$\boxed{\textbf{ECONOMETRICS 720}}$

Structural, Reduced and Final Forms

- $\boxed{\text{The prototype micro model}}$

Consider the following model (structural form):

$$
\begin{aligned}
q^D &= \gamma_1 p + \beta_1 I + \delta_1 + \epsilon^D \\
q^S &= \gamma_2 p + \beta_2 r + \delta_2 + \epsilon^S \\
q^D &= q^S
\end{aligned}
$$

with

q^D quantity demanded

q^S quantity supplied

I exogenous income

r exogenous rainfall

which can be written as

$$q = \gamma_1 p + \beta_1 I + \delta_1 + \epsilon^D$$

$$q = \gamma_2 p + \beta_2 r + \delta_2 + \epsilon^S$$

or in matrix notation:

$$\begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\gamma_1 & -\gamma_2 \end{pmatrix} + \begin{pmatrix} I & r & 1 \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & -\beta_2 \\ -\delta_1 & -\delta_2 \end{pmatrix} = \begin{pmatrix} \epsilon^D & \epsilon^S \end{pmatrix}$$

which is of the form: $y\Gamma + x\beta = \epsilon$

with

y the vector of endogenous variables

Γ the coeffcients matrix of the endogenous variables

x the vector of exogenous variables (predetermined)

β the coefficient matrix of the exogenous variables

ϵ the vector of stochastic disturbance terms

Solving the system above for the unknown endogenous variables yield the following reduced form:

$y = \Gamma^{-1}(\epsilon - x\beta)$, which for this example is

$$(q \quad p) = (I \quad r \quad l)\begin{pmatrix} \frac{\gamma_2\beta_1}{\gamma_2-\gamma_1} & \frac{\beta_1}{\gamma_2-\gamma_1} \\ \frac{-\gamma_1\beta_2}{\gamma_2-\gamma_1} & \frac{-\beta_2}{\gamma_2-\gamma_1} \\ \frac{\gamma_2\delta_1-\gamma_1\delta_2}{\gamma_2-\gamma_1} & \frac{\delta_1-\delta_2}{\gamma_2-\gamma_1} \end{pmatrix}$$

$$+\left(\frac{\gamma_2\epsilon^D-\gamma_1\epsilon^S}{\gamma_2-\gamma_1} \qquad \frac{\epsilon^D-\epsilon^S}{\gamma_2-\gamma_1}\right)$$

or in the usual form:

$$p = \frac{\beta_1}{\gamma_2 - \gamma_1} I - \frac{\beta_2}{\gamma_2 - \gamma_1} r + \frac{\delta_1 - \delta_2}{\gamma_2 - \gamma_1} + \frac{\epsilon^D - \epsilon^S}{\gamma_2 - \gamma_1}$$

$$q = \frac{\gamma_2\beta_1}{\gamma_2 - \gamma_1} I - \frac{\gamma_1\beta_2}{\gamma_2 - \gamma_1} r + \frac{\gamma_2\delta_1 - \gamma_1\delta_2}{\gamma_2 - \gamma_1} + \frac{\gamma_2\epsilon^D - \gamma_1\epsilon^S}{\gamma_2 - \gamma_1}$$

The comparative static results (selected) are:

$$\frac{\partial p}{\partial I} = \frac{\beta_1}{\gamma_2 - \gamma_1}$$

$$\frac{\partial q}{\partial I} = \frac{\gamma_2\beta_1}{\gamma_2 - \gamma_1}$$

$$\frac{\partial p}{\partial r} = \frac{-\beta_2}{\gamma_2 - \gamma_1}$$

$$\frac{\partial q}{\partial r} = \frac{-\gamma_1\beta_2}{\gamma_2 - \gamma_1}$$

- $\boxed{\text{The prototype macro model}}$

Consider the following model:

$$
\begin{aligned}
C_t &= \gamma_1 Y_t + be_1 + \epsilon_t^C \\
I_t &= \gamma_2 Y_t + \beta_3 Y_{t-1} + \beta_3 + \epsilon_t^I \\
Y_t &= C_t + I_t + G_t
\end{aligned}
$$

	C_t	endogenous consumption
	I_t	endogenous investment
with	Y_t	endogenous national icome in year t respectively
	G_t	exogenous government spending
	Y_{t-1}	lagged national income (predetermined)

The structural form in matrix notation is:

$$\begin{pmatrix} C_t & Y_t \end{pmatrix} \begin{pmatrix} -1 & \dfrac{1}{1-\gamma_2} \\ \gamma_1 & -1 \end{pmatrix} + \begin{pmatrix} Y_{t-1} & G_t & 1 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{\beta_2}{1-\gamma_2} \\ 0 & \dfrac{1}{1-\gamma_2} \\ \beta_1 & \dfrac{\beta_3}{1-\gamma_2} \end{pmatrix}$$

$$= \begin{pmatrix} -\epsilon_t^C & \dfrac{-\epsilon_t^I}{1-\gamma_2} \end{pmatrix}$$

The reduced form is given by:

$$
\begin{aligned}
Y_t &= \frac{\beta_2}{1-\gamma_1-\gamma_2} Y_{t-1} + \frac{1}{1-\gamma_1-\gamma_2} G_t \\
&\quad + \frac{\beta_1+\beta_3}{1-\gamma_1-\gamma_2} + \frac{\epsilon_t^C + \epsilon_t^I}{1-\gamma_1-\gamma_2} \\
Y_t &= \pi_1 Y_{t-1} + \pi_2 G_t + \pi_3 + u_t^Y
\end{aligned}
$$

$$
\begin{aligned}
C_t &= \frac{\gamma_1\beta_2}{1-\gamma_1-\gamma_2} Y_{t-1} + \frac{\gamma_1}{1-\gamma_1-\gamma_2} G_t \\
&\quad \frac{\gamma_1\beta_3 + (1-\gamma_2)\beta_1}{1-\gamma_1-\gamma_2} + \frac{\gamma_1\epsilon_t^I + (1-\gamma_2)\epsilon_t^C}{1-\gamma_1-\gamma_2} \\
C_t &= \pi_4 Y_{t-1} + \pi_5 G_t + \pi_6 + u_t^C
\end{aligned}
$$

$$
\text{with}\quad
\begin{aligned}
\pi_1 &= \frac{\beta_2}{1-\gamma_1-\gamma_2}\\
\pi_2 &= \frac{1}{1-\gamma_1-\gamma_2}\\
\pi_3 &= \frac{\beta_1+\beta_3}{1-\gamma_1-\gamma_2}\\
\pi_4 &= \frac{\gamma_1\beta_2}{1-\gamma_1-\gamma_2}\\
\pi_5 &= \frac{\gamma_1}{1-\gamma_1-\gamma_2}\\
\pi_6 &= \frac{\gamma_1\beta_3+(1-\gamma_2)\beta_1}{1-\gamma_1-\gamma_2}\\
u_t^Y &= \frac{\epsilon_t^C+\epsilon_t^I}{1-\gamma_1-\gamma_2}\\
u_t^C &= \frac{\gamma_1\epsilon_t^I+(1-\gamma_2)\epsilon_t^C}{1-\gamma_1-\gamma_2}
\end{aligned}
$$

Consider the reduced form equation for Y_t: (in the form of a difference equation)

$$Y_t = \pi_1 Y_{t-1} + \pi_2 G_t + \pi_3 + u_t^Y$$

$$Y_{t-1} = \pi_1 Y_{t-2} + \pi_2 G_{t-1} + \pi_3 + u_{t-1}^Y$$

$$Y_t = \pi_1 \left(\pi_1 Y_{t-2} + \pi_2 G_{t-1} + \pi_3 + u_{t-1}^Y\right) + \pi_2 G_t + \pi_3 + u_t^Y$$

$$Y_t = \pi_1^2 Y_{t-2} + \pi_2 (G_t + \pi_1 G_{t-1}) + \pi_3 (1 + \pi_1) + (u_t^Y + \pi_1 u_{t-1}^Y)$$

$$Y_t = \pi_1^3 Y_{t-3} + \pi_2 (G_t + \pi_1 G_{t-1} + \pi_1^2 G_{t-2}) + \pi_3 (1 + \pi_1 + \pi_1^2) + (u_t^Y + \pi_1 u_{t-1}^Y + \pi_1^2 u_{t-2}^Y)$$

Continuing this process of iteration back to the base year, $t = 0$ yields

$$
\begin{aligned}
Y_t = {} & \pi_1^t Y_0 + \pi_2(G_t + \pi_1 G_{t-1} + \pi_1^2 G_{t-2} + \ldots + \pi_1^{t-1} G_1) \\
& + \pi_3(1 + \pi_1 + \pi_1^2 + \ldots \pi_1^{t-1}) \\
& + (u_t^Y + \pi_1 u_{t-1}^Y + \pi_1^2 u_{t-2}^Y + \ldots + \pi_1^{t-1} u_1^Y)
\end{aligned}
$$

The solution to the difference equation as given above is the **Final form** of the Y_t equation.

From the equation above all multipliers for income, both short and long term, can be calculated.

$$
-\frac{\partial Y_t}{\partial G_t} = \pi_2 = \frac{1}{1 - \gamma_1 - \gamma_2}
$$

– $\dfrac{\partial Y_t}{\partial G_{t-1}} = \pi_2 \pi_1$

– Adding the previous two multipliers gives the effect of change in G over both the current and preceding periods.

$$\frac{\partial Y_t}{\partial G_t}\Big|_{\Delta G_{t-1}=\Delta G_t} = \pi_2(1+\pi_1) = \frac{1-\gamma_1-\gamma_2+\beta_2}{1-\gamma_1-\gamma_2}$$

– Similarly: $\dfrac{\partial Y_t}{\partial G_t}\Big|_{\Delta G_{t-2}=\Delta G_{t-1}=\Delta G_t} = \pi_2(1+\pi_1+\pi_1^2)$

– Similarly: $\dfrac{\partial Y_t}{\partial G_t}\Big|_{Long \quad term} = \pi_2(1+\pi_1+\pi_1^2+\ldots) = \dfrac{\pi_2}{1-\pi_1}$