Taylor series expansion

EKT 720
Introduction to Statistical learning
September 2019

1 Taylor Series Expansion

A function $f(x)$ is approximated by a polynomial, expanded in a specific point, $x = a$. The general expression for a Taylor series expansion of the function $f(x)$ in a is

$$\begin{aligned} f(x) &= f(a) + f^1(a)(x-a) + \frac{f^2}{2!}(x-a)^2 + + \frac{f^3}{3!}(x-a)^3 + \ldots + \frac{f^n}{n!}(x-a)^n + R_n \\ &= \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n + R_n \end{aligned} \tag{1}$$

with R_n the remainder if a polynomial of degree n is used, $f^n(a)$ the n^{th} derivative of $f(x)$ evaluated in the point $x = a$, $f^0(a) = f(a)$ the function value of $f(x)$ in the point $x = a$.

2 Examples

2.1 Simple second degree polynomial

Consider the function (simple function to illustrate the principle):

$$f(x) = 5 + 2x - 4x^2 \tag{2}$$

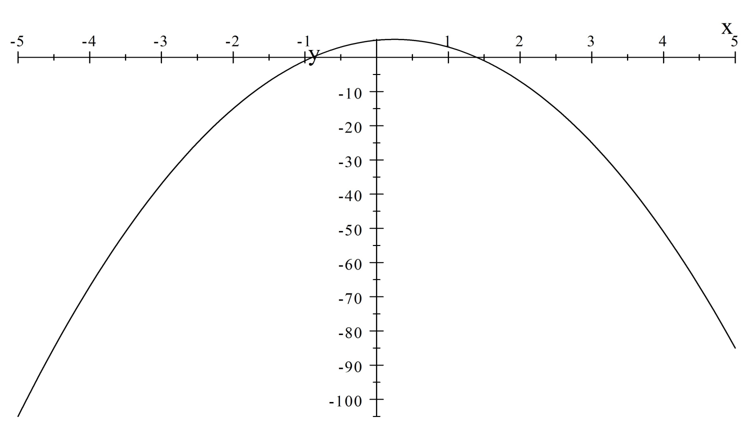

Figure 1: Graphical representation of the function $f(x)$.

The derivatives of $f(x)$ are

$$
\begin{aligned}
f^1(x) &= 2-8x \\
f^2(x) &= -8 \\
f^3(x) &= 0
\end{aligned}
$$

The second degree Taylor series expansion (1) in the point $x=3$ (chosen arbitrarily) is

$$
\begin{aligned}
f(3) &= -25 \\
f^1(3) &= 2-8\times 3 = -22 \\
f^2(3) &= -8 \\
f(x) &= \sum_{n=0}^{2} \frac{f^n(a)}{n!}(x-a)^n = -25 + (-22)(x-3) + \left(-\frac{8}{2}\right)(x-3)^2 \\
&= 41 - 22x - 4\left(x^2-6x+9\right) \\
&= -4x^2 + 2x + 5
\end{aligned}
$$

From the above it is clear the a 2^{nd} degree Taylor series expansion in the point $x=3$ yields the exact polynomial (2) (that we started with).

2.2 Non-linear function

Consider the function:

$$A(t) = 100e^{\left(\sqrt{t}-0.08t\right)} \tag{3}$$

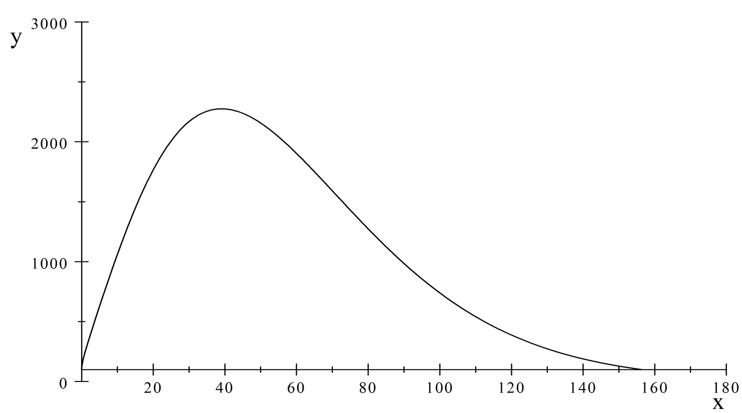

Figure 2: Graphical representation of the function $A(t)$.

The derivatives of $A(t)$ are

$$
\begin{aligned}
A^1(t) &= 100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)\\
A^2(t) &= 100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)^2-\frac{25}{t^{\frac{3}{2}}}e^{\sqrt{t}-0.08t}\\
A^3(t) &= \frac{75}{2t^{\frac{5}{2}}}e^{\sqrt{t}-0.08t}+100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)^3-\frac{75}{t^{\frac{3}{2}}}e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)\\
A^4(t) &= \frac{75}{4t^3}e^{\sqrt{t}-0.08t}-\frac{375}{4t^{\frac{7}{2}}}e^{\sqrt{t}-0.08t}+100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)^4\\
&\quad+\frac{150}{t^{\frac{5}{2}}}e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)-\frac{150}{t^{\frac{3}{2}}}e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)^2\\
A^5(t) &= \frac{2625}{8t^{\frac{9}{2}}}e^{\sqrt{t}-0.08t}-\frac{375}{4t^4}e^{\sqrt{t}-0.08t}+100e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)^5\\
&\quad+\frac{375}{4t^3}e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)-\frac{1875}{4t^{\frac{7}{2}}}e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)\\
&\quad-\frac{250}{t^{\frac{3}{2}}}e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)^3+\frac{375}{t^{\frac{5}{2}}}e^{\sqrt{t}-0.08t}\left(\frac{1}{2\sqrt{t}}-0.08\right)^2
\end{aligned}
$$

Expanding the function $A(t)$ in the point $t=60$

$$
\begin{aligned}
A(60) &= 100e^{\left(2\sqrt{15}-4.8\right)}=1902.9\\
A^1(60) &= 100e^{\sqrt{60}-4.8}\left(\frac{1}{120}\sqrt{60}-0.08\right)=-29.4\\
A^2(60) &= 100e^{\sqrt{60}-4.8}\left(\frac{1}{120}\sqrt{60}-0.08\right)^2-\frac{1}{144}\sqrt{60}e^{\sqrt{60}-4.8}=-0.569\,36\\
A^3(60) &= 6.601\,6\times10^{-2}\\
A^4(60) &= -2.353\,5\times10^{-3}\\
A^5(60) &= -2.352\,8\times10^{-5}
\end{aligned}
$$

Different degrees of expansion follows,

First degree:

$$A(t) = A(60) + A'(60)(t-60) = 3666.9 - 29.4t$$

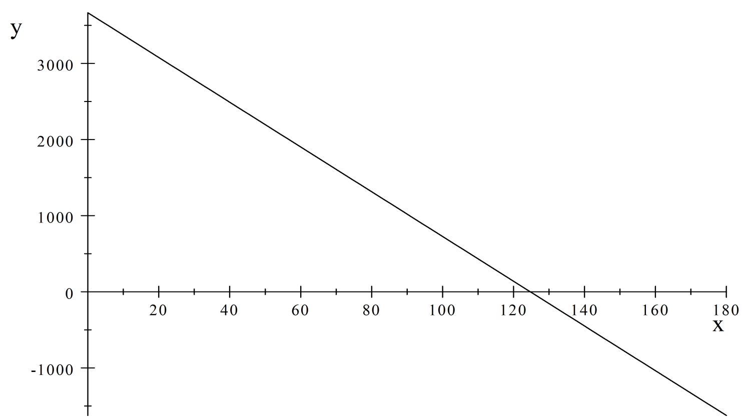

Figure 3: Graphical representation of the first degree approximation of $A(t)$.

From Figure 3 it is clear that a first degree approximation is not appropriate.

Second degree:

$$\begin{aligned} A(t) &= A(60) + A'(60)(t-60) + \frac{A''(60)}{2!}(t-60)^2 \\ &= 3666.9 - 0.28468\,(t-60.0)^2 - 29.4t \end{aligned}$$

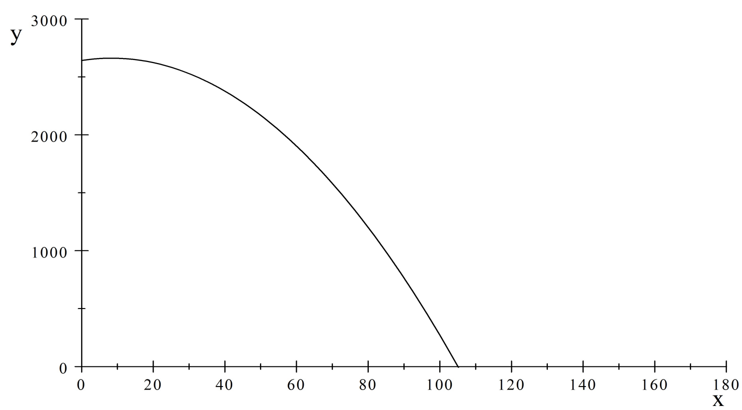

Figure 4: Graphical representation of the second degree approximation of $A(t)$.

From Figure 4 it is clear that a second degree approximation is a better approximation of $A(t)$.

Third degree:

$$\begin{aligned}A(t) &= A(60) + A'(60)(t-60) + \frac{A''(60)}{2!}(t-60)^2 + \frac{A'''(60)}{3!}(t-60)^3 \\ &= 1.1003 \times 10^{-2}(t-60.0)^3 - 0.28468(t-60.0)^2 - 29.4t + 3666.9\end{aligned}$$

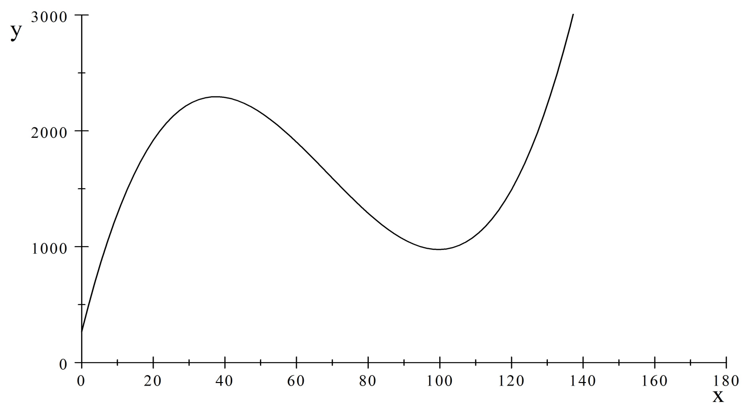

Figure 5: Graphical representation of the third degree approximation of $A(t)$.

Fourth degree:

$$\begin{aligned}A(t) &= A(60) + A'(60)(t-60) + \frac{A''(60)}{2!}(t-60)^2 + \frac{A'''(60)}{3!}(t-60)^3 + \frac{A''''(60)}{4!}(t-60)^4 \\ &= 1.1003 \times 10^{-2}(t-60.0)^3 - 0.28468(t-60.0)^2 - 29.4t - 9.8064 \times 10^{-5}(t-60.0)^4 + 3666.9\end{aligned}$$

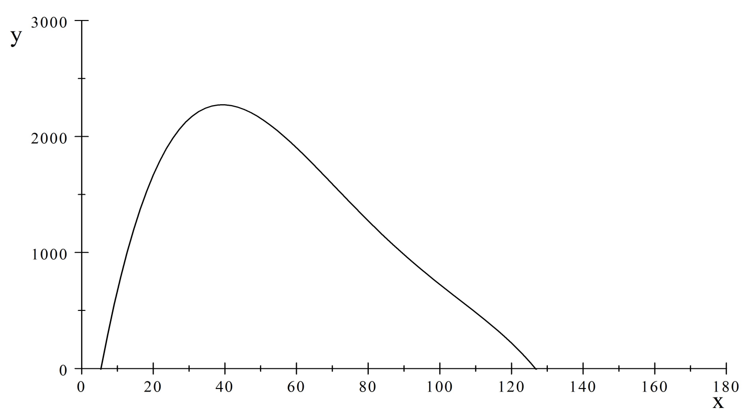

Figure 6: Graphical representation of the fourth degree approximation of $A(t)$.

Fifth degree:

$$
\begin{aligned}
A(t) &= A(60) + A'(60)(t-60) + \frac{A''(60)}{2!}(t-60)^2 + \frac{A'''(60)}{3!}(t-60)^3 + \frac{A''''(60)}{4!}(t-60)^4 \\
&\quad + \frac{A'''''(60)}{5!}(t-60)^5 \\
&= 1.1003 \times 10^{-2}(t-60.0)^3 - 0.28468(t-60.0)^2 - 29.4t - 9.8064 \times 10^{-5}(t-60.0)^4 \\
&\quad -1.9607 \times 10^{-7}(t-60.0)^5 + 3666.9
\end{aligned}
$$

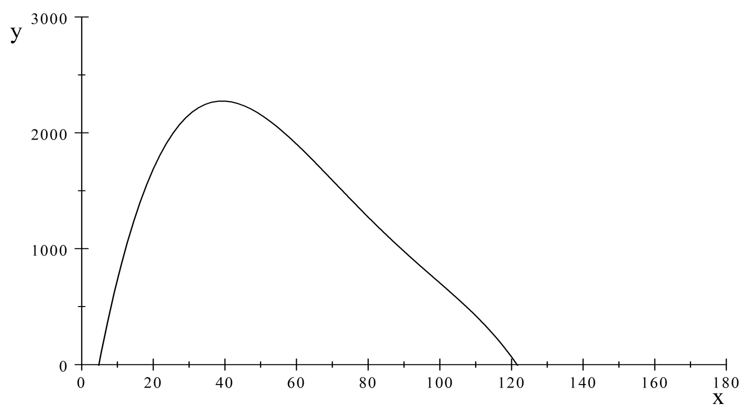

Figure 7: Graphical representation of the fifth degree approximation of $A(t)$.

Figure 8 shows a comparison between the original function $A(t)$ and the fifth degree approximation. From this it is apparent that the fifth order approximation is a fair representation of the function $A(t)$.

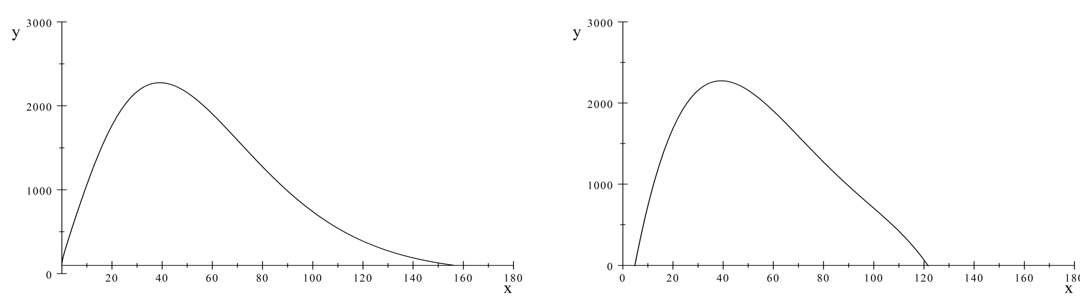

Figure 8: Graphical representation of $A(t)$ and the fifth degree approximation of $A(t)$.