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7.119

Let the three groups be : A_1, A_2, A_3, and their respective claim amounts be X_1, X_2, X_3

By law of total expectation, $E(X) = \sum_{i=1}^{3} E[X_i \mid A_i] \cdot P_A(A_i)$

$= 0.458 \cdot 1457 + 0.326 \cdot 2234 + 0.216 \cdot 2516 = 1939.046$

7.120

(a) Let the number of people in 1-hr period be R.V., $N \sim P(\lambda = 25.8)$

Let the avg_amount deposited perperson be $E(X \mid N) = 574$

$\therefore E(X) = E\left(E\left[\sum_{i=1}^{N} X_i \mid N\right]\right)$

By linearity principle, $= E\left(\sum_{i=1}^{N} E[X_i \mid N]\right)$

Based on the prompt, it's reasonable to assume that X_i and N are independent

$\therefore E(X) = E\left(\sum_{i=1}^{N} E(X_i)\right)$

$= E\left(E(X_i) \cdot \sum_{i=1}^{N} 1\right) = E(X_i) \cdot E(N) = 574 \cdot 25.8 = 14809.2$

(b) $Var(X) = E[Var(X \mid N)] + Var[E(X \mid N)]$

$= E[Var(X \mid N)] + \left(E\left[(E(X \mid N))^2\right] - \left[E(E(X \mid N))\right]^2\right)$

$= 25.8\,(3167)^2 + (574)^2 \cdot 25.8 - (14809.2)^2 \approx 47959212.36$

$\therefore \sigma = \sqrt{Var(X)} \approx 6925.26$

7.126 (a) $\because \bar{X}_n = E(X_i) = x$, and X_i, $i \in \{1 \dots n\}$ are identically distributed

$\therefore E(X_k \mid \bar{X}_n = x) = x$, where X_k represents any R.V. in X_i's

(b) $E(\bar{X}_n \mid \bar{X}_n = x) = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i \,\middle|\, \bar{X}_n = x\right)$

$\because$ indentically distributed $\therefore$ $= \frac{1}{n} \sum_{k=1}^{n} E(X_k \mid \bar{X}_n = x) = E(X_k \mid \bar{X}_n = x) = x$

8.3 $X = \tan(x) \Rightarrow X \in \mathbb{R}, \quad x \in (-\pi/2, \pi/2)$

$P(X = a) = P(\tan(x) = a) = P(x = \tan^{-1}(a))$

$= \frac{|x = \tan^{-1}(a)|}{|\Omega|} = \frac{0}{\pi} = 0 \Rightarrow X$ is a continuous R.V.

8.7 (a) $\because X$ is a continuous R.V.

$\therefore \forall k \in K, \ P(X = k) = 0$

$\Rightarrow P(X \in K) = \sum_{k \in K} (X = k) = 0$

(b) By definition 5.2 on P 178, X is a discrete R.V. if $P(X \in K) = 1$

$\because P(X \in K) = 0 \quad \therefore X$, a continuous R.V., can't be a discrete R.V.

8.16 (a): Let $X \sim U(-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow P_X(x) = \frac{1}{(\frac{\pi}{2}) - (-\frac{\pi}{2})} = \frac{1}{\pi}$; $Y = \tan(x)$

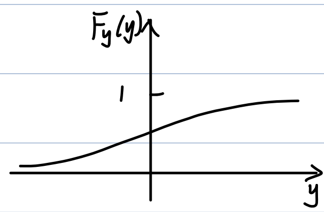
$F_Y(Y \leq y) = P(Y \leq y) = P(\tan(x) \leq y) = P(x \leq \tan^{-1} y) = F_X(\tan^{-1} y)$

By definition of cdf: $F_X(\tan^{-1} y) = \int_{-\frac{\pi}{2}}^{\tan^{-1}(y)} P_X(x)\, dx$

$F_Y(y) = \int_{-\frac{\pi}{2}}^{\tan^{-1}(y)} \frac{1}{\pi}\, dx = \frac{1}{\pi}[x]_{-\frac{\pi}{2}}^{\tan^{-1}(y)} = \frac{1}{\pi}(\tan^{-1}(y) + \frac{\pi}{2}) = \frac{1}{\pi}\tan^{-1}(y) + \frac{1}{2}$

(b) (a) By definition of $\tan^{-1}()$, $F_Y(y)$ is a non-decreasing function.

(b)



As shown to the left, a sketch of $F_Y(y)$, It's always right-continuous.

(c)/(d): As shown above, $\lim_{y \to -\infty} F_Y(y) = 0$ and $\lim_{y \to \infty} F_Y(y) = 1$.

(c) $\because F_Y(y)$ is definend everywhere on $\mathbb{R}$

$\therefore F_Y(y)$ is continuous everywhere

$\therefore$ By proposition 8.3 (P413), Y is a continuous R.V.

8.21 (a) $\because$ X is discrete

$\therefore$ PMF:

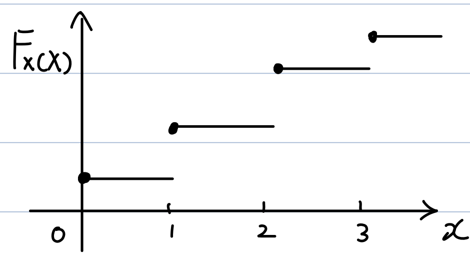
x	0	1	2	3
$p(X=x)$	$\binom{6}{3}\binom{5}{0}/\binom{11}{3} = \frac{4}{33}$	$\binom{6}{2}\binom{5}{1}/\binom{11}{3} = \frac{15}{33}$	$\binom{6}{1}\binom{5}{2}/\binom{11}{3} = \frac{12}{33}$	$\binom{6}{0}\binom{5}{3}/\binom{11}{3} = \frac{2}{33}$

$\therefore$ CDF:

$$F_x(X) = \begin{cases} 0 & x<0 \\ 4/33 & 0 \le x < 1 \\ 19/33 & 1 \le x < 2 \\ 31/33 & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

(b) (a) As shown, $F_x(x)$ is non-decreasing

(b)



As shown graphically, $F_x(X)$ is always right-continuous.

(c) $\because F_x(x) = 0$, $x<0$ $\therefore \lim_{x\to-\infty} F_x(x) = 0$

(d) $\because F_x(x) = 1$, $x \geq 3$ $\therefore \lim_{x\to\infty} F_x(x) = 1$

(c) By proposition 8.3 (p413), X is not a continuous R.V. because $F_x(x)$ is clearly not continuous.

8.33 (a) $F_Y(y) = P(Y \le y) = p(\max\{X_1, \dots X_m\} \le y) = p(X_1 \le y, \dots, X_m \le y)$

$= F_{X_1}(y) \cdot \dots \cdot F_{X_m}(y) = [F_X(y)]^m$

$\because$ Each X is identically distributed $\therefore F_Y(y) = [F_X(y)]^m$

(b) $F_Z(z) = P(Z \le z) = P(\min\{X_1 \dots X_m\} \le z) = 1 - P(X_1 \geq z \dots X_m \geq z)$

$= 1 - [(1 - F_{X_1}(z)) \cdot \dots \cdot (1 - F_{X_m}(z))]$

$\because$ Each X is identically distributed $\therefore F_Z(z) = 1 - [1 - F_Z(z)]^m$

8.43 (a) $F_X(X) = \frac{1}{\pi}\tan^{-1}(X) + \frac{1}{2}$

$\Rightarrow f_x(X) = \frac{1}{\pi}\left(\frac{1}{1+X^2}\right), \quad X \in \mathbb{R}$

(b) $f_x(x)$ has a Larger value when $X \to 0$.

(c) $p(X \leq 1) = \frac{1}{\pi}\int_{-\infty}^{1} \frac{1}{1+x^2}\, dx$

$= \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{-\infty}^{1}$

$= \frac{1}{\pi}\cdot\left[\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{\pi}\cdot\frac{3\pi}{4} = \frac{3}{4}$

(d) $F_X(1) = \frac{1}{\pi}\left(\tan^{-1}(1)\right) + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

8.67 (a) Guess that the distribution of Y is also uniform

(b) $g(x) = c + dX \Rightarrow g^{-1}(y) = \frac{Y-c}{d}$; $\because X \sim U(0,1) \therefore P_x(x) = 1, \forall x$

$F_y(y) = p(Y \leq y) = p(X \leq g^{-1}(y)) = p\left(X \leq \frac{Y-c}{d}\right) = F_x\left(\frac{Y-c}{d}\right)$

By definition, $F_x\left(\frac{Y-c}{d}\right) = \int_{-\infty}^{\frac{Y-c}{d}} P_x(x)\, dx$

$F_y(y) = \int_0^{Y-c/d} 1\, dx = [x]_0^{Y-c/d} = \frac{Y-c}{d}$

$\Rightarrow f_y(y) = F_y'(y) = \frac{1}{d} \Rightarrow Y \sim U(c, c+d)$

(c) $\because Y = c + dX \sim U(c, c+d) \sim U(a,b)$

$\therefore c = a, \quad d = b - c = b - a$

8.72 (a) $g(x) = bX \Rightarrow g^{-1}(y) = \frac{Y}{b}$; $X \sim Exp(\lambda = 1)$

$F_y(y) = p(Y \leq y) = p(X \leq g^{-1}(y)) = P\left(X \leq \frac{Y}{b}\right) = F_x\left(\frac{Y}{b}\right) = \int_0^{\frac{Y}{b}} 1 e^{-x} dx = \left[-e^{-x}\right] = -e^{-\frac{Y}{b}} + 1$

$P_Y(y) = \frac{d}{dy}\left(-e^{-\frac{Y}{b}} + 1\right) = \frac{Y}{b} e^{-\frac{Y}{b}}$

(b) $\int_0^x \lambda e^{-\lambda x}\, dx$ $\quad\therefore$ For $bX \sim Exp(\lambda)$, $-\frac{1}{b} = -\lambda$ must be true

$= \left[-e^{-\lambda x}\right]_0^x \quad\Rightarrow b = \frac{1}{\lambda}$

$= -e^{-\lambda x} + 1$