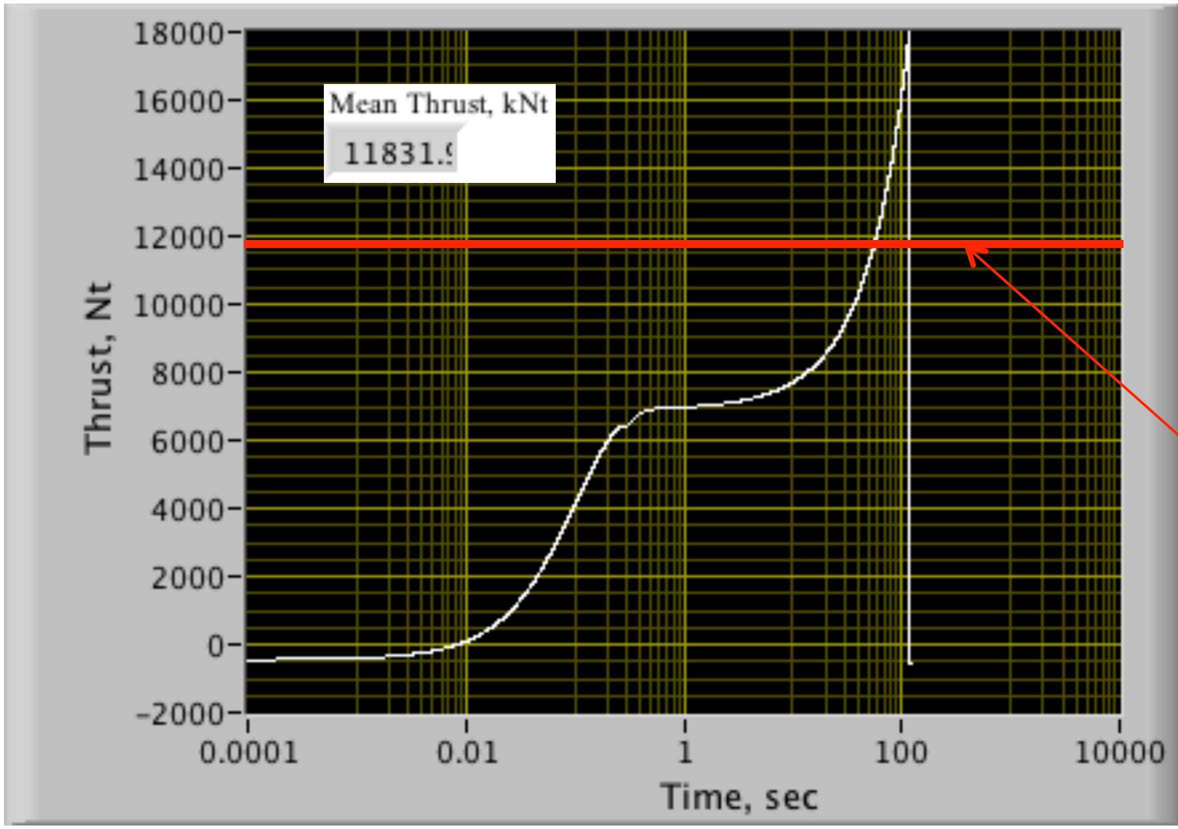


- "Progressive Burn Pattern"

Burn Time: 121 seconds

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\ n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

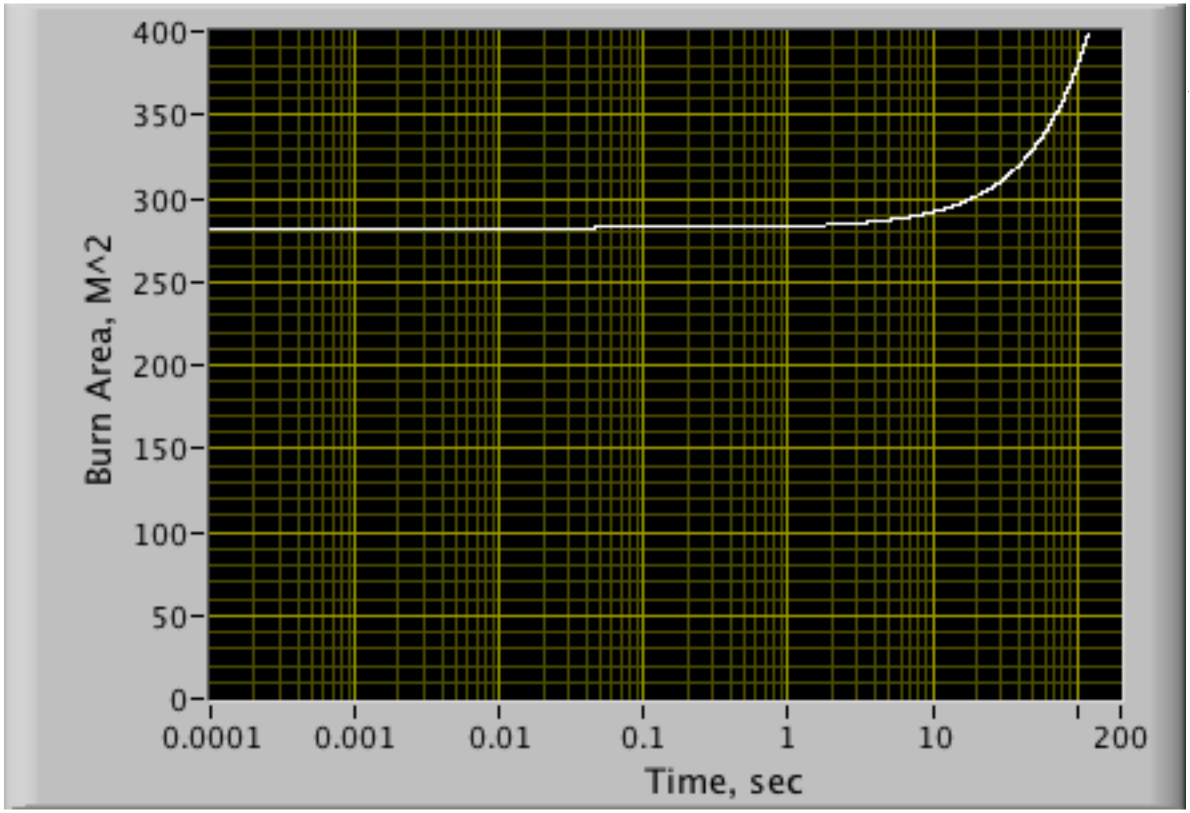
SRB Burn Time History (cont'd)

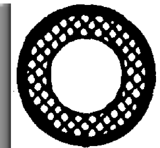


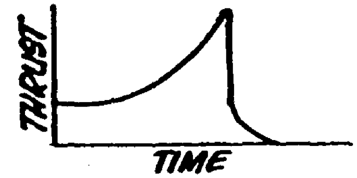
- "Progressive Burn Pattern"
- Quoted "Nominal Thrust" of SRB (11,780 kNt)

So our model is "pretty good"

Burn Area Revisited







- With this siimple Model Uniform Cylindrical Propellant grain necessarily leads to Progressive burn

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0 \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

Burn Area Revisited (cont'd)

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] =$$

$$\boxed{\rightarrow C^* = \left(\frac{P_0 A^*}{\dot{m}_{exit}}\right) = \frac{\sqrt{\gamma R_g T_0}}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}}$$

$$\boxed{\boxed{\frac{\partial P_0}{\partial t} = R_g T_0\left(\frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p - \rho_0\right] - \left(\frac{P_0 A^*}{V_c C^*}\right)\right)}}$$

Burn Area Revisited (cont'd)

$$\boxed{\frac{\partial P_0}{\partial t} = R_g T_0 \left(\frac{A_{burn} a P_o^{\,n}}{V_c} \left[\rho_p - \rho_0 \right] - \left(\frac{P_0 A^*}{V_c C^*} \right) \right)}$$

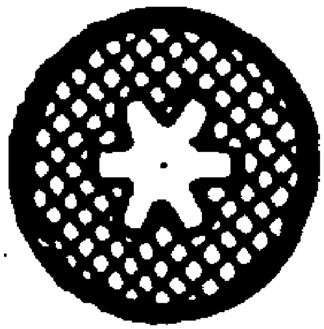
$$\rho_p >> \rho_0 \rightarrow \boxed{\frac{\partial P_0}{\partial t} \approx R_g T_0 \left(\frac{A_{burn} a P_o^{\,n}}{V_c} \rho_p - \left(\frac{P_0 A^*}{V_c C^*} \right) \right)}$$

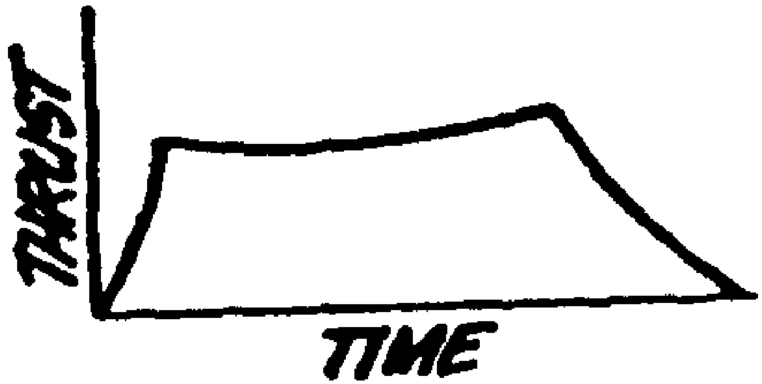
- Cylindrical Port …

$$\frac{A_{burn}}{V_{burn}} = \frac{2\pi \cdot r \cdot L}{\pi \cdot r^2 \cdot L} = \frac{2}{r}$$

$$\boxed{\frac{\partial P_0}{\partial t} \approx R_g T_0 \left(\frac{2 a P_o^{\,n}}{r} \rho_p - \left(\frac{P_0 A^*}{\pi \cdot r^2 \cdot L \cdot C^*} \right) \right)}$$

Burn Area Revisited (cont'd)

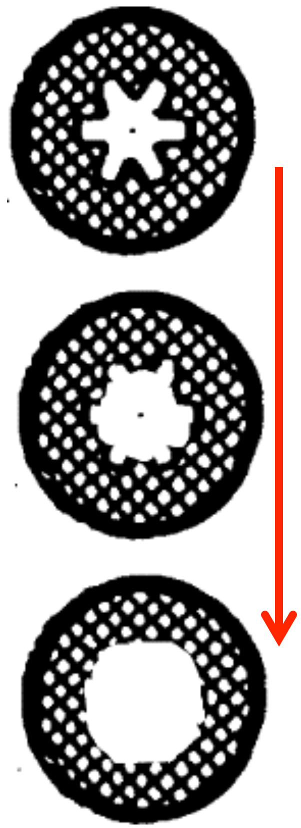




- What happens to the burn area with this pattern?

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

Burn Area Revisited (cont'd)



time

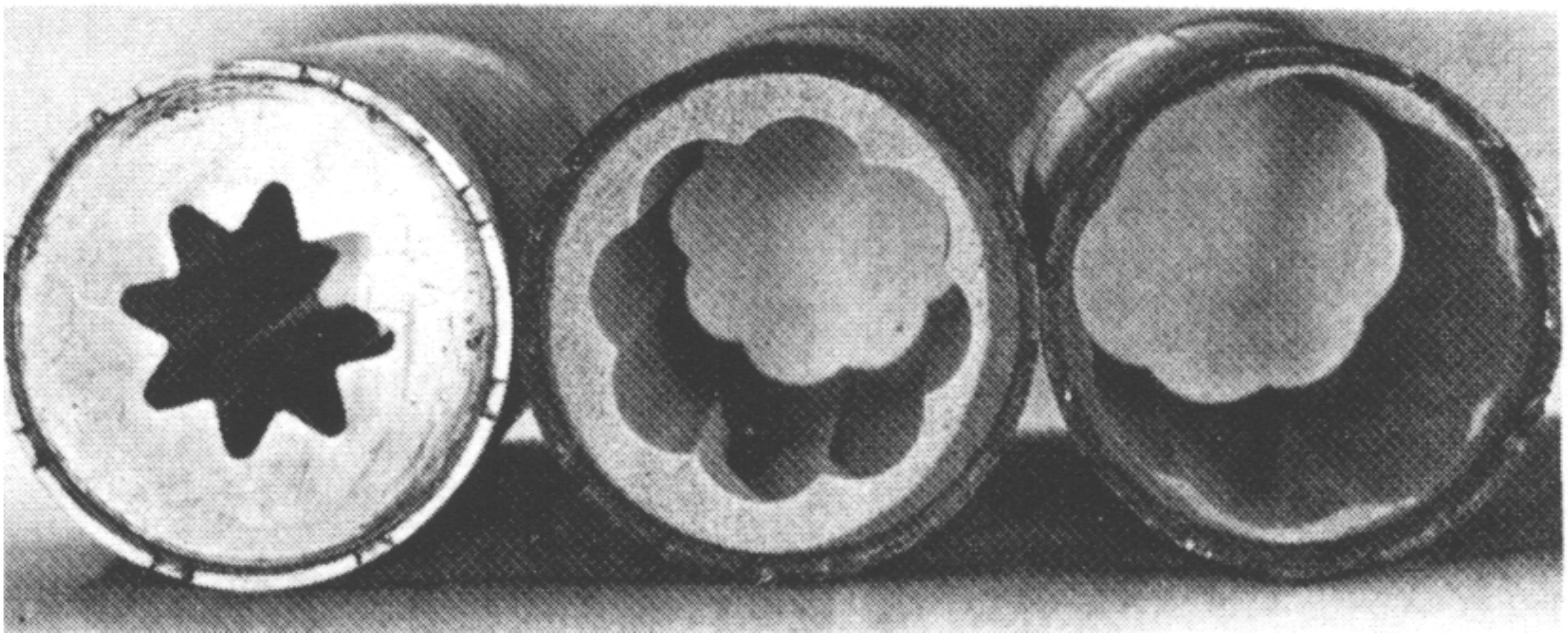
"tips burn first"

- Burn Area stays relatively constant
- Burn Volume Goes Down
- Ratio of Burn Area to Chamber Volume goes ….. *Down! Fast!*
- *Result is a more shaped burn profile*

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0 \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

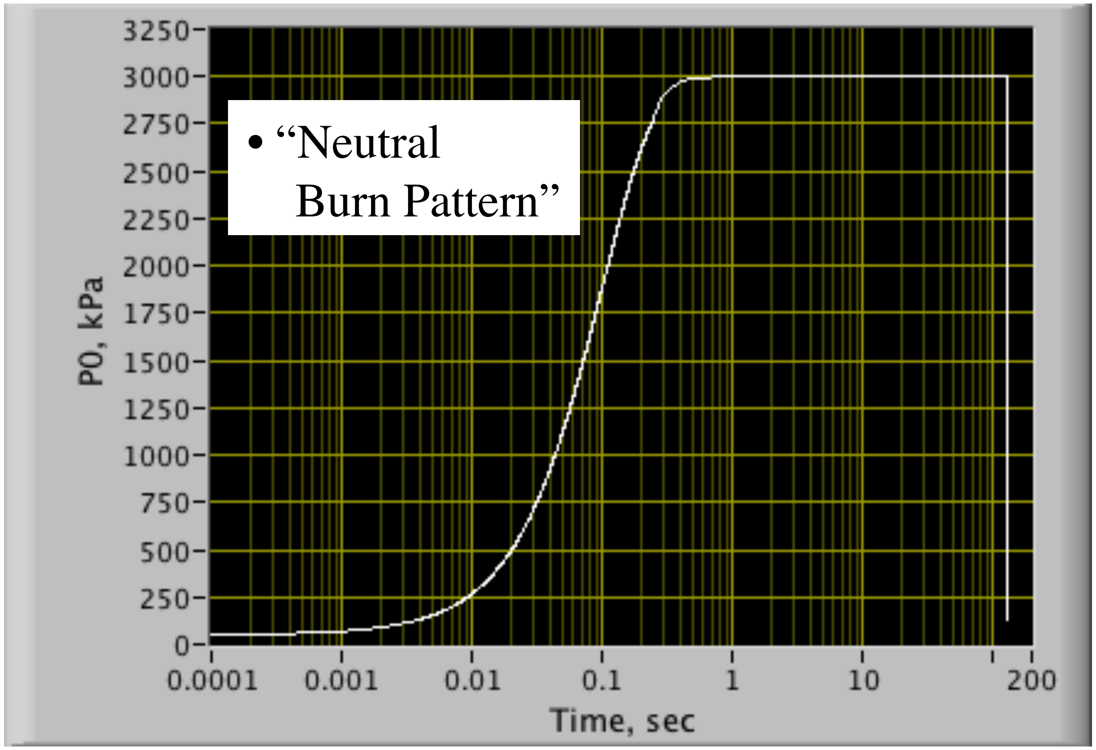
SHAPE OF PROPELLANT GRAINS QUENCHED AT DIFFERENT TIMES

Life History of Solid Motor Shown

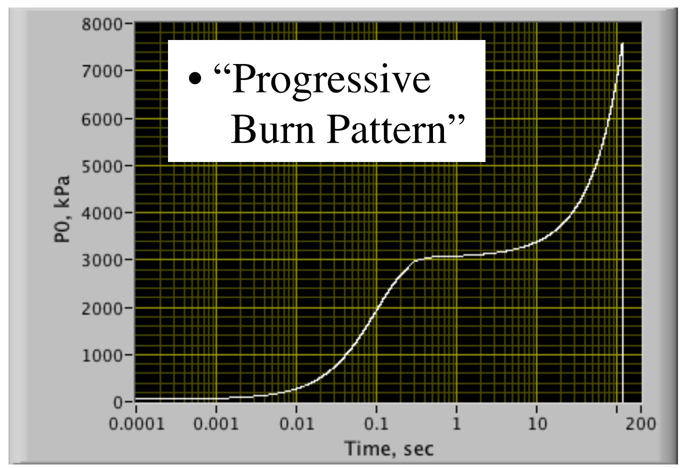


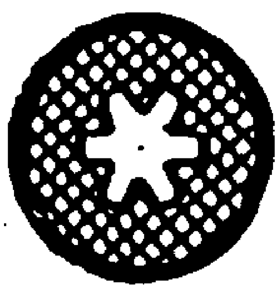
Start condition Quenched at 1.5 s Quenched at 2.5 s

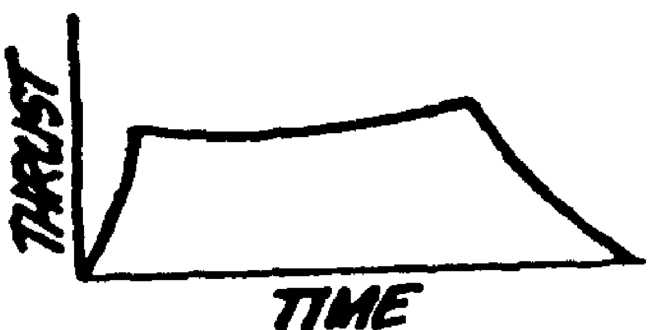
Modified SRB Burn Time History



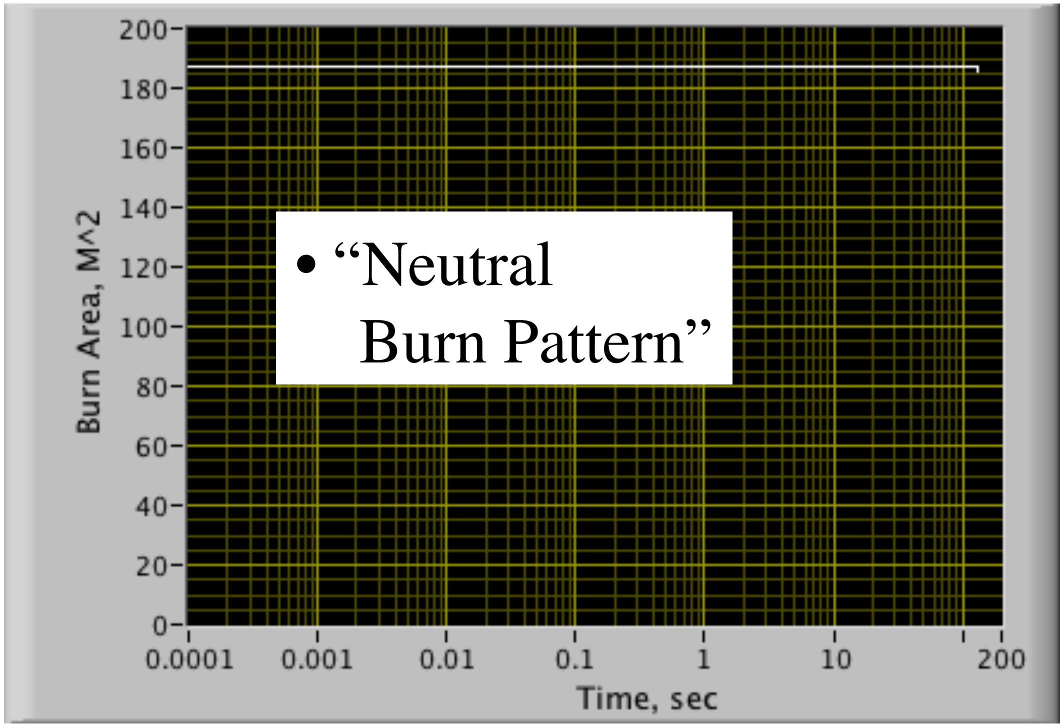
Burn Time: 129 seconds



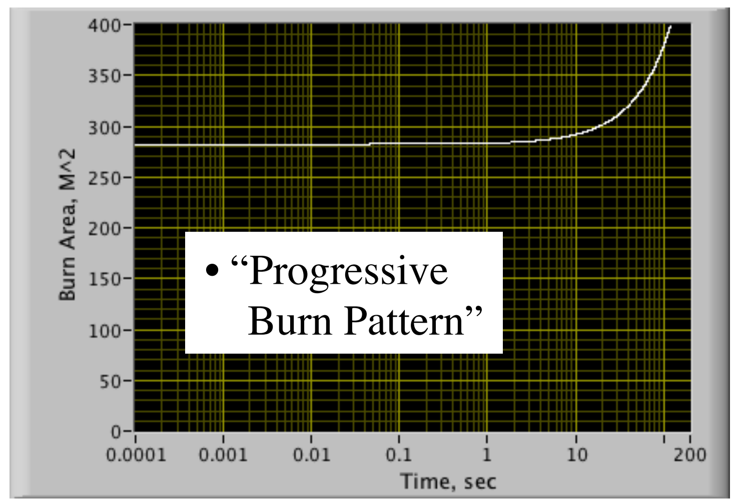


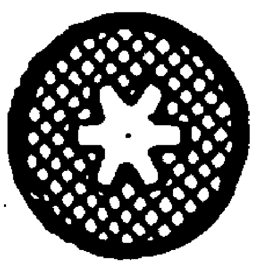


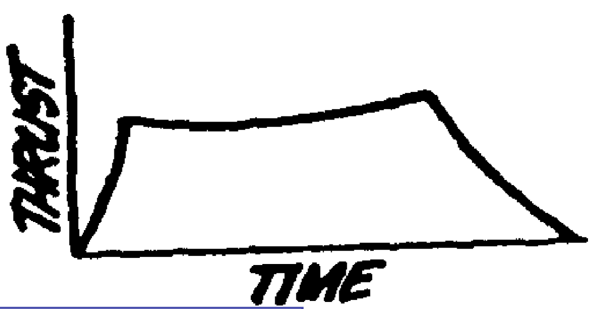
Modified SRB Burn Time History (cont'd)



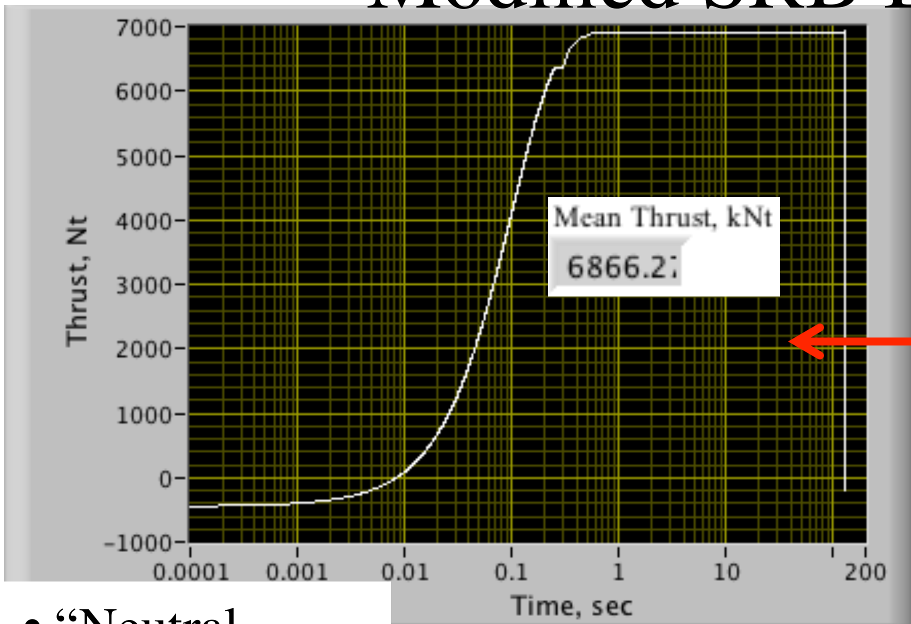
Burn Time: 129 seconds







Modified SRB Burn Time History (cont'd)



$$\int_0^{t_{burn}} F_{thrust_{SL}}\, dt =$$

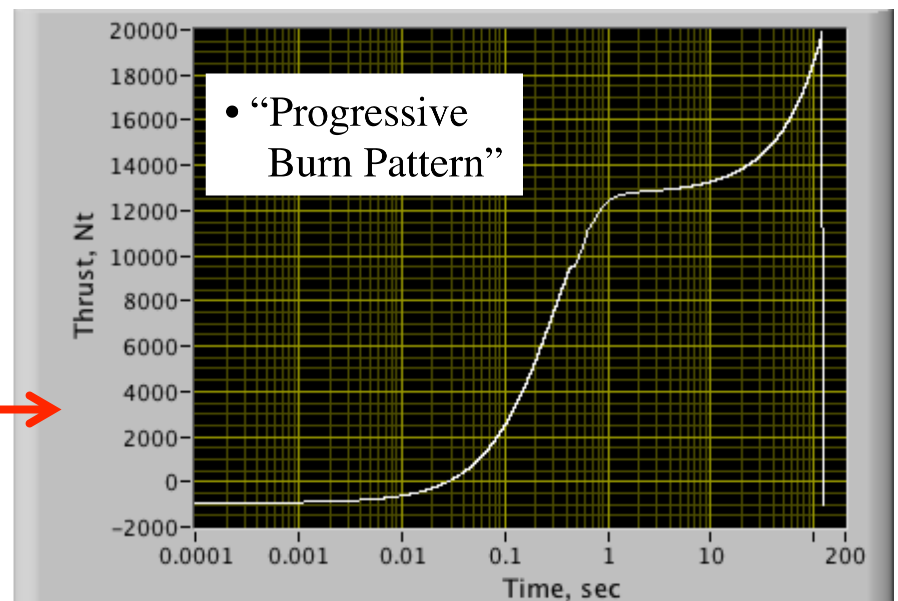
$$6.86627 \cdot 129 = 887.75_{MNt-\mathrm{sec}}$$

- Significantly Less delivered impulse

- "Neutral Burn Pattern"

$$\int_0^{t_{burn}} F_{thrust_{SL}}\, dt =$$

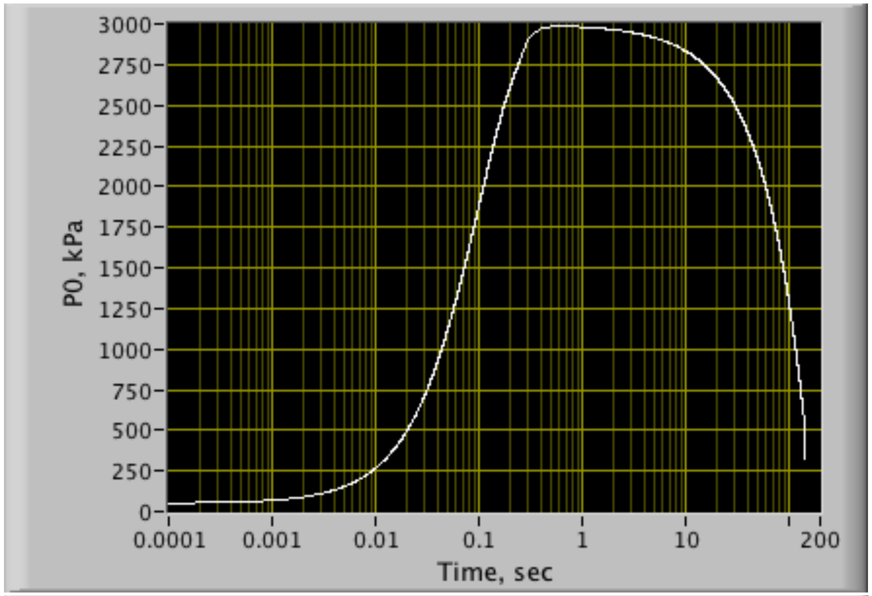
$$11.832 \cdot 121 = 1431.7_{MNt-\mathrm{sec}}$$

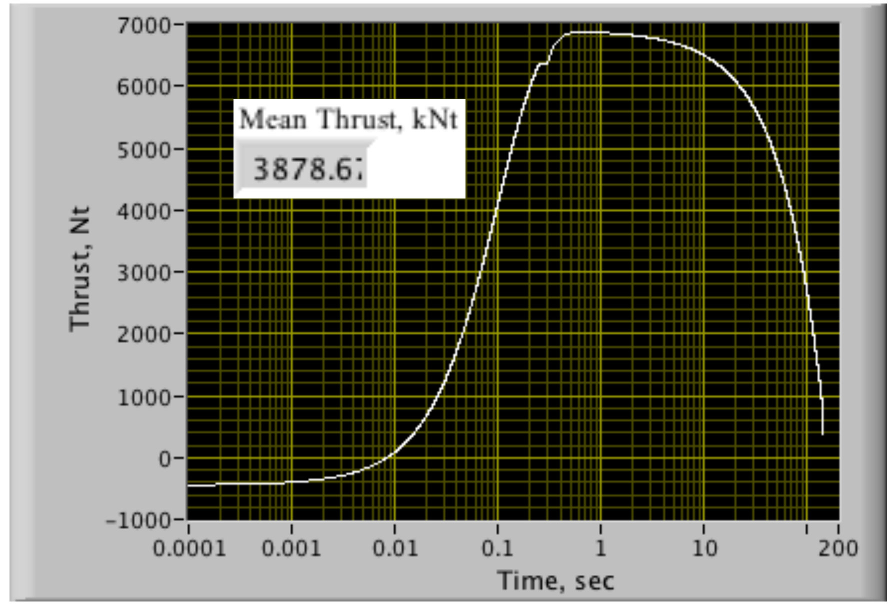


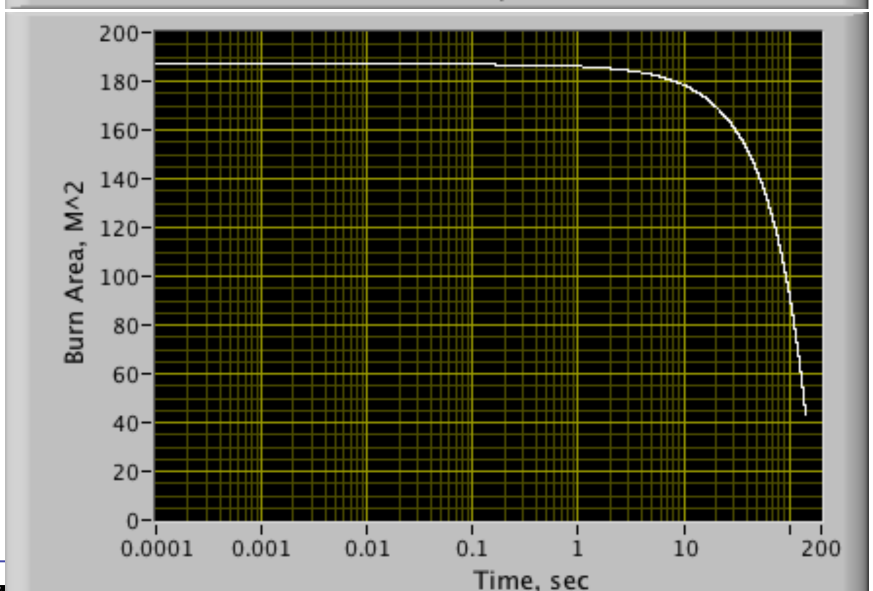
Finally Look at grain pattern

- "*Regressive Grain pattern*" … Burn surface area actually shrinks As propellant is burned

Modified SRB for Regressive Burn Pattern





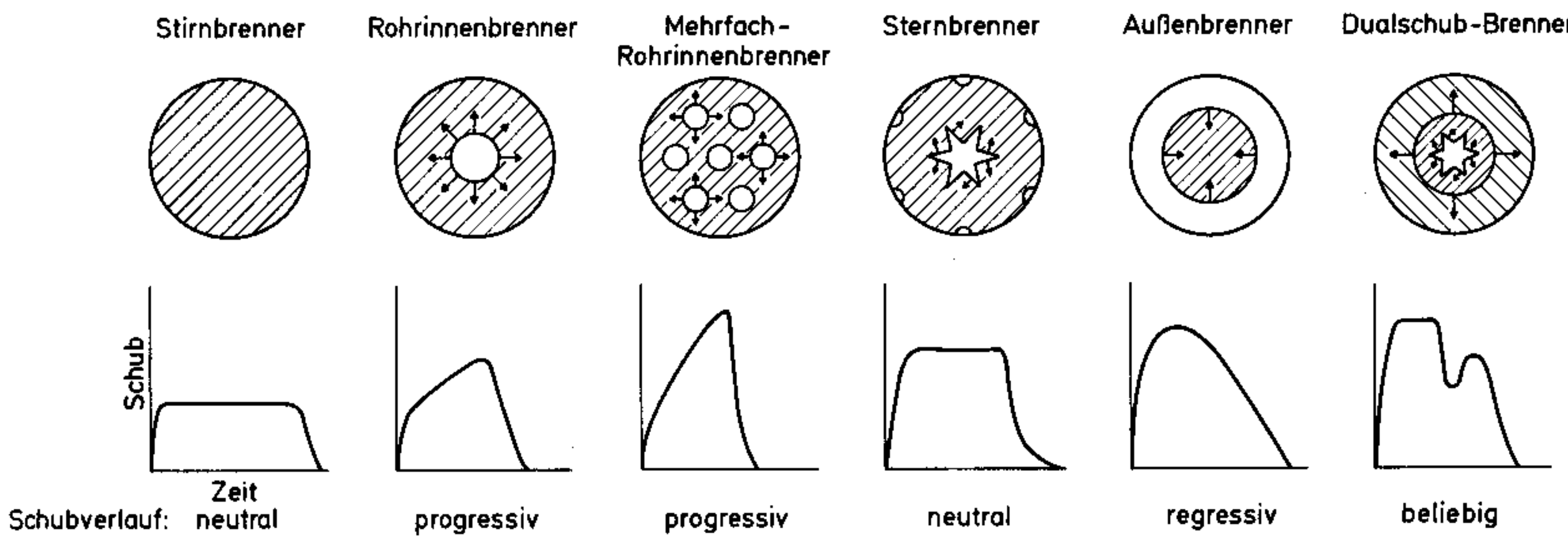


$$\int_0^{t_{burn}} F_{thrust_{SL}}\, dt =$$

$$3.87867 \cdot 142 = 550.77_{MNt-\sec}$$

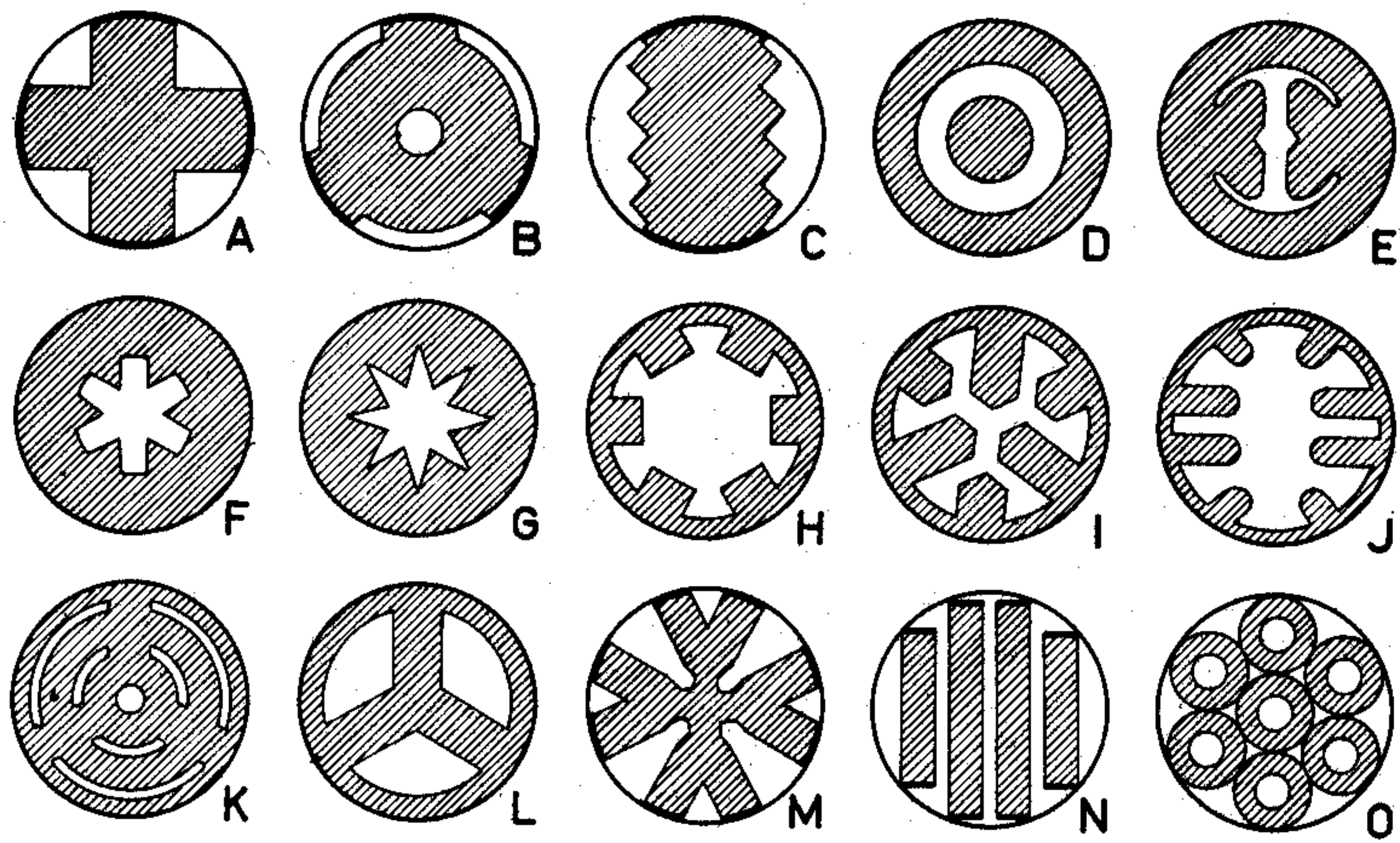
Burn Time: 142 seconds

SOLID ROCKET MOTOR GRAIN DESIGNS



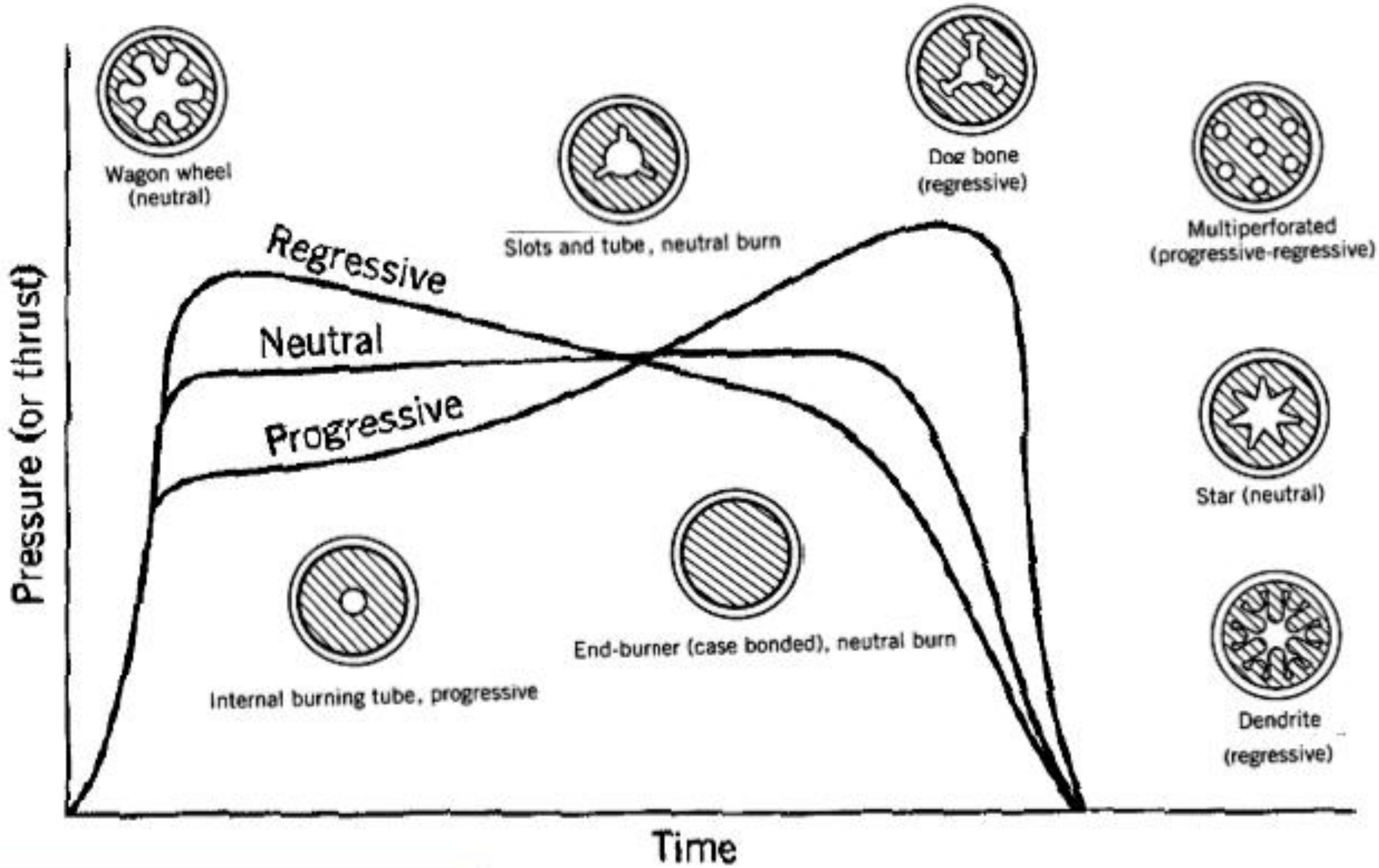
Source: Köhler, Feststoffraketenantriebe, Vol. 1, p. 122 (1972)

EXAMPLES OF SOLID ROCKET MOTOR UNUSUAL GRAIN DESIGNS



Source: Barrere et al., Raketenantriebe, p. 321, Fig 6.1 (1961)

Solid Rocket Burn Summary



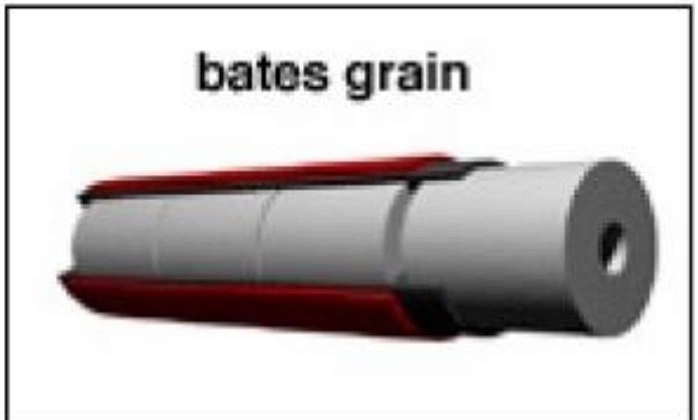
SOLID ROCKET MOTOR GRAIN DESIGN PROGRAMS

- **Grain Design Program (GDP-Light)**
- **http://home.vianetworks.nl/users/aed/gdp/gdp.htm**

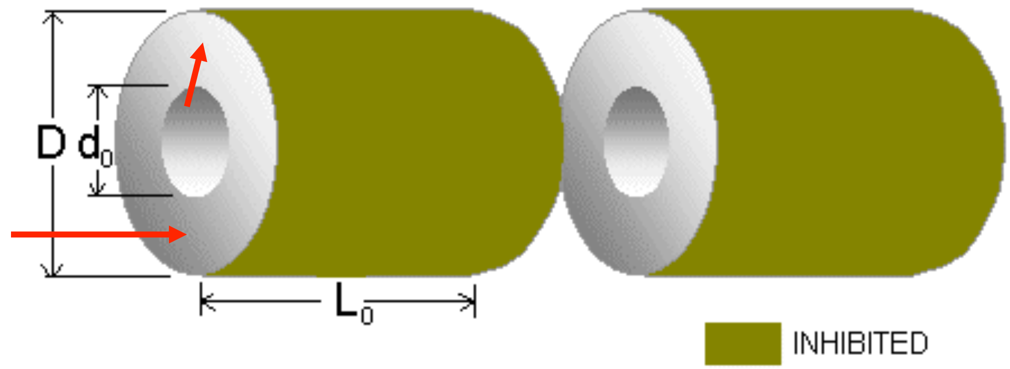
- **Useful Code to test new and unusual grain shapes to achieve certain thrust profiles or minimize slivers and residual burning.**

The "Bates Grain" Geometry

Simple Modification to Cylindrical Port to Give More Even Burn Pattern

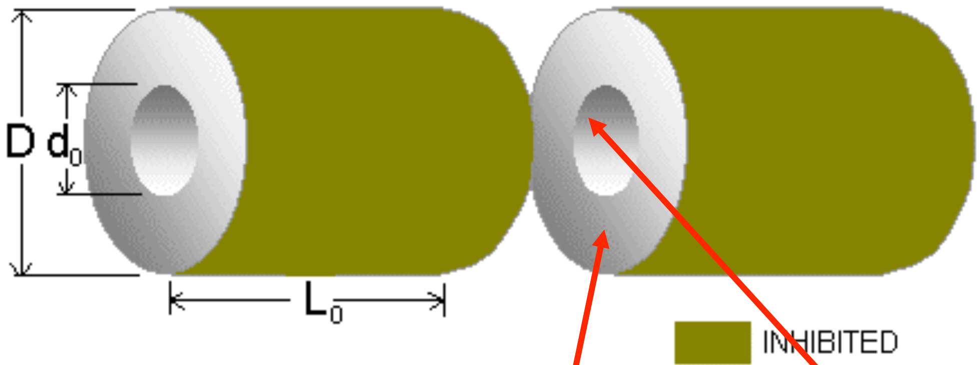


the bates grain uses multi-ple grain segments



Grain segments burn from "inside" and along the "ends"

The "Bates Grain" Geometry (2)



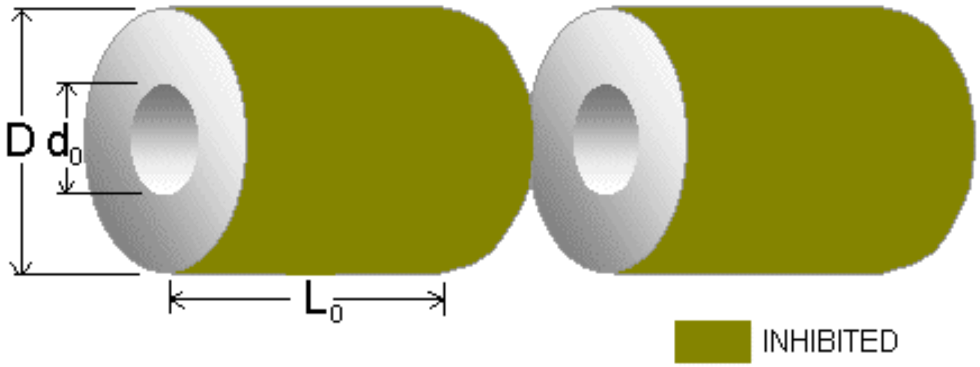
Look at Burn evolution of $\boxed{\dfrac{A_{burn}}{V_{burn}}}$

$$s_{regression} = \int_t \dot{r} \cdot dt$$

$$\rightarrow \boxed{A_{burn} = \underbrace{2\pi \cdot \left(\frac{D_0^{\,2} - d^2}{4} \right)}_{\text{end of segment}} + \underbrace{L \cdot \pi \cdot d}_{\text{interior of grain}}}$$

For each grain segment

The "Bates Grain" Geometry (3)



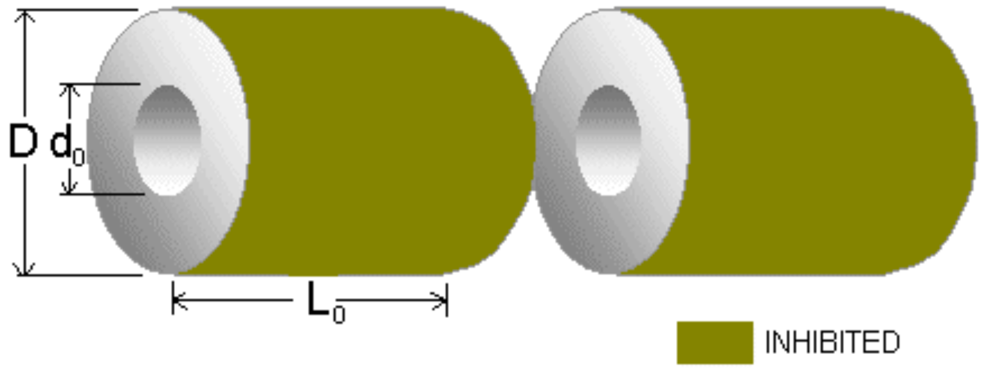
Look at Burn evolution of $\boxed{\dfrac{A_{burn}}{V_{burn}}}$

$$\left.\begin{array}{l} \textit{regressing interior} \\ \textit{surface diameter +} \\ \textit{ends of segment} \end{array}\right| \rightarrow \left.\begin{array}{l} d = d_0 + 2 \cdot s \\ L = L_0 - 2 \cdot s \end{array}\right|$$

For each grain segment

$$\boxed{\begin{array}{l} A_{burn} = 2\pi \cdot \left(\dfrac{D_0^{\ 2} - d^2}{4} \right) + L \cdot \pi \cdot d = \\ \dfrac{\pi}{2} \cdot \left(D_0^{\ 2} - \left(d_0 + 2 \cdot s \right)^2 \right) + \pi \cdot \left(L_0 - 2 \cdot s \right) \cdot \left(d_0 + 2 \cdot s \right) \end{array}}$$

The "Bates Grain" Geometry (4)



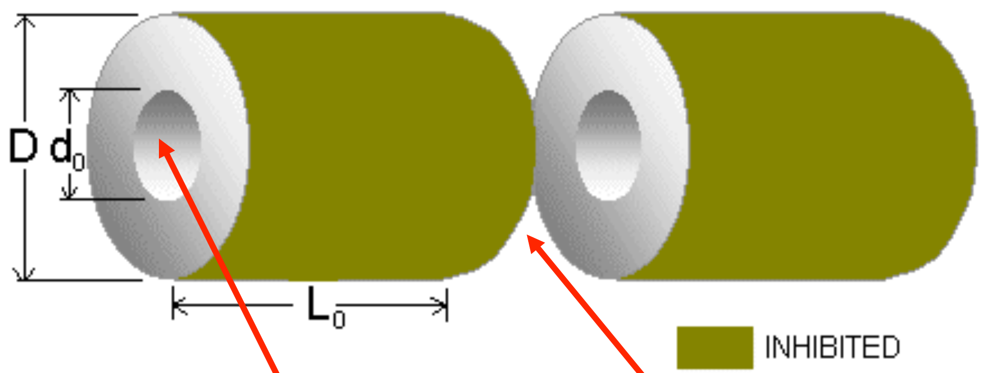
Look at Burn evolution of $\boxed{\dfrac{A_{burn}}{V_{burn}}}$

regressing interior surface diameter + ends of segment $\rightarrow$ $d = d_0 + 2 \cdot s$, $L = L_0 - 2 \cdot s$

For N grain segments

$$\boxed{\left(A_{burn}\right)_{total} = N \cdot \pi \cdot \left[\frac{\left(D_0^{\ 2} - \left(d_0 + 2 \cdot s\right)^2\right)}{2} + \left(L_0 - 2 \cdot s\right) \cdot \left(d_0 + 2 \cdot s\right)\right]}$$

The "Bates Grain" Geometry (5)



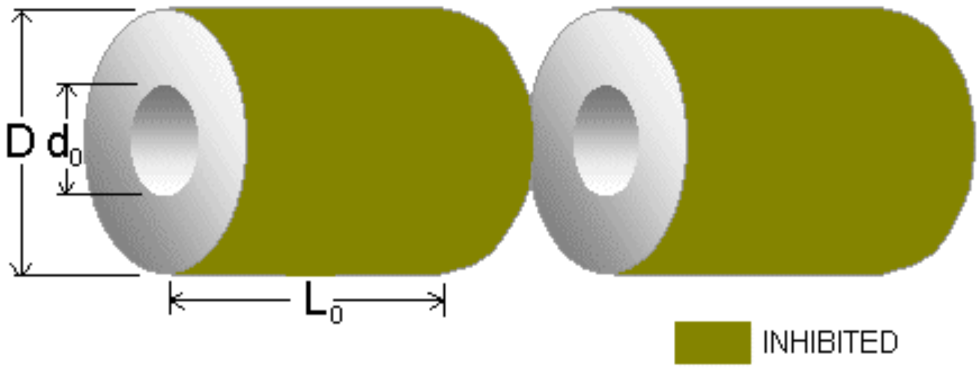
Look at Burn evolution of $\boxed{\dfrac{A_{burn}}{V_{burn}}}$

Look at total chamber burn volume (empty space) at any time

$$\left(V_{ol}\right)_{total} = N \cdot \left[\pi \cdot \frac{d^2}{4} \cdot L \quad + \quad \pi \frac{D_0^{\ 2}}{4} \cdot (2 \cdot s)\right]$$

Port Volume …. "burn end" volume

The "Bates Grain" Geometry (6)



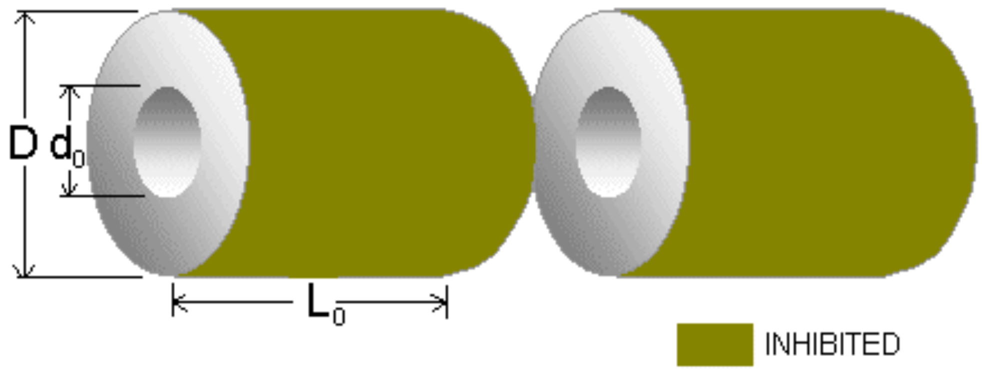
Look at Burn evolution of $\boxed{\dfrac{A_{burn}}{V_{burn}}}$

Allowing for regression from original geometry

$$\left(V_{ol}\right)_{total} = N \cdot \left[\pi \cdot \frac{d^2}{4} \cdot L + \pi \frac{D_0^{\ 2}}{4} \cdot \left(2 \cdot s\right)\right] =$$

$$\frac{N \cdot \pi}{4}\left[\left(d_0 + 2 \cdot s\right)^2 \cdot \left(L_0 - 2 \cdot s\right) + D_0^{\ 2} \cdot \left(2 \cdot s\right)\right]$$

The “Bates Grain” Geometry (7)



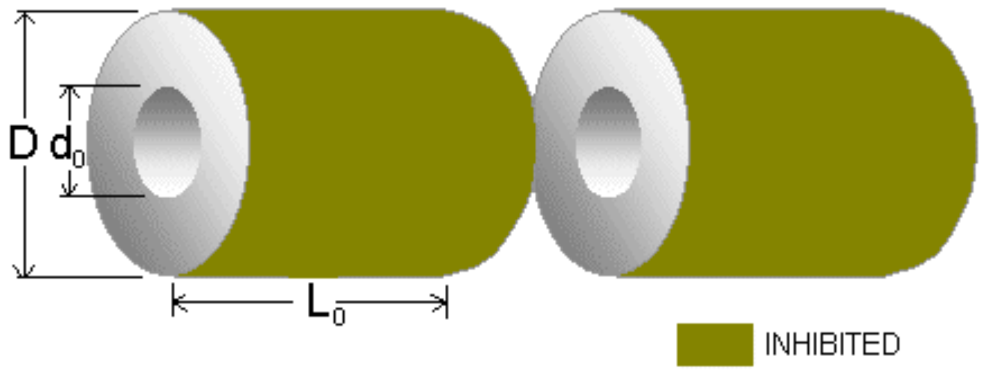
Look at Burn evolution of $\boxed{\dfrac{A_{burn}}{V_{burn}}}$

$$\left(A_{burn}\right)_{total} = N\cdot\pi\cdot\left[\frac{\left(D_0^{\,2} - \left(d_0 + 2\cdot s\right)^2\right)}{2} + \left(L_0 - 2\cdot s\right)\cdot\left(d_0 + 2\cdot s\right)\right]$$

$$\left(V_{ol}\right)_{total} = \frac{N\cdot\pi}{4}\left[\left(d_0 + 2\cdot s\right)^2\cdot\left(L_0 - 2\cdot s\right) + D_0^{\,2}\cdot\left(2\cdot s\right)\right]$$

$$\frac{A_{burn}}{V_{burn}} = \frac{N\cdot\pi\cdot\left[\dfrac{\left(D_0^{\,2} - \left(d_0 + 2\cdot s\right)^2\right)}{2} + \left(L_0 - 2\cdot s\right)\cdot\left(d_0 + 2\cdot s\right)\right]}{\dfrac{N\cdot\pi}{4}\left[\left(d_0 + 2\cdot s\right)^2\cdot\left(L_0 - 2\cdot s\right) + D_0^{\,2}\cdot\left(2\cdot s\right)\right]} = \frac{4\left[\dfrac{\left(1 - \left(\dfrac{d_0 + 2\cdot s}{D_0}\right)^2\right)}{2} + \left(1 - 2\cdot\dfrac{s}{L_0}\right)\cdot\left(\dfrac{d_0 + 2\cdot s}{D_0}\right)\right]}{\left[\left(\dfrac{d_0 + 2\cdot s}{D_0}\right)^2\cdot\left(1 - 2\cdot\dfrac{s}{L_0}\right) + \left(2\cdot\dfrac{s}{L_0}\right)\right]}$$

The "Bates Grain" Geometry (8)



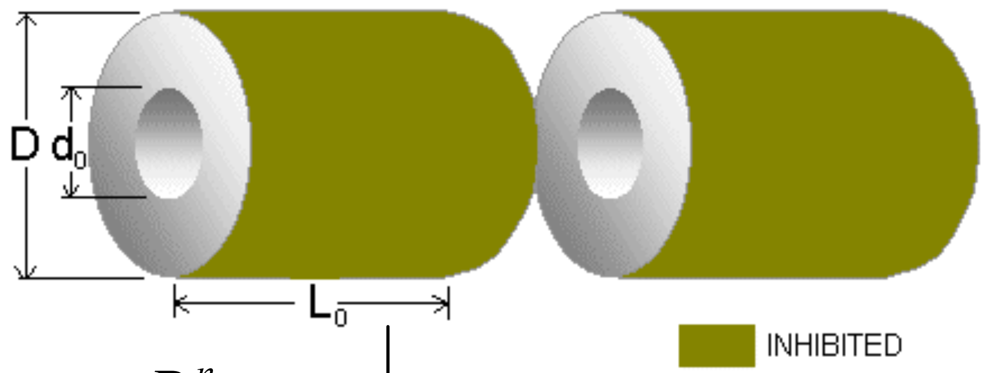
Look at Burn evolution of $\boxed{\dfrac{A_{burn}}{V_{burn}}}$

$$\left(A_{burn}\right)_{total} = N \cdot \pi \cdot \left[\frac{\left(D_0^2 - \left(d_0 + 2\cdot s\right)^2\right)}{2} + \left(L_0 - 2\cdot s\right)\cdot\left(d_0 + 2\cdot s\right)\right]$$

$$\left(V_{ol}\right)_{total} = \frac{N\cdot\pi}{4}\left[\left(d_0 + 2\cdot s\right)^2\cdot\left(L_0 - 2\cdot s\right) + D_0^2\cdot\left(2\cdot s\right)\right]$$

$$\frac{A_{burn}}{V_{burn}} = \frac{N \cdot \pi \cdot \left[\dfrac{\left(D_0^2 - \left(d_0 + 2\cdot s\right)^2\right)}{2} + \left(L_0 - 2\cdot s\right)\cdot\left(d_0 + 2\cdot s\right)\right]}{\dfrac{N\cdot\pi}{4}\left[\left(d_0 + 2\cdot s\right)^2\cdot\left(L_0 - 2\cdot s\right) + D_0^2\cdot\left(2\cdot s\right)\right]} = \frac{4\left[\dfrac{\left(1 - \left(\dfrac{d_0 + 2\cdot s}{D_0}\right)^2\right)}{2} + \left(1 - 2\cdot\dfrac{s}{L_0}\right)\cdot\left(\dfrac{d_0 + 2\cdot s}{D_0}\right)\right]}{\left[\left(\dfrac{d_0 + 2\cdot s}{D_0}\right)^2\cdot\left(1 - 2\cdot\dfrac{s}{L_0}\right) + \left(2\cdot\dfrac{s}{L_0}\right)\right]}$$

The "Bates Grain" Geometry (9)



Summary of Algorithm

$$\dot{r} = a \cdot P_o^{\,n}$$

$$s_{regression} = \int_t \dot{r} \cdot dt$$

$\rightarrow$

$$\left(A_{burn}\right)_{total} = N \cdot \pi \cdot \left[\frac{\left(D_0^{\,2} - \left(d_0 + 2 \cdot s\right)^2\right)}{2} + \left(L_0 - 2 \cdot s\right) \cdot \left(d_0 + 2 \cdot s\right) \right]$$

$$\left(V_{ol}\right)_{total} = \frac{N \cdot \pi}{4} \left[\left(d_0 + 2 \cdot s\right)^2 \cdot \left(L_0 - 2 \cdot s\right) + D_0^{\,2} \cdot \left(2 \cdot s\right) \right]$$

Erosive Burning

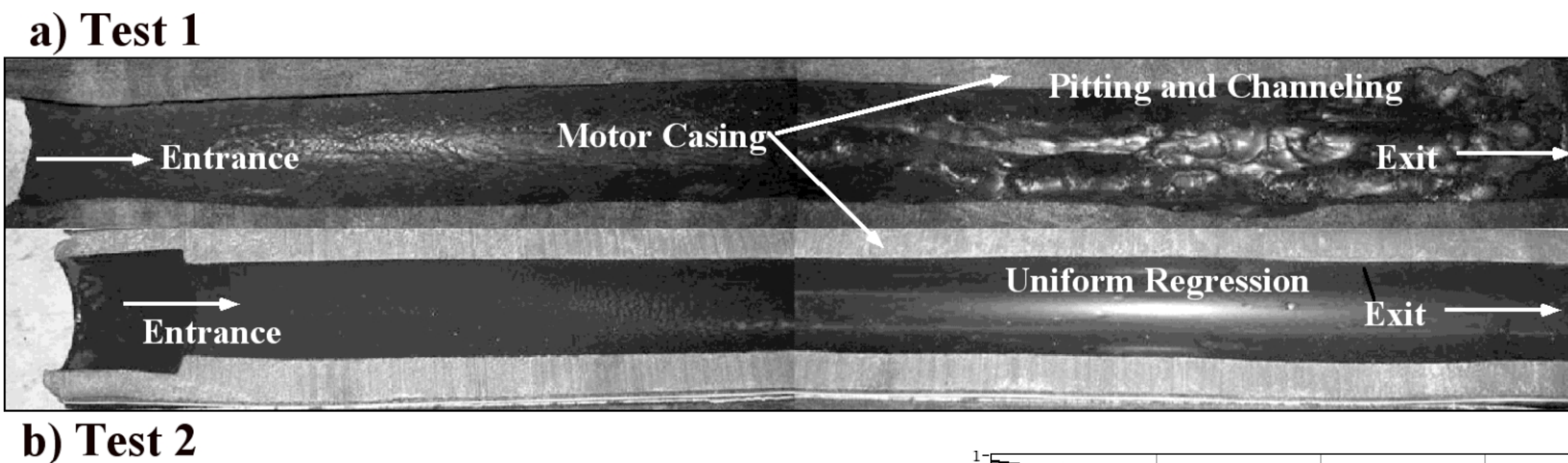
1) **When high velocity or high mass flow hot gas from upstream combustion passes over** a downstream burning surface in a solid rocket motor local, chaotic increase in propellant burning rate results; phenomenon referred to as erosive burning.

2) **Two types of erosive burning;** velocity-based erosive burning and mass flux-based erosive burning. AP/composite propellants are more sensitive to the effect of the hot gas velocity flowing past burning propellant surface, some propellants (hybrids in particular) are more sensitive to the effect of the mass flux of the hot gas over the burning surface

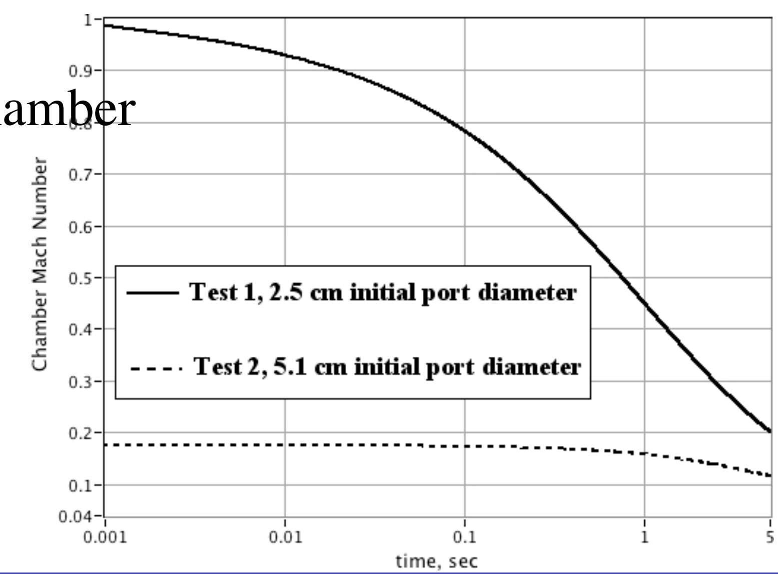
3) **Distinct thresholds for core combustion gas velocity and core mass flux** for the onset of velocity-based erosive burning and mass flux-based erosive burning.

Erosive Burning

… St. Roberts Law strictly only "works" for non-erosive grain burns … Erosive burns are very complex to predict and analyze



Erosion Properties Highly Dependent on Chamber Flow Velocity

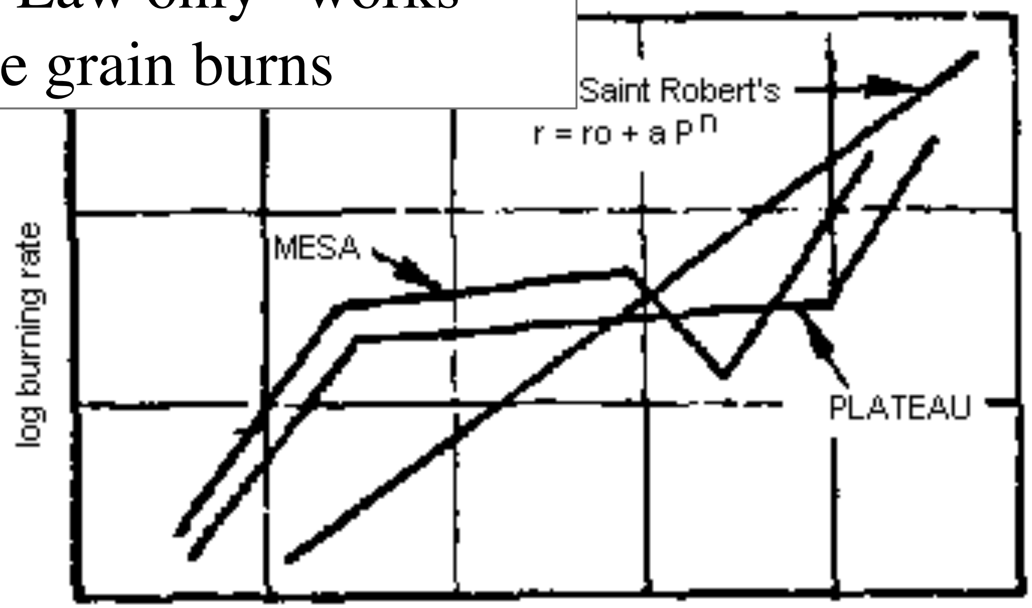


Eilers/Whitmore, JPC 2007,

AIAA-2007-5349

Erosive Burning (2)

… St. Roberts Law only “works” for non-erosive grain burns



(Ref. NASA SP-8039)

“Plateau effects” are result different surface regression rates --- pressure dependent

-- condensed phase combustion products also "pool" and retard heat transfer to the surface at elevated pressures.

Erosive Burning (4)

-- most propellants have certain levels of combustion gas velocity (Mach number) that leads to an increased burning rate.

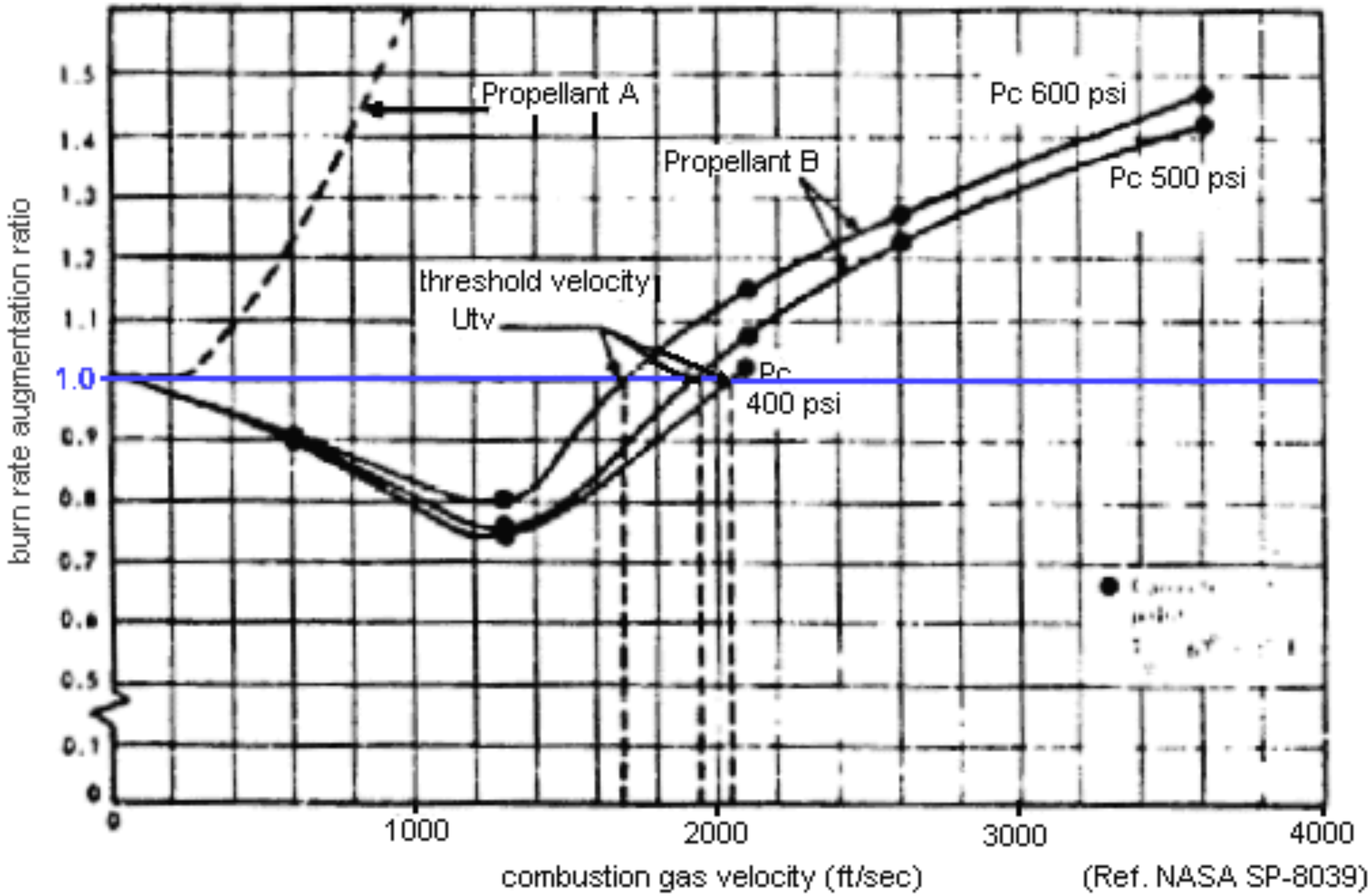
-- "augmentation" of burn rate is referred to as *erosive burning*, chaotic and difficult to predict

-- physical mechanism -- increased convective heat transfer to the propellant surface – resulting from flow turbulence

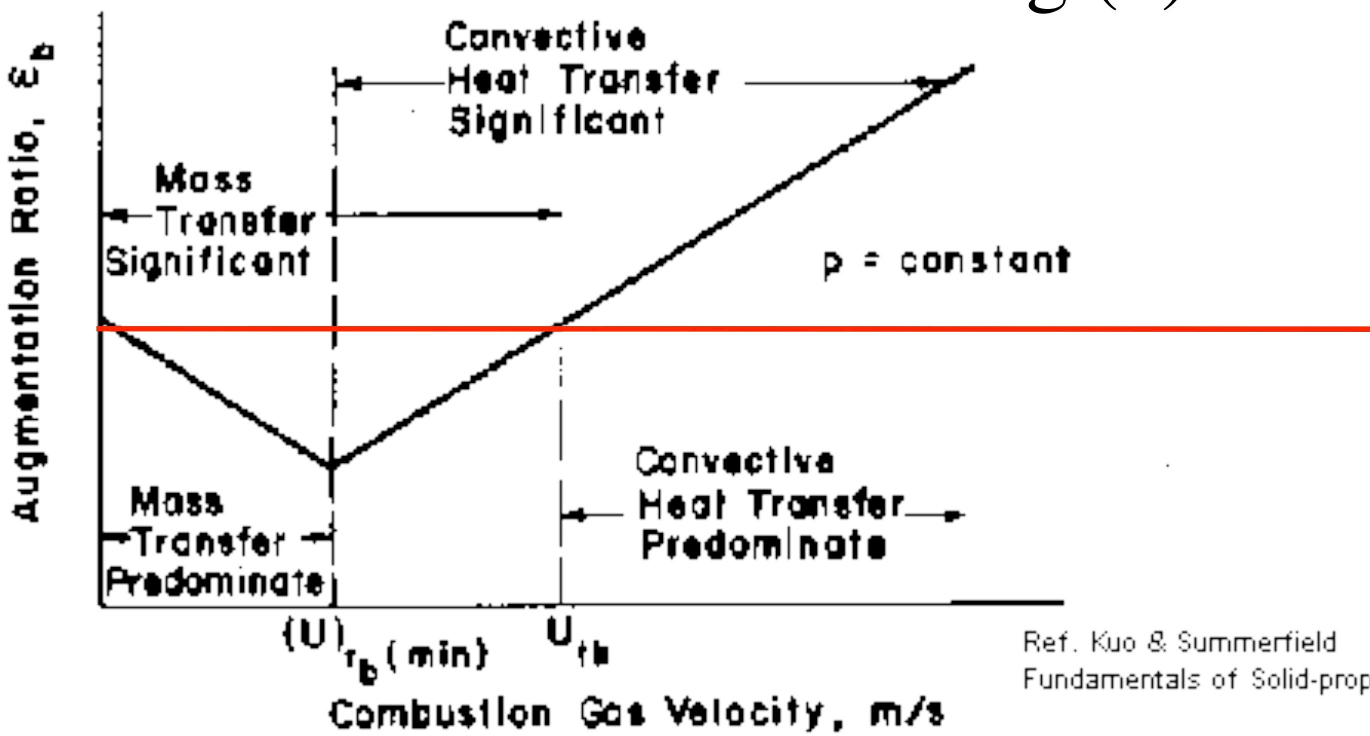
-- For many propellants, a *threshold* Mach number occurs.

Below this flow level, no augmentation occurs, or a *decrease* in burn rate is experienced (negative erosive burning). *(Slag buildup)*

Erosive Burning (4)



Erosive Burning (5)



Ref. Kuo & Summerfield
Fundamentals of Solid-propellant Combustion

..Effects of erosive burning can be minimized by designing the motor with a sufficiently large *port-to-throat area* ratio (A_{port}/A^*).

-- “Rule of thumb”, the ratio should be a minimum of 2, for a "typical" grain L/D ratio of 6.

-- A greater A_{port}/A^* ratio should be used for grains with larger L/D ratios.

Erosive Burning (7)

“ Rules of Thumb” for Erosive Burning (Charles E. Rodgers, NASA DFRC)

- Non-Erosive:

Mach Number

Core Mach Number < 0.70

For $\gamma = 1.2$; $A_{port}/A^* > 1.10$

Mass Flux

$P_0 = 400\text{-}600$ *psia; Core Mass Flux* < 2.0 *lb/sec-in*^{2}

$P_0 = 800$ *psia; Core Mass Flux* < 2.5 *lb/sec-in*^{2}

$P_0 = 1400$ *psia; Core Mass Flux* < 32.0 *lb/sec-in*^{2}

Erosive Burning (8)

" Rules of Thumb" for Erosive Burning (Charles E. Rodgers, NASA DFRC)

- Maximum Recommended Allowable Erosion Rate:

Mach Number

Core Mach Number < 0.50

For $\gamma = 1.2$; $A_{port}/A^* > 1.36$

Mass Flux

P_0 = 400-600 psia; Core Mass Flux < 1.0 lb/sec-in^2

P_0 = 800 psia; Core Mass Flux < 1.75 lb/sec-in^2

P_0 = 1400 psia; Core Mass Flux <ı 2.0 lb/sec-in^2

Core Mass Flux limits for Max Recommended Erosivity should not be exceeded unless erosive Burning Characterization Tests are performed for propellant.

Design Steps to Mitigate Erosive Burning

Combined Core Mach Number/Core Mass Flux Erosive Burning Design Criteria for Motors with Constant Diameter Cores

Step 1:

Initial Core Diameter Sized Based On Core Mach Number

Non-Erosive; $M_a = 0.50$
$\gamma = 1.2$; $A_p/A_{th} = 1.36$

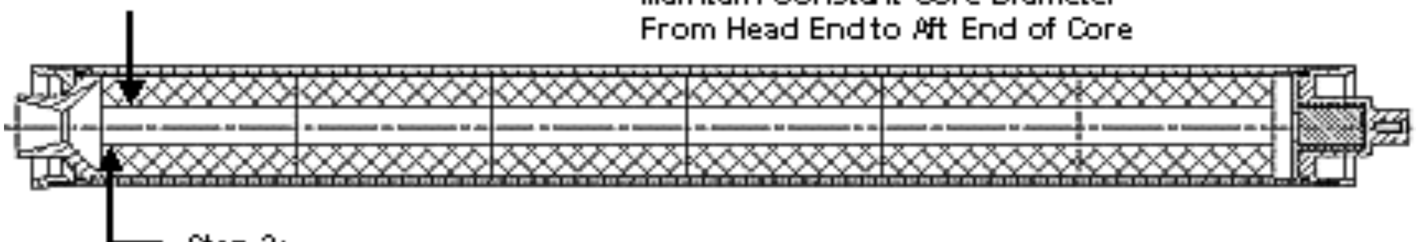
Max Erosive; $M_a = 0.70$
$\gamma = 1.2$; $A_p/A_{th} = 1.10$

Core Mach Number and Core Mass Flux Conditions are at Motor Ignition

Core Mass Flux Values Based on Non-Erosive Propellant Burn Rate

(Charles E. Rodgers, NASA DFRC)

Maintain Constant Core Diameter From Head End to Aft End of Core



Step 2:

Check Core Mass Flux at Aft End of Core

If Core Mass Flux at Aft End of Core Higher Than Design Point Core Mass Flux, Increase Core Diameter to Reduce Core Mass Flux to Design Point Value.

Design Point Core Mass Flux (Recommended Values)

Non-Erosive;	p_c = 400-600 psia	Core Mass Flux ≤ 1.0 lb/sec-in^2
	p_c = 800 psia	Core Mass Flux ≤ 1.75 lb/sec-in^2
	p_c = 1400 psia	Core Mass Flux ≤ 2.0 lb/sec-in^2
Max Erosive;	p_c = 400 psia	Core Mass Flux = 2.0 lb/sec-in^2
	p_c = 600 psia	Core Mass Flux = 2.5 lb/sec-in^2
	p_c ≥ 800 psia	Core Mass Flux = 3.0 lb/sec-in^2

Design Steps to Mitigate Erosive Burning (2)

Constant Core Mass Flux Core Design

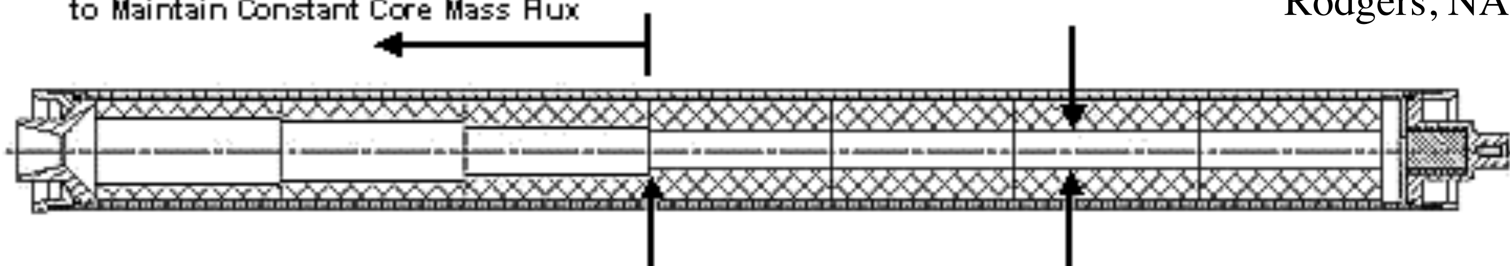
Core Mach Number and Core Mass Flux Design Point Conditions are at Motor Ignition

Core Mass Flux Values Based on <u>Non-Erosive</u> Propellant Burn Rate

Provides Maximum Motor Length, Minimum Motor Core Diameter, Maximum Propellant Loading for a Given Level (Design Point) of Erosive Burning

(Charles E. Rodgers, NASA DFRC)

Core Diameter Increased Past This Point to Maintain Constant Core Mass Flux



Design Point Core Mass Flux Achieved

Initial Core Diameter Based On Design Point Core Mach Number

Design Point Core Mass Flux (Recommended Values)

Non-Erosive;	p_c = 400-600 psia	Core Mass Flux ≤ 1.0 lb/sec-in²
	p_c = 800 psia	Core Mass Flux ≤ 1.75 lb/sec-in²
	p_c = 1400 psia	Core Mass Flux ≤ 2.0 lb/sec-in²
Max Erosive;	p_c = 400 psia	Core Mass Flux = 2.0 lb/sec-in²
	p_c = 600 psia	Core Mass Flux = 2.5 lb/sec-in²
	p_c ≥ 800 psia	Core Mass Flux = 3.0 lb/sec-in²

Non-Erosive; M_a = 0.50
γ = 1.2; A_p/A_{th} = 1.36

Max Erosive; M_a = 0.70
γ = 1.2; A_p/A_{th} = 1.10

Erosive Burning (6)

.. Simple Erosive Burn Model

$$\dot{r} = a P_0^n \frac{\left(1 + k \left(M/M_{crit} \right) \right.}{(1 + k)}$$

k … *empirical scale factor*

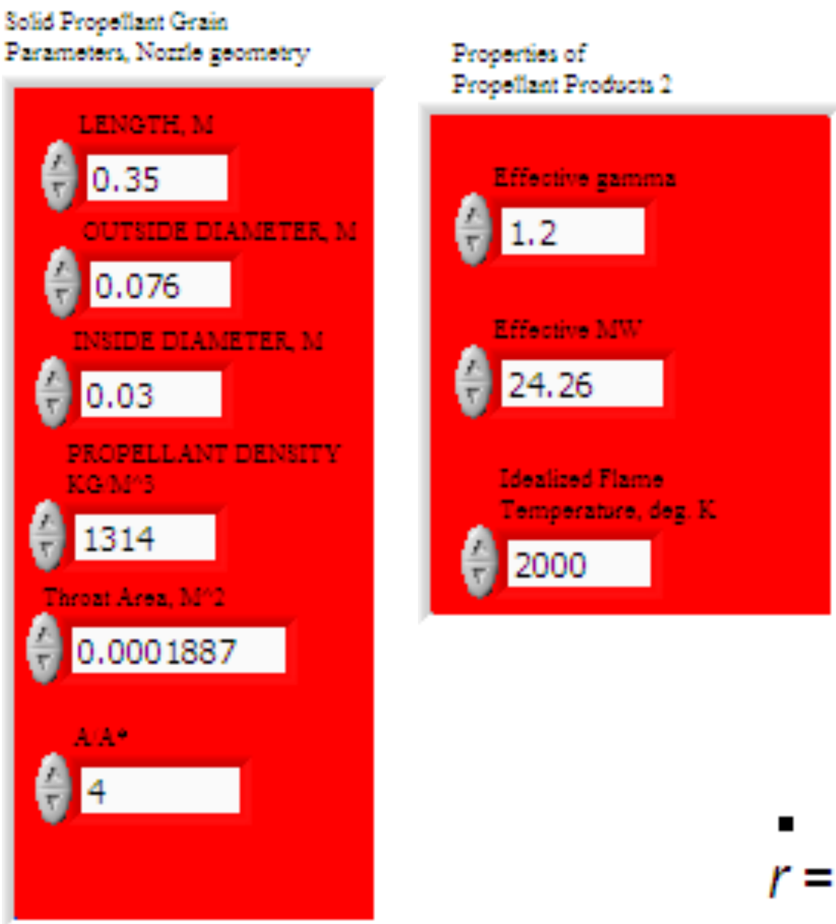
M … *chamber mach number based on* A_{port}/A^*

M_{crit} … *critical or thresh hold Mach number*

Below M_{crit} .. Burn rate is reduced, above .. enhanced

Erosive Burn Model

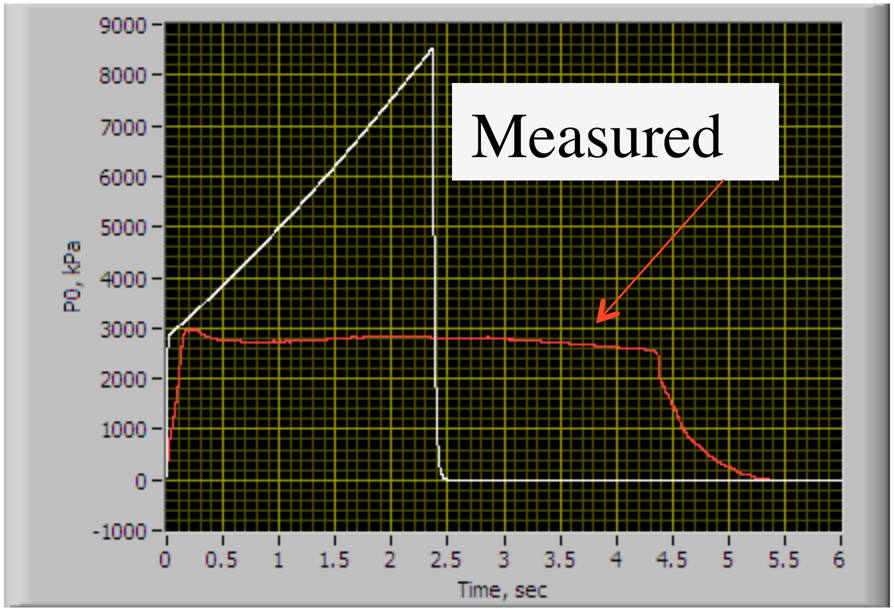
-- Non Erosive Burning

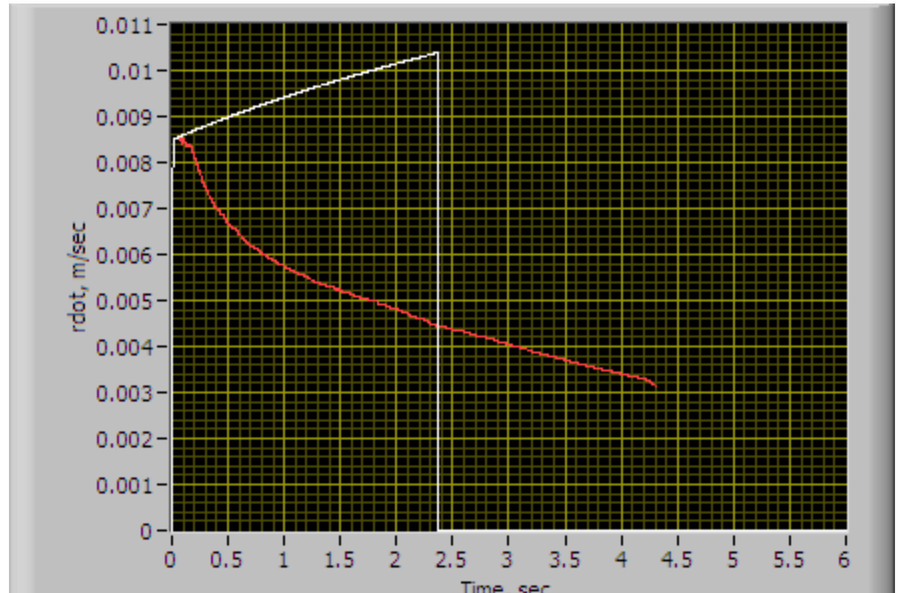


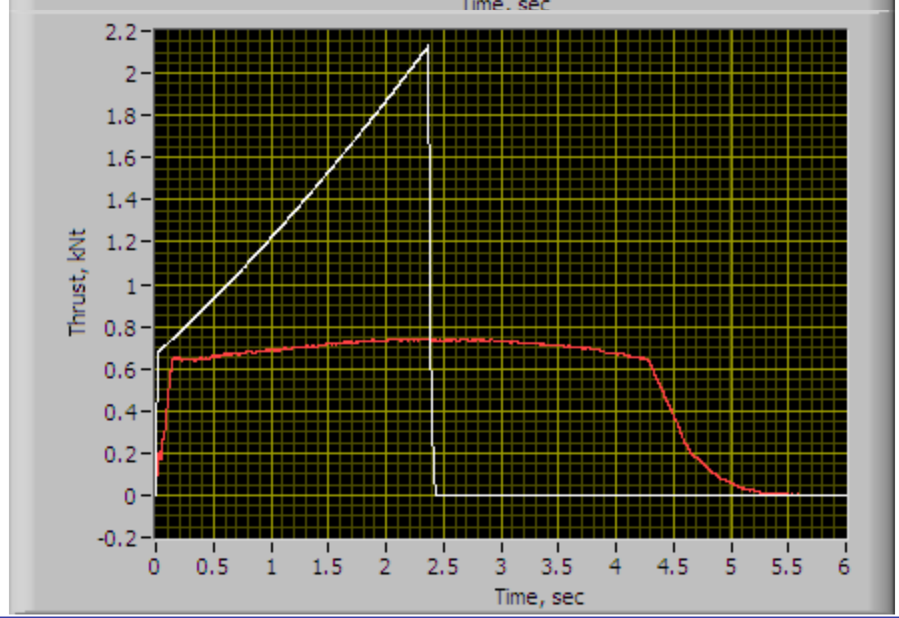
Set to zero

$$\dot{r} = a P_0^n \frac{(1 + k\,(M/M_{crit}))}{(1 + k)}$$

Erosive Burn Model (2)







-- Non Erosive Burning

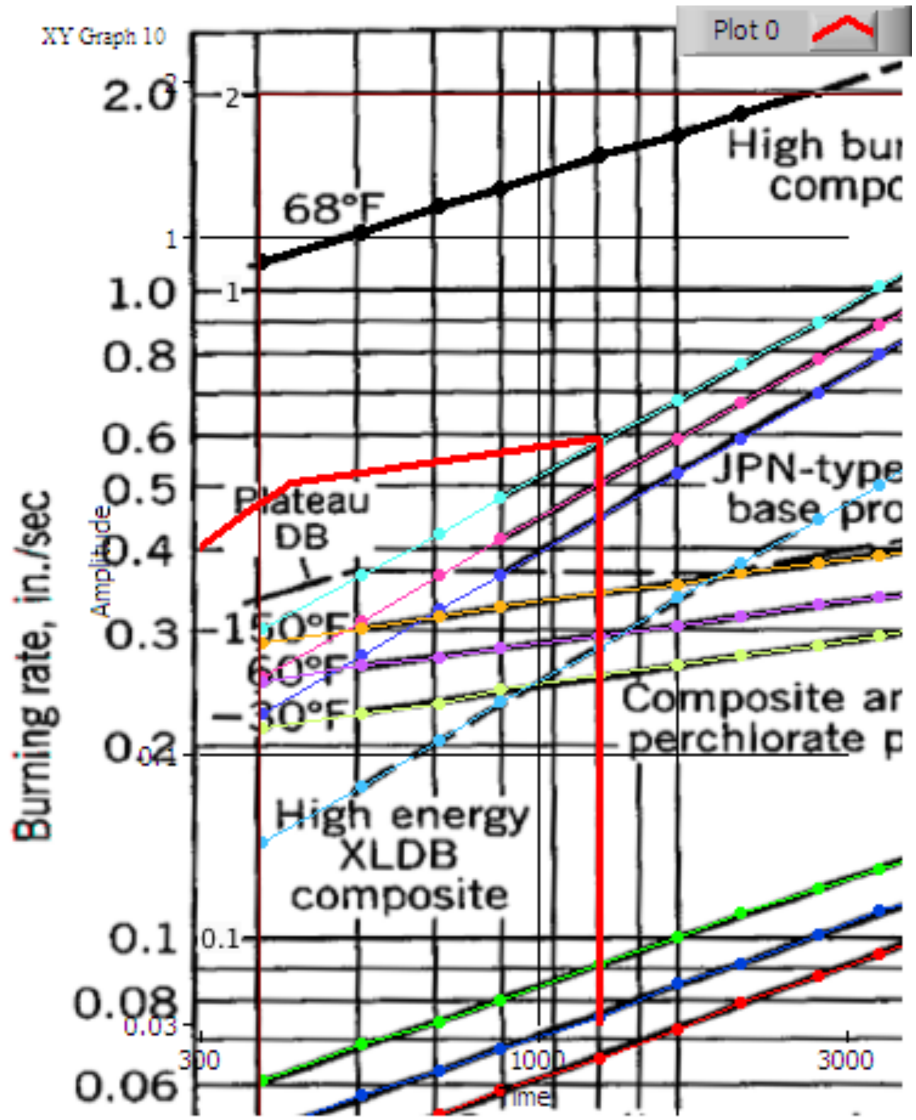
… "not so good!"

Erosive Burn Model (3)

-- Non Erosive Burning

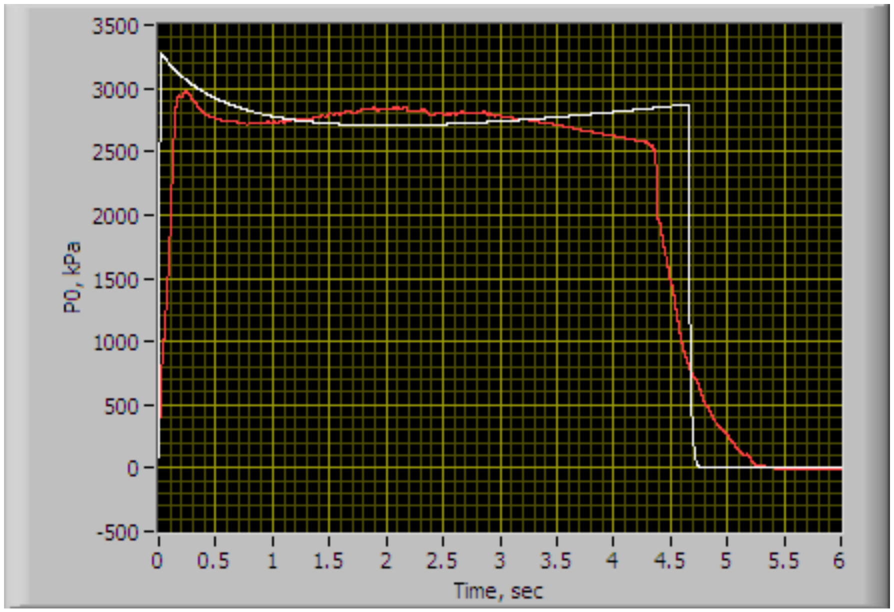
… "burn rates become very large As fuel port "opens up"

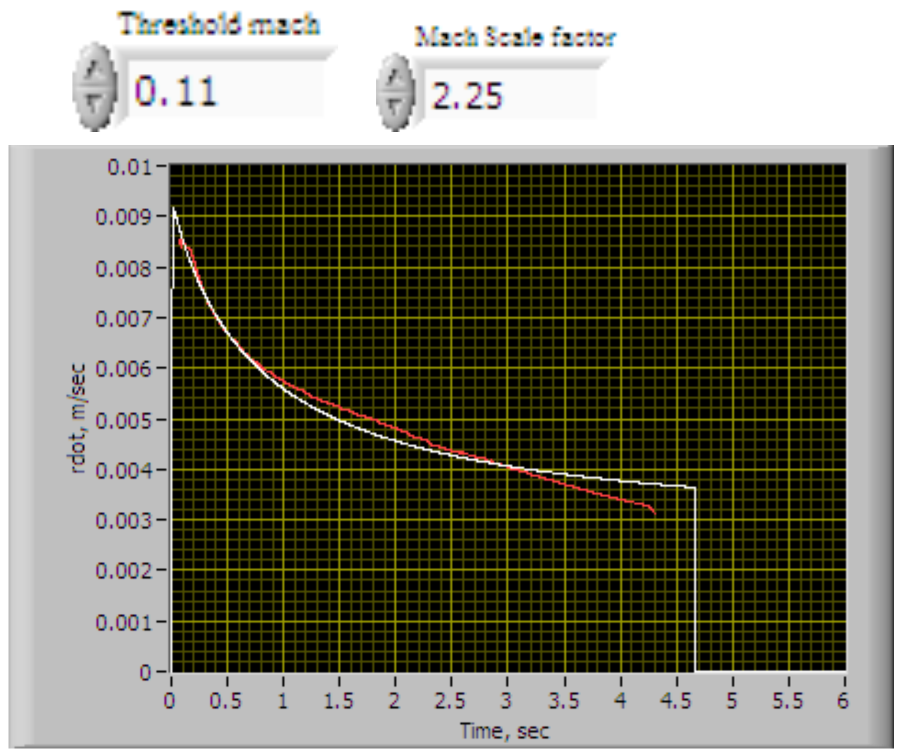
.. Predicted burn time only\ 2.5 seconds!



Erosive Burn Model (4)

-- Erosive Burning Model

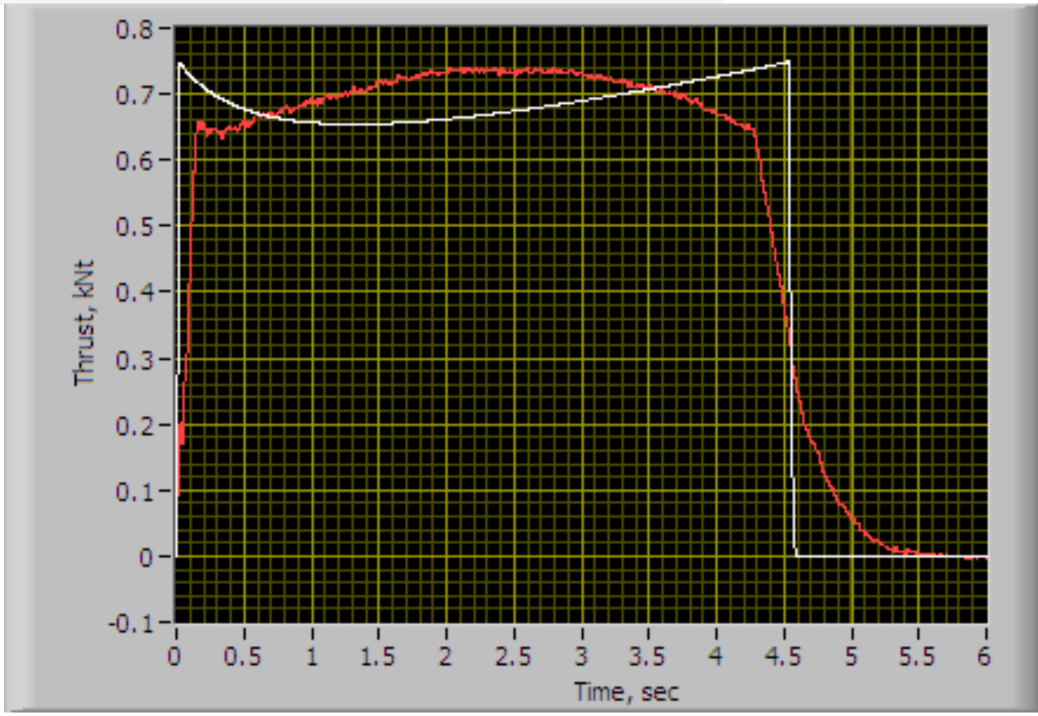




Much Better .. Tail Off in Measured chamber pressure curve
Likely due to change in St Roberts parameters as grain erodes

Erosive Burn Model (5)

-- Erosive Burning Model



Threshold mach

0.11

Mach Scale factor

2.25

Total Impulse, Nt-sec

3128.48

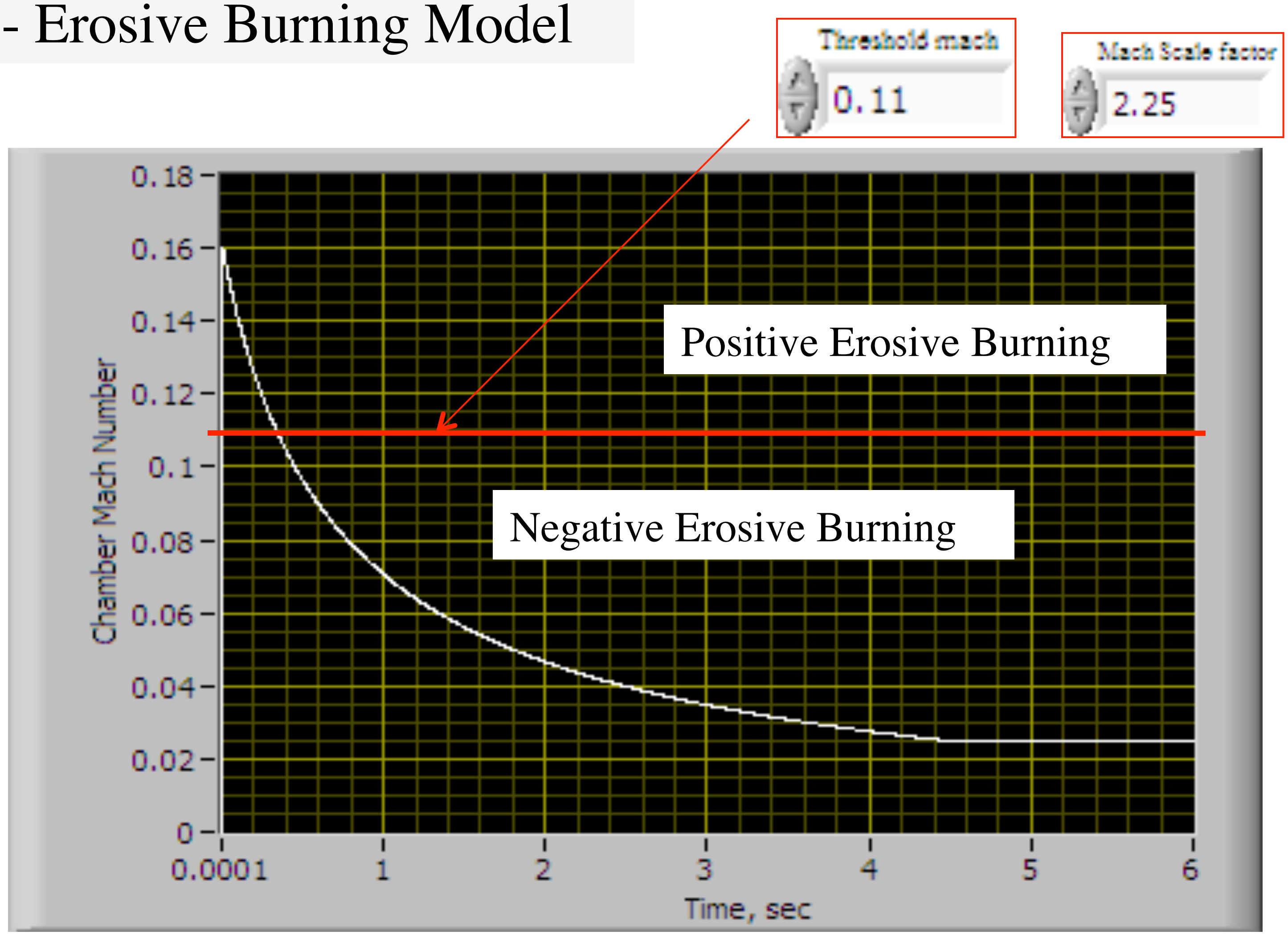
Measured Total Impulse

3126.67

Much Better .. “Break” in curve slope likely due to Changing Saint-Roberts Burn properties as Grain Heats Up and becomes “gooey” .. Melting before burning

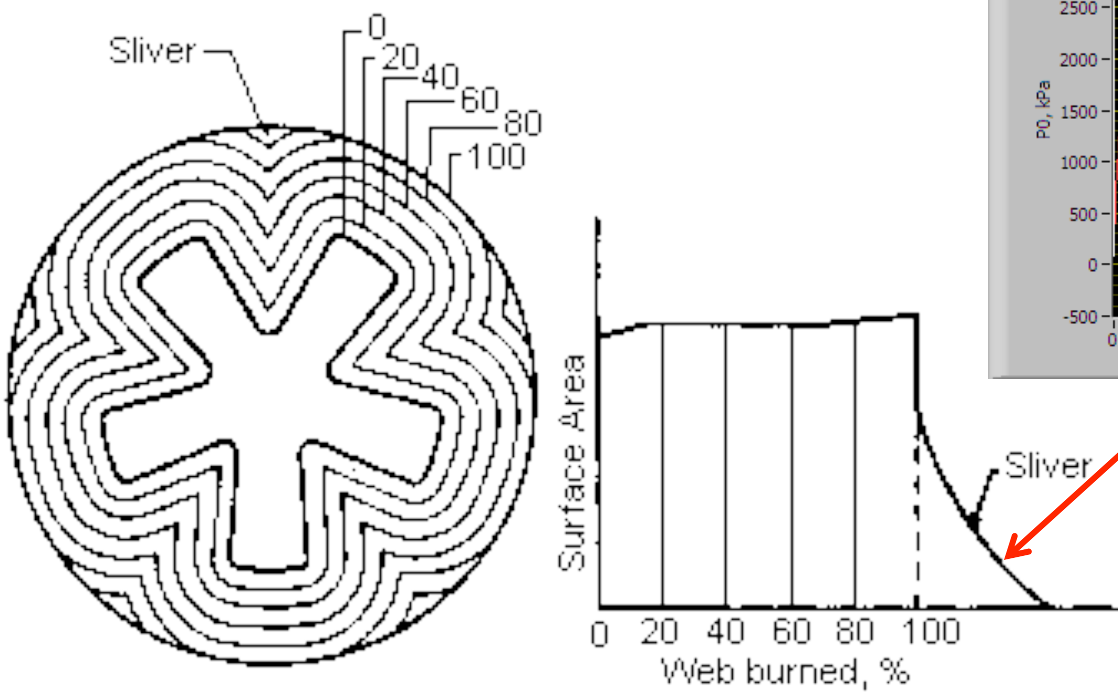
Erosive Burn Model (6)

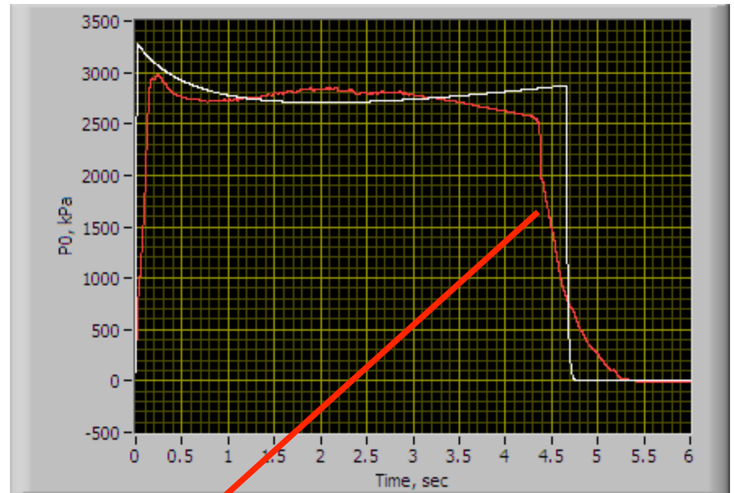
-- Erosive Burning Model



Erosive Burn Model (6)

-- Erosive Burning Model





.. Tail Off in Measured chamber pressure curve
Likely due to left-over Propellant "slivers" burning

Erosive Burn Model (6)

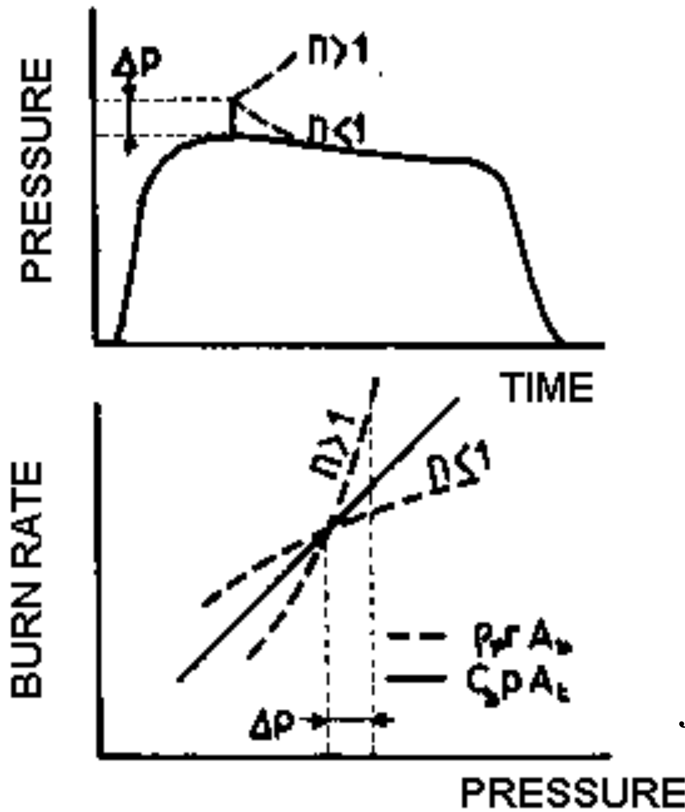
-- Erosive Burning Model

Erosion Corrected Burn rate

$$\dot{r}_{adjusted} = \frac{\dot{r}_{true}}{\left[\frac{1 + k(M / M_{crit})}{1 + k}\right]}$$

"True" Burn rate

Exponent Effect on Burn Rate (Pressure Excursion)



Source: Barrere et al., Raketenantriebe, Fig 5.1 (1961)

- *High values of burn exponent (**n**) make for a propellant whose burn rate is sensitive to chamber pressure*

- *Solid propellant motors with high burn rate profiles specially susceptible to fuel grain cracks and fractures*

- *Erosive burning can precipitate grain fracture*

What happens when a solid propellant grain factures?

• Solid propellant grain fracture events produce detrimental effects on motor performance, and can sometimes be catastrophic

• As crack propagates in the grain or along the grain/case interface, it creates additional burning surfaces

• Augmented burn area produces an excess of hot gas

• Excess mass flow strongly affects the chamber pressure rise and can (depending on the burn exponent) couple with the regression rate to produce a runaway burn and catastrophic failure.

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What happens when a solid propellant grain factures? (2)

- A classical example of a catastrophic solid rocket failure resulting from propagation of a crack along the grain/case interface is the Titan IV accident August, 1998
- Aerodynamic effects associated with the grain shape near a slot and the interaction between core and cross flows resulted in a dramatic increase in the head end pressure of the motor.
- The crack extended to the propellant case bond and propagated along the interface between the fuel and case.
- This sequence of events eventually led to the choking of the core flow and resulted in the rocket exploding.

http://www.youtube.com/watch?v=ZFeZkrRE9wI

Project 2

Build Unsteady Model of "Pike" .. Use Integrator of your choice

Calculate:

Chamber pressure profile
Regression rate profile
Massflow rate (compare to choking massflow)
Mass depletion vs time
plot Thrust profile
plot Total Impulse profile
Effective Mean Specific Impulse

Allow:

St. Robert's Parameter Input
Variable Step Size
Variable Thermodynamic Properties ***(as inputs to the problem)***
Erosive burn model for cylindrical port (Not Bates grain)

Project 2 (2)

Part 1, cylindrical port

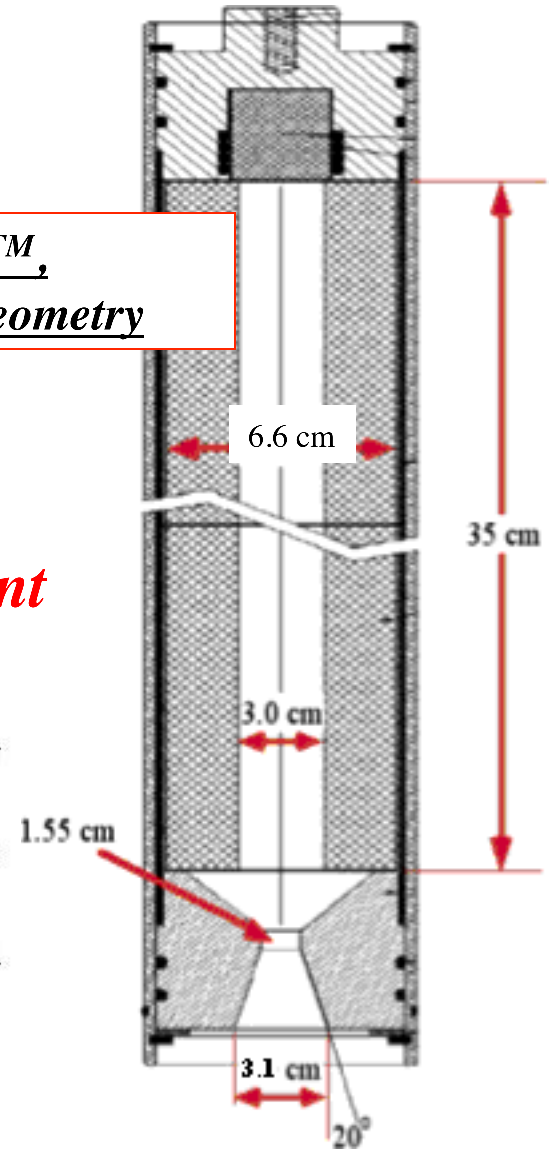
Fuel Grain Geometry

$L_0 = 35$ cm
$D_0 = 6.6$ cm
$D_0 = 3$ cm
$\rho_{propellant} = 1260$ kg/M^3

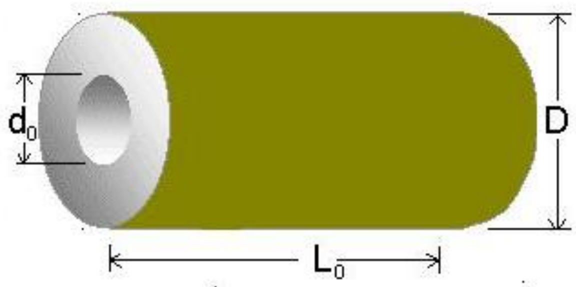
Nozzle Geometry

$A^* = 1.887$ cm ^{2}
$A_{exit}/A^* = 4.0$
$\theta_{exit} = 20$ deg.

Animal Works™, L700 Motor Geometry

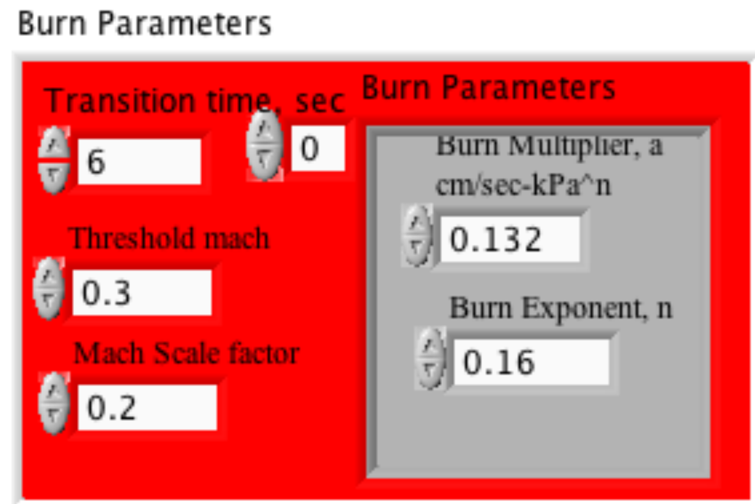


Single propellant segment



A*ssume ends are burn inhibited*

Project 2 (3)



Combustion Gas Properties

$\gamma = 1.18$

$M_W = 23$ *kg/kg-mol*

$T_0 = 2900$ *K*

Burn Parameters

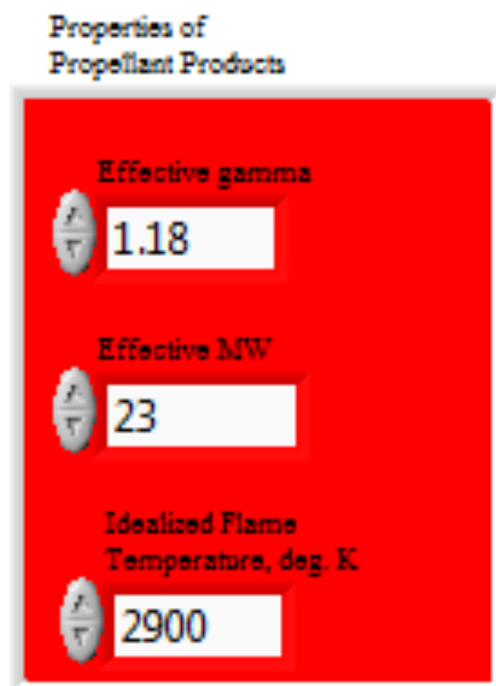
$a = 0.132$ *cm/(sec-kPa* ^{n}*)*

$n = 0.16$

$M^{crit} = 0.3$

$k = 0.2$

(cylindrical port only)



Project 2 (4)

Part 1, cylindrical port

Fuel Grain Geometry

$L_0 = 35$ cm

$D_0 = 6.6$ cm

$D_0 = 3$ cm

$\rho_{propellant} = 1260$ kg/M^3

Animal Works™, L700 Motor Geometry

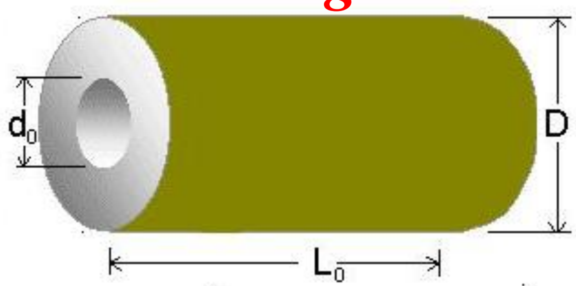
Repeat results Using bates grain With 3 segments

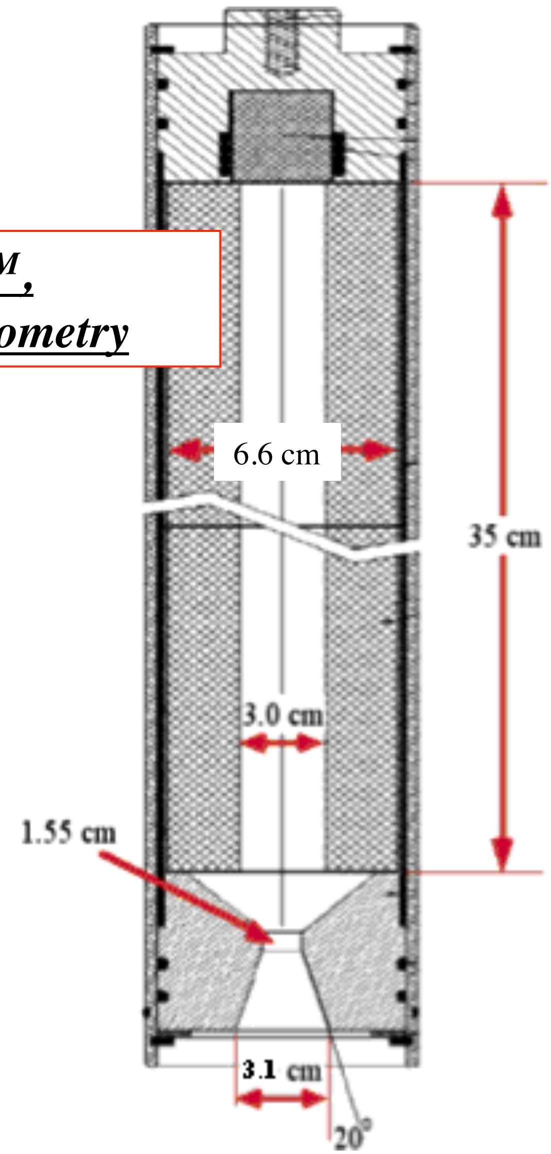
Nozzle Geometry

$A^* = 1.887$ cm ^{2}

$A_{exit}/A^* = 4.0$

$\theta_{exit} = 20$ deg.





Assume ends are not! burn inhibited

Project 2 (5)

Combustion Gas Properties

γ = 1.18

***M_W = 23** kg/kg-mol*

T_0= 2900 K

Burn Parameters

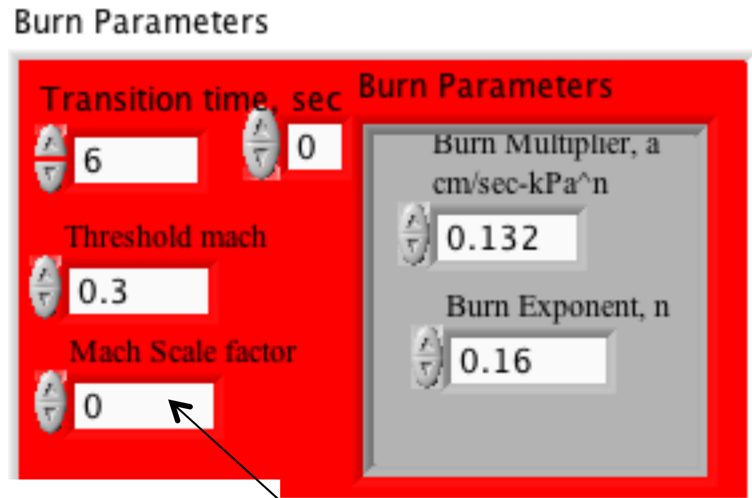
***a= 0.132** cm/(sec-kPa ^{n})*

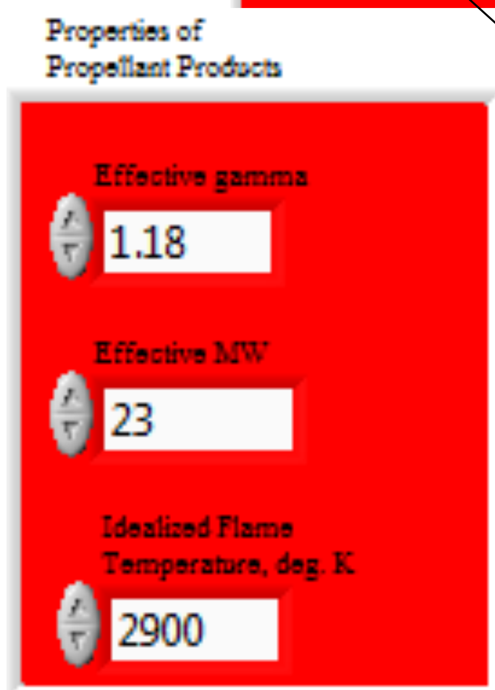
n=0.16

M^{crit} = 0.3

k = 0.0

(Bates grain only)





Set to Zero for Bates grain

Assume no erosive' burning

Project 2 (6)

Examine sensitivity of calculations to burn rate parameters, Critical Mach number (for erosion) … cylindrical port Only, Assume bates grain does not burn erosively

What is the effect of Flame temperature (T_0)

Plot Regression rate versus Chamber pressure

Prepare report stating your results and conclusions

State Equation Formulation of Problem

$$\begin{bmatrix} \dot{P}_0 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \left(\frac{A_{burn} \cdot \dot{r}}{V_c}\right) \cdot \left(\rho_{propellant} \cdot R_g \cdot T_0 - P_0\right) - \left(\frac{A^*}{V_c}\right) \cdot P_0 \cdot \sqrt{\gamma \cdot R_g \cdot T_0 \cdot \left(\frac{2}{\gamma+1}\right)^{\left(\frac{\gamma+1}{\gamma-1}\right)}} \\ a \cdot P_0^n \end{bmatrix}$$

$$\begin{bmatrix} P_0 \\ r \end{bmatrix}_{t=0} = \begin{bmatrix} P_{ambient} \\ \frac{d_0}{2} \end{bmatrix} \rightarrow \boxed{s(t) = \int_0^t \dot{r} \cdot dt \approx r(t) - \frac{d_0}{2}}$$

$$\begin{bmatrix} k \equiv Erosion\ Constant \ (\text{GRAIN DEPENDENT}) \\ M_{crit} \equiv Critical\ Port\ Mach\ Number \end{bmatrix}$$

$\rightarrow$ *State Equations for Erosive Burning :*

$$\begin{bmatrix} \dot{P}_0 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \left(\frac{A_{burn} \cdot \dot{r}}{V_c}\right) \cdot \left(\rho_{propellant} \cdot R_g \cdot T_0 - P_0\right) - \left(\frac{A^*}{V_c}\right) \cdot P_0 \cdot \sqrt{\gamma \cdot R_g \cdot T_0 \cdot \left(\frac{2}{\gamma+1}\right)^{\left(\frac{\gamma+1}{\gamma-1}\right)}} \\ \left(1 + k \cdot \frac{M_{port}}{M_{crit}}\right) \cdot a \cdot P_0^n \,/\, \left(1+k\right) \end{bmatrix}$$

$$\begin{bmatrix} P_0 \\ r \end{bmatrix}_{t=0} = \begin{bmatrix} P_{ambient} \\ \frac{d_0}{2} \end{bmatrix} \rightarrow \boxed{s(t) = \int_0^t \dot{r} \cdot dt \approx r(t) - \frac{d_0}{2}}$$

State Equation Formulation of Problem (2)

$\rightarrow$ *Cylindrical Port :*

$$\boxed{\begin{aligned} A_{burn} &= 2\cdot\pi\cdot r\cdot L_{port} \\ V_c &= \pi\cdot r^2\cdot L_{port}\end{aligned}} \rightarrow \begin{bmatrix} r \equiv Port\ Radius \\ L_{port} \equiv Port\ Length \end{bmatrix}$$

$\rightarrow$ *Bates Grain :*

$$A_{burn} = N\cdot\pi\cdot\left\{\left[\frac{D_0^2-\left(d_0+2\cdot s\right)^2}{2}\right]+\left(L_0-2\cdot s\right)\cdot\left(d_0+2\cdot s\right)\right\}$$

$$V_c = \frac{N\cdot\pi}{4}\cdot\left\{\left(d_0+2\cdot s\right)^2\cdot\left(L_0-2\cdot s\right)+D_0^2\cdot 2s\right\}$$

Do NOT! Use Erosive Burning for Bates Grain

State Equation Formulation of Problem (3)

Calculating Chamber Mach Number

Erosive Burning

$$\rightarrow \frac{V_c / L_{port}}{A^*} = \frac{1}{M_{port}} \cdot \left[\left(\frac{2}{\gamma + 1} \right) \cdot \left(1 + \left(\frac{\gamma - 1}{2} \right) \cdot M_{port}^2 \right) \right]^{\left(\frac{\gamma+1}{2 \cdot (\gamma - 1)} \right)}.$$

... *Subsonic Branch Solution*!

Do NOT! Use Erosive Burning for Bates Grain

Cylindrical Port: Decoupled Model

- Use Trapezoidal rule or Runge-Kutta to integrate

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

- Recursive propagation of chamber diameter

$$R_{burn_{\,k+1}} = R_{i_{\,initial}} + \int_0^{(k+1)\Delta t} \dot{r}\,dt = R_{i_{\,initial}} + \int_0^{(k)\Delta t} \dot{r}\,dt + \int_{(k)\Delta t}^{(k+1)\Delta t} \dot{r}\,dt \rightarrow$$

$$R_{burn_{\,k+1}} = R_{burn_{\,k}} + \int_{(k)\Delta t}^{(k+1)\Delta t} \dot{r}\,dt \approx R_{burn_{\,k}} + \dot{r}\,\Delta t = R_{burn_{\,k}} + a {P_o^{\,n}}_{k}\,\Delta t$$

Bates grain Port: Decoupled Model

- Use Trapezoidal rule or Runge-Kutta to integrate

$$\frac{\partial P_0}{\partial t} = \frac{A_{burn} a P_o^{\,n}}{V_c}\left[\rho_p R_g T_0 - P_0\right] - P_0\left[\frac{A^*}{V_c}\sqrt{\gamma R_g T_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right]$$

$$\dot{r} = a \cdot P_o^{\,n}$$

$$s_{regression} = \int_t \dot{r} \cdot dt \quad \rightarrow$$

$$\left(A_{burn}\right)_{total} = N \cdot \pi \cdot \left[\frac{\left(D_0^{\,2} - \left(d_0 + 2\cdot s\right)^2\right)}{2} + \left(L_0 - 2\cdot s\right)\cdot\left(d_0 + 2\cdot s\right)\right]$$

$$\left(V_{ol}\right)_{total} = \frac{N\cdot\pi}{4}\left[\left(d_0 + 2\cdot s\right)^2\cdot\left(L_0 - 2\cdot s\right) + D_0^{\,2}\cdot\left(2\cdot s\right)\right]$$