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9.9

(a) $F_V(v) = P(V \le v) = P(\max\{X, Y\} \le v) = P(X \le v,\ Y \le v) = F_{X,Y}(v, v)$

(b) $F_U(u) = P(U \le u) = P(\min\{X, Y\} \le u) = P[(X \le u) \cup (Y \le u)]$

$= F_X(u) + F_Y(u) - F_{X,Y}(u, u)$

(c) $F_U(u) = F_{X,Y}(u, \infty) + F_{X,Y}(\infty, u) - F_{X,Y}(u, u)$

9.21 Joint PDF of $X, Y = f_{X,Y}(x,y)$; let $U = g(X) = X^2$; $V = h(Y) = Y^2$

$U = \sqrt{X}$; $V = \sqrt{Y}$

$\therefore f_{U,V}(u,v) = \frac{1}{J(X,Y)} f_{X,Y}(x,y)$

$J(X,Y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} 2X & 0 \\ 0 & 2Y \end{vmatrix} = 4XY$

$= \frac{1}{4XY} \cdot f_{X,Y}(x,y)$

$= \frac{1}{4\sqrt{uv}} \cdot f_{X,Y}(\pm\sqrt{u}, \pm\sqrt{v})$

9.25 According to the prompt, $X_1 \dots X_n \sim Exp(\lambda)$

$X = \min\{X_1 \dots X_n\}$, $Y = \max\{X_1 \dots X_n\}$

First find $P(X > x, Y \le y) = P(\min\{X_1 \dots X_n\} > x,\ \max\{X_1 \dots X_n\} \le y)$

$= P(x < X_1 \le y \dots x < X_n \le y) = (F(y) - F(x))^n$

$P(Y \le y) = P(X > x, Y \le y) + P(X \le x, Y \le y)$

$P(Y \le y) = P(\max\{X_1 \dots X_n\} \le y) = (F(y))^n$

$\Rightarrow F_{X,Y}(x,y) = P(X \le x, Y \le y) = (F(y))^n - (F(y) - F(x))^n$

$\therefore f_{X,Y} = \frac{\partial}{y}\left(\frac{\partial}{\partial x} F_{X,Y}(x,y)\right) = \frac{\partial}{y}\left(n f(x) \cdot (F(y) - F(x))^{n-1}\right)$

$= n(n-1) \cdot f_X(x) \cdot f(y) \cdot (F(y) - F(x))^{n-2}$

$= n(n-1) \cdot \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} \cdot (1 - e^{-\lambda y} - (1 - e^{-\lambda x}))^{n-2}$, $0 < x < y$

$f_{X,Y} = 0$, otherwise

9.45 (a) $p = \int_0^{\frac{1}{2}} \int_x^{1-x} 6(1-x-y)\, dy\, dx$

$= \int_0^{\frac{1}{2}} 6 \int_x^{1-x} 1-x-y \; dy\, dx = \int_0^{\frac{1}{2}} 6 \left[y - xy - \frac{1}{2}y^2 \right]_x^{1-x} dx$

$= 6 \int_0^{\frac{1}{2}} \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right] - \left[x - x^2 - \frac{1}{2}x^2 \right] dx$

$= 6 \int_0^{\frac{1}{2}} 1 - x - x + x^2 - \frac{1}{2} + x - \frac{1}{2}x^2 - x + x^2 + \frac{1}{2}x^2 \; dx$

$= 6 \int_0^{\frac{1}{2}} \frac{1}{2} - 2x + 2x^2 dx = 6 \left[\frac{1}{2}x - x^2 + \frac{2}{3}x^3 \right]_0^{\frac{1}{2}} = \frac{1}{2}$

(b) $p = \int_0^{0.2} \int_0^{1-x} 6(1-x-y)\, dy\, dx$

$= 6 \int_0^{0.2} 1 - x - x + x^2 - \frac{1}{2} + x - \frac{1}{2}x^2 \; dx$

$= 6 \int_0^{0.2} \frac{1}{2} - x + \frac{1}{2}x^2 \; dx = 6 \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right]_0^{0.2} = 0.488$

9.64 $P(X > 0.9 \mid Y = 0.8) = 1 - P(X < 0.9 \mid Y = 0.8) = 1 - F_{X|Y}(X < 0.9 \mid Y = 0.8)$

(a) $f_{(x|y)}(x|y) = \frac{f_{X,Y}(x,y)}{f(y)} = \frac{1}{\int_0^1 1 \; dx} = 1$

$F_{X|Y}(X < 0.9 \mid Y = 0.8) = \int_0^{0.9} f_{X|y}(x|y)\, dx = 0.9$

$\therefore P(X > 0.9 \mid Y = 0.8) = 1 - 0.9 = 0.1$

(b) $f_{X|y}(x|y=0.8) = \frac{x+y}{\int_0^1 x+y \; dx} = \frac{x+y}{\left[\frac{1}{2}x^2 + yx\right]_0^1} = \frac{x+0.8}{\frac{1}{2}+0.8} = \frac{x+0.8}{1.3}$

$F_{X|Y}(x < 0.9 \mid Y = 0.8) = \frac{1}{1.3} \int_0^{0.9} x + 0.8 \; dx = \frac{1}{1.3} \left[\frac{1}{2}x^2 + 0.8x \right]_0^{0.9} = \frac{45}{52}$

$\therefore P(X > 0.9 \mid Y = 0.8) = 1 - \frac{45}{52} = \frac{7}{52} \approx 0.1346$

(c) $f_{X|y}(x|y=0.8) = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}\int_0^1 x^2+y^2 dx} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}\left[\frac{1}{3}x^3 + y^2x\right]_0^1} = \frac{x^2+0.8^2}{\frac{1}{3}+0.8^2} = \frac{75}{73}(x^2+0.8^2)$

$F_{X|Y}(x < 0.9 \mid Y = 0.8) = \frac{75}{73} \int_0^{0.9} x^2 + 0.64 = \frac{75}{73} \left[\frac{1}{3}x^3 + 0.64x \right]_0^{0.9} = \frac{2457}{2920}$

$\therefore P(X > 0.9 \mid Y = 0.8) = 1 - \frac{2457}{2920} = \frac{463}{2920} \approx 0.1586$

9.93 For X, Y to be independent, $f_Y(y) = f_{Y|X}(y|x)$

i.e. $$\frac{1}{\sqrt{2\pi}\cdot\sqrt{1-\rho^2}} \cdot e^{-(y-\rho x)^2/2(1-\rho^2)} = \frac{1}{\sqrt{2\pi}} \cdot e^{-y^2/2}$$

$\Rightarrow 1-\rho^2 = 1 \wedge \rho x = 0 \quad \Rightarrow \rho = 0$ must be true

9.118 Let $X, Y \sim \mathrm{Exp}(\lambda)$, Let $U = \frac{X}{X+Y}$, $V = X$

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) \Rightarrow f_{X,Y}(x,y) = \begin{cases} \lambda^2 \cdot e^{-\lambda x} \cdot e^{-\lambda y}, & x, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Set $g(x,y) = \frac{x}{x+y}$, $h(x,y) = x$

$$\Rightarrow J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \\ 1 & 0 \end{vmatrix} = \frac{x}{(x+y)^2}$$

$U = \frac{X}{X+Y}$

$u(x+y) = x$

$x + y = \frac{x}{u}$

$y = \frac{x}{u} - x = \frac{v}{u} - v$

$$\Rightarrow f_{u,v} = \frac{(x+y)^2}{x} \cdot \lambda^2 \cdot e^{-\lambda x} \cdot e^{-\lambda y}$$

$$f_{u,v} = \frac{(v+\frac{v}{u}-v)^2}{v} \cdot \lambda^2 e^{-\lambda v} \cdot e^{-\lambda(\frac{v}{u}-v)} = \frac{v}{u^2} \cdot \lambda^2 e^{-\lambda(\frac{v}{u})}$$

$$\therefore f_u = \frac{1}{u^2}\int v e^{-\lambda(\frac{v}{u})} \, dv = \frac{u^2}{\lambda^2}$$

$$F_u = \frac{1}{\lambda^2}\int_0^u u^2 \, du = \frac{1}{3\lambda^2} \cdot u^3$$

9.121 Let basic wait time $X \sim \mathrm{Exp}(\frac{1}{2})$, deluxe wait time $Y \sim \mathrm{Exp}(\frac{1}{3})$

(a) $f_{X,Y}(x,y) = (\frac{1}{2} e^{-x/2}) \cdot (\frac{1}{3} e^{-y/2}) = \frac{1}{6} e^{-x/2} \cdot e^{-y/3} = e^{-x/2 - y/3}$

$$P(Y < X) = \int_0^\infty \int_0^x \frac{1}{6} e^{-x/2 - y/3} \, dy \, dx$$

$u = -\frac{y}{3} - \frac{x}{3}$

$$= \frac{1}{6}\int_0^\infty -3 \int e^u \, du = \frac{1}{6}\int_0^\infty -3\left[e^{-y/3 - x/2}\right]_0^x dx$$

$\frac{du}{dy} = -\frac{1}{3} \Rightarrow dy = -3du$

$$= \frac{1}{6}\int_0^\infty 3e^{-\frac{x}{2}} - 3e^{-\frac{5x}{6}} \, dx = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}} dx - \frac{1}{2}\int e^{-\frac{5x}{6}} dx$$

$= 1 - 0.6 = 0.4$

(b) by proposition 9.11, $\min\{X, Y\} \sim Exp\left(\frac{1}{2}+\frac{1}{3}\right) = Exp\left(\frac{5}{6}\right)$

$\therefore P(Y = \min\{X,Y\}) = \frac{1/3}{5/6} = 0.4$