The Ski Slope

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September 19, 2024

Abstract

The document addresses three main problems related to finding slopes and angles of a curve defined by the function $y = 0,5e^{-x^2}$ and its generalized form $y = 0,5e^{-ax^2}$.

First, it calculates the angle of inclination of the curve at the specific point $x = 0,8$ by differentiating the function using the chain rule.

The second part extends this concept to find where the curve is steepest, requiring the second derivative of the function which is found using the chain rule and quotient rule

Finally, the task involves selecting a constant aa that positions the steepest point at $x = 1,0$.

1 Introduction

This report will solve and describe tasks related to a ski slope with a vertical drop of 500 meters. The slope can be written as a function with a relation between y (the height in km.)and x (the length in km.), which can be written as:
$y = 0.5e^{-x^2} \quad \text{where} \quad 0 \leq x \leq 2.5$

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2 The angle of a slope

Our first task is:

> Determine the angle at which the curve of the function is inclined at the point $x = 0,8$.

To do this, we first need to calculate the derivative $\left(\frac{dy}{dx}\right)$ of the function, which will give us the slope of the tangent line at that point. From the slope, we can then calculate the angle using the arctangent function. Because the function involves an exponential expression, we will use the **chain rule**.

2.1 Splitting the function

The function $y = 0,5e^{-x^2}$ can be viewed as a composition of two functions:

- The outer function $f(u) = 0,5e^u$, where $u = -x^2$.
- The inner function $g(x) = -x^2$.

According to the chain rule, the derivative of the composite function is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

First, we differentiate the outer function $y = 0,5e^u$ with respect to u. The derivative of e^u with respect to u is simply e^u, and since there is a constant 0,5 in front, we have:

$$\frac{dy}{du} = 0,5e^u \tag{1}$$

Next, we differentiate the inner function $u = -x^2$ with respect to x using the basic power rule differentiation:

$$\frac{du}{dx} = \frac{d}{dx}\left(-x^2\right) = -2x \tag{2}$$

Now, we apply the chain rule by multiplying the results from Step 1 and Step 2:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 0,5e^{-x^2} \cdot (-2x) \\ &= -x \cdot e^{-x^2}\end{aligned} \tag{3}$$

Thus we have the derivative of y, which means that we can now calculate the slope of the curve at the point x. By substituting $x = 0,8$ into the function we get:

$$\begin{aligned}y'(0,8) &= -0,8 \cdot e^{-(0,8)^2} \\ &= -0,8 \cdot e^{-(0,64)} \\ &\approx -0,8 \cdot 0,5272 \approx -0,42\end{aligned} \tag{4}$$

So the answer is that the slope of the tangent line at $x = 0,8$ is approximately $-0,42$.

3 The Steepest Point

Here our task is:

Find an equation where the angle is the steepest using this function:

$$y = 0,5e^{-ax^2}$$

Where a is a constant and x is in the range $0 \leq x \leq 2,5$. Curves are steepest where their second derivative is zero so we'll need to find the second derivative. But first, we need to calculate the first derivative y'.

3.1 First Derivative

Just like earlier we will make use of the **chain rule** for this, so, again:

- The outer function is $f(u) = 0.5e^u$, where $u = -ax^2$
- The inner function is $g(x) = -ax^2$.

The outer function's derivative with respect to u is, once again, e^u, so we have:

$$\frac{dy}{du} = 0,5e^u \tag{5}$$

And the inner function's derivative with respect to x using the power rule, we get:

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}\left(-ax^2\right)\\ &= -2ax\end{aligned} \tag{6}$$

Now we can apply the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= 0,5e^{-ax^2} \cdot (-2ax)\\ &= 0,5e^{-ax^2} \cdot (-2ax)\\ &= -ax \cdot e^{-ax^2}\end{aligned} \tag{7}$$

3.2 Finding the Second Derivative

Finally we can work toward finding the Second Derivative $\frac{d^2y}{dx^2}$. To do this we can use the **quotient rule** and, once again, the **chain rule**.

The Quotient Rule

The quotient rule states:

$$\frac{f(x)}{g(x)} = \frac{f(x) \cdot g'(x) - f'(x) \cdot g(x)}{g(x)^2}$$

And in our case:

- $f(x) = -ax$
- $g(x) = e^{-ax^2}$
- $f'(x) = -a$
- $g'(x) = e^{ax^2} \cdot (2ax)$

So we can use this by rewriting the first derivative to make it suitable for the quotient rule:

$$y' = \frac{-ax}{e^{ax^2}}$$

Then we end up with:

$$\begin{aligned} y'' &= \frac{\left(-a \cdot e^{ax^2}\right) - \left(-ax\left(2ax \cdot e^{ax^2}\right)\right)}{\left(e^{ax^2}\right)^2} \\ &= \frac{-ae^{ax^2} + 2a^2x^2e^{ax^2}}{\left(e^{ax^2}\right)^2} \end{aligned} \tag{8}$$

Now if we factor out and have it equal 0:

$$\frac{ae^{ax^2}\left(-1 + 2ax^2\right)}{\left(e^{ax^2}\right)^2} = 0 \tag{9}$$

Since this is division, only the numerator can be 0, otherwise we would be dividing by 0. So because of that we can find where the curve is steepest:

$$\begin{aligned} 2ax^2 - 1 = 0 &\Rightarrow 1 = 2ax^2 \\ &\Rightarrow x^2 = \frac{1}{2a} \\ &\Rightarrow x = \sqrt{\frac{1}{2a}} \end{aligned}$$

4 Pick Constant for Steepest Angle

This task reads:

Pick a constant for a which turns the slope in a way that turns the steepest angle to be at point $x = 1,0$

We can make use of the work we did in the last task where we found

$$x = \sqrt{\frac{1}{2a}}$$

and by backtracking one step we can find the constant a where $x = \pm 1$:

$$\begin{aligned} x = \sqrt{\frac{1}{2a}} &\Rightarrow x^2 = \frac{1}{2a} \\ &\Rightarrow a = \frac{1}{2x^2} \\ &\Rightarrow a = \frac{1}{2} = 0,5 \end{aligned} \tag{10}$$

5 Conclusion

The document effectively demonstrates how differentiation techniques, including the chain rule and quotient rule, are used to solve problems related to the slope and steepness of an exponential function. By calculating the derivative, it identifies the slope at a specific point and locates the steepest point on the curve. The selection of an appropriate constant shows how parameters can be adjusted to fit certain criteria, such as positioning the steepest point at a desired location. This approach highlights the practical application of calculus in analyzing the behavior of curves.

Results

1. **Slope at** $x = 0.8$:
   After calculating the derivative $y'(x) = -x \cdot e^{-x^2}$, substituting $x = 0.8$, the slope of the curve at that point is approximately -0.42. This indicates that the curve is decreasing at this point, with a relatively moderate rate of decline.

2. **Steepest Point**:
   By solving the second derivative $y''(x) = 0$, the curve is found to be steepest at $x = \sqrt{\frac{1}{2a}}$. This shows that the steepest point is influenced by the parameter a, and the steepness occurs closer to the origin as a increases.

3. **Determining the Constant** a:
   To ensure the steepest point occurs at $x = 1,0$, the constant a is calculated to be $0,5$. This result reflects the importance of parameter tuning in mathematical models and highlights how the shape of the function can be controlled by varying a.

References

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