Example 1- simple national income model

Structural form:

$Y = C + I_o + G_o$
$C = \alpha + \beta(Y - T), \qquad (a > 0; 0 < \beta < 1)$
$T = \gamma + \delta Y, \qquad (\gamma > 0; 0 < \delta < 1)$
or

$$\begin{array}{ccccccccc} Y & - & C & & & = & I_o & + & G_o \\ -\beta Y & + & C & + & \beta T & = & \alpha & & \\ -\delta Y & & & + & T & = & \gamma & & \end{array}$$

or
$Ax = d$
with

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{pmatrix}, x = \begin{pmatrix} Y \\ C \\ T \end{pmatrix}, \text{ and } d = \begin{pmatrix} I_o + G_o \\ \alpha \\ \gamma \end{pmatrix}$$

Reduced form:

Solution to the system as indicated above (to address the problem of simultaneous equation bias)
$x = A^{-1}d$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{pmatrix}, \text{ determinant: } |A| = \beta\delta - \beta + 1$$

$$x = A^{-1}d = \begin{pmatrix} \frac{\alpha}{-\beta+\beta\delta+1} - \beta\frac{\gamma}{-\beta+\beta\delta+1} + \frac{G_o+I_o}{-\beta+\beta\delta+1} \\ \frac{\alpha}{-\beta+\beta\delta+1} - \beta\frac{\gamma}{-\beta+\beta\delta+1} + (G_o + I_o)\frac{\beta-\beta\delta}{-\beta+\beta\delta+1} \\ \alpha\frac{\delta}{-\beta+\beta\delta+1} + \delta\frac{G_o+I_o}{-\beta+\beta\delta+1} + \gamma\frac{-\beta+1}{-\beta+\beta\delta+1} \end{pmatrix} = \begin{pmatrix} Y \\ C \\ T \end{pmatrix}$$

$$\begin{pmatrix} Y \\ C \\ T \end{pmatrix} = \begin{pmatrix} \frac{\alpha-\beta\gamma+G_o+I_o}{1-\beta+\beta\delta} \\ \frac{\alpha-\beta\gamma+(G_o+I_o)(\beta-\beta\delta)}{1-\beta+\beta\delta} \\ \frac{\alpha\delta+\delta(G_o+I_o)+\gamma(1-\beta)}{1-\beta+\beta\delta} \end{pmatrix} = \begin{pmatrix} \frac{\alpha-\beta\gamma}{1-\beta+\beta\delta} + \frac{1}{1-\beta+\beta\delta}G_o + \frac{1}{1-\beta+\beta\delta}I_o \\ \frac{\alpha-\beta\gamma}{1-\beta+\beta\delta} + \frac{(\beta-\beta\delta)}{1-\beta+\beta\delta}G_o + \frac{(\beta-\beta\delta)}{1-\beta+\beta\delta}I_o \\ \frac{\alpha\delta+\gamma(1-\beta)}{1-\beta+\beta\delta} + \frac{\delta}{1-\beta+\beta\delta}G_o + \frac{\delta}{1-\beta+\beta\delta}I_o \end{pmatrix}$$

or

$$\begin{pmatrix} Y \\ C \\ T \end{pmatrix} = \begin{pmatrix} \pi_{11} + \pi_{21}G_0 + \pi_{31}I_0 \\ \pi_{12} + \pi_{22}G_0 + \pi_{32}I_0 \\ \pi_{13} + \pi_{23}G_0 + \pi_{33}I_0 \end{pmatrix}, \text{with}$$

$\pi_{11} = \frac{\alpha-\beta\gamma}{1-\beta+\beta\delta}, \pi_{21} = \frac{1}{1-\beta+\beta\delta}, \pi_{31} = \frac{1}{1-\beta+\beta\delta}$
$\pi_{12} = \frac{\alpha-\beta\gamma}{1-\beta+\beta\delta}, \pi_{22} = \frac{(\beta-\beta\delta)}{1-\beta+\beta\delta}, \pi_{32} = \frac{(\beta-\beta\delta)}{1-\beta+\beta\delta}$
$\pi_{13} = \frac{\alpha\delta+\gamma(1-\beta)}{1-\beta+\beta\delta}, \pi_{23} = \frac{\delta}{1-\beta+\beta\delta}, \pi_{33} = \frac{\delta}{1-\beta+\beta\delta}$

The objective is to estimate the structural form parameters via estimating the reduced form. We wil have estimates for $\pi_{11}, \ldots \pi_{33}$ and need to use these to estimate the structural form parameters.

How?
For instance: $\delta = \frac{\pi_{33}}{\pi_{31}}$ or $\frac{\pi_{23}}{\pi_{21}}$, $\beta =$??? is it possible?????
The answer to the above lies in the identification of the system and the individual equations

Comparative static analysis:
Comparative static results with respect to G_0:

$$\begin{pmatrix} \frac{\partial Y}{\partial G_o} \\ \frac{\partial C}{\partial G_o} \\ \frac{\partial T}{\partial G_o} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\beta+\beta\delta} \\ \frac{\beta-\beta\delta}{1-\beta+\beta\delta} \\ \frac{\delta}{1-\beta+\beta\delta} \end{pmatrix}$$

Comparative static results with respect to I_0:

$$\begin{pmatrix} \frac{\partial Y}{\partial I_o} \\ \frac{\partial C}{\partial I_o} \\ \frac{\partial T}{\partial I_o} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\beta+\beta\delta} \\ \frac{\beta-\beta\delta}{1-\beta+\beta\delta} \\ \frac{\delta}{1-\beta+\beta\delta} \end{pmatrix}$$

Example 2- identified system

Consider the following structural form:

$y_1 = a + bx_1 + cy_2 + u_1$

$y_2 = d + ex_2 + fy_1 + u2$

with y_1 and y_2 endogenous, and x_1 and x_2 exogenous

or

$$\begin{array}{ccccccccc} y_1 & - & cy_2 & = & a & + & bx_1 & + & u_1 \\ -fy_1 & + & y_2 & = & d & + & ex_2 & + & u_2 \end{array}$$

or

$$A = \begin{pmatrix} 1 & -c \\ -f & 1 \end{pmatrix}, |A| = \begin{vmatrix} 1 & -c \\ -f & 1 \end{vmatrix} = 1 - cf$$

The reduced form only exists if $cf \neq 1$

$g = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $r = \begin{pmatrix} a + bx_1 + u_1 \\ d + ex_2 + u_2 \end{pmatrix}$

Thus $Ag = r$ and thefore $g = A^{-1}r$

(Note that $g = A^{-1}r$ is the reduced form)

$$A = \begin{pmatrix} 1 & -c \\ -f & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} -\frac{1}{cf-1} & -\frac{c}{cf-1} \\ -\frac{f}{cf-1} & -\frac{1}{cf-1} \end{pmatrix}$$

$$g = A^{-1}\begin{pmatrix} a + bx_1 + u_1 \\ d + ex_2 + u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-cf}(a + u_1 + bx_1) + \frac{c}{1-cf}(d + u_2 + x_2 e) \\ \frac{1}{1-cf}(d + u_2 + x_2 e) + \frac{f}{1-cf}(a + u_1 + bx_1) \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{a+bx_1+u_1}{1-cf} + c\frac{d+ex_2+u_2}{1-cf} \\ \frac{d+ex_2+u_2}{1-cf} + f\frac{a+bx_1+u_1}{1-cf} \end{pmatrix} = \begin{pmatrix} \frac{a+u_1}{1-cf} + \frac{bx_1}{1-cf} + c\frac{d+u_2}{1-cf} + c\frac{ex_2}{1-cf} \\ \frac{d+u_2}{1-cf} + \frac{ex_2}{1-cf} + f\frac{a+u_1}{1-cf} + f\frac{bx_1}{1-cf} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a+u_1}{1-cf} + c\frac{d+u_2}{1-cf} + \frac{bx_1}{1-cf} + c\frac{ex_2}{1-cf} \\ \frac{d+u_2}{1-cf} + f\frac{a+u_1}{1-cf} + f\frac{bx_1}{1-cf} + \frac{ex_2}{1-cf} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a+u_1+c(d+u_2)}{1-cf} + \frac{bx_1}{1-cf} + c\frac{ex_2}{1-cf} \\ \frac{d+u_2+f(a+u_1)}{1-cf} + f\frac{bx_1}{1-cf} + \frac{ex_2}{1-cf} \end{pmatrix} = \begin{pmatrix} \frac{a+cd}{1-cf} + \frac{b}{1-cf}x_1 + \frac{ce}{1-cf}x_2 + \frac{u_1+cu_2}{1-cf} \\ \frac{d+fa}{1-cf} + \frac{fb}{1-cf}x_1 + \frac{e}{1-cf}x_2 + \frac{u_2+fu_1}{1-cf} \end{pmatrix}$$

$$= \begin{pmatrix} \pi_{11} + \pi_{21}x_1 + \pi_{31}x_2 + u_1^* \\ \pi_{12} + \pi_{22}x_1 + \pi_{32}x_2 + u_2^* \end{pmatrix}$$

with

$\pi_{11} = \frac{a+cd}{1-cf}, \pi_{21} = \frac{b}{1-cf}, \pi_{31} = \frac{ce}{1-cf}, \pi_{12} = \frac{d+fa}{1-cf}, \pi_{22} = \frac{fb}{1-cf}, \pi_{32} = \frac{e}{1-cf}$

The comparative static results for a change in x_1 is given by

$$\frac{\partial y_1}{\partial x_1} = \pi_{21} = \frac{b}{1-cf}$$
$$\frac{\partial y_2}{\partial x_1} = \pi_{22} = \frac{fb}{1-cf}$$

The comparative static results for a change in x_2 is given by

$$\frac{\partial y_2}{\partial x_1} = \pi_{31} = \frac{ce}{1-cf}$$
$$\frac{\partial y_2}{\partial x_2} = \pi_{32} = \frac{e}{1-cf}$$

The structural form parameters are estimated by the following:

$f = \frac{\pi_{22}}{\pi_{21}}, c = \frac{\pi_{31}}{\pi_{32}}$
$b = \pi_{21}(1-cf) = \pi_{21}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}),$
$e = \pi_{32}(1-cf) = \pi_{32}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}),$
$\pi_{11} = \frac{a+cd}{1-cf} = \frac{a+\frac{\pi_{31}}{\pi_{32}}d}{1-\frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}}$, and
$\pi_{12} = \frac{d+fa}{1-cf} = \frac{d+\frac{\pi_{22}}{\pi_{21}}a}{1-\frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}}$

define
$z_1 = \pi_{11}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}})$, and $z_2 = \pi_{12}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}})$
$z_1 = a + cd$, or $a = z_1 - cd$

$z_2 = d + fa = d + f(z_1 - cd) = d + fz_1 - fcd$
$d - fcd = z_2 - fz_1$

$$d(1-fc) = z_2 - fz_1$$
$$d = \frac{z_2 - fz_1}{1-fc} = \frac{\pi_{12}(1-\frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}) - \pi_{11}\frac{\pi_{22}}{\pi_{21}}(1-\frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}})}{1-\frac{\pi_{22}}{\pi_{21}}\frac{\pi_{31}}{\pi_{32}}} = \pi_{12} - \pi_{11}\frac{\pi_{22}}{\pi_{21}}$$

$$a = \pi_{11}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}) - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{12}(1-\frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}) - \pi_{11}\frac{\pi_{22}}{\pi_{21}}(1-\frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}})}{1-\frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}}$$
$$= \pi_{11}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}) - \frac{\pi_{31}}{\pi_{32}}(\pi_{12} - \pi_{11}\frac{\pi_{22}}{\pi_{21}})$$
$$= \pi_{11} - \frac{\pi_{31}}{\pi_{32}}\pi_{12}$$

How do we estimate the standard errors of the structural form parameters?

Example 3- **SAS Example (Example 2 continued)**

The REG Procedure
Model: MODEL1
Dependent Variable: y1

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	39244291	19622145	137.30	<.0001
Error	97	13862788	142915		
Corrected Total	99	53107079			

Root MSE	378.04146	R-Square	0.7390
Dependent Mean	3977.51000	Adj R-Sq	0.7336
Coeff Var	9.50448		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	1008.85165	210.44972	4.79	<.0001
x1	1	20.63740	1.32274	15.60	<.0001
x2	1	9.76651	1.08785	8.98	<.0001

The REG Procedure
Model: MODEL1
Dependent Variable: y2

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	273834386	136917193	293.59	<.0001
Error	97	45236258	466353		
Corrected Total	99	319070645			

Root MSE	682.90056	R-Square	0.8582
Dependent Mean	4459.52000	Adj R-Sq	0.8553
Coeff Var	15.31332		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	2668.42257	380.15998	7.02	<.0001
x1	1	50.72257	2.38941	21.23	<.0001
x2	1	-12.89939	1.96512	-6.56	<.0001

The REG Procedure
Model: MODEL1
Dependent Variable: y1

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	39244291	19622145	137.30	<.0001
Error	97	13862788	142915		
Corrected Total	99	53107079			

Root MSE	378.04146	R-Square	0.7390
Dependent Mean	3977.51000	Adj R-Sq	0.7336
Coeff Var	9.50448		

Parameter Estimates

Variable	Label	DF	Parameter Estimate	Standard Error	t Value
Intercept	Intercept	1	3029.19452	109.52448	27.66
x1		1	59.04098	4.75556	12.42
y2h	Predicted Value of y2	1	-0.75713	0.08433	-8.98

Parameter Estimates

Variable	Label	DF	Pr > |t|
Intercept	Intercept	1	<.0001
x1		1	<.0001
y2h	Predicted Value of y2	1	<.0001

The REG Procedure
Model: MODEL2
Dependent Variable: y2

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	273834386	136917193	293.59	<.0001
Error	97	45236258	466353		
Corrected Total	99	319070645			

Root MSE	682.90056	R-Square	0.8582
Dependent Mean	4459.52000	Adj R-Sq	0.8553
Coeff Var	15.31332		

Parameter Estimates

Variable	Label	DF	Parameter Estimate	Standard Error	t Value
Intercept	Intercept	1	188.86879	463.24042	0.41
x2		1	-36.90351	2.03272	-18.15
y1h	Predicted Value of y1	1	2.45780	0.11578	21.23

Parameter Estimates

Variable	Label	DF	Pr > |t|
Intercept	Intercept	1	0.6844
x2		1	<.0001
y1h	Predicted Value of y1	1	<.0001

Therefore:

$\pi_{11} = 1008.85165$
$\pi_{21} = 20.63740$
$\pi_{31} = 9.76651$
$\pi_{12} = 2668.42257$
$\pi_{22} = 50.72257$
$\pi_{32} = -12.89939$
yielding estimates:

$f = \frac{\pi_{22}}{\pi_{21}} = 2.4578$
$c = \frac{\pi_{31}}{\pi_{32}} = -0.75713$
$b = \pi_{21}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}) = 59.041$
$e = \pi_{32}(1 - \frac{\pi_{31}}{\pi_{32}}\frac{\pi_{22}}{\pi_{21}}) = -36.904$
$d = \pi_{12} - \pi_{11}\frac{\pi_{22}}{\pi_{21}} = 188.87$
$a = \pi_{11} - \frac{\pi_{31}}{\pi_{32}}\pi_{12} = 3029.2$
The estimates of the structural model equations are:
$\hat{y}_1 = a + bx_1 + cy_2 = +3029.2 + 59.041x_1 - 0.75713y_2$

$\hat{y}_2 = d + ex_2 + fy_1 = 188.87 - 36.904x_2 + 2.4578y_1$

This method of estimation is known as Indirect Least Squares (ILS)