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Section B3

6.1

(a) $P(X=Y) = \sum\sum_{x=y} p_{x,y}(x,y)$

$= \sum_{x=1}^{\infty}\sum_{y=x}^{\infty} p^2(1-p)^{x+y-2}$

$\because x=y$ $= p^2(1-p)^{-2} \sum_{y}^{\infty} (1-p)^y \sum_{y}^{\infty} (1-p)^y$

$= p^2(1-p)^{-2} \cdot \frac{(1-p)^2}{1-(1-2p+p^2)}$

$= \frac{p^2}{p(2-p)}$

$= \frac{p}{2-p}$

$P(X=Y)$ is the probability of the two components having the exact same life span.

(b) $P(X>Y) = \sum\sum_{x>y} p^2(1-p)^{x+y-2}$

$= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty} p^2(1-p)^{x+y-2}$

$= p^2 \cdot (1-p)^{-2} \sum_{y=1}^{\infty} (1-p)^y \sum_{x=y+1}^{\infty} (1-p)^x$

$= p^2 \cdot (1-p)^{-2} \sum_{y=1}^{\infty} (1-p)^y \cdot \frac{(1-p)^{y+1}}{1-(1-p)}$

$= p \cdot (1-p)^{-2} \sum_{y=1}^{\infty} (1-p)^{2y+1}$

$= p \cdot (1-p)^{-1} \sum_{y=1}^{\infty} (1-p)^{2y}$

$= p \cdot (1-p)^{-1} \cdot \frac{(1-p)^2}{1-(1-p)^2} = \frac{p(1-p)}{p(2-p)} = \boxed{\frac{1-p}{2-p}}$

6.13

(a) $P_x(x) = \sum_{y}^{\infty} e^{-(\lambda+\mu)} \frac{\lambda^x \cdot \mu^y}{x!\, y!}$

$= e^{-(\lambda+\mu)} \cdot \frac{\lambda^x}{x!} \sum_{y=0}^{\infty} \frac{\mu^y}{y!}$ (taylor Series: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$)

$= e^{-(\lambda+\mu)} \cdot \frac{\lambda^x}{x!} \cdot e^{\mu} = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$

$P_y(y) = \sum_{x}^{\infty} e^{-(\lambda+\mu)} \frac{\lambda^x \cdot \mu^y}{x!\, y!}$

$= e^{-(\lambda+\mu)} \cdot \frac{\mu^y}{y!} \cdot e^{\lambda} = e^{-\mu} \cdot \frac{\mu^y}{y!}$

(b) $\because P_{xy}(x,y) = e^{-(\lambda+\mu)} \frac{\lambda \cdot \mu^y}{x!\, y!} = P_x(x) \cdot P_y(y)$

$\therefore$ Joint PMF is a product of the marginal PMFs

6.28 Let X be the number of good items, Y be the number of salvageable items.

$P_{X,Y}(x=48, y=1) = P_X(x=48) \cdot P_Y(y=1)$

$= \left[\binom{50}{48} (0.97)^{48} \cdot (0.03)^2 \right] \cdot \left[\binom{2}{1} \cdot \left(\frac{2}{3}\right)^1 \cdot \left(\frac{1}{3}\right)^1 \right]$

$= 0.256 \cdot \frac{4}{9} \approx 0.114$

6.46

(a) $P_{Y|X}(y \mid X=x) = \frac{P_{X,Y}(Y=y, X=x)}{P_X(X=x)}$

$= \frac{\frac{1}{9} \cdot \frac{1}{10}}{\frac{1}{10}} = \frac{1}{9}$

$\therefore$ PMF $= \frac{1}{|S|-1} = \frac{1}{9}$

(b) $P_{X|Y}(x \mid Y=y) = \frac{P_{X,Y}(X=x, Y=y)}{P_Y(Y=y)}$

$P_{X,Y}(X=x, Y=y) = P_{Y|X}(y \mid X=x) \cdot P_X(X=x) = \frac{1}{9} \cdot \frac{1}{10} = \frac{1}{90}$

$P_Y(Y=y) = \sum_{X=1}^{9} P_{X,Y}(X=x, Y=y) = \frac{1}{90} \cdot 9 = \frac{1}{10}$

$\therefore P_{X|Y}(x \mid Y=y) = \dfrac{\frac{1}{90}}{\frac{1}{10}} = \dfrac{1}{9}$

(c) $P(3 \le X \le 4 \mid Y=2) = \sum_{x=3}^{4} \frac{1}{9} = \frac{2}{9}$

6.52

(a) $P_{Z,X|Y}(z, x \mid 0) = \dfrac{P_{X,Y,Z}(x, 1, z)}{P_Y(0)} = \dfrac{(x+2+z)/63}{P_Y(0)}$

From 6.7, P274, $P_Y(y) = (4y+3)/21 \Rightarrow P_Y(1) = \frac{1}{3}$

$P_{Z,X|Y}(z, x \mid 1) = \dfrac{(x+2+z)/63}{\frac{1}{3}} = \boxed{\dfrac{(x+2+z)}{21}}$

(b) $P(z+x \le 2 \mid Y=1) = \sum_{z+x \le 2}\sum P_{X,Z|Y}(z, x \mid 0)$

$= P_{X,Z}(0,0 \mid 1) + P_{X,Z}(0,1 \mid 1) + P_{X,Z}(1,0 \mid 1)$

$= \frac{2}{21} + \frac{3}{21} + \frac{3}{21} = \frac{8}{21}$

(c) $P_{Z|X,Y}(z \mid 0,1) = \dfrac{P_{X,Y,Z}(0,1,z)}{P_{X,Y}(0,1)} = \dfrac{2+z\ /63}{P_{X,Y}(0,1)}$

From 6.7 P275, $P_{X,Y}(x,y) = (x+2y+1)/21 \Rightarrow P_{X,Y}(0,1) = \frac{3}{21} = \frac{1}{7}$

$P_{Z|X,Y}(z \mid 0,1) = \dfrac{(2+z)/63}{\frac{1}{7}} = \boxed{\dfrac{2+z}{9}}$

(d) $P(z \le 1 \mid 0,1) = \sum_{z \le 1} P_{Z|X,Y}(z \mid 0,1)$

$= P_Z(0 \mid 0,1) + P_Z(1 \mid 0,1) = \frac{5}{9}$

6.63 X, Y are independenit, if $P(X, Y) = P(X) \cdot P(Y)$

Possible (x,y): $\Omega = (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$

$(2,3)\ (2,4)\ (2,5)\ (2,6)$

$(3,4)\ (3,5)\ (3,6)$

$(4,5)\ (4,6)$

$(5,6)$

$P_X(X=1) = \frac{6}{36} = \frac{1}{6}$

$P_Y(Y=2) = \frac{1}{36}$, yet $P_{X,}(1,2) = \frac{1}{36}$

$\because P_{XY}(x,y) \neq P(x) \cdot P(y)$

$\therefore$ X, Y are dependent.

6.74 Let $X \sim Poi(\lambda)$, X is the number of ppl entering the bank.

$\therefore P_X(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$

Let Y be the number of ppl making deposits.

$P_Y(y=k) = \binom{x}{k} p^k \cdot (1-p)^{x-k}$

$\because$ X, Y are independent

$\therefore P_{X,Y}(X=x, Y=k) = P_X(x) \cdot P_Y(y)$

$= e^{-\lambda} \cdot \frac{\lambda^x}{x!} \cdot \binom{x}{k} p^k \cdot (1-p)^{x-k}$

6.90 (a) Let $Z = \text{sum}(X,Y) = X+Y$

$\therefore P_Z(Z) = \sum\sum_{x+y=z} P_{X,Y}(x,y) = \sum\sum_{x+y=z} P_X(x) \cdot P_Y(y)$

$= \sum_{x=0}^{N} \sum_{y=z-x}^{z-x} \frac{1}{N^2} = \sum_{x=0}^{N} \frac{z-x+1}{N^2}$

$= N \cdot \left(\frac{z-x+1}{N^2}\right) = \frac{z-x+1}{N}$

(b) $P_{\min\{X,Y\}}(z) = \sum\sum_{\min\{x,y\}=z} \frac{1}{N^2} = 1 - P(X > z, Y > z)$

6.103 Let $X \sim G(p)$, $Y \sim G(p)$, $Z = X + Y$

$$P_Z(z) = \sum_x P_X(x) \cdot P_Y(z-x)$$

$$= \sum_x p(1-p)^{x-1} \cdot p(1-p)^{z-x-1}$$

$$= p^2 \cdot (1-p)^z \cdot \sum_z (1-p)^{x-1} \cdot (1-p)^{-x-1}$$

$$= p^2(1-p)^z \cdot \sum_x (1-p)^{-z}$$

$$= (z-1)\, p^2 \cdot (1-p)^{z-2}$$

$$= \binom{z-1}{2-1} p^2 (1-p)^{z-2}$$, which is not a geometric distribution.

6.108 $\because X \sim P(\lambda)$ $\because Y \sim P(\mu)$

$\therefore P_X(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$ $\therefore P_Y(y) = e^{-\mu} \cdot \frac{\mu^y}{y!}$

$$P_{X|X+Y=z}(Y=k \mid z) = \frac{P((Y=k \cap X+Y=z))}{P(X+Y=z)}$$

$$= \frac{P(X=z-k, Y=k)}{P(X+Y=z)}$$

$\because X, Y$ are independent

$$= \frac{P(X=z-k) \cdot P(Y=k)}{P(X+Y=z)}$$

$$= \frac{e^{-\lambda} \cdot \frac{\lambda^{z-k}}{(z-k)!} \cdot e^{-\mu} \cdot \frac{\mu^k}{k!}}{e^{-\lambda-\mu} \cdot \frac{(\lambda+\mu)^z}{z}}$$