To see why, we first observe that, because the possible values of X are 2, 3, and 4, the events $\{X = 2\}$, $\{X = 3\}$, and $\{X = 4\}$ form a partition of the sample space. Therefore, by the law of partitions (Proposition 2.8 on page 68), we have

$$p_Y(3) = P(Y = 3) = \sum_{x=2}^{4} P(X = x,\ Y = 3) = \sum_{x=2}^{4} p_{X,Y}(x, 3),$$

as required. Thus we can obtain each marginal probability by summing the appropriate row or column of joint probabilities. More generally, we have Proposition 6.2 whose proof, which we leave to you as Exercise 6.18, is essentially the same as the argument just given.

◆◆◆ **Proposition 6.2 Obtaining Marginal PMFs From the Joint PMF**

Let X and Y be discrete random variables defined on the same sample space. Then

$$p_X(x) = \sum_{y} p_{X,Y}(x, y), \qquad x \in \mathcal{R}, \tag{6.3}$$

and

$$p_Y(y) = \sum_{x} p_{X,Y}(x, y), \qquad y \in \mathcal{R}. \tag{6.4}$$

In words, we can obtain the (marginal) PMF of X by summing on y the joint PMF of X and Y and, likewise, we can obtain the (marginal) PMF of Y by summing on x the joint PMF of X and Y.

AMPLE 6.2 ***Obtaining Marginal PMFs From the Joint PMF***

Lifetimes of Electrical Components Consider two identical electrical components. Let X and Y denote the respective lifetimes of the two components observed at discrete time units (e.g., every hour). Assume that the joint PMF of X and Y is

$$p_{X,Y}(x, y) = p^2(1 - p)^{x+y-2}, \qquad x, y \in \mathcal{N},$$

and $p_{X,Y}(x, y) = 0$ otherwise, where $0 < p < 1$. Find and identify the marginal PMFs of X and Y.

Solution The possible values of X are the positive integers, so we can concentrate on values of $x \in \mathcal{N}$. From Equation (6.3), the fact that $p_{X,Y}(x, y) = 0$ unless y is a positive integer, and a geometric series formula, we obtain

$$\begin{aligned} p_X(x) &= \sum_{y} p_{X,Y}(x, y) = \sum_{y=1}^{\infty} p^2(1 - p)^{x+y-2} = p^2(1 - p)^{x-2} \sum_{y=1}^{\infty} (1 - p)^y \\ &= p^2(1 - p)^{x-2} \cdot \frac{1 - p}{1 - (1 - p)} = p(1 - p)^{x-1}, \end{aligned}$$

for each $x \in \mathcal{N}$. Consequently, $p_X(x) = p(1 - p)^{x-1}$ for $x = 1, 2, \ldots$, and $p_X(x) = 0$ otherwise. Thus X has the geometric distribution with parameter p.

Likewise, applying Equation (6.4) yields $p_Y(y) = p(1 - p)^{y-1}$ for $y = 1, 2, \ldots$, and $p_Y(y) = 0$ otherwise. Thus Y also has the geometric distribution with parameter p. ■

Solution As previously, we let X and Y denote the number of bedrooms and number of bathrooms, respectively, of the home obtained.

a) We want to find $P(X = Y)$. From the FPF and the joint PMF of X and Y given in Table 6.2 on page 262,

$$
\begin{aligned}
P(X = Y) &= \sum_{x=y}\sum p_{X,Y}(x, y) \\
&= p_{X,Y}(2, 2) + p_{X,Y}(3, 3) + p_{X,Y}(4, 4) \\
&= 0.06 + 0.24 + 0.10 \\
&= 0.40.
\end{aligned}
$$

The probability is 0.40 that the home obtained has the same number of bedrooms and bathrooms. In terms of percentages, 40% of the 50 homes have the same number of bedrooms and bathrooms.

b) We want to find $P(X > Y)$. From the FPF and the joint PMF of X and Y given in Table 6.2,

$$
\begin{aligned}
P(X > Y) &= \sum_{x>y}\sum p_{X,Y}(x, y) \\
&= p_{X,Y}(3, 2) + p_{X,Y}(4, 2) + p_{X,Y}(4, 3) \\
&= 0.28 + 0.04 + 0.22 \\
&= 0.54.
\end{aligned}
$$

The probability is 0.54 that the home obtained has more bedrooms than bathrooms. In terms of percentages, 54% of the 50 homes have more bedrooms than bathrooms. ■

EXAMPLE 6.5 ***The Fundamental Probability Formula***

Lifetimes of Electrical Components Refer to Example 6.2 on page 263, where we considered the lifetimes X and Y of two identical electrical components. Determine the probability that

a) both electrical components last longer than 4 time units.

b) one of the electrical components lasts at least twice as long as the other.

Solution We recall that the joint PMF of X and Y is

$$
p_{X,Y}(x, y) = p^2(1-p)^{x+y-2}, \qquad x, y \in \mathcal{N},
$$

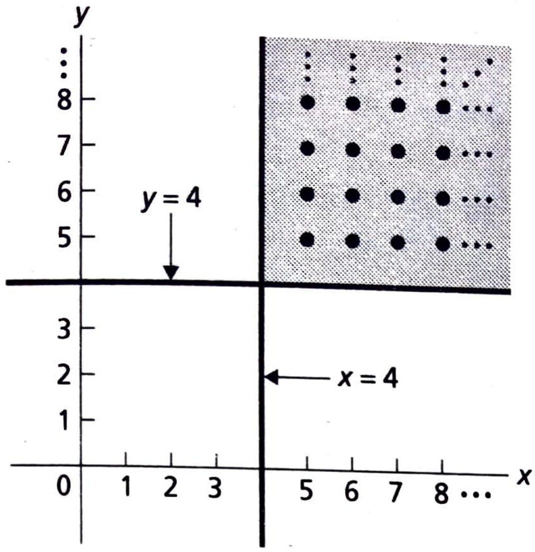
and $p_{X,Y}(x, y) = 0$ otherwise, where $0 < p < 1$.

a) Here we want $P(X > 4,\ Y > 4)$. Applying the FPF, we get

$$
P(X > 4,\ Y > 4) = \sum_{x>4,\ y>4}\sum p_{X,Y}(x, y). \tag{6.6}
$$

Figure 6.1 shows the set over which the double summation is taken.

Figure 6.1



There are several ways to evaluate the double sum in Equation (6.6). Here is the most straightforward way, although not the quickest. Iterating the sums and applying geometric series formulas twice, we get

$$\begin{aligned}
\sum_{x>4,\, y>4}\sum p_{X,Y}(x, y) &= \sum_{x=5}^{\infty}\left(\sum_{y=5}^{\infty} p^2(1-p)^{x+y-2}\right) \\
&= \frac{p^2}{(1-p)^2}\sum_{x=5}^{\infty}(1-p)^x\left(\sum_{y=5}^{\infty}(1-p)^y\right) \\
&= \frac{p^2}{(1-p)^2}\sum_{x=5}^{\infty}(1-p)^x \cdot \frac{(1-p)^5}{1-(1-p)} \\
&= p(1-p)^3\sum_{x=5}^{\infty}(1-p)^x \\
&= p(1-p)^3 \cdot \frac{(1-p)^5}{1-(1-p)} \\
&= (1-p)^8.
\end{aligned}$$

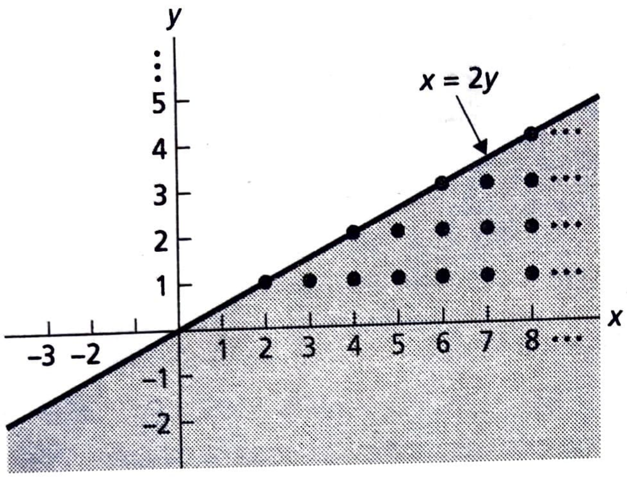
The probability is $(1-p)^8$ that both components last longer than 4 time units.

b) The event that one of the electrical components lasts at least twice as long as the other can be expressed as $\{X \geq 2Y\} \cup \{Y \geq 2X\}$. The two events in the union are mutually exclusive and, by symmetry of the PMF, have the same probability. Hence the required probability equals $2P(X \geq 2Y)$. Applying the FPF yields

$$P(X \geq 2Y) = \sum_{x \geq 2y}\sum p_{X,Y}(x, y).$$

Figure 6.2 shows the set over which the double summation is taken.

Figure 6.2



In view of Figure 6.2, we can evaluate the double sum as follows:

$$\sum_{x \geq 2y}\sum p_{X,Y}(x, y) = \sum_{y=1}^{\infty}\left(\sum_{x=2y}^{\infty} p^2(1-p)^{x+y-2}\right)$$

$$= \frac{p^2}{(1-p)^2}\sum_{y=1}^{\infty}(1-p)^y\left(\sum_{x=2y}^{\infty}(1-p)^x\right)$$

$$= \frac{p^2}{(1-p)^2}\sum_{y=1}^{\infty}(1-p)^y \cdot \frac{(1-p)^{2y}}{1-(1-p)}$$

$$= \frac{p}{(1-p)^2}\sum_{y=1}^{\infty}((1-p)^3)^y = \frac{p}{(1-p)^2}\cdot\frac{(1-p)^3}{1-(1-p)^3}$$

$$= \frac{p(1-p)}{1-(1-p)^3} = \frac{1-p}{3-3p+p^2}.$$

The probability that one of the components lasts at least twice as long as the other is $2(1-p)/(3-3p+p^2)$. ■

EXERCISES 6.1 Basic Exercises

6.1 Refer to Examples 6.2 and 6.5 on pages 263 and 266, respectively, where X and Y denote the lifetimes of two identical electrical components observed at discrete time units. The joint PMF of X and Y is

$$p_{X,Y}(x, y) = p^2(1-p)^{x+y-2}, \qquad x, y \in \mathcal{N},$$

and $p_{X,Y}(x, y) = 0$ otherwise, where $0 < p < 1$.

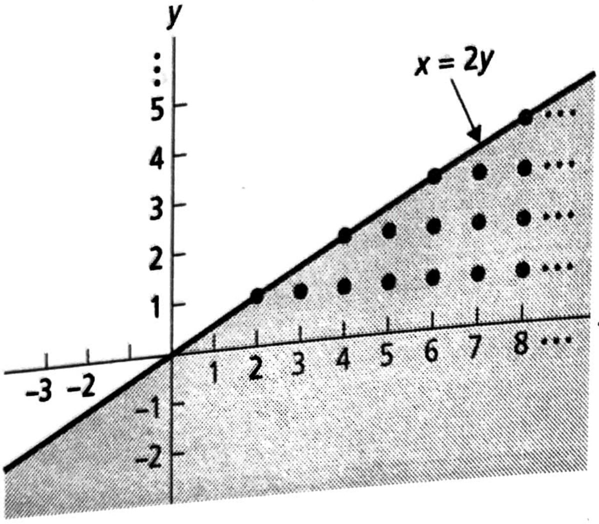
a) Determine and interpret $P(X = Y)$.

b) Use the FPF to determine $P(X > Y)$. Interpret your answer.

c) Without doing any computations, explain why $P(X > Y) = P(X < Y)$.

d) Use the results of parts (a) and (c) to obtain $P(X > Y)$. Compare your answer to that obtained in part (b).

Figure 6.2



In view of Figure 6.2, we can evaluate the double sum as follows:

$$\sum_{x \ge 2y}\sum p_{X,Y}(x,y) = \sum_{y=1}^{\infty}\left(\sum_{x=2y}^{\infty} p^2(1-p)^{x+y-2}\right)$$
$$= \frac{p^2}{(1-p)^2}\sum_{y=1}^{\infty}(1-p)^y\left(\sum_{x=2y}^{\infty}(1-p)^x\right)$$
$$= \frac{p^2}{(1-p)^2}\sum_{y=1}^{\infty}(1-p)^y \cdot \frac{(1-p)^{2y}}{1-(1-p)}$$
$$= \frac{p}{(1-p)^2}\sum_{y=1}^{\infty}\left((1-p)^3\right)^y = \frac{p}{(1-p)^2}\cdot\frac{(1-p)^3}{1-(1-p)^3}$$
$$= \frac{p(1-p)}{1-(1-p)^3} = \frac{1-p}{3-3p+p^2}.$$

The probability that one of the components lasts at least twice as long as the other is $2(1-p)/(3-3p+p^2)$. ∎

EXERCISES 6.1 **Basic Exercises**

6.1 Refer to Examples 6.2 and 6.5 on pages 263 and 266, respectively, where X and Y denote the lifetimes of two identical electrical components observed at discrete time units. The joint PMF of X and Y is

$$p_{X,Y}(x,y) = p^2(1-p)^{x+y-2}, \qquad x, y \in \mathcal{N},$$

and $p_{X,Y}(x,y) = 0$ otherwise, where $0 < p < 1$.

a) Determine and interpret $P(X = Y)$.
b) Use the FPF to determine $P(X > Y)$. Interpret your answer.
c) Without doing any computations, explain why $P(X > Y) = P(X < Y)$.
d) Use the results of parts (a) and (c) to obtain $P(X > Y)$. Compare your answer to that obtained in part (b).

h) Obtain the marginal PMF of Y by expressing Y as a function of an appropriate binomial random variable and then applying Corollary 5.1 on page 247.
i) Without doing any computations, find the PMF of $Y - X$.
j) Solve part (i) by using the FPF to obtain $P(Y - X = z)$ for each possible value z of $Y - X$.

6.11 Repeat Exercise 6.10 if the probability of a head is p.

6.12 Repeat Exercise 6.10 if X and Y represent Jan's total winnings after the third and fourth tosses, respectively, where losses are negative winnings.

6.13 Arrivals at the emergency rooms of two hospitals occur at average hourly rates of λ and μ, respectively. The joint PMF of the numbers of arrivals during any particular hour is

$$p_{X,Y}(x, y) = e^{-(\lambda+\mu)} \frac{\lambda^x \mu^y}{x!y!}, \qquad x, y = 0, 1, 2, \ldots,$$

and $p_{X,Y}(x, y) = 0$ otherwise.
a) Determine and identify the marginal PMFs of X and Y.
b) What is the relationship between the joint PMF and the marginal PMFs?

6.14 Let X and Y be discrete random variables. Show that the function $p: \mathcal{R}^2 \to \mathcal{R}$ defined by $p(x, y) = p_X(x)p_Y(y)$ is a joint PMF by verifying that it satisfies properties (a)–(c) of Proposition 6.1 on page 262. *Hint:* A subset of a countable set is countable.

6.15 Let X and Y be discrete random variables. Suppose that the joint PMF of X and Y can be factored into a function of x alone and a function of y alone; that is, there exist two real-valued functions, q and r, of one real variable such that $p_{X,Y}(x, y) = q(x)r(y)$ for all real numbers x and y.
a) Obtain the marginal PMFs of X and Y.
b) Verify that $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all real numbers x and y. Interpret this result.
c) Under what conditions are q and r the marginal PMFs of X and Y, respectively?

Theory Exercises

6.16 Prove Proposition 6.1 on page 262 by showing that the joint PMF of two discrete random variables X and Y satisfies properties (a), (b), and (c) of that proposition. *Hint:* For property (b), use the facts that the Cartesian product of two countable sets is countable (Exercise 1.41) and that a subset of a countable set is countable (Exercise 1.40).

6.17 Show that a function $p: \mathcal{R}^2 \to \mathcal{R}$ that satisfies properties (a), (b), and (c) of Proposition 6.1 is the joint PMF of some pair of discrete random variables. *Hint:* Consider the sample space $\Omega = \{ (x, y) \in \mathcal{R}^2 : p(x, y) \neq 0 \}$ and refer to Proposition 2.3 on page 43.

6.18 Prove Proposition 6.2 on page 263 by verifying Equations (6.3) and (6.4).

6.19 Prove the FPF for two discrete random variables, Proposition 6.3 on page 265.

Advanced Exercises

6.20 A number is chosen at random from the first N positive integers—call it X. Subsequently, a number is chosen at random from the first X positive integers—call it Y.
a) Obtain the joint PMF of X and Y. *Hint:* Use the general multiplication rule.
b) Obtain the marginal PMF of Y and verify that its values sum to 1.
c) Determine $P(Y = X)$.
d) Show that $P(Y = X) \sim (\ln N)/N$ as $N \to \infty$, where "$\sim$" means that the limit of the ratio of the two sides is 1. *Hint:* Use the fact that $\sum_{n=1}^{N} n^{-1} \sim \ln N$ as $N \to \infty$.

d) Without doing any computations, identify and obtain the PMF of each of the three random variables $X + Y$, $X + Z$, and $Y + Z$.
e) Without doing any computations, identify and obtain the joint PMF of $X + Y$ and Z.
f) Use the FPF and the result of part (a) to find $P(1 \leq X + Z \leq 2)$.
g) Solve part (f) by using the result of part (d).
h) Without doing any computations, obtain the PMF of $X + Y + Z$.

6.26 In Example 5.1 on page 176, we considered the number of siblings for each of 40 students in one of Professor Weiss's classes. That class contains 6 freshmen, 15 sophomores, 12 juniors, and 7 seniors. Suppose that a random sample of 5 students is taken with replacement. Let X_1, X_2, X_3, and X_4 denote the number of freshmen, sophomores, juniors, and seniors, respectively, obtained.
a) Identify and determine the joint PMF of X_1, X_2, X_3, and X_4.
Use part (a) and the FPF to determine the probability that the sample contains
b) no freshmen, 3 sophomores, 1 junior, and 1 senior.
c) at least 3 sophomores.
d) at least 3 sophomores and at least 1 junior.

6.27 A balanced die is tossed 18 times. Determine the probability that each face appears
a) exactly 3 times.
b) at most 3 times.
c) at least 3 times.

6.28 Items from an assembly line are classified as either good or defective. On average, the assembly line produces 3% defective items. Two-thirds of the defective items are salvageable and can be repaired, but the other defective items must be scrapped. If 50 items are sampled from a week's output, what is the probability that the sample contains 48 good items and 1 each of salvageable and scrapped items?

6.29 Let $X_1, \ldots, X_5$ have the multinomial distribution with parameters n and $p_1, \ldots, p_5$. Without doing any computations, identify and determine the PMFs of each of the following random variables. Explain your reasoning.
a) X_2 **b)** $X_3 + X_5$ **c)** $X_1 + X_4 + X_5$
Without doing any computations, identify and determine the joint PMFs of each of the following collections of random variables. Explain your reasoning.
d) $X_1 + X_2$ and $X_3 + X_4 + X_5$
e) X_1, $X_2 + X_3$, and $X_4 + X_5$
f) X_1, $X_2 + X_3 + X_4$, and X_5

6.30 Let q and r be two nonnegative real numbers whose sum is less than 1 and let n be a positive integer. Verify the identity

$$\binom{n}{x} = \frac{1}{(q+r)^{n-x}} \sum_{y=0}^{n-x} \binom{n}{x,\ y,\ n-x-y} q^y r^{n-x-y}, \qquad x = 0, 1, \ldots, n,$$

by using
a) the binomial theorem.
b) a probabilistic argument based on the multinomial distribution.

6.31 Multiple hypergeometric distribution: Consider a finite population of size N in which each member is classified as having one of m mutually exclusive and exhaustive attributes, say, $a_1, \ldots, a_m$. For each $j = 1, \ldots, m$, let p_j denote the proportion of the population that has attribute a_j. Suppose that a random sample of size n is taken without replacement.

6.3 Basic Exercises

Note: Several of the exercises in this section are continuations of exercises presented in Sections 6.1 and 6.2.

6.44 Refer to Example 6.12 on page 283.

a) Obtain the conditional PMF of the number of bedrooms for each number of bathrooms. Construct a table similar to Table 6.6 on page 284.

b) Given that a randomly selected home (from among the 50 homes) has two bathrooms, what is the probability that it has at least three bedrooms?

c) Interpret your answer in part (b) in terms of percentages.

6.45 In Exercise 6.2 on page 269, you considered the number of siblings, X, and the number of sisters, Y, for a randomly selected student from one of Professor Weiss's classes.

a) For each number of siblings, determine and interpret the conditional PMF of the number of sisters.

b) Construct a table like Table 6.6 on page 284 that displays the marginal PMF of the number of sisters and the conditional PMF of the number of sisters for each number of siblings.

c) What percentage of students in the class with two siblings have at least one sister?

d) For each number of sisters, determine and interpret the conditional PMF of the number of siblings.

e) Construct a table that displays the marginal PMF of the number of siblings and the conditional PMF of the number of siblings for each number of sisters.

f) What percentage of students in the class with one sister have at least two siblings?

6.46 Let X be a number chosen according to the discrete uniform distribution on the set S of 10 decimal digits and let Y be a number chosen according to the discrete uniform distribution on S with X removed.

a) Determine and identify the conditional PMF of Y given $X = x$.

b) Determine and identify the conditional PMF of X given $Y = y$.

c) Find $P(3 \leq X \leq 4 \mid Y = 2)$.

6.47 Let E and F be events of a sample space.

a) Determine the conditional PMF of I_F given $I_E = x$ for each possible value x of I_E. Construct a table similar to Table 6.6 on page 284 that displays these conditional PMFs and the marginal PMF of I_F.

b) In this special case of indicator random variables, what result guarantees that the values in each row of your table in part (a) sum to 1?

6.48 Let X and Y denote the smaller and larger of the two faces, respectively, when a balanced die is tossed twice.

a) Determine each conditional PMF of Y given $X = x$ and construct a table similar to Table 6.6 on page 284 that displays these conditional PMFs and the marginal PMF of Y.

b) Repeat part (a) for the conditional PMFs of X given $Y = y$ and the marginal PMF of X.

6.49 Consider a sequence of Bernoulli trials with success probability p. Let X denote the number of trials up to and including the first success and let Y denote the number of trials up to and including the second success.

a) Determine the conditional PMF of Y given $X = x$ without doing any computations. Explain your reasoning.

b) Determine the conditional PMF of Y given $X = x$ by referring to Exercise 6.9 on page 270 and applying Definition 6.3 on page 283. Compare your result with that in part (a).

c) Determine and identify the conditional PMF of X given $Y = y$ without doing any computations but, rather, by using a symmetry argument.

d) Determine the conditional PMF of X given $Y = y$ by referring to Exercise 6.9 on page 270 and applying Definition 6.3 on page 283. Compare your result with that in part (c).

e) Determine and identify the conditional PMF of $Y - X$ given $X = x$ without doing any computations. Explain your reasoning.

f) Determine the conditional PMF of $Y - X$ given $X = x$ by using the conditional probability rule and referring to Exercise 6.9 on page 270.

6.50 Arrivals at the emergency rooms of two hospitals occur at average hourly rates of λ and μ, respectively. The joint PMF of the numbers of arrivals during any particular hour is

$$p_{X,Y}(x, y) = e^{-(\lambda+\mu)} \frac{\lambda^x \mu^y}{x!y!}, \qquad x, y = 0, 1, 2, \ldots,$$

and $p_{X,Y}(x, y) = 0$ otherwise.

a) Determine and identify each conditional distribution of Y given $X = x$.

b) Determine and identify each conditional distribution of X given $Y = y$.

c) What is the relationship between each conditional distribution of Y given $X = x$ and the marginal distribution of Y?

d) What is the relationship between each conditional distribution of X given $Y = y$ and the marginal distribution of X?

6.51 Suppose that X and Y are discrete random variables such that their joint PDF equals the product of their marginal PDFs.

a) Obtain the conditional PMF of Y given $X = x$ and compare it to the marginal PMF of Y.

b) Obtain the conditional PMF of X given $Y = y$ and compare it to the marginal PMF of X.

6.52 Refer to the automobile insurance illustration of Example 6.15 on page 287.

a) Determine the conditional joint PMF of X and Z given $Y = 1$.

b) Given that $Y = 1$ during a particular year, what is the probability that all the family's automobile-accident losses will be reimbursed?

c) Determine the conditional PMF of Z given $X = 0$ and $Y = 1$.

d) Given that $X = 0$ and $Y = 1$ during a particular year, what is the probability that all the family's automobile-accident losses will be reimbursed?

6.53 Let X, Y, and Z have the multinomial distribution with parameters n and p, q, r.

a) Obtain and identify the conditional joint PMF of Y and Z given $X = x$.

b) Obtain the conditional PMF of Z given $X = x$ and $Y = y$.

6.54 Let $X_1, \ldots, X_m$ have the multinomial distribution with parameters n and $p_1, \ldots, p_m$.

a) Determine and identify the conditional distribution of $X_1, \ldots, X_{m-1}$ given $X_m = x_m$.

b) Now refer to the roulette illustration of Example 6.10 on page 278. Apply the result of part (a) to determine and identify the conditional distribution of the number of times that red and black occur in 10 plays at the roulette wheel, given that green occurs once.

6.55 Refer to Exercise 6.31 on page 280. Suppose that $X_1, \ldots, X_m$ have the multiple hypergeometric distribution with parameters N, n, and $p_1, \ldots, p_m$.

a) Determine and identify the conditional distribution of $X_1, \ldots, X_{m-1}$ given $X_m = x_m$.

b) Now refer to Exercise 6.33 on page 281. Apply the result of part (a) to determine and identify the conditional distribution of the number of freshmen, sophomores, and juniors obtained, given that exactly one senior is obtained.

6.4 Basic Exercises

Note: Several of the exercises in this section are continuations of exercises presented in Sections 6.1–6.3.

6.60 Let X and Y be discrete random variables defined on the same sample space. Suppose that the joint PMF of X and Y can be factored into a function of x alone and a function of y alone—that is, suppose that there exist two real-valued functions q and r of one real variable such that $p_{X,Y}(x, y) = q(x)r(y)$ for all real numbers x and y.

a) Show that X and Y are independent random variables.

b) Is it necessarily true that q and r are the marginal PMFs of X and Y, respectively? If not, under what conditions are they?

c) Generalize to the multivariate case of m discrete random variables $X_1, \ldots, X_m$.

6.61 In Exercise 6.2 on page 269, you were asked to determine the joint and marginal PMFs of the number of siblings (X) and the number of sisters (Y) of a randomly selected student from among 40 students in one of Professor Weiss's classes. Are X and Y independent random variables? Explain your answer.

6.62 Let S consist of the 10 decimal digits. Suppose that a number X is chosen according to the discrete uniform distribution on S and then a number Y is chosen according to the discrete uniform distribution on S with X removed. Are X and Y independent random variables? Explain your answer.

6.63 A balanced die is tossed twice. Let X and Y denote the smaller and larger of the two faces, respectively. Are X and Y independent random variables? Explain your answer.

6.64 Refer to Exercise 6.9 on page 270, where, in a sequence of Bernoulli trials with success probability p, we let X denote the number of trials up to and including the first success and we let Y denote the number of trials up to and including the second success.

a) Without doing any computations, explain why X and Y aren't independent.

b) Use Proposition 6.10 on page 292 to show that X and Y aren't independent.

c) Use Proposition 6.11 on page 295 to show that X and Y aren't independent.

d) Without doing any computations, explain why X and $Y - X$ are independent.

e) Use Proposition 6.10 to show that X and $Y - X$ are independent.

f) Use Proposition 6.11 to show that X and $Y - X$ are independent.

6.65 Suppose that $X_1, \ldots, X_m$ are random variables defined on the same sample space. Further suppose that some two of the m random variables aren't independent. Is it possible that $X_1, \ldots, X_m$ are independent random variables? Explain your answer.

6.66 Refer to Exercise 6.26 on page 280. There, X_1, X_2, X_3, and X_4 denote the number of freshmen, sophomores, juniors, and seniors, respectively, obtained when a random sample of 5 students is taken with replacement from the 40 students in one of Professor Weiss's classes in which there are 6 freshmen, 15 sophomores, 12 juniors, and 7 seniors.

a) Without doing any computations, explain why X_1 and X_2 aren't independent.

b) Deduce from part (a) that X_1, X_2, X_3, and X_4 aren't independent.

6.67 Repeat Exercise 6.66 if the sampling is without replacement.

6.68 Let $X_1, \ldots, X_m$ have the multinomial distribution with parameters n and $p_1, \ldots, p_m$. Without doing any computations, explain why these random variables aren't independent.

6.69 A point (X, Y) is chosen randomly from $\{\, (x, y) : x, y \in \{0, 1, \ldots, 9\} \,\}$. Are X and Y independent random variables? Explain your answer.

6.70 A point (X, Y) is chosen randomly from $\{(x, y) : x, y \in \{0, 1, \ldots, 9\} \text{ and } x \geq y\}$.
a) Without doing any computations, explain why X and Y aren't independent.
b) Use Proposition 6.10 on page 292 to show that X and Y aren't independent.
c) Use Proposition 6.11 on page 295 to show that X and Y aren't independent.

6.71 A university gives separate placement exams in mathematics and verbal skills to incoming students. To the nearest minute, the time it takes a student to complete the mathematics exam has a geometric distribution with parameter 0.020, and the time it takes a student to complete the verbal exam has a geometric distribution with parameter 0.025. Assume that completion times for the two exams are independent random variables.
a) Determine the probability that a student takes longer to complete the mathematics exam than the verbal exam.
b) Determine the probability that a student takes at most twice as long to complete the mathematics exam as the verbal exam.
c) Determine the PMF of the total time that a student takes to complete both exams.
d) If students are allowed a total of 2 hours to complete both exams, what percentage finish in time?

6.72 Suppose that $X_1, \ldots, X_7$ are independent random variables and that $t \in \mathcal{R}$. In each part, decide whether the random variables are necessarily independent and explain your answers.
a) $\sqrt{X_1^2 + X_2^2 + X_3^2}$ and $\sqrt{X_4^2 + X_5^2 + X_6^2}$
b) $\sin(X_1 X_2 X_3)$ and $\cos(X_3 X_5)$
c) $\sin(X_1 X_2 X_5)$ and $\cos(X_3 X_4)$
d) $X_1 - X_2$ and $X_1 + X_2$
e) $\min\{X_1, X_2, X_7\}$ and $\max\{X_3, X_4\}$
f) $\sum_{k=1}^{5} X_k \sin(kt)$ and $\sum_{k=6}^{7} X_k \sin(kt)$

6.73 At a large hospital, an average of 6.9 patients arrive at the hospital emergency room each hour. The number who arrive during any particular hour has a Poisson distribution. At a smaller hospital, an average of 2.6 patients arrive at the hospital emergency room each hour. The number who arrive during any particular hour has a Poisson distribution. Assume that the number of patients arriving at the two emergency rooms are independent.
a) Determine and identify the PMF of the total number of patients that arrive each hour at both emergency rooms combined. *Hint:* Use the FPF and the binomial theorem.
b) On average, how many patients per hour arrive at the two hospital emergency rooms combined? Explain your answer.
c) Use your answer from part (a) to determine the probability that, during a one-hour period, at most eight patients arrive at the two hospital emergency rooms combined.

6.74 The number of people entering a bank during a specified time interval has a Poisson distribution with parameter λ. Each person entering makes a deposit with probability p, independent of the other people entering and the number of people entering. Determine and identify the PMF of the number of people making a deposit during the specified time interval.

6.75 The number of people entering a bank during a specified time interval has a Poisson distribution with parameter λ. Each arriving person is classified by type as follows.

- Type 1: drives to the bank and walks in
- Type 2: drives to the bank and uses the drive-thru facility
- Type 3: walks to the bank

An arriving person has probability p_i of being Type i, independent of other arrivals and the number of arrivals. Let X_i denote the number of Type i people that arrive during the specified time interval. Show that X_1, X_2, and X_3 are independent Poisson random variables with parameters λp_1, λp_2, and λp_3, respectively.

b) difference between the number of bedrooms and the number of bathrooms.
c) maximum of the number of bedrooms and the number of bathrooms.
d) minimum of the number of bedrooms and the number of bathrooms.

6.86 In Exercise 6.2 on page 269, you were asked to determine the joint PMF of the number of siblings, X, and the number of sisters, Y, of a randomly selected student from among 40 students in one of Professor Weiss's classes.
a) Determine the PMF of $X - Y$.
b) What does the random variable $X - Y$ represent?

6.87 A balanced die is tossed twice. Let X and Y denote the smaller and larger of the two faces, respectively.
a) Determine the PMF of $Y - X$. *Note:* You were asked to obtain the joint PMF of X and Y in Exercise 6.6 on page 270.
b) What does the random variable $Y - X$ represent?

6.88 Two people agree to meet at a specified place between 3:00 P.M. and 4:00 P.M. Suppose that you measure time to the nearest minute relative to 3:00 P.M. so that, for instance, time 40 represents 3:40 P.M. Further suppose that each person arrives according to the discrete uniform distribution on $\{0, 1, \ldots, 60\}$ and that the two arrival times are independent.
a) Determine the PMF of the number of minutes that the first person to arrive waits for the second person to arrive.
b) Use the result of part (a) to find the probability that the first person to arrive waits no longer than 10 minutes for the second person to arrive.
c) Determine the PMF of the time of the first person to arrive.
d) Determine the PMF of the time of the second person to arrive.

6.89 Suppose that X and Y are independent random variables, each having the discrete uniform distribution on $\{0, 1, \ldots, N\}$. Determine the PMF of the magnitude of the difference between X and Y by using
a) combinatorial analysis. b) Proposition 6.16.

6.90 Let X and Y be as in Exercise 6.89. Determine the PMF of the
a) sum of X and Y. b) minimum of X and Y. c) maximum of X and Y.

6.91 Let X and Y be independent random variables that have geometric distributions with parameters p and q, respectively. Find the PMF of $\min\{X, Y\}$
a) directly, as in Example 6.22(b) on page 304. b) by using tail probabilities.
c) Verify that your answers in parts (a) and (b) agree with that in Example 6.22(b) in the case where $p = q$.

Theory Exercises

6.92 Prove Proposition 6.17 on page 305, which provides the PMF of a real-valued function of $X_1, \ldots, X_m$ in terms of the joint PMF of $X_1, \ldots, X_m$.

6.93 Suppose that X and Y are discrete random variables defined on the same sample space. Use Proposition 6.16 on page 302 to derive the following formulas for the sum and difference of X and Y.
a) $p_{X+Y}(z) = \sum_x p_{X,Y}(x, z - x)$ b) $p_{Y-X}(z) = \sum_x p_{X,Y}(x, x + z)$
c) Specialize the formulas in parts (a) and (b) to the case where X and Y are independent random variables.

6.94 Derive the formulas in Exercise 6.93 by applying the law of partitions, Proposition 2.8 on page 68.

Basic Exercises

6.100 In Example 6.1, we considered the number of bedrooms and number of bathrooms of a randomly selected home among 50 homes currently for sale. The joint PMF of those two random variables is provided in Table 6.2 on page 262. Determine the PMF of the total number of bedrooms and bathrooms by applying Equation (6.30) on page 307.

6.101 A balanced die is tossed twice. Let X and Y denote the smaller and larger of the two faces, respectively. You were asked to obtain the joint PMF of X and Y in Exercise 6.6 on page 270.

a) Interpret the random variable $M = (X + Y)/2$.

b) Determine the PMF of M by first applying Proposition 6.18 on page 307 and then Corollary 5.1 on page 247.

6.102 Suppose that X and Y are independent random variables each having the discrete uniform distribution on $\{0, 1, \ldots, N\}$. Use Proposition 6.19 on page 309 to obtain the PMF of $X + Y$. Compare your result with that obtained in Exercise 6.90(a) on page 306.

6.103 Is the sum of two independent geometric random variables with the same success-probability parameter a geometric random variable? If not, what is its distribution?

6.104 During any particular week, the probability is $2^{-(n+1)}$ that an insurance agency receives n claims ($n = 0, 1, \ldots$). Furthermore, the numbers of claims received are independent from one week to the next. Determine the PMF of the total number of claims received in two consecutive weeks by using Proposition 6.19 on page 309.

6.105 In this exercise, you are asked to solve Exercise 6.104 without using Proposition 6.19, but by proceeding as follows. Let X and Y denote the number of claims received in weeks one and two, respectively.

a) Identify the common PMF of $X + 1$ and $Y + 1$.

b) Without doing any computations, use your result from part (a) to identify the PMF of the random variable $X + Y + 2$.

c) Use your result from part (b) and Corollary 5.1 on page 247 to obtain the PMF of $X + Y$, the total number of claims received in two consecutive weeks.

6.106 Interpret each of parts (a), (b), and (c) of Proposition 6.20 on page 311 in words.

6.107 Suppose that $X_1, X_2, \ldots, X_r$ are independent random variables each having the geometric distribution with parameter p. Without doing any computations, identify the probability distribution of $X_1 + X_2 + \cdots + X_r$.

6.108 Let X and Y be independent random variables with $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$. Determine and identify the conditional PMF of X given $X + Y = z$.

6.109 Let X and Y be independent random variables with $X \sim \mathcal{NB}(r, p)$ and $Y \sim \mathcal{NB}(s, p)$.

a) Determine the conditional PMF of X given $X + Y = z$.

b) Without doing any computations, use the result of part (a) to obtain the combinatorial identity

$$\sum_{x=r}^{z-s} \binom{x-1}{r-1}\binom{z-x-1}{s-1} = \binom{z-1}{r+s-1},$$

for $z = r + s, r + s + 1, \ldots$.

c) The combinatorial identity in part (b) can be obtained without using probability. Use Proposition 6.19 and that identity to prove that $X + Y \sim \mathcal{NB}(r + s, p)$.