3.31 An economics professor is using a new method to teach a junior-level course with an enrollment of 42 students. The professor wants to conduct in-depth interviews with the students to get feedback on the new teaching method but doesn't want to interview all 42 of them. She decides to interview a sample of 5 students from the class. How many different samples are possible?

3.32 The Powerball® is a multistate lottery that was introduced in April 1992 and is now sold in 24 states, the District of Columbia, and the Virgin Islands. To play the game, a player first selects five numbers from the numbers 1–53 and then chooses a Powerball number, which can be any number between 1 and 42, inclusive. How many possibilities are there?

3.33 In the game of *keno,* there are 80 balls, numbered 1–80. From these 80 balls, 10 are selected at random.

a) How many different outcomes are possible?

b) If a player specifies 20 numbers, in how many ways can he get all 10 numbers selected?

3.34 A club has 14 members.

a) How many ways can a governing committee of size 3 be chosen?

b) How many ways can a president, vice president, and treasurer be chosen?

c) How many ways can a president, vice president, and treasurer be chosen if two specified club members refuse to serve together?

3.35 How many license plates are there consisting of three digits and three letters if there is no restriction on where the digits and letters are placed?

3.36 A five-card draw poker hand consists of 5 cards dealt from an ordinary deck of 52 playing cards. The order in which the cards are received is unimportant. Note that, in sequence, an ace can play as either the lowest or highest card. In other words, the hierarchy of card denominations, from lowest to highest, is ace, 2, 3, . . . , 10, jack, queen, king, ace. Determine the number of possible hands of the specified type.

a) Straight flush: five cards of the same suit in sequence

b) Four of a kind: $\{w, w, w, w, x\}$, where w and x are distinct denominations

c) Full house: $\{w, w, w, x, x\}$, where w and x are distinct denominations

d) Flush: five cards of the same suit, not all in sequence

e) Straight: five cards in sequence, not all of the same suit

f) Three of a kind: $\{w, w, w, x, y\}$, where w, x, and y are distinct denominations

g) Two pair: $\{w, w, x, x, y\}$, where w, x, and y are distinct denominations

h) One pair: $\{w, w, x, y, z\}$, where w, x, y, and z are distinct denominations

3.37 Repeat Exercise 3.36 for the game of five-card stud, where the order in which the cards are received matters.

3.38 Refer to the inclusion–exclusion principle, Proposition 2.10 on page 73. Determine the number of summands in each sum.

3.39 The U.S. Senate consists of 100 senators, 2 from each state. A committee consisting of 5 senators is to be formed.

a) How many different committees are possible?

b) How many are possible if no state can have more than 1 senator on the committee?

3.40 Refer to Example 3.17 on page 105. How many possible results are there in which the United States has exactly two finishers in the top three and one in the bottom three?

3.41 Without doing any calculations, explain why

a) $\binom{n}{k} = \binom{n}{n-k}$. **b)** $\binom{n}{k} = \binom{n}{k,\, n-k}$.

3.3 Basic Exercises

3.56 Provide the details for the solution of Example 3.18(b) on page 111.

3.57 Four cards are dealt from an ordinary deck of 52 playing cards. What is the probability that the denominations (face values) of the cards are

a) all the same? **b)** all different?

3.58 In a small lottery, 10 tickets—numbered $1, 2, \ldots, 10$—are sold. Two numbers are drawn at random for prizes. You hold tickets numbered 1 and 2. What is the probability that you win at least one prize?

3.59 An ordinary deck of 52 playing cards is shuffled and dealt. What is the probability that

a) the seventh card dealt is an ace? **b)** the first ace occurs on the seventh card dealt?

3.60 From an urn containing M red balls and $N - M$ black balls, a random sample of size n is taken without replacement. Find the probability that exactly j black balls are in the sample.

3.61 The birthday problem: A probability class has 38 students.

a) Find the probability that at least 2 students in the class have the same birthday. For simplicity, assume that there are always 365 days in a year and that birth rates are constant throughout the year. *Hint:* Use the complementation rule.

b) Repeat part (a) if the class has N students.

c) Evaluate the probability obtained in part (b) for $N = 1, 2, \ldots, 70$. Use a computer or calculator to do the number crunching.

d) What is the smallest class size for which the probability that at least 2 students in the class have the same birthday exceeds 0.5?

3.62 An urn contains four red balls and six black balls. Balls are drawn one at a time at random until three red balls have been drawn. Determine the probability that a total of seven balls is drawn if the sampling is

a) without replacement. **b)** with replacement.

3.63 Four mathematicians, three chemists, and five physicists are seated randomly in a row. Find the probability that all the members of each discipline sit together.

3.64 Suppose that a random sample of size n without replacement is taken from a population of size N. For $k = 1, 2, \ldots, n$, determine the probability that k specified members of the population will be included in the sample.

3.65 Suppose that a random sample of size n with replacement is taken from a population of size N.

a) Determine the probability that no member of the population is selected more than once.

b) Show that the probability in part (a) approaches 1 as $N \to \infty$. Interpret this result.

3.66 Refer to Example 3.19 on page 111. Determine the probability that the number of defective TVs selected is

a) exactly one. **b)** at most one. **c)** at least one.

3.67 Refer to Example 3.20(b) on page 113. In this exercise, you are to obtain $P(E)$—the probability that a specified member of the population will be included in the sample—in two additional ways.

a) Compute $N(E)$ directly and then apply Equation (3.4) on page 110 to determine $P(E)$.

b) For $k = 1, 2, \ldots, n$, let A_k denote the event that the kth member selected is the specified member. Without doing any computations, explain why $P(A_k) = 1/N$, for $k = 1, 2, \ldots, n$. Conclude that $P(E) = n/N$. Explain your reasoning.

f) Interpret your answers in parts (a)–(e) in terms of percentages.
g) Construct a table for the conditional probability distributions of age by living arrangement and the marginal probability distribution of age.
h) Construct a table for the conditional probability distributions of living arrangement by age and the marginal probability distribution of living arrangement.

4.9 Four cards are dealt at random without replacement from an ordinary deck of 52 cards.
a) If it is known that the four cards have different face values, what is the probability that the first card is a spade?
b) If it is known that the four cards come from different suits, what is the probability that the first card is a face card?

4.10 A balanced die is tossed 12 times. Given that a 3 occurs at least once, what is the probability that it occurs four times or more?

4.11 A king has two children. What is the probability that both children are boys, given that
a) the first child born is a boy? b) at least one child is a boy?

4.12 A balanced die is rolled until the first 6 occurs. If that happens on the eighth toss, what is the probability that there are exactly two 4s among the first seven tosses?

4.13 As reported by the U.S. Federal Bureau of Investigation (FBI) in *Crime in the United States*, 4.9% of property crimes are committed in rural areas and 1.9% of property crimes are burglaries committed in rural areas. What percentage of property crimes committed in rural areas are burglaries?

4.14 A balanced coin is tossed eight times. Find the probability that
a) no heads appear on the first four tosses.
b) no heads appear on the first four tosses, given that exactly three heads appear in all eight tosses.
c) at least one head appears in the first four tosses, given that exactly three heads appear in all eight tosses.

4.15 A dart is thrown randomly at the unit square $\{ (x, y) : 0 \le x \le 1,\ 0 \le y \le 1 \}$, and the point at which it hits is observed.
a) Determine the probability that the x coordinate of the point exceeds 1/2.
b) Determine the probability that the x coordinate of the point exceeds its y coordinate.
c) Determine the conditional probability that the x coordinate of the point exceeds its y coordinate, given that the x coordinate of the point exceeds 1/2.

4.16 Repeat Exercise 4.15 if the dart is thrown at the unit disk, $\{ (x, y) : x^2 + y^2 < 1 \}$, instead of the unit square.

Theory Exercises

4.17 Let A and B be events of a sample space with positive probability. Prove that
a) $P(B \mid A) > P(B)$ if and only $P(A \mid B) > P(A)$. (In this case, events A and B are said to be *positively correlated.*)
b) $P(B \mid A) < P(B)$ if and only $P(A \mid B) < P(A)$. (In this case, events A and B are said to be *negatively correlated.*)
c) $P(B \mid A) = P(B)$ if and only $P(A \mid B) = P(A)$. (In this case, events A and B are said to be *independent.*)
d) Interpret the results in parts (a)–(c).

4.18 Prove Proposition 4.1 on page 134 by showing that $P_A(\cdot) = P(\cdot \mid A)$ is a probability measure—that is, prove that $P_A(\cdot)$ satisfies the three Kolmogorov axioms.

4.19 Let A, B, and C be events of a sample space and assume that $A \cap B$ has positive probability. Prove that

$$P(C \mid A \cap B) = \frac{P(B \cap C \mid A)}{P(B \mid A)}.$$

4.20 Let A and B be events with $P(A \cap B) > 0$. Prove that $P_A(C \mid B) = P(C \mid A \cap B)$ for each event C. Interpret this result. *Hint:* Use Exercise 4.19.

Advanced Exercises

4.21 Suppose that the inhabitants of a certain community are classified according to their religion as Religion 1 (R_1), Religion 2 (R_2), or Religion 3 (R_3), and according to their occupation as white-collar (W), blue-collar (B), or other (O). Assume that the conditional probability distributions of occupation by religion are as follows.

$$\begin{array}{lll} P(W \mid R_1) = 0.12 & P(W \mid R_2) = 0.18 & P(W \mid R_3) = 0.13 \\ P(B \mid R_1) = 0.81 & P(B \mid R_2) = 0.72 & P(B \mid R_3) = 0.75 \\ P(O \mid R_1) = 0.07 & P(O \mid R_2) = 0.10 & P(O \mid R_3) = 0.12 \end{array}$$

Assume in addition that the marginal probability distribution of religion is $P(R_1) = 0.35$, $P(R_2) = 0.60$, and $P(R_3) = 0.05$.

a) Use the information provided to determine the nine joint probabilities for religion and occupation. *Hint:* Algebraically manipulate the conditional probability rule.
b) Find the marginal probability distribution of occupation.
c) Obtain the conditional probability distributions of religion by occupation and construct a table similar to Table 4.2 on page 130.
d) If you are told that a randomly chosen member of this community is a white-collar worker, is it more likely or less likely that the person is of Religion 1 than it would be if you didn't know that person's occupational classification? Explain.

4.22 Each student is allowed two attempts to pass an exam. Experience shows that 60% of the students pass on the first try and that, for those who don't, 80% pass on the second try.
a) What is the probability that a student passes the exam?
b) If a student passed, what is the probability that he or she passed on the first attempt?

4.2 The General Multiplication Rule

As we demonstrated in Section 4.1, the conditional probability rule is used to obtain conditional probabilities in terms of unconditional probabilities:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}. \tag{4.1}$$

More precisely, the conditional probability rule provides a formula for computing conditional probabilities in terms of joint probabilities and marginal probabilities.

Multiplying both sides of Equation (4.1) by $P(A)$, we obtain a formula, called the **general multiplication rule,** for computing joint probabilities in terms of marginal and

4.2 Basic Exercises

4.23 An article in *Science News* (2000, Vol. 157, p. 135), reported on research by InterSurvey and the Quantitative Study of Society on the effects of regular Internet usage. According to the article, 36% of Americans with Internet access are regular Internet users, meaning that they log on for at least 5 hours per week. Among regular Internet users, 25% say that the Web has reduced their social contacts.

a) Determine the probability that a randomly selected American with Internet access is a regular Internet user who feels that the Web has reduced his or her social contact.

b) Interpret your answer in part (a) in terms of percentages.

4.24 A woman has agreed to participate in an ESP experiment. She's asked to pick, randomly, two numbers between 1 and 6. The second number must be different from the first. Determine the probability that

a) the first number picked is 3 and the second number picked exceeds 4.

b) both numbers picked are less than 3; greater than 3.

4.25 Refer to Table 1.1 on page 4, which gives the regions of the states in the United States. Suppose that two U.S. states are selected at random without replacement.

a) Find the probability that the first is in the Northeast and the second is in the West.

b) Find the probability that both are in the South.

c) Draw a tree diagram for this problem similar to the one shown in Figure 4.1 on page 141.

d) What is the probability that the two states selected are in the same region?

e) What is the probability that one of the states selected is in the Midwest and the other is in the West?

4.26 Suppose that, in Exercise 4.25, three states (instead of two) are selected at random without replacement.

a) Find the probability that the first is in the Northeast and the second two are in the South.

b) Find the probability that all three are in the West.

c) Without drawing a tree diagram for this problem, determine how many branches it would have.

d) What is the probability that the three states selected are all from different regions?

e) What is the probability that two of the states selected are in the South and one is in the Midwest?

4.27 Students are given three chances to pass a basic skills exam for permission to enroll in Calculus I. Sixty percent of the students pass on the first try; of those that fail on the first try, 54% pass on the second try; and, of those remaining, 48% pass on the third try.

a) What is the probability that a student passes on the second try?

b) What is the probability that a student passes on the third try?

c) What percentage of students pass?

4.28 Suppose that you have a key ring with N keys, exactly one of which is your house key. Further suppose that you get home after dark and can't see the keys on the key ring. You randomly try one key at a time until you get the correct key, being careful not to mix the keys you have already tried with the ones you haven't. Let n be an integer between 1 and N, inclusive. Determine the probability that you get the correct key on the nth try by using

a) combinatorial analysis.

b) the general multiplication rule.

c) a symmetry argument.

4.29 The National Sporting Goods Association collects and publishes data on participation in selected sports activities. For Americans 7 years old or older, 17.4% of males and 4.5% of females play golf. And, according to the U.S. Census Bureau's *Current Population Reports,* of Americans 7 years old or older, 48.6% are male and 51.4% are female. From among those Americans who are 7 years old or older, one is selected at random. Find the probability that the person selected plays golf. Interpret your answer in terms of percentages.

4.30 A survey conducted by TELENATION/Market Facts, Inc., combined with information from the U.S. Census Bureau's *Current Population Reports,* yielded the following table. The first two columns provide a percentage distribution of adults by age group; the third column gives the percentage of people in each age group who go to the movies at least once a month—people referred to here as *moviegoers.* Find the percentage of adult moviegoers.

Age	Percentage of adults	Percentage moviegoers
18–24	12.7	83
25–34	20.7	54
35–44	22.0	43
45–54	16.5	37
55–64	10.9	27
65 & over	17.2	20

4.31 A balanced die is rolled and then a balanced coin is tossed the number of times that the die shows. Find the probability of obtaining exactly two heads.

4.32 Refer to Pólya's urn scheme on page 141. Determine the probability that
a) the second ball drawn is red.
b) the third ball drawn is red.
c) the first ball drawn is red, given that the second ball drawn is red.
d) the second ball drawn is black and the third ball drawn is red.

4.33 Urn I contains two white balls and one black ball; Urn II contains one white ball and five black balls. A ball is randomly chosen from Urn I and placed in Urn II. If a ball is then randomly chosen from Urn II, what is the probability it is black?

4.34 Choose a number at random from $\{0, 1, 2, \ldots, 9\}$, call it j. Next, choose a number at random from $\{j, j+1, \ldots, 9\}$. Find the probability that the second number chosen is k, where $k = 0, 1, \ldots, 9$.

4.35 In the student-selection process in Example 4.8 on page 139, find the conditional probability that the first student is female, given that the second is male.

Theory Exercises

4.36 Prove Equation (4.5) on page 138 by using mathematical induction.

Advanced Exercises

4.37 Research by B. Hatchwell et al. on divorce rates among the long-tailed tit (*Aegithalos caudatus*) appeared in *Science News* (2000, Vol. 157, No. 20, p. 317). Tracking birds in Yorkshire from one breeding season to the next, the researchers noted that 63% of pairs

Mutually Exclusive Versus Independent Events

The terms *mutually exclusive* and *independent* refer to different concepts. Mutually exclusive events are those that can't occur simultaneously. Independent events are those for which the occurrence of some doesn't affect the probabilities of the others occurring. If two or more events are mutually exclusive, the occurrence of one precludes the occurrence of the others. Hence, two or more events with positive probabilities can't be both mutually exclusive and independent. See Exercise 4.55 for more on this issue.

EXERCISES 4.3 Basic Exercises

4.43 Verify the following statements made on page 148.

a) If event B is independent of event A in the sense of Definition 4.3, then the two events are independent in the sense of Definition 4.4.

b) If $P(A) > 0$ and events A and B are independent in the sense of Definition 4.4, then event B is independent of event A in the sense of Definition 4.3.

4.44 The U.S. National Center for Health Statistics compiles data on injuries and publishes the information in *Vital and Health Statistics.* A contingency table for injuries in the United States, by circumstance and sex, is as follows. Frequencies are in millions.

Sex	Circumstance: Work C_1	Home C_2	Other C_3	Total
Male S_1	8.0	9.8	17.8	35.6
Female S_2	1.3	11.6	12.9	25.8
Total	9.3	21.4	30.7	61.4

a) Are events C_1 and S_2 independent? Explain.

b) Is the event that an injured person is male independent of the event that an injured person was hurt at home? Explain.

4.45 Refer to the joint probability distribution in Exercise 4.8 on page 135 for living arrangement and age of U.S. citizens 15 years of age and older. Are events A_2 and L_1 independent? Interpret your answer.

4.46 In the game of Yahtzee, five balanced dice are rolled. What is the probability

a) of rolling all 2s?

b) that all the dice come up the same number?

c) of getting a full house—three of one number and two of another?

4.47 Let A be an event of a sample space. Verify the following statements.

a) If $P(A) = 0$ or $P(A) = 1$, then, for each event B of the sample space, A and B are independent events.

b) If A and A are independent events, then $P(A) = 0$ or $P(A) = 1$.