b) Based on your five tosses, what estimate would you give for the probability of a head when this coin is tossed once? Explain your answer.

c) Now toss the coin five more times and continue recording in the table so that you now have entries for tosses 1–10. Based on your 10 tosses, what estimate would you give for the probability of a head when this coin is tossed once? Explain your answer.

d) Now toss the coin 10 more times and continue recording in the table so that you now have entries for tosses 1–20. Based on your 20 tosses, what estimate would you give for the probability of a head when this coin is tossed once? Explain your answer.

e) In view of your results in parts (b)–(d), explain why the frequentist interpretation can't be used as the definition of probability—that is, why the probability of an event E can't be defined as $P(E) = \lim_{n\to\infty} n(E)/n$.

1.9 In 2004, according to the National Governors Association, 28 of the state governors were Republicans. Suppose that on each day of 2004, one U.S. state governor was randomly selected to read the invocation on a popular radio program. On approximately how many of those days should we expect that a Republican was chosen?

1.10 Use the frequentist interpretation of probability to interpret each statement.

a) The probability is 0.314 that the gestation period of a woman will exceed 9 months.

b) The probability is 2/3 that the favorite in a horse race finishes in the money (first, second, or third place).

c) The probability is 0.40 that a traffic fatality involves an intoxicated or alcohol-impaired driver or nonoccupant.

1.11 Refer to Exercise 1.10.

a) In 4000 human gestation periods, roughly how many will exceed 9 months?

b) In 500 horse races, roughly how many times will the favorite finish in the money?

c) In 389 traffic fatalities, roughly how many will involve an intoxicated or alcohol-impaired driver or nonoccupant?

Advanced Exercises

Odds. Closely related to probabilities are *odds*. Newspapers, magazines, and other popular publications often express likelihood in terms of odds instead of probabilities, and odds are used much more than probabilities in gambling contexts. If the probability that an event occurs is p, the odds that the event occurs are p to $1 - p$. This fact is also expressed by saying that the odds are p to $1 - p$ *in favor of the event* or that the odds are $1 - p$ to p *against the event.* Conversely, if the odds in favor of an event are a to b (or, equivalently, the odds against it are b to a), the probability that the event occurs is $a/(a + b)$. For example, if an event has probability 0.75 of occurring, the odds that the event occurs are 0.75 to 0.25, or 3 to 1; if the odds against an event are 3 to 2, the probability that the event occurs is $2/(2 + 3)$, or 0.4.

1.12 An American roulette wheel contains 38 numbers, of which 18 are red, 18 are black, and 2 are green. When the roulette wheel is spun, the ball is equally likely to land on any of the 38 numbers. For a bet on red, the house pays even odds (i.e., 1 to 1). What should the odds actually be to make the bet fair?

1.13 Fusaichi Pegasus, the winner of the 2000 Kentucky Derby, was the heavy favorite to win the Preakness on May 20, 2000, with odds at 3 to 5 (against). The second favorite and actual winner, Red Bullet, posted odds at 9 to 2 (against) to win the race. Based on the posted odds, determine the probability that

a) Fusaichi Pegasus would win the race.

b) Red Bullet would win the race.

1.14 In Example 1.5(d) on page 16, we showed that it is possible for the sets in a collection to not be pairwise disjoint even though the intersection of all the sets is empty. If the sets in a collection are pairwise disjoint, must the intersection of all the sets be empty?

1.15 Draw a Venn diagram showing three subsets, A, B, and C, such that no two are disjoint but that $A \cap B \cap C = \emptyset$.

1.16 Make a Venn diagram with four subsets, A, B, C, and D, such that $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$, $C \cap D \neq \emptyset$, and $A \cap D \neq \emptyset$ but that $A \cap B \cap C \cap D = \emptyset$.

1.17 Give an example of a collection of sets satisfying the following properties: The collection contains at least four sets, the sets are not pairwise disjoint, and every three sets have an empty intersection.

1.18 Let $U = \mathcal{R}$ and, for each $n \in \mathcal{N}$, define $A_n = [0, 1/n]$.

a) Determine $\bigcap_{n=1}^{4} A_n$ and $\bigcup_{n=1}^{4} A_n$. **b)** Determine $\bigcap_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} A_n$.

1.19 Express each of the following sets in a simple form in which "$\bigcup$" and "$\bigcap$" don't occur.

a) $\bigcup_{n=1}^{\infty}[1 + 1/n,\ 2 - 1/n]$ **b)** $\bigcup_{n=1}^{\infty}[1,\ 2 - 1/n]$ **c)** $\bigcap_{n=1}^{\infty}(1 - 1/n,\ 2 + 1/n)$
d) $\bigcap_{n=1}^{\infty}(3 - 1/n,\ 3 + 1/n)$ **e)** $\bigcap_{n=1}^{\infty}(n, \infty)$ **f)** $\bigcap_{n=1}^{\infty}(5 - 1/n,\ 5)$
g) $\bigcap_{n=1}^{\infty}(5 - 1/n,\ 6)$

1.20 Let the universal set U be $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$.

a) List the members of $\{1, 2, 3\} \times \{3, 4, 5\}$.
b) List the members of $(\{1, 2, 3\} \times \{3, 4, 5\}) \cup (\{3, 4, 5\} \times \{1, 2, 3\})$.
c) List the members of $A = \big((\{1, 2, 3\} \times \{3, 4, 5\}) \cup (\{3, 4, 5\} \times \{1, 2, 3\})\big)^c$.
d) Write the set A in part (c) as the union of two Cartesian products—that is, in the form (some set $\times$ some set) $\cup$ (some set $\times$ some set).

1.21 List the members of each of the following sets.

a) $\{0, 1\}^3 = \{0, 1\} \times \{0, 1\} \times \{0, 1\}$ **b)** $\{0, 1\} \times \{0, 1\} \times \{1, 2\}$
c) $(\{a, b\} \cup \{c, d, e\}) \times \{f, g, h\}$ **d)** $(\{a, b\} \times \{f, g, h\}) \cup (\{c, d, e\} \times \{f, g, h\})$

1.22 Simplify the expression $([0, 2] \times [0, 2]) \cap ([1, 3] \times [1, 3])$, writing your answer in the form (some set $\times$ some set).

1.23 Find a countable collection of subintervals $I_1, I_2, \ldots$ of the interval $[0, 1]$ such that $I_j \cap I_k \neq \emptyset$ for all j and k, but $\bigcap_{i=1}^{\infty} I_i = \emptyset$.

1.24 Find a countable collection of intervals $I_1, I_2, \ldots$ of $\mathcal{R}$ such that $\bigcap_{i=1}^{\infty} I_i$ consists of a single point (i.e., is a singleton set).

1.25 Express $\mathcal{R}$ as a countably infinite union of pairwise disjoint intervals, each of length 1.

Theory Exercises

1.26 Refer to De Morgan's laws, as given in Proposition 1.1 on page 11.

a) Verify part (b) of De Morgan's laws by using Venn diagrams.
b) Prove part (b) of De Morgan's laws mathematically in a manner similar to that done for part (a) of De Morgan's laws on page 11.
c) Prove part (b) of De Morgan's laws by using part (a) of De Morgan's laws.

1.27 Refer to the distributive laws, as given in Proposition 1.2 on page 12.

a) Verify parts (a) and (b) by using Venn diagrams.
b) Prove parts (a) and (b) mathematically.

1.28 Refer to the associative and commutative laws, as given in Proposition 1.3 on page 12.
a) Verify parts (a)–(d) by using Venn diagrams.
b) Prove parts (a)–(d) mathematically.

1.29 Let A, B, and C be subsets of U. Prove each of the following statements.
a) $A \cup \emptyset = A$ **b)** $A \subset A \cup B$ **c)** $A = A \cup B$ if and only if $B \subset A$

1.30 Let A, B, and C be subsets of U. Prove each of the following statements.
a) $A \cap \emptyset = \emptyset$ **b)** $A \supset A \cap B$ **c)** $A = A \cap B$ if and only if $B \supset A$

1.31 Let A and B be subsets of U. Verify each of the following statements.
a) $A = (A \cap B) \cup (A \cap B^c)$ **b)** $A \cap B = \emptyset \Rightarrow A \subset B^c$ **c)** $A \subset B \Rightarrow B^c \subset A^c$

1.32 Prove De Morgan's laws for countable collections of sets, Proposition 1.4 on page 14.

1.33 Prove the distributive laws for countable collections of sets, Proposition 1.5 on page 14.

1.34 Let $A_1, A_2, \ldots$ be a countable collection of subsets of U. Prove the following.
a) If $B \subset \bigcup_n A_n$, then $B = \bigcup_n (A_n \cap B)$.
b) If $\bigcup_n A_n = U$, then $E = \bigcup_n (A_n \cap E)$ for each subset E of U.
c) If $A_1, A_2, \ldots$ are pairwise disjoint, so are $A_1 \cap E, A_2 \cap E, \ldots$ for each subset E of U.
d) We say that $A_1, A_2, \ldots$ form a *partition* of U if they are pairwise disjoint and their union is U. Conclude from parts (b) and (c) that, if $A_1, A_2, \ldots$ form a partition of U, each subset E of U can be expressed as a disjoint union of the sets $A_1 \cap E, A_2 \cap E, \ldots$.

Advanced Exercises

1.35 Let $A_1, A_2, \ldots$ be a countably infinite collection of subsets of U.
a) Prove that

$$\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right) \subset \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right).$$

The set on the left is called the *limit inferior* of the A_ns, denoted $\liminf_{n\to\infty} A_n$; the set on the right is called the *limit superior* of the A_ns, denoted $\limsup_{n\to\infty} A_n$.
b) Describe in words the limit inferior and limit superior of the A_ns, and use that description to interpret the relation in part (a). *Hint:* To how many A_ns must a point belong?
c) Let $U = \mathcal{R}$ and define

$$A_n = \begin{cases} [0, 1 + 1/n], & \text{if } n \text{ is an even positive integer;} \\ [-1 - 1/n, 0], & \text{if } n \text{ is an odd positive integer.} \end{cases}$$

Determine $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$.

Finite and infinite sets; countability: Two sets are said to be *equivalent* if there is a one-to-one function from one set onto the other. A set E is *finite* if it is either empty or equivalent to the first N positive integers for some $N \in \mathcal{N}$; E is *infinite* if it is not finite; E is *countably infinite* if it is equivalent to $\mathcal{N}$; E is *countable* if it is either finite or countably infinite; E is *uncountable* if it is not countable.

1.36 Prove that $\mathcal{Z}$ is countable. *Hint:* Define a function that maps the even positive integers onto the positive integers and the odd positive integers onto the nonpositive integers.

1.37 Prove that $\mathcal{N}^2$ is countable. *Hint:* Define $f\colon \mathcal{N}^2 \to \mathcal{N}$ by $f(m, n) = 2^{m-1}(2n - 1)$.

1.38 Prove that a nonempty set is countable if and only if it is the range of an infinite sequence.

2.6 An urn contains 10 balls, numbered 0, 1, 2, ..., 9. Three balls are removed, one at a time, without replacement.

a) Obtain the sample space for this random experiment.

b) Determine, as a subset of the sample space, the event that an even number of odd-numbered balls are removed from the urn.

2.7 Refer to Example 2.3 on page 27 where two dice are rolled, one black and one gray. For $i = 2, 3, \ldots, 12$, determine explicitly as a subset of the sample space the event A_i that the sum of the faces is i.

2.8 Consider the following random experiment: First a die is rolled and you observe the number of dots facing up; then a coin is tossed the number of times that the die shows and you observe the total number of heads.

a) Determine the sample space for this random experiment.

b) Determine the event that the total number of heads is even.

2.9 George and Laura take turns tossing a coin. The first person to get a tail wins. George goes first. *Note:* You may assume that eventually a tail will be tossed.

a) Describe the sample space for this random experiment.

b) Determine, as a subset of the sample space, the event that Laura wins.

2.10 This exercise considers two random experiments involving the repeated tossing of a coin. *Note:* You may assume that eventually a head will be tossed.

a) If the coin is tossed until the first time a head appears, find the sample space.

b) If the coin is tossed until the second time a head appears, find the sample space.

c) For the experiment in part (a), express the event that the coin is tossed exactly six times in the form $\{\ldots\}$, where in place of "..." you list all of the outcomes in that event.

d) Repeat part (c) for the experiment described in part (b).

2.11 From 10 men and 8 women in a pool of potential jurors, 12 are chosen at random to constitute a jury. Suppose that you observe the number of men who are chosen for the jury. Let A be the event that at least half of the 12 jurors are men and let B be the event that at least half of the 8 women are on the jury.

a) Determine the sample space for this random experiment.

b) Find $A \cup B$, $A \cap B$, and $A \cap B^c$, listing all the outcomes for each of those three events.

c) Are A and B mutually exclusive? A and B^c? A^c and B^c? Explain your answers.

2.12 Let A and B be events of a sample space.

a) Show that, if A and B^c are mutually exclusive, then B occurs whenever A occurs.

b) Show that, if B occurs whenever A occurs, then A and B^c are mutually exclusive.

2.13 Let A, B, and C be events of a sample space. Write a mathematical expression for each of the following events.

a) A occurs, but B doesn't occur.

b) Exactly one of A and B occurs.

c) Exactly one of A, B, and C occurs.

d) At most two of A, B, and C occur.

2.14 Refer to Example 2.17 on page 34, but now suppose that two cards are selected at random, one after the other, without replacement.

a) What is Ω for this random experiment?

b) Let A be the event that at least one of the cards is a face card and let B be the event that at least one of the cards is an ace. Are A and B mutually exclusive? Why or why not?

2.24 Let Ω be the sample space for a random experiment and let P be a probability measure on Ω. Use the Kolmogorov axioms to verify the following for events A and B.

a) $P(B) = P(B \cap A) + P(B \cap A^c)$

b) $P(A \cup B) = P(A) + P(B \cap A^c)$

c) Suppose that at least one of events A and B must occur—that is, $A \cup B = \Omega$. Show that the probability that both events occur is $P(A) + P(B) - 1$.

2.25 Consider the experiment of rolling two dice. The possible outcomes are shown in Figure 2.1 on page 27.

a) Assign each outcome a probability of 1/36. Show that this probability assignment is legitimate.

b) Based on the probability assignment in part (a), determine the probability of the event A_i that the sum of the faces is i, for each $i = 2, 3, \ldots, 12$.

c) Provide another probabilistically legitimate assignment to the 36 possible outcomes, and then repeat part (b) for that assignment.

d) Assuming that the die is balanced, is your probability assignment in part (c) reasonable? What about the one in part (a)? Explain your answers.

2.26 A number is chosen at random from the integers $1, 2, \ldots, 100$. The sample space is the set $\Omega = \{1, 2, \ldots, 100\}$, and each outcome is assigned probability 0.01.

a) Show that this probability assignment is legitimate.

b) Let A be the event that the number chosen is even, B be the event that the number chosen is at most 10π, and C be the event that the number chosen is prime. Determine the probabilities of events A, B, and C.

2.27 An urn contains four balls numbered 1, 2, 3, and 4. A ball is chosen at random, its number noted, and the ball is replaced in the urn. This process is repeated one more time.

a) Determine the sample space Ω.

b) If each outcome is assigned the same probability, what is that common probability?

c) Using the probability assignment in part (b), find the probability that the two numbers chosen are different.

2.28 Suppose that Ω is a finite sample space—say, with N possible outcomes. Further suppose that those N possible outcomes are equally likely.

a) What common probability should be assigned to each possible outcome?

b) Determine the probability of an event that consists of m outcomes.

Theory Exercises

2.29 A special case of a relation called *Boole's inequality* is that, for each positive integer n,

$$P(A_1 \cup \cdots \cup A_n) \leq P(A_1) + \cdots + P(A_n), \tag{$*$}$$

for all events $A_1, \ldots, A_n$ of a sample space.

a) If $P(A_i) = 1/6$ for each i and $n = 10$, Relation $(*)$ is trivial. However, if $P(A_i) = 1/60$ for each i and $n = 10$, or if $P(A_i) = 1/6$ for each i and $n = 4$, Relation $(*)$ conveys more substantial information than it does in the trivial case. Explain the difference between the trivial case and the two cases in which more substantial information is conveyed. Why is one case "trivial" while the other two are "more substantial"?

b) Prove Relation $(*)$. *Hint:* Use Exercise 2.24, the nonnegativity axiom, and mathematical induction.

Suppose that a player on the New England Patriots is selected at random.

f) Describe the events Y_3, W_2, and $W_1 \cap Y_2$ in words.

g) Compute the probability of each event in part (f). Interpret your answers in terms of percentages.

h) Construct a joint probability distribution similar to Table 2.6 on page 55.

i) Verify that the sum of each row and column of joint probabilities equals the marginal probability in that row or column. *Note:* Rounding may cause slight deviations.

2.40 Refer to Example 2.3 on page 27, where two dice are rolled, and assume that the dice are balanced.

a) Is a classical probability model appropriate here? Explain your answer.

Find the probability of each of the following events.

b) The sum of the two numbers is 4.
c) The sum of the two numbers is 5.
d) The minimum of the two numbers is 4.
e) The maximum of the two numbers is 4.

2.41 Two Republicans and two Democrats sit on a committee in the Senate. A subcommittee of two senators is chosen at random from among these four. Find the probability that the number of Republicans on the subcommittee is

a) two. **b)** one. **c)** zero.

2.42 A commuter train arrives punctually at a station every half hour. Each morning, a commuter named John leaves his house and casually strolls to the train station. Find the probability that John waits for the train

a) between 10 and 15 minutes. **b)** at least 10 minutes.

2.43 Refer to Example 2.24 on page 57, where a petri dish of unit radius, containing nutrients upon which bacteria can multiply, is smeared with a uniform suspension of bacteria. Determine the probability of each of the following events.

a) The distance from the center to the first spot is less than half the distance from the center to the rim.

b) The distance from the center to the first spot is more than half the distance from the center to the rim.

c) The center of the first spot to appear is not in the inscribed square centered at the origin with vertices on the circle $x^2 + y^2 = 1$ and sides parallel to the coordinate axes.

d) The center of the first spot to appear is on the strip $\{ (x, y) : -1/4 < x < 1/4 \}$.

2.44 A point is chosen at random in the unit square $\Omega = \{ (x, y) : 0 \le x \le 1,\ 0 \le y \le 1 \}$. Describe each of the following events in words and determine the probability of each.

a) $A = \{ (x, y) \in \Omega : x > 1/3 \}$
b) $B = \{ (x, y) \in \Omega : y \le 0.7 \}$
c) $C = \{ (x, y) \in \Omega : x + y > 1.2 \}$
d) $D = \{ (x, y) \in \Omega : |y - x| < 1/10 \}$
e) $E = \{ (x, y) \in \Omega : x = y \}$

2.45 Refer to Exercise 2.44. Determine the probability that, for the point obtained,

a) the x coordinate is less than the y coordinate.

b) the smaller of the two coordinates is less than 1/2.

c) the smaller of the two coordinates exceeds 1/2.

d) the larger of the two coordinates is less than 1/2.

e) the larger of the two coordinates exceeds 1/2.

f) the sum of the two coordinates is between 1 and 1.5.

g) the sum of the two coordinates is between 1.5 and 2.

2.46 If a number is chosen at random from the interval (0, 1), find the probability that

a) the first digit of its decimal expansion is 7.

The probability that a head will eventually be tossed is 1. In other words, we will eventually get a head when we repeatedly toss a balanced coin. As we demonstrate in Example 4.17, this result holds regardless of whether the coin is balanced, provided only that the probability is not 0 of getting a head when the coin is tossed once. ■

EXERCISES 2.4 Basic Exercises

2.58 Let A and B be events of a sample space. Provide an example where, as sets, A is a proper subset of B, but $P(A) = P(B)$.

2.59 Give an example to show that the converse of the domination principle fails.

2.60 A person is selected at random from among the inhabitants of a state. Which is more probable: that the person so chosen is a lawyer, or that the person so chosen is a Republican lawyer? Explain your answer.

2.61 Refer to Exercise 2.41 on page 62. Use the complementation rule to find the probability that at least one Republican will be on the subcommittee. Why would use of the complementation rule for this problem make things easier than if that rule weren't used?

2.62 Refer to Exercise 2.20 on page 45. Determine the probability that an oil spill in U.S. navigable and territorial waters doesn't occur in the Gulf of Mexico
a) without use of the complementation rule **b)** by using the complementation rule.
c) Compare the work done in your solutions in parts (a) and (b).

2.63 Refer to Exercise 2.39 on page 61. Suppose that a player on the New England Patriots is selected at random. Determine the probability that the player obtained
a) has at least 1 year of experience. **b)** weighs at most 300 lb.
c) is either a rookie or weighs more than 300 lb. Solve this problem both with and without use of the general addition rule and compare your work.

2.64 If a point is selected at random from the unit square $\{ (x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1 \}$, find the probability that the magnitude of the difference between the x and y coordinates of the point obtained is at most 1/4. Solve this problem both with and without use of the complementation rule and compare your work.

2.65 According to *Current Population Reports,* published by the U.S. Bureau of the Census, 51.0% of U.S. adults are female, 7.1% are divorced, and 4.1% are divorced females. Determine the probability that a U.S. adult selected at random is
a) either female or divorced. **b)** a male.
c) a female but not divorced. **d)** a divorced male.

2.66 Let A and B be events such that $P(A) = 1/4$, $P(B) = 1/3$, and $P(A \cup B) = 1/2$.
a) Are events A and B mutually exclusive? Explain your answer.
b) Determine $P(A \cap B)$.

2.67 Let A and B be events such that $P(A) = 1/3$, $P(A \cup B) = 5/8$, and $P(A \cap B) = 1/10$. Determine
a) $P(B)$. **b)** $P(A \cap B^c)$. **c)** $P(A \cup B^c)$. **d)** $P(A^c \cup B^c)$.

2.68 Gerald Kushel, Ed.D., was interviewed by *Bottom Line/Personal* on the secrets of successful people. To study success, Kushel questioned 1200 people, among whom were lawyers, artists, teachers, and students. He found that 15% enjoy neither their jobs nor their

3.3 The menu at a restaurant has five choices of beverage, three salads, six entrees and four desserts. How many complete meals are available with beverage, salad, entree, and dessert?

3.4 An advertisement for *i*Dolls™ states: "Choose from 69 billion combinations to create a one-of-a-kind doll." The ad goes on to say that there are 39 choices for hairstyle, 19 for eye color, 8 for hair color, 6 for face shape, 24 for lip color, 5 for freckle pattern, 5 for line of clothing, 6 for blush color, and 5 for skin tone. Exactly how many possibilities are there for these options?

3.5 An identification number consists of an ordered arrangement of eight decimal digits. How many identification numbers can be formed if

a) there are no restrictions? **b)** no digit can occur twice?
c) no digit can agree with its predecessor?
d) no digit can agree with either of its two immediate predecessors?

3.6 How many batting orders are possible for the nine starting players on a baseball team?

3.7 The author of this book spoke with a representative of the United States Postal Service and obtained the following information about zip codes. A five-digit zip code consists of five digits of which the first three give the sectional center and the last two the post office or delivery area. In addition to the five-digit zip code, there is a trailing *plus four zip code.* The first two digits of the plus four zip code give the sector or several blocks and the last two the segment or side of the street. For the five-digit zip code, the first four digits can be any of the digits 0–9 and the fifth any of the digits 1–8. For the plus four zip code, the first three digits can be any of the digits 0–9 and the fourth any of the digits 1–9.

a) How many possible five-digit zip codes are there?
b) How many possible plus four zip codes are there?
c) How many possibilities are there including both the five-digit zip code and the plus four zip code?

3.8 Telephone numbers in the United States consist of a three-digit area code followed by a seven-digit local number. Suppose that neither the first digit of an area code nor the first digit of a local number can be a zero but that all other choices are acceptable.

a) How many different area codes are possible?
b) For a given area code, how many local telephone numbers are possible?
c) How many telephone numbers are possible?

3.9 In Example 2.3 on page 27, we considered the random experiment of rolling two dice. Use the BCR to determine the number of possible outcomes

a) for this random experiment. **b)** in which the sum of the dice is 5.
c) in which doubles are rolled. **d)** in which the sum of the dice is even.

3.10 In Example 2.4 on page 27, we considered the random experiment of observing a mechanical or electrical unit consisting of five components and determining which components are working and which have failed. Use the BCR to find the number of possible outcomes

a) for this random experiment.
b) in which exactly one of the five components is not working.
c) in which at least one of the five components is not working.
d) in which at most one of the five components is not working.

3.11 An alphabet has six letters a, b, c, d, e, and f. How many four-letter words (i.e., ordered arrangements of four of the six letters) can be formed if

a) there are no restrictions? **b)** no letter can occur twice?
c) the first letter must be a or b? **d)** the letter c must occur at least once?

3.12 Let A and B be two finite sets with the same number of elements—say, n.
a) How many functions are there from A to B?
b) How many of these functions are one-to-one?

3.13 Five people—say, a, b, c, d, and e—are arranged in a line. How many arrangements are there in which
a) c is before d? **b)** d is not first?

3.14 Suppose that you and your best friend are among n people to be arranged in a line. How many arrangements are possible in which exactly k people are between you and your friend?

3.15 Four married couples attend a banquet.
a) How many ways can they be seated on one side of a straight (i.e., noncircular) table in such a way that each husband sits next to his wife?
b) Repeat part (a) for a circular table.

3.16 Consider a domino that consists of two subrectangles, each marked with a number from 1 to n. How many such dominos are possible? *Note:* A number pair is not ordered.

3.17 If n people attend a party and each pair of people shake hands, how many handshakes will there be?

3.18 A poker hand consists of 5 cards dealt from an ordinary deck of 52 playing cards.
a) In five-card stud, the order in which the cards are dealt matters. How many five-card stud hands are possible?
b) In five-card draw, the order in which the cards are dealt doesn't matter. How many five-card draw hands are possible?

3.19 In how many ways can n distinguishable balls be arranged in n distinguishable boxes so that
a) no box is empty? **b)** exactly one box is empty? **c)** at least one box is empty?

Theory Exercises

3.20 Use mathematical induction to complete the proof of the BCR, as given in Proposition 3.1 on page 87.

Advanced Exercises

3.21 In this exercise, you are to obtain the number of subsets of a finite set, Ω.
a) Suppose that Ω consists of three elements—say, $\{a, b, c\}$. List the possible subsets of Ω and, from your list, determine the number of subsets.
b) In part (a), determine the number of subsets by using the BCR. *Hint:* For a given subset, each element of Ω either is a member of that subset or it isn't.
c) Suppose that Ω consists of n elements. Determine the number of subsets of Ω.

3.22 This exercise provides an alternative derivation of the result in Exercise 3.21(c). The eight subsets of $\{a, b, c\}$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. Although listing all 16 subsets of $\{a, b, c, d\}$ is relatively easy, you are to answer the following questions by other methods, which can be applied in more complicated problems where you may not be able to write a complete list easily.
a) To create a subset of the set $\{a, b, c, d\}$, first choose a subset of $\{a, b, c\}$, and then decide whether to add d to that subset. Use that idea and the BCR to find the number of subsets of $\{a, b, c, d\}$.

We can think of the possible results as partitioning the eight finishing places into three groups, one of size 3 for the U.S. athletes, one of size 2 for the Canadian athletes, and one of size 3 for the Mexican athletes. For instance, the three groups of finishing places $\{2, 5, 7\}$, $\{3, 4\}$, and $\{1, 6, 8\}$ would correspond to the three U.S. athletes finishing in second, fifth, and seventh places; the two Canadian athletes finishing in third and fourth places; and the three Mexican athletes finishing in first, sixth, and eighth places. Hence, by the partitions rule,

$$\binom{8}{3,\,2,\,3} = \frac{8!}{3!\,2!\,3!} = 560$$

results are possible. ■

Note that multinomial coefficients reduce to binomial coefficients in the case of two groups ($k = 2$). Indeed, if m_1 and m_2 are nonnegative integers whose sum is m, then

$$\binom{m}{m_1,\, m_2} = \frac{m!}{m_1!\,m_2!} = \frac{m!}{m_1!\,(m - m_1)!} = \binom{m}{m_1}.$$

This result makes sense from a conceptual point of view: The number of possible ordered partitions of m objects into two distinct groups of sizes m_1 and m_2 equals the number of ways that we can choose m_1 objects from the m objects to constitute the first group, which is $\binom{m}{m_1}$. The remaining $m - m_1 = m_2$ objects must constitute the second group.

Basic Exercises

3.23 Show for $k, j \in \mathcal{N}$, with $j \le k$, that $k! = k(k-1)\cdots(k-j+1)(k-j)!$.

3.24 Determine the value of

a) $(7)_3$. **b)** $(5)_2$. **c)** $(8)_4$. **d)** $(6)_0$. **e)** $(9)_9$.

3.25 At a movie festival, a team of judges is to pick the first, second, and third place winners from the 18 films entered. Use permutation notation to express the number of possibilities and then evaluate that expression.

3.26 Investment firms usually have a large selection of mutual funds from which an investor can choose. One such firm has 30 mutual funds. Suppose that you plan to invest in 4 of these mutual funds, 1 during each quarter of next year. Use permutation notation to express the number of possibilities and then evaluate that expression.

3.27 The sales manager of a clothing company needs to assign seven salespeople to seven different territories. How many possibilities are there for the assignments?

3.28 Determine the value of

a) $\binom{7}{3}$. **b)** $\binom{5}{2}$. **c)** $\binom{8}{4}$. **d)** $\binom{6}{0}$. **e)** $\binom{9}{9}$.

3.29 The Internal Revenue Service (IRS) decides that it will audit the returns of 3 people from a group of 18. Use combination notation to express the number of possibilities and then evaluate that expression.

3.30 At a lottery, 100 tickets were sold and three prizes are to be given. How many possible outcomes are there if

a) the prizes are equivalent? **b)** there is a first, second, and third prize?