7.5 Expected value as center of gravity: Let X be a random variable with finite range, say, $\{x_1, \ldots, x_m\}$. Set $p_k = p_X(x_k)$ for $k = 1, 2, \ldots, m$. Think of the x-axis as a seesaw and each p_k as a mass placed at point x_k on the seesaw. The *center of gravity* of these masses is defined to be the point $\bar{x}$ on the x-axis at which a fulcrum could be placed to balance the seesaw. Relative to the center of gravity, the torque acting on the seesaw by the mass p_k is proportional to the product of that mass and the signed distance of the point x_k from $\bar{x}$. Show that the center of gravity equals the expected value of X; that is, $\bar{x} = \mathcal{E}(X)$. *Hint:* To balance, the total torque acting on the seesaw must be 0.

7.6 Consider a random experiment with a finite number of possible outcomes and let Ω be the sample space.

a) For a random variable X defined on Ω, show that $\mathcal{E}(X) = \sum_{\omega \in \Omega} X(\omega) P(\{\omega\})$.

b) Show that, if the outcomes are equally likely, the expression in part (a) reduces to the arithmetic average of the possible observations of the random variable X.

7.7 A variable of a finite population has mean 82.4.

a) Let X denote the value of the variable for a randomly selected member of the population. Find the expected value of X.

b) If a random sample of size n is taken from the population with replacement and the value of the variable is observed each time, roughly what will be the sum of the observed values?

7.8 As part of a screening exam for prospective insured, a physician conducts tests for acute problems. On average, the first positive test occurs with the 25th person tested by the physician.

a) Determine the probability that the first positive test occurs by the sixth person tested.

b) On average, how many people must the physician test until obtaining the fifth positive test?

7.9 Pinworm infestation, commonly found in children, can be treated with the drug pyrantel pamoate. According to the *Merck Manual,* the treatment is effective in 90% of cases. If 20 children with pinworm infestation are treated with pyrantel pamoate, what is the expected number cured?

7.10 According to the *Daily Racing Form,* the probability is about 0.67 that the favorite in a horse race will finish in the money (first, second, or third place). Determine the smallest number of races required so that the expected number of times that the favorite finishes in the money is at least 10.

7.11 Sickle cell anemia is an inherited blood disease that occurs primarily in blacks. In the United States, about 15 of every 10,000 black children have sickle cell anemia. The red blood cells of an affected person are abnormal; the result is severe chronic anemia, which causes headaches, shortness of breath, jaundice, increased risk of pneumococcal pneumonia and gallstones, and other severe problems. Sickle cell anemia occurs in children who inherit an abnormal type of hemoglobin, called hemoglobin S, from both parents. If hemoglobin S is inherited from only one parent, the person is said to have sickle cell trait and is generally free from symptoms. There is a 50% chance that a person who has sickle cell trait will pass hemoglobin S to an offspring. If two people who have sickle cell trait have five children, how many children should they expect to have sickle cell anemia?

7.12 An upper-level probability class has six undergraduate students and four graduate students. A random sample of three students is taken from the class. Determine the expected number of undergraduate students selected if the sampling is

a) without replacement. **b)** with replacement.

c) Why are your results in parts (a) and (b) the same?

7.13 Refer to Example 5.14 on page 208, where X denotes the number of defective TVs obtained when 5 TVs are randomly selected without replacement from a lot of 100 TVs in which 6 are defective. Determine and interpret the expected value of X.

7.14 Suppose that a random sample of size n is taken from a population of size N in which $100p\%$ of the members have a specified attribute. Determine the expected number of the members sampled that have the specified attribute if the sampling is
a) without replacement. **b)** with replacement.

7.15 Between 5:00 P.M. and 6:00 P.M., the number of cars that use the drive-up window at a fast-food restaurant has a Poisson distribution. Data show that, on average, 15 cars use the drive-up window during that hour.
a) What is the parameter of the Poisson distribution? Explain your answer.
b) Roughly what percentage of days are the number of cars that use the drive-up window between 5:00 P.M. and 6:00 P.M. within three, inclusive, of the average number?

7.16 Refer to Proposition 5.7 on page 220, which gives the Poisson approximation to the binomial distribution. What is the relationship between the mean of the binomial distribution and the mean of the approximating Poisson distribution?

7.17 The number of customers that enter the Downtown Coffee Shop in an hour has a Poisson distribution with mean 31. Thirty percent of the customers buy a café mocha with no whipped cream. What is the expected number of customers per hour who buy a café mocha with no whipped cream? State any assumptions that you make.

7.18 The number of eggs that the female grunge beetle lays in her nest has a $\mathcal{P}(4)$ distribution. Assuming that an empty nest can't be recognized, determine the expected number of eggs observed. *Note:* Exercise 5.85 on page 227 asks for the PMF of the number of eggs observed in a nest.

7.19 Use the formula for the expected value of a binomial random variable to obtain the expected value of an indicator random variable.

7.20 A baseball player has a batting average of .260. Suppose that you observe successive at-bats of the player and note for each at-bat whether the player gets a hit. Presuming that the assumption of Bernoulli trials is appropriate, how many at-bats, on average, does it take until the player gets his
a) first hit? **b)** second hit? **c)** tenth hit?

7.21 Simulation: Let X have the discrete uniform distribution on the set $\{0, 1, \ldots, 9\}$.
a) Use simulation to estimate the mean of the random variable X.
b) Use the definition of expected value to determine the mean of the random variable X. Compare your result with that in part (a).

7.22 Let X have the discrete uniform distribution on the set $\{1, 2, \ldots, N\}$, where N is a positive integer. Determine the expected value of the random variable X. *Hint:* Use the formula for the sum of the first N positive integers.

Theory Exercises

7.23 Let X be a hypergeometric random variable with parameters N, n, and p. Show that $\mathcal{E}(X) = np$ by using an argument similar to the one given for a binomial random variable in Example 7.2 on page 331.

Let X, Y, and Z denote the numbers of times that she hits regions A, B, and C, respectively. Also, let T denote the total score in the four shots.

a) Determine and identify the joint PMF of the random variables X, Y, and Z.

b) Express T in terms of X, Y, and Z, and then use the FEF and the result of part (a) to obtain $\mathcal{E}(T)$.

c) Determine and identify the univariate marginal PMFs of X, Y, and Z and, from them, find the expected values of X, Y, and Z.

d) Express T in terms of X, Y, and Z and then use the linearity property of expected value and the result of part (c) to obtain $\mathcal{E}(T)$.

e) Exercise 7.2 on page 333 asks for the expected score for one shot at the target. Use that result and the linearity property of expected value to obtain $\mathcal{E}(T)$.

f) Identify yet another way to obtain $\mathcal{E}(T)$.

7.35 N married couples are randomly seated at a rectangular table, the women on one side and the men on the other. What is the expected number of men who sit across from their wives? *Hint:* One way to solve this problem is to first determine the PMF of the number of men who sit across from their wives and then apply the definition of expected value. However, there is an easier way.

7.36 A random sample of size n is taken from a very large lot of items in which $100p_1\%$ have exactly one defect and $100p_2\%$ have two or more defects, where $0 < p_1 + p_2 < 1$. An item with exactly one defect costs \$1 to repair, whereas an item with two or more defects costs \$3 to repair. Determine the expected cost of repairing the defective items in the sample.

7.37 Determine the expected number of times a balanced die must be thrown to get all six possible numbers. *Hint:* For $1 \le k \le 6$, let X_k denote the number of throws from the appearance of the $(k-1)$th distinct number until the appearance of the kth distinct number.

7.38 Expected utility: One method for deciding among various investments involves the concept of *expected utility.* Economists describe the importance of various levels of wealth by using *utility functions.* For instance, in most cases, a single dollar is more important (has greater utility) for someone with little wealth than for someone with great wealth. Consider two investments—say, Investment A and Investment B. Measured in thousands of dollars, suppose that Investment A yields 0, 1, and 4 with probabilities 0.1, 0.5, and 0.4, respectively, and that Investment B yields 0, 1, and 16 with probabilities 0.5, 0.3, and 0.2, respectively. Let Y denote the yield of an investment. For the two investments, determine and compare

a) $\mathcal{E}(Y)$, the expected yield.

b) $\mathcal{E}(\sqrt{Y})$, the expected utility, using the utility function $u(y) = \sqrt{y}$. Interpret the utility function u.

c) $\mathcal{E}(Y^{3/2})$, the expected utility, using the utility function $v(y) = y^{3/2}$. Interpret the utility function v.

7.39 A lot contains 17 items, each of which is subject to inspection by two quality assurance engineers. Each engineer randomly and independently selects 4 items from the lot. Determine the expected number of items selected by

a) both engineers. b) neither engineer. c) exactly one engineer.

d) Without doing any computations, obtain the sum of the three expected values found in parts (a), (b), and (c). Explain your reasoning.

7.40 Consider two electrical components whose lifetimes observed at discrete time units (e.g., every hour) are independent geometric random variables with parameter p. Use tail probabilities to determine the expected time until

a) the first component to fail. b) the second component to fail.

7.41 Two people agree to meet at a specified place between 3:00 P.M. and 4:00 P.M. Suppose that you measure time to the nearest minute relative to 3:00 P.M. so that, for instance, time 40 represents 3:40 P.M. Further suppose that each person arrives according to the discrete uniform distribution on $\{0, 1, \ldots, 60\}$ and that the two arrival times are independent. Determine the expected time

a) until the first arrival. **b)** until the second arrival.

c) that the first person to arrive waits for the second person to arrive.

7.42 Show that $\mathcal{E}(X^2) \geq (\mathcal{E}(X))^2$. *Hint:* Consider the random variable $(X - \mathcal{E}(X))^2$.

7.43 Random variables X and Y defined on the same sample space are said to be *linearly uncorrelated* if $\mathcal{E}\big((X - \mathcal{E}(X))(Y - \mathcal{E}(Y))\big) = 0$.

a) Show that X and Y are linearly uncorrelated if and only if $\mathcal{E}(XY) = \mathcal{E}(X)\,\mathcal{E}(Y)$.

b) Are independent random variables (with finite expectations) linearly uncorrelated? Justify your answer.

c) Are linearly uncorrelated random variables independent? Justify your answer.

Theory Exercises

7.44 Let X have the negative binomial distribution with parameters r and p.

a) From Exercise 6.151 on page 322, we know that X can be expressed as the sum of r independent geometric random variables with common parameter p. Use this fact and Example 7.11 on page 348 to deduce that $\mathcal{E}(X) = r/p$.

b) Is the independence of the geometric random variables required for the result in part (a)? Explain your answer.

7.45 Let $X_1, \ldots, X_m$ be discrete random variables defined on the same sample space and having finite expectation, and let $c_1, \ldots, c_m$ be real numbers. Use Proposition 7.3 on page 342 and mathematical induction to prove that $\sum_{j=1}^m c_j X_j$ has finite expectation and that

$$\mathcal{E}\left(\sum_{j=1}^m c_j X_j\right) = \sum_{j=1}^m c_j \mathcal{E}(X_j).$$

7.46 Let $X_1, \ldots, X_m$ be independent discrete random variables having finite expectation.

a) Use Proposition 7.5 on page 346 and mathematical induction to prove that $\prod_{j=1}^m X_j$ has finite expectation and that

$$\mathcal{E}\left(\prod_{j=1}^m X_j\right) = \prod_{j=1}^m \mathcal{E}(X_j).$$

b) Use a direct argument to establish the results of part (a). *Hint:* Refer to Proposition 6.14 on page 297.

7.47 Let X be a random variable with finite expectation. Prove that $|\mathcal{E}(X)| \leq \mathcal{E}(|X|)$.

7.48 Bounded random variable: A random variable X is said to be *bounded* if there is a positive real number M such that $P(|X| \leq M) = 1$. Prove that a bounded discrete random variable, bounded by M, has finite expectation and that $|\mathcal{E}(X)| \leq M$. *Hint:* Refer to Exercise 7.47.

7.49 Markov's inequality: Let X be a nonnegative random variable with finite expectation. *Markov's inequality* is that $P(X \geq t) \leq \mathcal{E}(X)/t$ for all positive real numbers t.

a) Prove Markov's inequality for discrete random variables by applying the FPF.

b) Show that, for a nonnegative random variable X, we have $tI_{\{X \geq t\}} \leq X$ for all positive real numbers t.
c) Prove Markov's inequality by using part (b) and properties of expected value.

Advanced Exercises

7.50 A number X is chosen at random from the interval $(0, 1)$. Find the expected value of
a) the first digit of the decimal expansion of X.
b) the first digit of the decimal expansion of $\sqrt{X}$.
c) the second digit of the decimal expansion of $\sqrt{X}$.

7.51 Of the customers who enter a store, 60% are women and 40% are men. If 50 customers enter the store, what is the expected number of times that a woman is followed by a man? State any assumptions that you make.

7.52 Probability generating function: Let X be a nonnegative-integer valued random variable. The *probability generating function* of X, denoted P_X, is defined by $P_X(t) = \mathcal{E}(t^X)$. We use PGF as an abbreviation for "probability generating function."
a) Verify that P_X is well defined for $|t| \leq 1$ by showing that the random variable t^X has finite expectation for each t in that interval.
b) Prove that $P_X(t) = \sum_{n=0}^{\infty} p_X(n)t^n$.
c) Show that, if X has finite expectation, then $\mathcal{E}(X) = P'_X(1)$.

7.53 Refer to Exercise 7.52. For a random variable with each of the following probability distributions, determine the PGF in closed form and use the PGF to obtain the expected value.
a) Binomial with parameters n and p
b) Poisson with parameter λ
c) Geometric with parameter p
d) Negative binomial with parameters r and p
e) Discrete uniform on $\{1, 2, \ldots, N\}$

7.3 Variance of Discrete Random Variables

In this section, we apply the concept of expected value to obtain additional characteristics of random variables. Although we concentrate on discrete random variables, the ideas presented here apply to all types of random variables.

Recall that another term for *expected value* is *mean*. At this point, we frequently use the phrase "mean of X" instead of "expected value of X"; and we often say that "X has finite mean" rather than "X has finite expectation." Also recall that the symbol μ_X is an alternative to $\mathcal{E}(X)$.

To begin, we consider the *moments* of a random variable. Let r be a positive integer and let X be a random variable. We say that X has a **moment of order *r*** or that X has a **finite *r*th moment** if the random variable X^r has finite expectation. In that case, we define the ***r*th moment** of X to be the expected value of the random variable X^r:

$$r\text{th moment of } X = \mathcal{E}(X^r). \tag{7.26}$$

If X^r doesn't have finite expectation, we say that X doesn't have a moment of order r or that X doesn't have a finite rth moment.

7.63 Refer to Example 5.14 on page 208, where X denotes the number of defective TVs obtained when 5 TVs are randomly selected without replacement from a lot of 100 TVs in which 6 are defective. Determine the variance of X.

7.64 Suppose that a random sample of size n is taken from a population of size N in which $100p\%$ of the members have a specified attribute. Determine the variance of the number of members sampled that have the specified attribute if the sampling is
a) without replacement. **b)** with replacement.

7.65 Between 5:00 P.M. and 6:00 P.M., the number of cars that use the drive-up window at a fast-food restaurant has a Poisson distribution. Data show that, on average, 15 cars use the drive-up window during that hour. Find the variance of the number of cars that use the drive-up window between 5:00 P.M. and 6:00 P.M.

7.66 The number of customers that enter the Downtown Coffee Shop in an hour has a Poisson distribution with mean 31. Thirty percent of the customers buy a café mocha with no whipped cream. What is the variance of the number of customers per hour who buy a café mocha with no whipped cream? State any assumptions that you make.

7.67 The number of eggs that the female grunge beetle lays in her nest has a $\mathcal{P}(4)$ distribution. Assuming that an empty nest can't be recognized, determine the variance of the number of eggs observed. *Note:* Exercise 5.85 on page 227 asks for the PMF of the number of eggs observed in a nest.

7.68 A baseball player has a batting average of .260. Suppose that you observe successive at-bats of the player and note for each at-bat whether the player gets a hit. Presuming that the assumption of Bernoulli trials is appropriate, what is the variance of the number of at-bats until the player gets his
a) first hit? **b)** second hit? **c)** tenth hit?

7.69 Derive the formula for the variance of the indicator random variable of an event E, Equation (7.32) on page 356, by first noting that $I_E^2 = I_E$ and then applying the computing formula for the variance.

7.70 Use the formula for the variance of a binomial random variable to obtain the variance of an indicator random variable.

7.71 Let X be a random variable with finite nonzero variance. By using properties of expected value and variance, verify in detail that the mean and variance of X^*—the standardized random variable corresponding to X—are 0 and 1, respectively.

7.72 For each $n \in \mathcal{N}$, let X_n be a random variable that equals 0 with probability $1 - 1/n^2$ and equals $\pm n$ with probability $1/2n^2$ each.
a) Apply Chebyshev's inequality to obtain an upper bound for the probability that X_n is at least 3 in magnitude.
b) For $n \geq 3$, determine the exact probability that X_n is at least 3 in magnitude.
c) Compare your results in parts (a) and (b). How do they compare as $n \to \infty$?

7.73 Chebyshev's inequality is commonly stated in a form alternative to that given in Proposition 7.11 on page 360—namely,

$$P(|X - \mu_X| < k\sigma_X) \geq 1 - \frac{1}{k^2}. \tag{$*$}$$

a) State Relation $(*)$ in words.

7.98 An actuary is modeling the sizes of surgical claims and their associated hospital claims, measured in thousands of dollars. The first and second moments of a surgical claim are 5 and 27.4, respectively, the first and second moments of a hospital claim are 7 and 51.4, respectively, and the variance of the total of the surgical and hospital claims is 8. Let X and Y denote the sizes of a combined surgical and hospital claim before and after the application of a 20% surcharge on the hospital portion of the claim, respectively. Determine and interpret the covariance and correlation coefficient of X and Y.

7.99 Let X and Y be discrete random variables defined on the same sample space and having finite variances.

a) Prove that XY has finite mean. *Hint:* First show that $|xy| \leq (x^2 + y^2)/2$.

b) Prove that $X + Y$ has finite variance. *Hint:* First show that $(x + y)^2 \leq 2(x^2 + y^2)$.

7.100 Verify the computing formula for the covariance: $\text{Cov}(X, Y) = \mathcal{E}(XY) - \mathcal{E}(X)\mathcal{E}(Y)$.

7.101 Consider a finite population of size N and a variable defined thereon. Let σ^2 denote the population variance—that is, the arithmetic mean of all possible square deviations of the variable from the population mean for the entire population. Suppose that a random sample of size n is taken with replacement and let $\bar{X}_n$ denote the sample mean. Using results that we have already obtained, explain why $\text{Var}(\bar{X}_n) = \sigma^2/n$. *Note:* No calculations are required.

7.102 Use properties of covariance to show $\text{Cov}(X^*, Y^*) = \text{Cov}(X, Y)\,/\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}$ and explain why that quantity is unitless.

7.103 Let X have the discrete uniform distribution on $\{-1, 0, 1\}$ and let $Y = X^2$.

a) Show that $\rho(X, Y) = 0$, although X and Y are functionally related and hence associated.

b) Why doesn't the result of part (a) conflict with the correlation coefficient's role as a measure of association?

7.104 In Exercise 7.75 on page 364, we discussed how to best estimate a random variable Y by a constant, c, where "best" means choosing c to minimize the mean square error, $\mathcal{E}\left((Y - c)^2\right)$. Now determine how to best estimate Y by a linear function of a random variable X, again using minimum mean square error as the criterion for "best."

a) Obtain the values of a and b that minimize the mean square error $\mathcal{E}\left([Y - (a + bX)]^2\right)$. *Hint:* Use a calculus technique for finding extrema of functions of two variables.

b) Find the minimum mean square error.

Use the result of part (b) to

c) deduce part (a) of Proposition 7.16 on page 372.

d) deduce parts (c) and (d) of Proposition 7.16.

e) explain why the correlation coefficient is a measure of linear association.

Theory Exercises

7.105 Establish the properties of covariance presented in Proposition 7.12 on page 366.

7.106 Prove the bilinearity property of covariance, Equation (7.40) on page 366.

7.107 Use properties of expected value to establish the computing formula for the covariance, Equation (7.41) on page 366.

7.108 Complete the proof of Proposition 7.13 by verifying Equation (7.43) on page 366.

Advanced Exercises

7.109 Sample variance: Let X be a random variable with finite variance. Suppose that you don't know the variance of X and want to estimate it. You take a random sample, $X_1, \ldots, X_n$,