UNIVERSITY OF PRETORIA

FACULTY OF ECONOMIC AND MANAGEMENT SCIENCES

DEPARTMENT OF STATISTICS

ECONOMETRICS 723: SYSTEMS OF EQUATIONS

QUESTION

Consider the following stochastic demand/supply model:

$$\begin{aligned} q^d &= \gamma_1 p + \beta_1 I + \delta_1 + u_1 \\ q^s &= \gamma_2 p + \beta_2 r + \delta_2 + u_2 \\ q^d &= q^s \end{aligned}$$

with

$$\begin{aligned} q^d &= \text{quantity demanded} \\ q^s &= \text{quantity supplied} \\ p &= \text{price} \\ I &= \text{exogenous income} \\ r &= \text{exogenous rainfall} \\ u_i &= \text{stochastic disturbance terms} \end{aligned}$$

(a) Determine the reduced form.

$$\begin{aligned} q^d - \gamma_1 p &= \beta_1 I + \delta_1 + u_1 \\ q^s - \gamma_2 p &= \beta_2 r + \delta_2 + u_2 \\ q^d &= q^s = q \end{aligned}$$

$$A = \begin{pmatrix} 1 & -\gamma_1 \\ 1 & -\gamma_2 \end{pmatrix}$$

$$d = \begin{pmatrix} \beta_1 I + \delta_1 + u_1 \\ \beta_1 r + \delta_2 + u_2 \end{pmatrix}$$

$$|A| = -\gamma_2 + \gamma_1$$

$$A^{-1} = \begin{pmatrix} -\frac{\gamma_2}{-\gamma_2+\gamma_1} & \frac{\gamma_1}{-\gamma_2+\gamma_1} \\ -\frac{1}{-\gamma_2+\gamma_1} & \frac{1}{-\gamma_2+\gamma_1} \end{pmatrix}$$

$$y = A^{-1}d = \begin{pmatrix} -\frac{\gamma_2}{-\gamma_2+\gamma_1}(\beta_1 I + \delta_1 + u_1) + \frac{\gamma_1}{-\gamma_2+\gamma_1}(\beta_1 r + \delta_2 + u_2) \\ -\frac{1}{-\gamma_2+\gamma_1}(\beta_1 I + \delta_1 + u_1) + \frac{1}{-\gamma_2+\gamma_1}(\beta_1 r + \delta_2 + u_2) \end{pmatrix}$$

therefore:

$$\begin{aligned} \overline{q} &= -\frac{\gamma_2}{-\gamma_2+\gamma_1}(\beta_1 I + \delta_1 + u_1) + \frac{\gamma_1}{-\gamma_2+\gamma_1}(\beta_1 r + \delta_2 + u_2) \\ &= \frac{-\beta_1\gamma_2}{\gamma_1-\gamma_2} I + \frac{\beta_2\gamma_1}{\gamma_1-\gamma_2} r + \frac{\delta_2\gamma_1 - \delta_1\gamma_2}{\gamma_1-\gamma_2} + \upsilon_1 \\ &= \pi_1 I + \pi_2 r + \pi_3 + \upsilon_1 \end{aligned}$$

$$\begin{aligned} \overline{p} &= -\frac{1}{-\gamma_2+\gamma_1}(\beta_1 I + \delta_1 + u_1) + \frac{1}{-\gamma_2+\gamma_1}(\beta_1 r + \delta_2 + u_2) \\ &= \frac{-\beta_1}{\gamma_1-\gamma_2} I + \frac{\beta_2}{\gamma_1-\gamma_2} r + \frac{\delta 2-\delta 1}{\gamma_1-\gamma_2} + \upsilon_2 \\ &= \pi_4 I + \pi_5 r + \pi_6 + \upsilon_2 \end{aligned}$$

(b) Use the reduced form parameters to determine the values of the unknown structural equation parameters: $\gamma_1, \gamma_2, \beta_1, \beta_2, \delta_1$ and δ_2

$$\gamma_2 = \frac{\pi_1}{\pi_4} \quad \gamma_1 = \frac{\pi_2}{\pi_5}$$

$$\begin{aligned} \pi_4 &= \frac{-\beta_1}{(\gamma_1-\gamma_2)} \\ -\beta_1 &= \pi_4(\gamma_1 - \gamma_2) \\ -\beta_1 &= \pi_4(\tfrac{\pi_2}{\pi_5} - \tfrac{\pi_1}{\pi_4}) \\ \beta_1 &= \pi_4(\tfrac{\pi_1}{\pi_4} - \tfrac{\pi_2}{\pi_5}) \end{aligned}$$

$$\begin{aligned} \pi_5 &= \frac{\beta_2}{\gamma_1-\gamma_2} \\ \beta_2 &= \pi_5(\gamma_1 - \gamma_2) \\ \beta_2 &= \pi_5(\tfrac{\pi_2}{\pi_5} - \tfrac{\pi_1}{\pi_4}) \end{aligned}$$

$$\begin{aligned} \pi_3 &= \frac{\delta_2\gamma_1 - \delta_1\gamma_2}{\gamma_1 - \gamma_2} \\ &= \frac{\delta_2\frac{\pi_2}{\pi_5} - \delta_1\frac{\pi_1}{\pi_4}}{\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4}} \\ (\tfrac{\pi_2}{\pi_5} - \tfrac{\pi_1}{\pi_4})\pi_3 &= \delta_2\tfrac{\pi_2}{\pi_5} - \delta_1\tfrac{\pi_1}{\pi_4} \\ a &= b\delta_2 - c\delta_1 ----A \end{aligned}$$

$$\begin{aligned} \pi_6 &= \frac{\delta_2 - \delta_1}{\gamma_1 - \gamma_2} \\ &= \frac{\delta_2 - \delta_1}{\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4}} \\ (\tfrac{\pi_2}{\pi_5} - \tfrac{\pi_1}{\pi_4})\pi_6 &= \delta_2 - \delta_1 \\ d &= \delta_2 - \delta_1 ----B \end{aligned}$$

$$\begin{aligned} (A - bB) \quad a - bd &= 0 - c\delta_1 + b\delta_1 \\ &= \delta_1(b - c) \\ \delta_1 &= \frac{a-bd}{b-c} \end{aligned}$$

AND

$$\begin{aligned} \delta_2 &= d + \delta_1 \\ &= d + \frac{a-bd}{b-c} \\ &= \frac{d(b-c)+a-bd}{b-c} \\ &= \frac{a-cd}{b-c} \end{aligned}$$

Substitute the values for a,b,c,d:

$$\begin{aligned} \delta_2 &= \frac{(\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4})\pi_3 - \frac{\pi_1}{\pi_4}(\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4})\pi_6}{(\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4})} \\ &= \pi_3 - \tfrac{\pi_1}{\pi_4}.\pi_6 \end{aligned}$$

$$\begin{aligned} \delta_1 &= \frac{(\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4})\pi_3 - \frac{\pi_2}{\pi_5}(\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4})\pi_6}{(\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4})} \\ &= \pi_3 - \tfrac{\pi_2}{\pi_5}.\pi_6 \end{aligned}$$

EXAMPLE

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                          The REG Procedure
                            Model: MODEL1
                        Dependent Variable: q

                         Analysis of Variance

                                 Sum of          Mean
Source                  DF      Squares        Square  F Value  Pr > F

Model                    2        18642    9320.87788  24445.2  <.0001
Error                   97     36.98583       0.38130
Corrected Total         99        18679

                         Parameter Estimates

                     Parameter       Standard
Variable     DF       Estimate          Error    t Value    Pr > |t|

Intercept     1        3.90141        0.77279       5.05      <.0001
i             1        1.22340        0.00588     207.90      <.0001
r             1        1.16618        0.01248      93.45      <.0001


                            Model: MODEL2
                        Dependent Variable: p

                         Analysis of Variance

                                 Sum of          Mean
Source                  DF      Squares        Square  F Value  Pr > F

Model                    2    426.57600     213.28800  79158.6  <.0001
Error                   97      0.26136       0.00269
Corrected Total         99    426.83737

                         Parameter Estimates

                     Parameter       Standard
Variable     DF       Estimate          Error    t Value    Pr > |t|

Intercept     1        0.10662        0.06496       1.64      0.1040
i             1        0.15406     0.00049468     311.44      <.0001
r             1       -0.22975        0.00105    -219.02      <.0001
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The output above represents the estimate of the reduced form. (Why?)

$$\begin{aligned}
\overline{q} &= -\tfrac{\gamma_2}{-\gamma_2+\gamma_1}\left(\beta_1 I + \delta_1 + u_1\right) + \tfrac{\gamma_1}{-\gamma_2+\gamma_1}\left(\beta_1 r + \delta_2 + u_2\right) \\
&= \tfrac{-\beta_1\gamma_2}{\gamma_1-\gamma_2} I + \tfrac{\beta_2\gamma_1}{\gamma_1-\gamma_2} r + \tfrac{\delta_2\gamma_1-\delta_1\gamma_2}{\gamma_1-\gamma_2} + \upsilon_1 \\
&= \pi_1 I + \pi_2 r + \pi_3 + \upsilon_1 \\
\hat{q} &= 1.22340I + 1.16618r + 3.90141
\end{aligned}$$

$$\begin{aligned}
\overline{p} &= -\tfrac{1}{-\gamma_2+\gamma_1}\left(\beta_1 I + \delta_1 + u_1\right) + \tfrac{1}{-\gamma_2+\gamma_1}\left(\beta_1 r + \delta_2 + u_2\right) \\
&= \tfrac{-\beta_1}{\gamma_1-\gamma_2} I + \tfrac{\beta_2}{\gamma_1-\gamma_2} r + \tfrac{\delta 2-\delta 1}{\gamma_1-\gamma_2} + \upsilon_2 \\
&= \pi_4 I + \pi_5 r + \pi_6 + \upsilon_1 \\
\hat{p} &= 0.15406I - 0.22975r + 0.10662
\end{aligned}$$

The structural form parameters are:

$$\hat{\gamma}_2 = \frac{\pi_1}{\pi_4} = \frac{1.22340}{0.15406} = 7.941$$

$$\hat{\gamma}_1 = \frac{\pi_2}{\pi_5} = \frac{1.16618}{-0.22975} = -5.076$$

$$\begin{aligned}
\hat{\beta}_1 &= \pi_4\left(\frac{\pi_1}{\pi_4} - \frac{\pi_2}{\pi_5}\right) \\
&= 0.15406 \times \left(\frac{1.22340}{0.15406} - \frac{1.16618}{-0.22975}\right) \\
&= 2.005
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_2 &= \pi_5\left(\frac{\pi_2}{\pi_5} - \frac{\pi_1}{\pi_4}\right) \\
&= -0.22975 \times \left(\frac{1.16618}{-0.22975} - \frac{1.22340}{0.15406}\right) \\
&= 2.991
\end{aligned}$$

$$\begin{aligned}
\hat{\delta}_2 &= \pi_3 - \frac{\pi_1}{\pi_4}.\pi_6 \\
&= 3.90141 - \frac{1.22340}{0.15406} \times 0.10662 \\
&= 3.055
\end{aligned}$$

$$\begin{aligned}
\hat{\delta}_1 &= \pi_3 - \frac{\pi_2}{\pi_5}.\pi_6 \\
&= 3.90141 - \frac{1.16618}{-0.22975} \times 0.10662 \\
&= 4.443
\end{aligned}$$

The estimated model is:

$$\begin{aligned}
\hat{q}_d &= -5.076p + 2.005I + 4.443 \\
\hat{q}_s &= 7.941p + 2.991r + 3.055
\end{aligned}$$