EXERCISES 9.7 Basic Exercises

9.146 Suppose that, in Example 9.25 on page 547, the times it takes the first and second engineers to inspect the item have $\Gamma(\alpha, \lambda)$ and $\Gamma(\beta, \lambda)$ distributions, respectively.
a) Use the bivariate transformation theorem to obtain a joint PDF of S and T.
b) Use the result of part (a) to obtain and identify a marginal PDF of S.
c) Use the result of part (a) to obtain and identify a marginal PDF of T.
d) Are S and T independent random variables? Justify your answer.

9.147 Suppose that, as in Example 9.25 on page 547, the times it takes the first and second engineers to inspect the item are both exponentially distributed but that the parameters for the exponential distributions differ; say that they are λ and μ, respectively, where $\lambda \neq \mu$.
a) Use the bivariate transformation theorem to obtain a joint PDF of S and T.
b) Use the result of part (a) to obtain a marginal PDF of S.
c) Use the result of part (a) to obtain a marginal PDF of T.
d) Are S and T independent random variables? Justify your answer.

9.148 Let X and Y be continuous random variables with a joint PDF. Apply the bivariate transformation theorem to obtain a PDF of the random variable Y/X. Compare your result with Equation (9.48) on page 540.

9.149 In the petri-dish illustration of Example 9.8 on page 511, let R and Θ denote the polar coordinates of the center of the first spot (visible bacterial colony) to appear.
a) Obtain a joint PDF of R and Θ.
b) Obtain and identify marginal PDFs of R and Θ.
c) Decide whether R and Θ are independent random variables.

9.150 Let U and V be independent $\mathcal{U}(0, 1)$ random variables and let a, b, c, and d be real numbers with $a < b$ and $c < d$.
a) Obtain and identify a joint PDF of $X = a + (b - a)U$ and $Y = c + (d - c)V$.
b) Apply your result from part (a) to explain how to simulate the random (uniform) selection of a point from the rectangle $(a, b) \times (c, d)$ by using a basic random number generator.

9.151 Range and midrange of a random sample: Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with CDF F and PDF f. The *range* and *midrange* of the random sample are defined to be $R = Y - X$ and $M = (X + Y)/2$, respectively, where $X = \min\{X_1, \ldots, X_n\}$ and $Y = \max\{X_1, \ldots, X_n\}$.
a) Apply the bivariate transformation theorem to obtain a joint PDF of R and M. *Note:* Equation (9.20) on page 498 provides a joint PDF of X and Y.
b) Use the result of part (a) to obtain a marginal PDF of R. Compare your answer to that found in Example 9.6 on page 504.
c) Use the result of part (a) to obtain a marginal PDF of M. Compare your answer to that presented in Exercise 9.116 on page 542.
d) Are R and M independent random variables? Justify your answer.

9.152 Let X and Y be independent $\mathcal{N}(0, \sigma^2)$ random variables. Show that the random variables $X^2 + Y^2$ and Y/X are independent.

9.153 Let X and Y be independent random variables with $X \sim \Gamma(\alpha, \lambda)$ and $Y \sim \Gamma(\beta, \lambda)$. Are $X + Y$ and Y/X independent? Justify your answer.

9.154 Suppose that X and Y are continuous random variables with a joint PDF.
a) Use the bivariate transformation theorem to obtain a joint PDF of the random variables $X + Y$ and $X - Y$.

10.9 Suppose that the random variable X has the beta distribution with parameters α and β. Show that $\mathcal{E}(X) = \alpha/(\alpha + \beta)$.

10.10 For a certain manufactured item, the proportion of the annual production that requires service during the first 5 years of use has the beta distribution with parameters $\alpha = 2$ and $\beta = 3$. Determine and interpret the expected proportion of the annual production of these manufactured items that require service during the first 5 years of use.

10.11 Because the expected value, μ_X, of a random variable X is often considered the "center" of the probability distribution of X, it's tempting to think that $P(X \geq \mu_X) = 1/2$. By using beta random variables with appropriate parameters, provide examples where
a) $P(X \geq \mu_X) = 1/2$. **b)** $P(X \geq \mu_X) > 1/2$. **c)** $P(X \geq \mu_X) < 1/2$.

10.12 Suppose that the random variable X has the triangular distribution on the interval (a, b). Show that $\mathcal{E}(X) = (a + b)/2$.

10.13 An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose PDF is proportional to $(1 + x)^{-4}$ for $x > 0$. Determine the company's expected monthly claims.

10.14 A company agrees to accept the highest of four sealed bids on a property that it owns. The four bids are regarded as independent random variables with common CDF given by $F(x) = (1 + \sin \pi x)/2$ for $3/2 \leq x < 5/2$. Which expression represents the expected value of the accepted bid?

a) $\dfrac{\pi}{2} \displaystyle\int_{3/2}^{5/2} x \cos \pi x \, dx$ **b)** $\dfrac{1}{16} \displaystyle\int_{3/2}^{5/2} x(1 + \sin \pi x)^4 \, dx$

c) $\dfrac{\pi}{4} \displaystyle\int_{3/2}^{5/2} \cos \pi x (1 + \sin \pi x)^3 \, dx$ **d)** $\dfrac{\pi}{4} \displaystyle\int_{3/2}^{5/2} x \cos \pi x (1 + \sin \pi x)^3 \, dx$

10.15 Let X have the Cauchy distribution with parameters η and θ. Show that X doesn't have finite expectation.

10.16 Let T denote the interior of the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 1)$. Suppose that X and Y have joint PDF given by $f_{X,Y}(x, y) = 2xy$ for $(x, y) \in T$ and $f(x, y) = 0$ otherwise. Determine $\mathcal{E}(X)$ and $\mathcal{E}(Y)$.

10.17 Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. A joint density function of T_1 and T_2 is constant over the region $0 < t_1 < 6$, $0 < t_2 < 6$, $t_1 + t_2 < 10$, and zero otherwise. Determine the expected time between a car accident and payment of the claim. *Hint:* Use Proposition 9.12 on page 534 to obtain a PDF of the time between a car accident and payment of the claim.

10.18 A number X is chosen at random from the interval $(0, 1)$. Then another number Y is chosen at random from the interval $(0, X)$.
a) Without doing any calculations, guess the value of $\mathcal{E}(Y)$.
b) Calculate $\mathcal{E}(Y)$. *Note:* First obtain a PDF of Y and then apply Definition 10.1 on page 565.

Theory Exercises

10.19 Bounded random variable: A random variable X is said to be *bounded* if there is a positive real number M such that $P(|X| \leq M) = 1$. Prove that a bounded continuous random variable (with a PDF), bounded by M, has finite expectation and that $|\mathcal{E}(X)| \leq M$. *Hint:* First show that you can assume that $f_X(x) = 0$ for $|x| > M$.

Referring to Figure 10.1, interchanging the order of integration, and applying the FPF, we get

$$\int_0^\infty \left(\int_0^x 1\,dy\right) f_X(x)\,dx = \int_0^\infty \left(\int_y^\infty f_X(x)\,dx\right) dy = \int_0^\infty P(X > y)\,dy.$$

The required result now follows easily. ♦

Note the following.

- Proposition 10.5 holds for any type of random variable—discrete, continuous, or otherwise.
- A version of Proposition 10.5 exists for random variables without the restriction of nonnegativity. See Exercise 10.45.

EXAMPLE 10.8 ***Using Tail Probabilities to Obtain Expected Values***

Exponential Random Variables Use tail probabilities to obtain the expected value of an exponential random variable with parameter λ.

Solution Let $X \sim \mathcal{E}(\lambda)$. From Proposition 8.8 on page 433, $P(X > x) = e^{-\lambda x}$ for $x > 0$. Thus, by Proposition 10.5,

$$\mathcal{E}(X) = \int_0^\infty P(X > x)\,dx = \int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda}.$$

Compare this simple computation using tail probabilities to a direct computation using the definition of expected value, as required by Example 10.2 on page 566. ■

EXERCISES 10.2 Basic Exercises

10.23 The speed of a gas molecule is $S = \sqrt{X^2 + Y^2 + Z^2}$, where X, Y, and Z denote the velocity components, which are independent $\mathcal{N}(0, \sigma^2)$ random variables. In Example 10.7 on page 574, we presented two ways to determine the expected value of S. Solve that problem a third way as follows:

a) Without doing any computations, explain why $W = (X^2 + Y^2 + Z^2)/\sigma^2$ has the chi-square distribution with three degrees of freedom.

b) Use your result from part (a) and the FEF to obtain $\mathcal{E}(S)$.

10.24 Let Z have the standard normal distribution. Obtain the expected value of $|Z|$

a) by first obtaining a PDF of $|Z|$ and then applying the definition of expected value.

b) by using the FEF.

10.25 Let T denote the interior of the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 1)$. Let X and Y have joint PDF given by $f_{X,Y}(x, y) = 2xy$ for $(x, y) \in T$, and $f(x, y) = 0$ otherwise. In Exercise 10.16 on page 570, we asked for $\mathcal{E}(X)$ and $\mathcal{E}(Y)$, requiring you first to obtain marginal PDFs of X and Y. Apply the FEF to determine $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ without first obtaining marginal PDFs of X and Y.

10.26 Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim.

A joint density function of T_1 and T_2 is constant over the region $0 < t_1 < 6$, $0 < t_2 < 6$, $t_1 + t_2 < 10$, and zero otherwise. In Exercise 10.17 on page 570, we asked for the expected time between a car accident and payment of the claim, $\mathcal{E}(T_1 + T_2)$, requiring you first to obtain a PDF of $T_1 + T_2$. Determine $\mathcal{E}(T_1 + T_2)$ without first obtaining a PDF of $T_1 + T_2$.

10.27 An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss follows a distribution with density function $f(y) = 2/y^3$ for $y > 1$, and $f(y) = 0$ otherwise. What is the expected value of the benefit paid under the insurance policy?

10.28 Let $X \sim \mathcal{U}(1, 2)$.

a) Determine $\mathcal{E}(1/X)$ by first obtaining a PDF of $1/X$.

b) Determine $\mathcal{E}(1/X)$ by using the FEF.

c) Compare $\mathcal{E}(1/X)$ and $1/\mathcal{E}(X)$. Conclude that, in general, $\mathcal{E}(1/X) \neq 1/\mathcal{E}(X)$.

10.29 A piece of equipment, whose lifetime is exponentially distributed with mean 10 years, is being insured against failure. The insurance will pay $\$x$ if failure occurs during the first year, $\$0.5x$ if failure occurs during the second or third year, and nothing if failure occurs after the first 3 years. At what level must x be set if the expected payment made under this insurance is to be \$1000?

10.30 Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(0, 1)$. Set $X = \min\{X_1, \ldots, X_n\}$, $Y = \max\{X_1, \ldots, X_n\}$, $R = Y - X$ (the range of the random sample), and $M = (X + Y)/2$ (the midrange of the random sample). In Example 9.4(c) on page 497, we obtained a joint PDF of X and Y. Use that joint PDF and the FEF to determine

a) $\mathcal{E}(X)$. **b)** $\mathcal{E}(Y)$. **c)** $\mathcal{E}(R)$. **d)** $\mathcal{E}(M)$.

10.31 Refer to Exercise 10.30.

a) In Example 9.9(c) on page 512, we showed that X has the beta distribution with parameters 1 and n. Use that fact to obtain $\mathcal{E}(X)$.

b) In Exercise 9.70(c) on page 520, we asked you to obtain the marginal distribution of Y, which is the beta distribution with parameters n and 1. Use that fact to obtain $\mathcal{E}(Y)$.

c) Use properties of expected value and your results from parts (a) and (b) to obtain $\mathcal{E}(R)$.

d) Use properties of expected value and your results from parts (a) and (b) to obtain $\mathcal{E}(M)$.

10.32 Let X and Y be random losses, in thousands of dollars, with joint density function $f_{X,Y}(x, y) = 2x$ for $0 < x < 1$ and $0 < y < 1$, and $f_{X,Y}(x, y) = 0$ otherwise. An insurance policy with a deductible of 1 is written to cover the loss $X + Y$. What is the expected payment under this policy?

10.33 For purposes of quality assurance, an expensive item produced at a manufacturing plant is independently inspected by two engineers.

a) Assume that the amount of time it takes each engineer to inspect the item has the exponential distribution with parameter λ. Use the result of Example 9.25(b) on page 547 to show that, on average, 50% of the total inspection time is attributed to the first engineer.

b) Now assume only that the probability distribution of the amount of time it takes to inspect the item is the same for both engineers. Show that the result obtained in part (a) of this problem still holds.

10.34 A stick of length ℓ is randomly broken into two pieces.

a) Determine and identify a PDF of the length, X, of the shorter piece.

b) Use your result from part (a) and the FEF to find the expected value of the ratio, R, of the shorter piece to the longer piece.

c) Use your result from part (a) to obtain a PDF of R.

We presented several important properties of the correlation coefficient in Proposition 7.16 on page 372, which you should now review. One of those properties states that the correlation coefficient of independent random variables equals 0. Random variables that have zero correlation are said to be **uncorrelated random variables.** Although independent random variables are uncorrelated, uncorrelated random variables need not be independent. See, for instance, Exercise 10.52.

EXAMPLE 10.16 ***The Correlation Coefficient***

Obtain the correlation coefficient of the random variables X and Y from Example 10.15.

Solution From Example 10.15, we have $\operatorname{Var}(X) = \sigma_1^2$, $\operatorname{Var}(Y) = \sigma_2^2$, and $\operatorname{Cov}(X, Y) = \rho\sigma_1\sigma_2$. Therefore,

$$\rho(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X) \cdot \operatorname{Var}(Y)}} = \frac{\rho\sigma_1\sigma_2}{\sqrt{\sigma_1^2\sigma_2^2}} = \frac{\rho\sigma_1\sigma_2}{\sigma_1\sigma_2} = \rho.$$

This result shows that we can construct random variables with any specified correlation coefficient, ρ, where, of course, $|\rho| \leq 1$. ■

EXERCISES 10.3 **Basic Exercises**

10.51 Suppose that a tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive). How does that increase translate into the percentage increase in
a) the mean annual cost?
b) the variance of the annual cost?
c) the standard deviation of the annual cost?

10.52 Refer to the petri-dish illustration of Example 10.1 on page 565. Let X and Y denote the x and y coordinates, respectively, of the first spot (visible bacteria colony) to appear.
a) In Example 9.8(b) on page 511, we obtained a PDF of X. Use that PDF to find $\operatorname{Var}(X)$. *Note:* From Exercise 10.4 on page 569, $\mathcal{E}(X) = 0$.
b) Without doing further computations, find $\operatorname{Var}(Y)$. Explain your reasoning.
c) Obtain $\rho(X, Y)$.
d) From part (c), the random variables X and Y are uncorrelated. Does this result imply that X and Y are independent? Explain your answer.
e) Are X and Y independent random variables? Justify your answer.

10.53 A commuter train arrives punctually at a station every half hour. Each morning, a commuter named John leaves his house and casually strolls to the train station. Let X denote the amount of time, in minutes, that John waits for the train from the time he reaches the train station. In Example 8.10 on page 428, we found that a PDF of X is $f_X(x) = 1/30$ for $0 < x < 30$, and $f_X(x) = 0$ otherwise. Use that result to obtain the variance of the random variable X. *Note:* From Exercise 10.5 on page 569, $\mathcal{E}(X) = 15$.

10.54 Suppose that the random variable X has the uniform distribution on the interval (a, b).
a) Show that $\operatorname{Var}(X) = (b - a)^2/12$.
b) Apply your result from part (a) to obtain a quick solution of Exercise 10.53.

10.55 According to the text *Rhythms of Dialogue* by J. Jaffee and S. Feldstein (New York: Academic Press, 1970), the duration, in seconds, of a pause during a monologue has an exponential distribution with parameter 1.4. Determine the standard deviation of the duration of a pause during a monologue.

10.56 As reported in *Runner's World* magazine, the times of the finishers in the New York City 10 km run are normally distributed with $\mu = 61$ minutes and $\sigma = 9$ minutes. What is the standard deviation of the finishing times?

10.57 Suppose that the random variable X has the beta distribution with parameters α and β. Show that $\operatorname{Var}(X) = \alpha\beta/(\alpha+\beta)^2(\alpha+\beta+1)$.

10.58 For a certain manufactured item, the proportion of the annual production that requires service during the first 5 years of use has the beta distribution with parameters $\alpha = 2$ and $\beta = 3$. Determine the standard deviation of the proportion of the annual production of these manufactured items that require service during the first 5 years of use.

10.59 Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. A joint density function of T_1 and T_2 is constant over the region $0 < t_1 < 6$, $0 < t_2 < 6$, $t_1 + t_2 < 10$, and zero otherwise. Determine the standard deviation of the time between a car accident and payment of the claim
a) by first obtaining a PDF of the time between a car accident and payment of the claim.
b) without first obtaining a PDF of the time between a car accident and payment of the claim.

10.60 A number X is chosen at random from the interval $(0, 1)$. Then another number Y is chosen at random from the interval $(0, X)$. Obtain $\operatorname{Var}(Y)$ by first finding a PDF of Y and then applying the definition of the variance of a random variable.

10.61 For purposes of quality assurance, an expensive item produced at a manufacturing plant is independently inspected by two engineers. Let X and Y denote the proportions of the total inspection time attributed to the first and second engineers, respectively.
a) Use properties of variance to show that $\operatorname{Var}(Y) = \operatorname{Var}(X)$.
b) Use part (a) and properties of variance to show that $\operatorname{Cov}(X, Y) = -\operatorname{Var}(X)$.
c) Use properties of covariance to show that $\operatorname{Cov}(X, Y) = -\operatorname{Var}(X)$.
d) Conclude from parts (a) and (b) that $\rho(X, Y) = -1$.
e) Obtain the result of part (d) by using properties of the correlation coefficient.

10.62 The warranty on a machine whose mean lifetime is 3 years specifies that the machine will be replaced at failure or at age 4 years, whichever occurs first. Determine the standard deviation of the age of the machine at the time of replacement if the distribution of the lifetime of the machine is
a) uniform. **b)** triangular. **c)** exponential.

10.63 A joint density function of X and Y is given by $f_{X,Y}(x, y) = cx$ for $0 < x < 1$ and $0 < y < 1$, and $f_{X,Y}(x, y) = 0$ otherwise. Here c is a constant. Obtain $\operatorname{Cov}(X, Y)$
a) by using the joint PDF of X and Y.
b) without doing any computations.

10.64 The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of \$250. In the event that the automobile is damaged, repair costs (in dollars) can be modeled by a uniform random variable on the interval $(0, 1500)$. Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

10.65 A stock market analyst knows from experience that the mean daily sales revenues for Companies A and B are both 100. Furthermore, a daily sales revenue above 100 for

We emphasize that the results obtained in this section for continuous random variables hold in general:

> The law of total expectation, the law of total variance, and the prediction theorem are valid for any types or combination of types of random variables.

Conditional Expectation: Multivariate Case

Conditional expectation, conditional variance, and related concepts can be easily extended to the multivariate case. For instance, suppose that X, Y, and Z are continuous random variables with a joint PDF. Then the **conditional expectation of Z given $X = x$ and $Y = y$**, denoted $\mathcal{E}(Z \mid X = x,\ Y = y)$, is defined to be the expected value of Z relative to the conditional distribution of Z given $X = x$ and $Y = y$. Thus, if $f_{X,Y}(x, y) > 0$,

$$\mathcal{E}(Z \mid X = x,\ Y = y) = \int_{-\infty}^{\infty} z f_{Z \mid X,Y}(z \mid x, y)\, dz,$$

provided, of course, that $Z_{|X=x,Y=y}$ has finite expectation. If $f_{X,Y}(x, y) = 0$, we define $\mathcal{E}(Z \mid X = x,\ Y = y) = 0$ but don't refer to $\mathcal{E}(Z \mid X = x,\ Y = y)$ as a conditional expectation. The law of total expectation remains valid in the multivariate case. For instance, as you are asked to show in Exercise 10.92, $\mathcal{E}(Z) = \mathcal{E}\big(\mathcal{E}(Z \mid X, Y)\big)$.

EXERCISES 10.4 Basic Exercises

10.77 A number X is chosen at random from the interval $(0, 1)$. Then another number Y is chosen at random from the interval $(0, X)$.

a) Determine the conditional expectation of Y and the conditional variance of Y for each possible value of X.

b) In Exercises 10.18 and 10.60 on pages 570 and 593, we asked you to find $\mathcal{E}(Y)$ and $\mathrm{Var}(Y)$, respectively, by first obtaining a PDF of Y. Use the laws of total expectation and total variance to find $\mathcal{E}(Y)$ and $\mathrm{Var}(Y)$.

10.78 The stock prices of two companies at the end of any specified year are modeled with random variables X and Y that follow a distribution with joint density function $f_{X,Y}(x, y) = 2x$ for $x < y < x + 1$ and $0 < x < 1$, and $f_{X,Y}(x, y) = 0$ otherwise. Determine $\mathcal{E}(Y \mid X = x)$ and $\mathrm{Var}(Y \mid X = x)$ for all $x \in \mathcal{R}$.

10.79 Let T denote the interior of the triangle with vertices at $(0, 0)$, $(2, 0)$ and $(2, 1)$. Let X and Y have joint PDF given by $f_{X,Y}(x, y) = 2xy$ for $(x, y) \in T$, and $f_{X,Y}(x, y) = 0$ otherwise. Use the law of total expectation to determine $\mathcal{E}(X)$.

10.80 Consider two electrical components, A and B, with respective lifetimes X and Y. Assume that a joint PDF of X and Y is given by $f_{X,Y}(x, y) = \lambda\mu\, e^{-(\lambda x + \mu y)}$ for positive x and y, and $f_{X,Y}(x, y) = 0$ otherwise, where λ and μ are positive constants.

a) Determine $\mathcal{E}(Y \mid X = x)$ and $\mathrm{Var}(Y \mid X = x)$ for each possible value x of X.

b) Referring to part (a), why don't $\mathcal{E}(Y \mid X = x)$ and $\mathrm{Var}(Y \mid X = x)$ depend on x?

c) Use the law of total expectation to obtain $\mathcal{E}(Y)$.

d) Use the law of total variance to obtain $\mathrm{Var}(Y)$.