1. Let S be a positively oriented surface and let $c = \partial S$ (the boundary of S) be a positively oriented, piecewise smooth curve. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function on S (twice continuously differentiable). Show that,

$$\int_c f\nabla f \cdot d\mathbf{r} = 0 \tag{1}$$

Use this to evaluate the work done by,

$$\mathbf{F}(x, y, z) = 2(x^3 + xy^2 + xz^2, x^2y + y^3 + yz^2, x^2z + y^2z + z^3)$$

on any piecewise smooth, positively oriented closed curve c. *Hint: Use Stoke's theorem to show that (1) holds.*

We apply Stoke's theorem to the vector field $f\nabla f$. As such, we take the curl of $f\nabla f$. Realizing that $f\nabla f = (ff_x, ff_y, ff_z)$, we have,

$$\nabla \times f\nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ff_x & ff_y & ff_z \end{vmatrix}$$

Notice that $f \in C^2$, so mixed partials of f commute. As such, the x component of the curl is given by,

$$f_y f_z + ff_{zy} - f_z f_y - ff_{yz} = 0$$

A similar argument may be made for the y and z component of the curl. As such, $\nabla \times f\nabla f = \mathbf{0}$. This proves that (1) holds. Moreover, we may write,

$$\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)(2x, 2y, 2z)$$

Setting $f(x, y, z) = x^2 + y^2 + z^2$ shows that $\mathbf{F}(x, y, z) = f\nabla f$, so $\int_c \mathbf{F} \cdot d\mathbf{r} = 0$ for any piecewise smooth, positively oriented closed curve c.

2. Evaluate $\int_c \mathbf{F} \cdot dr$, where $\mathbf{F}(x, y, z) = (y, 2z, 3x)$ and c is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$ oriented counterclockwise as seen from above.

We apply Stoke's theorem to the disk induced by intersecting the plane $z = y$ with the given cylinder. I encourage you to draw/graph this if this is not clear. Now, computing the curl,

$$\nabla \times \mathbf{F} = (-2, -3, -1)$$

There are many ways to parameterize the induced surface. A trivial parameterization will do since the surface is is determined by a function. Letting $P = -2$, $Q = -3$, and $R = -1$, we set up,

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA = 2\int\int_D dA$$

When the plane intersects the cylinder, the constraints on x and y remain tied to the cylinder. Namely, $x^2 + y^2 \leq 1$ for every point on S. As such, D is a circle of radius 1, so,

$$2\int\int_D dA = 2\pi$$

3. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x + y^2 + 1, y + xz, 0)$ and S consists of the part of the cone $z^2 = x^2 + y^2$ bounded by the disk $0 \leq x^2 + y^2 \leq 1$, $z = 1$, and the disk $0 \leq x^2 + y^2 \leq 4$, $z = 2$.

The divergence theorem applies as S is the closed boundary of a solid. Computing the divergence gives $\nabla \cdot \mathbf{F} = 2$. We now must take a triple integral over the region bounded by S:

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_E \nabla \cdot \mathbf{F} dV = 2\int\int\int_E dV$$

We appeal to cylindrical coordinates to compute the integral above. Computing the volume element as $dV = rdzdrd\theta$, we divide the integral into pieces: the first piece is over the cylindrical region bounded by $x^2 + y^2 \leq 1$ and $1 \leq z \leq 2$. The second region is the rest of the solid, which resembles a torus. In the second region, the lower bound for z is given by the cone's formula: $z^2 = x^2 + y^2 = r^2$, so the lower bound for z in polar terms is r. The upper bound is the upper plane $z = 2$. The limits in the (r, θ) plane are given by an annulus in the second region. Again, I encourage you to draw a picture and think about the bounds we get here:

$$2\int\int\int_E dV = 2\int_0^{2\pi}\int_0^1\int_1^2 rdzdrd\theta + 2\int_0^{2\pi}\int_1^2\int_r^2 rdzdrd\theta = \frac{14\pi}{3}$$

4. Use the following guide to calculate $\int_c \mathbf{F} \cdot d\mathbf{r}$ where c is a path from $(0,0,0)$ to $(\pi, \pi/2, \pi/3)$ and $\mathbf{F}$ is defined as,

$$\mathbf{F}(x,y,z) = (\cos x \tan z, 1, \sin x \sec^2 z)$$

(a) Show that $\mathbf{F}$ is conservative in a suitable domain. $\mathbf{F}$ may be defined in a simply connected domain containing a path from our points listed above (for instance, the domain may look like a spheroid, or something similar). Moreover, the curl of $\mathbf{F}$ is $\mathbf{0}$, which you can easily check. It follows that $\mathbf{F}$ is conservative.

(b) Find a real-valued function f such that $\mathbf{F} = \nabla f$.

A function f that is defined in a domain where $\mathbf{F}$ is defined may be found as follows: we know that $\cos x \tan z$ should be f_x, so integrating with respect to x gives,

$$f(x,y,z) = \int \cos x \tan z dx = \sin x \tan z + g(y,z)$$

Moreover, $f_y = 1$, so we have,

$$f_y = 1 = g'(y,z) \quad \Longrightarrow \quad g(y,z) = y + h(x) \quad \Longrightarrow \quad f(x,y,z) = \sin x \tan z + y + h(x)$$

Finally, we solve for $h(x)$, knowing that $f_z = \sin x \sec^2 z$. But if $h(x)$ is constant, our candidate function $f(x,y,z)$ already satisfies $\nabla f = \mathbf{F}$. Thus, $f(x,y,z) = \sin x \tan z + y + C$ for a constant C. In the next part of this problem, we may take C to be zero without loss of generality, as is easily observed by evaluating $f(\mathbf{b}) - f(\mathbf{a})$.

(c) Since a potential function can be found, evaluate $\int_c \mathbf{F} \cdot d\mathbf{r}$ by taking $f(\mathbf{b}) - f(\mathbf{a})$, where $\mathbf{b}$ is the end point of c and $\mathbf{a}$ is the starting point of c.

Since $\mathbf{F}$ is independent of path in its domain of definition (discussed in part a)), we may evaluate the given line integral by simply taking $f(\pi, \pi/2, \pi/3) - f(0,0,0) = \pi/2$.