10.106

(a) According to example 10.22, $y = 34.9 + 0.537x$

$\Rightarrow$ When $x = 5'8'' = 68$ (inches), $y = 34.9 + 0.537 \cdot 68 = 71.416$ (inches)

(b) When $y = 68$ (inches), $X = \frac{68 - 34.9}{0.537} \approx 61.64$ (inches)

(c) $P(y \geq 72 \mid X = 62) = 1 - P(y \leq 72 \mid X = 62)$

$= 1 - P\left(\frac{y - \mu_{Y|x}}{\sigma_{Y|x}} \leq \frac{72 - (34.9 + 0.537 \cdot 62)}{\sqrt{6.31}} \mid X = 62\right)$

$= 1 - P(Z \leq 1.515 \mid X = 62) \approx 1 - 0.9345 = 0.0655$

(d) Based on the 68 - 95 - 99.7 (empirical rule), the bounds of the 95% prediction interval are 2 std. deviations away from μ.

Thus, the interval would be: $((34.9 + 0.537 \cdot 62) - 2 \cdot \sqrt{6.31}, (34.9 + 0.537 \cdot 62) + 2 \cdot \sqrt{6.31})$

$\approx (63.17, 73.22)$

11.3 $X \sim G(p)$

(a) $M_X(t) = \sum_x e^{tx} P_X(X = x)$

$= \sum_{x=1}^{\infty} e^{tx} \cdot p \cdot (1-p)^{x-1} = p \sum_{x=1}^{\infty} e^{tx} \cdot (1-p)^{x-1}$

Let $x - 1 = j \Rightarrow x = j + 1$

$= p \cdot \sum_{j=0}^{\infty} e^{t(j+1)} \cdot (1-p)^j = p \cdot e^t \sum_{j=0}^{\infty} e^{tj} \cdot (1-p)^j$

$= p \cdot e^t \cdot \frac{1}{1 - (1-p)e^t} = \boxed{p \cdot e^t \cdot (1 - (1-p)e^t)^{-1}}$

$\because$ converge iff $|(1-p)e^t| < 1$

$e^t < \frac{1}{(1-p)}$

$\Rightarrow t < \ln((1-p)^{-1})$

$\boxed{t < -\ln(1-p)}$

$$E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = pe^t(1-(1-p)e^t)^{-1} + pe^{2t}\cdot(1-p)(1-(1-p)e^t)^{-2}$$

$$= \frac{p}{p} + \frac{p(1-p)}{p^2} = \frac{p+1-p}{p} = \frac{1}{p}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \frac{d}{dt}\Big[pte^t(1-(1-p)e^t)^{-1} + pe^{2t}\cdot(1-(1-p)e^t)^{-2}\Big]\Big|_{t=0}$$

$$= pe^t(1-(1-p)e^t)^{-1} + p\cdot(1-p)\cdot e^{2t}(1-(1-p)e^t)^{-2} + 2pe^{2t}(1-p)(1-(1-p)e^t)^{-2} + 2pe^{2t}\cdot(1-p)^2\cdot(1-(1-p)e^t)^{-3}$$

$$= \frac{p}{p} + \frac{p(1-p)}{p^2} - \frac{2p(1-p)}{p^2} + \frac{2p(1-2p+p^2)}{p^3}$$

$$= 1 + \frac{1-p}{p} + \frac{2-2p}{p} + \frac{2-4p+2p^2}{p^2}$$

$$= \frac{\cancel{p^2} + p - \cancel{p^2} + 2p - \cancel{2p^2} + 2 - 4p + \cancel{2p^2}}{p^2}$$

$$= \frac{-p+2}{p^2}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{-p+2}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$$

11.25

(a) $M_{X,Y}(s,t) = (pe^s + qe^t)^n$

$\Rightarrow M_X(s) = M_X(s,0) = (pe^s + q)^n = (pe^s + (1-p))^n \Rightarrow X \sim Bin(n,p)$

$\Rightarrow M_Y(t) = M_Y(0,t) = (p + qe^t)^n = (qe^t + (1-q))^n \Rightarrow Y \sim Bin(n,q)$

(b) X, Y are independent if $M_X(s)\cdot M_Y(t) = M_{X,Y}(s,t)$

$M_X(s)\cdot M_Y(t) = (pe^s + q)^n\cdot(qe^t + p)^n \neq M_{X,Y}(s,t)$

$\Rightarrow$ X, Y are dependent R.V.s

(c) $M_{X+Y}(t) = M_{X,Y}(t,t)$

$= (pe^t + qe^t)^n = [(p+q)e^t]^n = e^{tn}$

$$\sum_{x=0}^{n}\sum_{y=0}^{n} (e^t)^{(x+y)}\cdot P(X=x, Y=y) = e^{tn}$$

$\Rightarrow P(x+y=k) = \begin{cases} 1 & \text{if } x+y=n \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow$ X+Y is likly to be a Dirac Delta function

11.40 By definition, an estimator is consistant if $\lim_{n\to\infty} P(|\hat{\theta}(x_n)-\theta| < \varepsilon) = 1$

By law of large numbers, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i^r = \overline{X_n}$,

and that, $\lim_{n\to\infty} P(|\bar{X}_n - E[X^r]| < \varepsilon) = 1$

$\Rightarrow (x_1^r + \dots + x_n^r)/n$ is a consistent estimator.

11.43 $\because X \sim \mathcal{U}(0,1) \therefore P_X(x) = \frac{1}{1-0} = 1$

(a) $E(g(x)) = \int_{-\infty}^{\infty} g(x)\cdot P_X(x)\,dx = \int_0^1 g(x)\,dx$

$\because$ $E(g(x))$ is defined by a definite integral, $E(g(x))$ is finite.

(b) $g(x_1) + \dots + g(x_n)$ is an estimator for $E(g(x))$;

As shown in part (a), $E(g(x)) = \int_0^1 g(x)\,dx$

By LLN, $\lim_{n\to\infty} P((g(x_1)+\dots g(x_n)) - E(g(x)) < \varepsilon) = 1$, i.e., $\lim_{n\to\infty} P((g(x_1)+\dots g(x_n))/n = \int_0^1 g(x)\,dx) = 1$

(c) As shown in part (b), $\lim_{n\to\infty} P((g(x_1)+\dots g(x_n))/n = \int_0^1 g(x)\,dx) = 1$

Thus, we generate n samples of X, where $n \to \infty$. Then we use the generate X as input for $g(x)$. We can estimate $\int_0^1 g(x)dx$ by finding the average of all the $g(X)$.

11.59 Let X be an R.V. representing battery lifetime. $\Rightarrow X \sim N(\mu{=}30, \sigma^2{=}25)$

$\Rightarrow P(X \geq 25) = P(Z \geq \frac{25-30}{5}) = P(Z \geq -1) = 1 - P(Z < -1) \approx 0.8413$

Let Y be an R.V. representing number of batteries that last longer than 25 hrs

Then, based on the prompt, $Y \sim Bin(500, 0.8413)$

By de Moirre-Laplace theorem, $P(Y \geq 400) = 1 - P(Y < 400)$

$= 1 - P(Y \leq 399) = 1 - P(Y \leq 399.5)$

$\approx 1 - (Z \leq \frac{399.5 - 500\cdot 0.8413}{\text{Sqrt}(500\cdot 0.8413\cdot 0.1587)}) = 1 - (Z \leq -2.589) \approx 1 - 0.0049 = 0.9951$

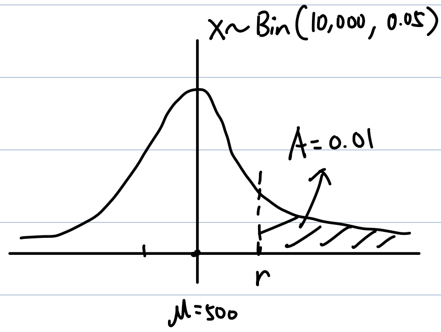
$\therefore$ The probability of having 80% of the batteries lasting longer than 25 hrs is 0.9951.

11.64 Let X be the number of screws that are out of the tolerance range.

Based on manufacturer's experience, $X \sim (n = 10{,}000,\ p = 0.05)$

(a) $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$, where $\mu = n \cdot p = 500$, $\sigma = \sqrt{n \cdot p \cdot (1-p)} \approx 21.8$

$\frac{1}{k^2} = 0.01 \Rightarrow k = 10 \Rightarrow P(|X - 500| \geq 218) \leq 0.01$



$\because$ Full refund only if $X > r$

$\therefore\ r = k\sigma + \mu = 218 + 500 \approx 718$

(b) $S_n = Bin(n = 10{,}000,\ p = 0.05)$

$Y_n = \frac{S_n - np}{\sqrt{n \cdot p \cdot (1-p)}} \approx Z \sim N(0,1)$

$P\left(Y_n > \frac{r - np}{\sqrt{np \cdot (1-p)}}\right) \leq 0.01$

$P\left(Y_n \geq \frac{r - np + 0.5}{\sqrt{np \cdot (1-p)}}\right) \leq 0.01$

$\Rightarrow \frac{r - np + 0.5}{\sqrt{np \cdot (1-p)}} = 2.3$

$\therefore\ r = 2.3 \cdot 21.8 + 500 - 0.5 \approx 549.64$