10.104 For a certain population, the blood pressure $(X, Y) =$ (systolic, diastolic) can be modeled as having a $\mathcal{BVN}(120, 400, 82, 225, 0.6)$ distribution.
a) Obtain a joint PDF of X and Y.
b) Identify and interpret the marginal distribution of X.
c) Identify and interpret the marginal distribution of Y.
d) Identify and interpret the correlation coefficient of X and Y.
e) Obtain, identify, and interpret the conditional distribution of Y given $X = x$ (for $x > 0$).
f) Find the predicted diastolic pressure for an individual whose systolic pressure is 123.
g) Find the probability that an individual with a diastolic pressure of 86 will have a systolic pressure exceeding 130.

10.105 Scientists concerned with environmental issues have long been interested in the sources of acid rain. Nitrates are a constituent of acid rain with arsenic as an accompanying element. Research by J. D. Scudlark and T. M. Church, published in the paper "The Atmospheric Deposition of Arsenic and Association with Acid Precipitation" (*Atmospheric Environment,* 1989, Vol. 22, pp. 937–943), suggests that the variables nitrate concentration (X), in micromoles per liter, and arsenic concentration (Y), in nanomoles per liter, have the parameters $\mu_X = 49.8$, $\sigma_X = 30.7$, $\mu_Y = 2.26$, $\sigma_Y = 1.18$, and $\rho = 0.732$. Suppose that you model the joint distribution of X and Y with a bivariate normal distribution.
a) Obtain a joint PDF of X and Y.
b) Identify and interpret the marginal distribution of X.
c) Identify and interpret the marginal distribution of Y.
d) Identify and interpret the correlation coefficient of X and Y.
e) Obtain, identify, and interpret the conditional distribution of Y given $X = x$ (for $x > 0$).
f) Determine and graph the regression equation for predicting arsenic concentration from nitrate concentration.
g) Predict the arsenic concentration for a nitrate concentration of 60 micromoles per liter.

10.106 Refer to Examples 10.21 and 10.22 on pages 611 and 615, respectively.
a) Predict the height of an eldest son whose mother is $5'\,8''$ tall.
b) Predict the height of a mother whose eldest son is $5'\,8''$ tall.
c) What is the probability that the eldest son of a $5'\,2''$-tall mother will be taller than $6'$?
d) Determine a 95% *prediction interval* for the height of the eldest son of a $5'\,2''$-tall mother. In other words, find an interval which is symmetric about the expected height of the eldest son of a $5'\,2''$-tall mother and is such that the probability is 0.95 that the son's height will lie in that interval.

10.107 Let X and Y be bivariate normal random variables.
a) Show that any nonzero linear combination of X and Y is also normally distributed. *Hint:* Refer to Equations (10.55) on page 608.
b) Use part (a) and properties of means, variances, and covariances to determine the probability distributions of the random variables $X + Y$ and $Y - X$.

10.108 Refer to Example 10.21 on page 611, where we considered the joint distribution of the heights of mothers and eldest sons.
a) Identify the probability distribution of the average height of a mother and her eldest son. *Hint:* Refer to Exercise 10.107.
b) Identify the probability distribution of the difference between the height of an eldest son and his mother.
c) On average, how much taller is an eldest son than his mother?
d) Find the probability that an eldest son is taller than his mother.
e) Find the probability that an eldest son's height exceeds 110% of his mother's height.

c) Suppose that X and Y are independent random variables with $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$. Use moment generating functions to show that $X + Y \sim \mathcal{P}(\lambda + \mu)$.

d) Extend the result of part (c) for m independent Poisson random variables.

11.3 Let X have the geometric distribution with parameter p.

a) Determine the MGF of X, including where it's defined.

b) Use the result of part (a) to obtain the mean and variance of X.

11.4 Let X have the uniform distribution on the interval (a, b).

a) Determine the MGF of X.

b) Use the result of part (a) to obtain the mean and variance of X.

11.5 A company insures homes in three cities. Because sufficient distance separates the cities, it's reasonable to assume that the losses occurring among the three cities are independent random variables. The moment generating functions for the three loss distributions are $(1-2t)^{-3}$, $(1-2t)^{-2.5}$, and $(1-2t)^{-4.5}$. Determine the third moment of the combined losses from the three cities.

11.6 Let X be a Cauchy random variable. Does there exist an open interval containing 0 for which M_X is defined? Explain your answer.

11.7 In Example 9.20 on page 536, we showed that, if X and Y are independent random variables with $X \sim \Gamma(\alpha, \lambda)$ and $Y \sim \Gamma(\beta, \lambda)$, then $X + Y \sim \Gamma(\alpha + \beta, \lambda)$.

a) Use MGFs to establish the result referred to and compare your work with that required in Example 9.20.

b) Generalize the result of part (a) to m independent gamma random variables with the same second parameter, thus providing a simple proof of Proposition 9.13 on page 537.

11.8 Suppose that Y has the lognormal distribution with parameters μ and σ^2, which means that $\ln Y \sim \mathcal{N}(\mu, \sigma^2)$. Determine a formula for the nth moment of Y by

a) using a PDF of Y.

b) using the FEF and a PDF of a normal random variable.

c) recalling the formula for the MGF of a normal random variable.

d) For what values of t is the MGF of Y defined? Justify your answer.

e) As we mentioned on page 634, if the MGF of a random variable is defined (exists) in some open interval containing 0, the random variable has moments of all orders. Is the converse of this statement true? Explain your answer.

11.9 For each $n \in \mathcal{N}$, let X_n have the discrete uniform distribution on $\{-1 + 1/n, 1 - 1/n\}$.

a) Heuristically, what is the probability distribution of a random variable X to which $\{X_n\}_{n=1}^{\infty}$ converges in distribution?

b) Mathematically verify your heuristics in part (a) by showing that $M_{X_n}(t) \to M_X(t)$ as $n \to \infty$ for all $t \in \mathcal{R}$.

c) Mathematically verify your heuristics in part (a) by showing that $F_{X_n}(x) \to F_X(x)$ as $n \to \infty$ for all $x \in \mathcal{R}$ at which F_X is continuous.

11.10 Let $X_1, X_2, \ldots$ be independent random variables, each having a $\mathcal{U}(0, 1)$ distribution. For each $n \in \mathcal{N}$, let $U_n = \min\{X_1, \ldots, X_n\}$ and let $V_n = \max\{X_1, \ldots, X_n\}$.

a) Heuristically, what happens to the probability distribution of U_n as $n \to \infty$? Explain your reasoning.

b) Show that $\{U_n\}_{n=1}^{\infty}$ converges in distribution and identify its limiting distribution.

c) Repeat parts (a) and (b) for $\{V_n\}_{n=1}^{\infty}$.

11.11 Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, each having mean μ, variance σ^2, and moment generating function M defined for $-t_0 < t < t_0$.

EXERCISES 11.2 Basic Exercises

11.21 Let X and Y denote the x and y coordinates of a point selected at random from the vertices of the unit square.

a) Determine the joint MGF of X and Y.

b) Use your result from part (a) to determine the means and variances of X and Y and their covariance.

c) Use your result from part (a) to obtain the marginal MGFs of X and Y.

d) Use your results from parts (a) and (c) to decide whether X and Y are independent random variables.

e) Use your result from part (c) to identify the marginal probability distributions of X and Y.

11.22 Let X and Y denote the x and y coordinates of a point selected at random from the unit square.

a) Determine the joint MGF of X and Y.

b) Use your result from part (a) to determine the means and variances of X and Y and their covariance.

c) Use your result from part (a) to obtain the marginal MGFs of X and Y.

d) Use your results from parts (a) and (c) to decide whether X and Y are independent.

e) Use your result from part (c) to identify the marginal probability distributions of X and Y.

11.23 In Example 9.19 on page 535, we showed that the sum of two independent uniform random variables on the interval $(0, 1)$ has the triangular distribution on the interval $(0, 2)$. Use that result and results from Exercise 11.22 to show that the MGF of a triangular distribution on the interval $(0, 2)$ is given by $2e^t t^{-2}(\cosh t - 1)$, where $\cosh t = (e^t + e^{-t})/2$, the *hyperbolic cosine* of t.

11.24 Suppose that X and Y are random variables defined on the same sample space and set $\psi_{X,Y}(s, t) = \ln M_{X,Y}(s, t)$.

a) Show that $\operatorname{Cov}(X, Y) = \frac{\partial^2 \psi_{X,Y}}{\partial s \partial t}(0, 0)$.

b) Show that, if X and Y are bivariate normal, then $\operatorname{Cov}(X, Y) = \frac{\partial^2 \psi_{X,Y}}{\partial s \partial t}(s, t)$ for all s and t.

11.25 Suppose that the joint MGF of X and Y is $M_{X,Y}(s, t) = (pe^s + qe^t)^n$, where p and q are positive numbers whose sum is 1 and n is a positive integer.

a) Obtain the marginal MGFs of X and Y and use them to identify the (marginal) probability distributions of X and Y.

b) Use your results from part (a) to decide whether X and Y are independent.

c) Use MGFs to identify the probability distribution of $X + Y$.

11.26 Let X and Y be bivariate normal random variables.

a) Use MGFs to show that any nonzero linear combination of X and Y is normally distributed.

b) Use the MGF of a nonzero linear combination of X and Y, as obtained in part (a), to identify its mean and variance.

11.27 Let X and Y have joint PDF given by $f_{X,Y}(x, y) = |x - y|e^{-(x+y)}$ for positive x and y, and $f_{X,Y}(x, y) = 0$ otherwise.

a) Obtain the joint MGF of X and Y.

b) Use your result from part (a) to determine the means and variances of X and Y and their covariance.

c) Use your result from part (a) to obtain the marginal MGFs of X and Y.

d) Use your results from parts (a) and (c) to decide whether X and Y are independent.

e) Use your result from part (a) to find and identify the probability distribution of $X + Y$.

11.40 Consistency of sample moments: Let X be a random variable with finite rth moment. Suppose that you don't know the rth moment of X but want to estimate it. To do so, you take a random sample, $X_1, \ldots, X_n$, from the distribution of X and form the *rth sample moment*, $(X_1^r + \cdots + X_n^r)/n$. Show that the rth sample moment is a consistent estimator of the rth moment.

11.41 A particle of initial size s is subjected to repeated impacts. After impact j, a proportion X_j of the particle remains; that is, if Y_j denotes the size of the particle after impact j, then $Y_j = X_j Y_{j-1}$. Assume that $X_1, X_2, \ldots$ are independent random variables all with the same probability distribution as a random variable X whose natural logarithm has finite mean.

a) Show that $\lim_{n\to\infty}(Y_n/s)^{1/n} = e^{\mathcal{E}(\ln X)}$ with probability 1.

b) Deduce from part (a) that, roughly, $Y_n \approx s e^{n\mathcal{E}(\ln X)}$ for large n.

c) Specialize the results of parts (a) and (b) to the cases where $X \sim \mathcal{U}(0, 1)$, $X \sim \text{Beta}(2, 1)$, $X \sim \text{Beta}(1, 2)$, and $X \sim \text{Beta}(2, 2)$.

11.42 Empirical distributions: Let $X_1, X_2, \ldots$ be independent random variables, all having the same probability distribution as a random variable X. For each $n \in \mathcal{N}$, we define the *empirical distribution function* based on the sample of size n to be

$$\hat{F}_n(x) = \frac{N(\{\, 1 \le j \le n : X_j \le x \,\})}{n}, \qquad x \in \mathcal{R},$$

where $N(S)$ denotes the number of elements of a finite set S. Show that, for each $x \in \mathcal{R}$, $\lim_{n\to\infty} \hat{F}_n(x) = F_X(x)$ with probability 1. Interpret this result.

11.43 Monte Carlo integration: Suppose that you want to obtain the value of $\int_0^1 g(x)\,dx$, where g is a Riemann integrable function on the interval $[0, 1]$. Further assume that no simple formula exists for the antiderivative of g.

a) Let $X \sim \mathcal{U}(0, 1)$. Explain why the random variable $g(X)$ has finite expectation.

b) Let $X_1, X_2, \ldots$ be independent and identically distributed $\mathcal{U}(0, 1)$ random variables. Show that $\lim_{n\to\infty} \left(g(X_1) + \cdots + g(X_n)\right)/n = \int_0^1 g(x)\,dx$ with probability 1.

c) Explain how to use a basic random number generator—that is, a random number generator that simulates a $\mathcal{U}(0, 1)$ random variable—to approximate the value of $\int_0^1 g(x)\,dx$.

d) **Simulation:** This part requires access to a computer or graphing calculator. Apply your procedure from part (c) to estimate $\int_0^1 x^3\,dx$ based on 1000 uniform random numbers. Compare your answer to the exact value of the integral.

e) **Simulation:** This part requires access to a computer or graphing calculator. Let Z have the standard normal distribution. Apply your procedure from part (c) to estimate $P(0 \le Z \le 1)$ based on 1000 uniform random numbers. Compare your answer to that obtained by consulting Table I in the Appendix.

f) **Simulation:** Repeat parts (d) and (e) based on 10,000 uniform random numbers.

11.44 Monte Carlo integration (continued): Suppose that you want to determine the value of $\int_a^b g(x)\,dx$, where g is a Riemann integrable function on the interval $[a, b]$. Further assume that no simple formula exists for the antiderivative of g.

a) Explain how to use a basic random number generator—that is, a random number generator that simulates a $\mathcal{U}(0, 1)$ random variable—to approximate the value of $\int_a^b g(x)\,dx$. *Hint:* Refer to Exercise 11.43.

b) **Simulation:** This part requires access to a computer or graphing calculator. Apply your procedure from part (a) to estimate $\int_1^3 x^3\,dx$ based on 1000 uniform random numbers. Compare your answer to the exact value of the integral.

Other Central Limit Theorems

In this section, we examined the central limit theorem in its classical form. Many other versions of the central limit theorem have now been established and can be found in the extensive literature on the subject. These versions weaken, in one way or another, the independence and/or identically-distributed assumptions of the classical central limit theorem, thus providing normal approximations in contexts where the random variables under consideration may not be independent or may not be identically distributed.

EXERCISES 11.4 Basic Exercises

11.55 As reported by a spokesperson for Southwest Airlines, the no-show rate for reservations is 16%—that is, the probability is 0.16 that a person making a reservation will not take the flight. For a certain flight, 42 people have reservations. For each part, determine and compare the exact probability by using the appropriate binomial PMF and an approximate probability by using the integral De Moivre–Laplace theorem (in the form of Proposition 11.11 on page 661). The probability that the number of people who don't take the flight is
a) exactly 5. **b)** between 9 and 12, inclusive. **c)** at least 1. **d)** at most 2.

11.56 In Exercise 8.108 on page 449, you were asked to determine and compare the exact probability of each event in Exercise 11.55 by using the appropriate binomial PMF and an approximate probability by using the local De Moivre–Laplace theorem (in the form of Proposition 8.12 on page 445).
a) If you didn't previously do Exercise 8.108, do it now.
b) Compare the results for all three methods of obtaining the probabilities by constructing a table similar to Table 11.2 on page 663.
c) Discuss your results from part (b) in light of the rule of thumb for using the normal approximation to the binomial distribution.

11.57 Refer to the roulette illustration of Example 11.20 on page 670. Use the integral De Moivre–Laplace theorem to estimate the probability that the gambler will be ahead after
a) 100 bets. **b)** 1000 bets. **c)** 5000 bets.

11.58 Some boys want to play football at a park that doesn't have a formal football field. In an attempt to approximate the length of a football field, one of the boys tries to step-off 100 yards. In actuality, his steps are independent and identically distributed random variables with mean 0.95 yard and standard deviation 0.08 yard. Determine the approximate probability that the distance stepped-off by the boy is within
a) 4 yards of the length of a football field.
b) 6 yards of the length of a football field.

11.59 A brand of flashlight battery has normally distributed lifetimes with a mean of 30 hours and a standard deviation of 5 hours. A supermarket purchases 500 of these batteries from the manufacturer. What is the probability that at least 80% of them will last longer than 25 hours? Use the integral De Moivre–Laplace theorem.

11.60 The checkout times at the local food market are independent random variables having mean 3.5 minutes and standard deviation 1.5 minutes.
a) Determine the probability that it will take at least 6 hours to check out 100 customers.
b) Find the probability that the mean checkout time for 100 customers is less than 3.4 minutes.

11.61 In a large city, annual household incomes have a mean of \$35,216 and a standard deviation of \$3,134.

a) What is the probability that the average income of 160 randomly chosen households is below \$34,600?
b) Strictly speaking, in part (a), are the assumptions for the central limit theorem satisfied? Explain your answer.
c) Why is it permissible to use the central limit theorem to solve part (a)?

11.62 The claim amount for a health insurance policy follows a distribution with density function $f(x) = (1/1000)e^{-x/1000}$ for $x > 0$, and $f(x) = 0$ otherwise. The premium for the policy is set at 100 over the expected claim amount. Suppose that 100 policies are sold and that claim amounts are independent of one another.

a) Identify the exact probability distribution of the total claim amount for the 100 policies.
b) Use your answer from part (a) to obtain an expression that gives the exact probability that the insurance company will have claims exceeding the premiums collected.
c) Use the central limit theorem to obtain the approximate probability that the insurance company will have claims exceeding the premiums collected.
d) If you have access to statistical software, use it and your answer from part (a) to obtain the probability that the insurance company will have claims exceeding the premiums collected. Compare this probability to that found in part (c).

11.63 Let X and Y denote the number of hours that a randomly selected person watches movies and sporting events, respectively, during a 3-month period. Assume that $\mathcal{E}(X) = 50$, $\mathcal{E}(Y) = 20$, $\mathrm{Var}(X) = 50$, $\mathrm{Var}(Y) = 30$, and $\mathrm{Cov}(X, Y) = 10$. If 100 people are randomly selected and observed for 3 months, what is the probability that the total number of hours that they watch movies or sporting events is at most 7100?

11.64 A hardware manufacturer knows from experience that 95% of the screws produced by his company are within tolerance specifications. Each shipment of 10,000 screws comes with a warranty that promises a complete refund if more than r screws aren't within tolerance specifications. How small can r be chosen so that no more than 1% of shipments will require complete refunds? Solve this problem by

a) using Chebyshev's inequality. b) using the central limit theorem.
c) If you have access to statistical software, use it to obtain the exact value of r and compare your answer to those found in parts (a) and (b).

11.65 An air-conditioning contractor plans to offer service contracts on the brand of compressor used in all of the units her company installs. First she must estimate how long those compressors last, on average. To that end, the contractor consults records on the lifetimes of 250 previously used compressors. She plans to use the sample mean lifetime of those compressors as her estimate for the mean lifetime of all such compressors. If the lifetimes of this brand of compressor have a standard deviation of 40 months, what is the probability that the contractor's estimate will be within 5 months of the true mean?

11.66 A particle of initial size s is subjected to repeated impacts. After impact j, a proportion X_j of the particle remains; that is, if Y_j denotes the size of the particle after impact j, then $Y_j = X_j Y_{j-1}$. The random variables $X_1, X_2, \ldots$ are independent, all having the same probability distribution as a random variable X.

a) Use the central limit theorem to identify the approximate probability distribution of Y_n for large n. State explicitly any assumptions that you make.
b) Specialize the results obtained in part (a) to the cases where $X \sim \mathcal{U}(0, 1)$, $X \sim \mathrm{Beta}(2, 1)$, $X \sim \mathrm{Beta}(1, 2)$, and $X \sim \mathrm{Beta}(2, 2)$.