EKT 720

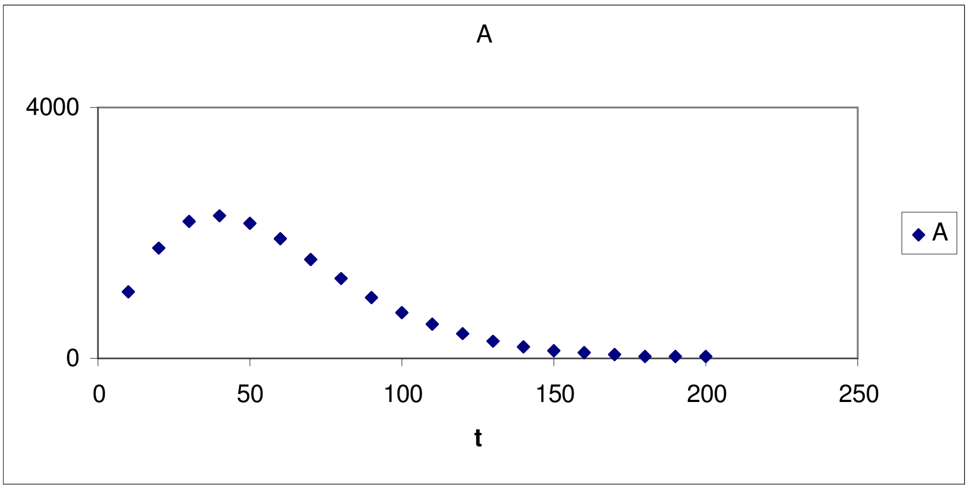
Example - Taylor series expansion and Newton Raphson algorithm

Taylor series expansion

Consider the following non-linear function:

$A(t) = 100e^{(\sqrt{t}-0.08t)}$

with graphical representation:



Objective: Search for the value of t that will maximise A. The maximum value can be determined as follows:

$A'(t) = 100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}} - 0.08\right)$

$100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}} - 0.08\right) = 0$, Solution is: 39.063 (Courtesy of Scientific Workplace)

$A''(t) = 100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}} - 0.08\right)^2 - \frac{25}{t^{\frac{3}{2}}}e^{\sqrt{t}-0.08t}$

$A''(39.063) = -2.3306$

Yielding a maximum value for A at a t value of 39.063.

Optimisation using a second degree polinomial taylor series expansion (Chiang):

$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + R_n$, or

$A(t) = \frac{f(t_0)}{0!} + \frac{f'(t_0)}{1!}(t-t_0) + \frac{f''(t_0)}{2!}(t-t_0)^2 + R_n$, or

$A(t) = \frac{A(t_0)}{0!} + \frac{A'(t_0)}{1!}(t-t_0) + \frac{A''(t_0)}{2!}(t-t_0)^2 + Rn$

For this example:
$A'(t) = 100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}} - 0.08\right)$

$A''(t) = 100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}} - 0.08\right)^2 - \frac{25}{t^{\frac{3}{2}}}e^{\sqrt{t}-0.08t}$

Step 1:

Need an initial value to start expansion, $t_0 = 55$:

$A(55) = 2041.4$
$A'(55) = -25.68$
$A''(55) = -0.92811$

$B(t) = A(55) + A'(55)(t-55) + \frac{A''(55)}{2}(t-55)^2$
$= 3453.8 - 0.46405(t-55.0)^2 - 25.68t$
$B'(t) = 25.366 - 0.92811t = 0$, Solution is: 27.331
$B''(27.331) = -0.92811$

Step 2:

Repeat the process with the solution from the previous step: $t = 27.331$
$A(27.331) = 2093.5$

$A'(27.331) = 32.744$
$A''(27.331) = -3.1508$

$C(t) = A(27.331) + A'(27.331)(t-27.331) + \frac{A''(27.331)}{2}(t-27.331)^2$
$= 32.744t - 1.5754(t-27.331)^2 + 1198.6$

$C'(t) = 118.86 - 3.1508t = 0$, Solution is: 37.724
$C''(37.724) = -3.1508$

Step 3:

Repeat the process with the solution from the previous step: $t = 37.724$
$A(37.724) = 2273.9$

$A'(37.724) = 3.1991$
$A''(37.724) = -2.4490$

$E(t) = A(37.724) + A'(37.724)(t-37.724) + \frac{A''(37.724)}{2}(t-37.724)^2$

$= 3.1991t - 1.2245(t - 37.724)^2 + 2153.2$
$E'(t) = 95.583 - 2.4490t = 0$, Solution is: 39.029
$E''(t) = -2.4490$

Step 4:

Repeat the process with the solution from the previous step: $t = 39.029$
$A(39.029) = 2276.0$

$A'(39.029) = 7.8126 \times 10^{-2}$
$A''(39.029) = -2.3336$

$F(t) = A(39.029) + A'(39.029)(t - 39.029) + \frac{A''(39.029)}{2}(t - 39.029)^2$
$= 7.8126 \times 10^{-2}t - 1.1668(t - 39.029)^2 + 2272.9$
$F'(t) = 91.157 - 2.3336t = 0$, Solution is: 39.063
$F''(t) = -2.3336$

Step 5:

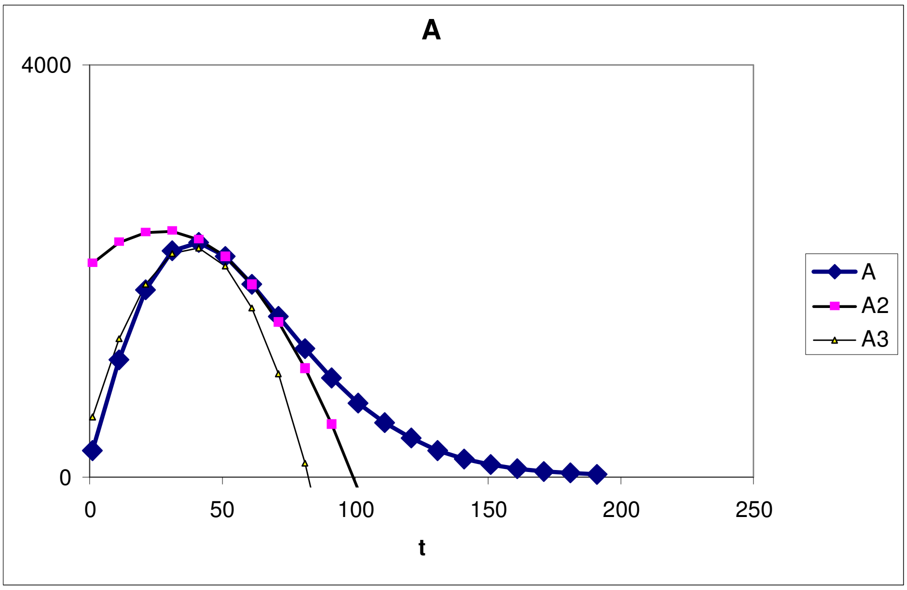
Repeat the process with the solution from the previous step: $t = 39.063$
$A(39.063) = 2276.0$

$A'(39.063) = -1.1653 \times 10^{-3}$
$A''(39.063) = -2.3306$

$G(t) = A(39.063) + A'(39.063)(t - 39.063) + \frac{A''(39.063)}{2}(t - 39.063)^2$
$= 2276.0 - 1.1653(t - 39.063)^2 - 1.1653 \times 10^{-3}t$
$G'(t) = 91.038 - 2.3306t = 0$, Solution is: 39.062
$G''(t) = -2.3306$

Summary:

Graphical representation of this process:



Newton Raphson algorithm

$$A(t) = \frac{A(t_0)}{0!} + \frac{A'(t_0)}{1!}(t-t_0) + \frac{A''(t_0)}{2!}(t-t_0)^2 + Rn$$

Differentiate with respect to t

$$A'(t) = A'(t_0) + \frac{A''(t_0)}{2!}2(t-t_0) = 0$$

$$A''(t_0)(t-t_0) = -A'(t_0)$$

$$(t-t_0) = -A'(t_0)/A''(t_0)$$

$$t = t_0 - A'(t_0)/A''(t_0)$$

$$t_{new} = t_{old} - A'(t_{old})/A''(t_{old})$$

Step 1:

Need an initial value to start expansion, $t_0 = 55$:

$A(55) = 2041.4$
$A'(55) = -25.68$
$A''(55) = -0.92811$

NR

$$
\begin{aligned}
t_{new} &= t_{old} - (A'(55))^{-1}(A'(55) \\
&= 55 - (-0.92811)^{-1}(-25.68) \\
&= 27.331
\end{aligned}
$$

Step 2:

Repeat the process with the solution from the previous step: $t = 27.331$
$A(27.331) = 2093.5$

$A'(27.331) = 32.744$
$A''(27.331) = -3.1508$

NR

$$
\begin{aligned}
t_{new} &= t_{old} - (A'(27.331))^{-1}(A'(27.331) \\
&= 27.331 - (-3.1508)^{-1}(32.744) \\
&= 37.724
\end{aligned}
$$

and so on