Vector Field with No Potential

1. Let $\mathbf{F}$ be the vector field defined as,

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Show that, if we define,

$$P(x,y) = \frac{-y}{x^2+y^2} \qquad Q(x,y) = \frac{x}{x^2+y^2}$$

Then wherever $\mathbf{F}$ is defined, we have,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1}$$

Show however that $\mathbf{F}$ fails to admit a function f such that,

$$\mathbf{F} = \nabla f$$

The derivatives are given by

$$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

Thus, equation (1) holds. But $\mathbf{F}$ is not defined at $(0,0)$. Thus, $\mathbf{F}$ is defined at every point on $\mathbb{R}^2$ except the origin. This is not a simply connected region since a loop around the origin cannot be shrunk to a point (it would hit the origin), so $\mathbf{F}$ cannot be written as the gradient of a function.

We now observe that the line integral of $\mathbf{F}$ along a closed curve around the origin is non-zero. Indeed, let $\mathbf{r}(t) = (\cos(t), \sin(t))$ be a parameterization of the circle of radius 1 in the plane, with $t \in [0, 2\pi]$. Then we use the formula for evaluating the line integral along this curve, given on the next page.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

$$= \int_0^{2\pi} \left(\frac{-\sin(t)}{\sin^2(t) + \cos^2(t)}, \frac{\cos(t)}{\sin^2(t) + \cos^2(t)} \right) \cdot (-\sin(t), \cos(t)) \, dt$$

$$= \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt = \int_0^{2\pi} dt = 2\pi$$