Will Zong

7.5

$\because$ Sum of torques $= 0$

$\therefore \sum_{k=1}^{k=m} P_k \cdot (\bar{X} - X_k) = 0$ $\qquad E(X) = x \cdot P(x)$

$\sum_{k=1}^{k=m} P_k \cdot \bar{X} - P_k \cdot X_k = 0$

$\sum_{k=1}^{k=m} P_k \cdot \bar{X} - \sum_{k=1}^{k=m} P_k \cdot X_k = 0 \Rightarrow \sum_{k=1}^{k=m} P_k = \sum_{k=1}^{k=m} P_k \cdot X_k$

$\bar{X} \sum_{k=1}^{k=m} P_k = \bar{X} = E(X)$, Q.E.D.

7.19 Let $X \sim Bin(n, p)$, I_E be the indicator R.V. for event E.

$E(X) = n \cdot p$ According to textbook proposition 5.12 (238), $I_E \sim Bin(1, P(E))$

$\therefore E(I_E) = 1 \cdot P(E)$

7.36 Let X_k be the R.V. representing the cost of repair for the k th item.

$E(X) = \sum_{k=1}^{n} X_k \cdot P(X_k)$

$= \sum_{k=1}^{n} 1 \cdot P_1 + 3 \cdot P_2 = nP_1 + 3nP_2$

7.42 Let Y be a discrete R.V., $Y = (X - E(X))^2$

$E(Y) = E\left[(X - E(X))^2\right]$ $\qquad \because \forall X,\ Y = (X - E(X))^2 \geq 0$

$= E\left[X^2 - 2 \cdot X \cdot E(X) + (E(X))^2\right]$ $\qquad \therefore E(Y) \geq 0$

$= E(X^2) - 2E(X) \cdot E(X) + (E(X))^2$ $\qquad \Rightarrow E(X^2) - (E(X))^2 \geq 0$

$= E(X^2) - 2(E(X))^2 + (E(X))^2$ $\qquad \Rightarrow E(X^2) \geq (E(X))^2$, QED.

$= E(X^2) - (E(X))^2$

7.47 By definition, $E(|X|) = \sum_x |x| \cdot P_X(x)$

$E(X) = \sum_x x \cdot P_X(x)$

$\because |x| \geq x$

$\therefore \sum_x |x| \cdot P_X(x) \geq \sum_x x \cdot P_X(x) \Rightarrow E(|X|) \geq E(X)$

$\Rightarrow E(|X|) \geq |E(X)|$ Q.E.D.

7.49 (a) $\because X \geq t,\ X \geq 0$

$\therefore E(X) = \sum_{x=0}^{x=t-1} x \cdot P_X(x) + \sum_{x=t}^{\infty} x \cdot P_X(x)$

$\because \sum_{t=0}^{x=t-1} x \cdot P_X(x) \geq 0$

$\therefore E(X) \geq \sum_{x=t}^{\infty} x \cdot P_X(x)$

$E(X) \geq t \sum_{x=t}^{\infty} P_X(x) \Rightarrow \frac{E(X)}{t} \geq \sum_{x=t}^{\infty} P_X(x)$

$\Rightarrow \frac{E(X)}{t} \geq P(X \geq t)$ by FPF definition. QED

(b) By proposition 5.12 (p.238), $I_{\{X \geq t\}} = E(I_{\{X \geq t\}}) = P(X \geq t)$

$\Rightarrow$ LHS: $t \cdot I_{\{X \geq t\}} = t \cdot P(X \geq t)$

$\because P(X < t) + P(X \geq t) = 1 \quad \therefore P(X \geq t) \leq 1$

$\Rightarrow$ LHS $= t \cdot P(X \geq t) \leq t$

$\because$ RHS $= X \geq t$

$\therefore t \cdot P(X \geq t) \leq X \Rightarrow t \cdot I_{\{X \geq t\}} \leq X,\ \forall t > 0$, Q.E.D.

(c) $\because$ Expectation is a linear operator

$\therefore E(t I_{\{X \geq t\}}) \leq E(X)$

$t E(I_{\{X \geq t\}}) \leq E(X)$

$E(I_{\{X \geq t\}}) \leq \frac{E(X)}{t}$

$P(X \geq t) \leq \frac{E(X)}{t}$, Q.E.D.

7.64. (a) $\because$ choose without replacement, $\therefore$ Let $X \sim H(N, n, p)$

According to textbook table 7.10 (p.358),

$Var(X) = \left(\frac{N-n}{N-1}\right) \cdot np \cdot (1-p)$

(b) $\because$ Choose with replacement, $\therefore$ Let $Y \sim Bin(n, p)$

According to text book table 7.10 (p.358),

$Var(Y) = np \cdot (1-p)$

7.72 (a) $Var(X_n) = E(X_n^2) - (E(X_n))^2$

$E(X_n^2) = 0^2 \cdot (1-\frac{1}{n^2}) + 2 \cdot n^2 \cdot (\frac{1}{2n^2}) = 1$ $\qquad \{X_1\ X_2\ X_3\ X_4\ X_5$

$(E(X_n))^2 = \left[0 \cdot (1-\frac{1}{n^2}) + n \cdot (\frac{1}{2n^2}) - n \cdot \frac{1}{2n^2}\right]^2 = 0$

$\therefore Var(X_n) = 1 - 0 = 1$

$\Rightarrow P(X_n \geq 3) \leq \frac{Var(X_n)}{9}$

$P(X_n \geq 3) \leq \frac{1}{9}$

(b) $P(X_n \geq 3) = 1 - P(X_n < 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=-1) + P(X=2) + P(X=-2)]$

$= 1 - \left[(1-\frac{1}{n^2}) + \frac{1}{2n^2} + \frac{1}{2n^2} + \frac{1}{2n^2} + \frac{1}{2n^2}\right]$

$= 1 - (1+\frac{1}{n^2}) = -\frac{1}{n^2}$?

(c) $\qquad \frac{1}{4} + \frac{1}{9}$

7.98 Let A be an R.V. representing the sizes of surgical claims

B be an R.V. representing the sizes of hospital claims.

Given: $E(A) = 5$, $E(A^2) = 27.4$; $E(B) = 7$, $E(B^2) = 51.4$

$X = A + B$; $Y = A + 1.2B$; $Var(A+B) = Var(X) = 8$

Find: $Cov(X,Y)$, $\rho(X,Y)$

$Cov(X,Y) = Cov(A+B,\ A+1.2B)$

$= E[(A+B)(A+1.2B)] - E(A+B)\cdot E(A+1.2B)$

$= E[A^2 + 2.2AB + 1.2B^2] - E(A+B)\cdot E(A+1.2B)$

$= E(A^2) + 2.2E(AB) + E(B^2) - [(E(A)+E(B))\cdot(E(A)+1.2E(B))]$

$= E(A^2) + 2.2E(AB) + 1.2E(B^2) - [(E(A))^2 + 2.2E(A)E(B) + 1.2(E(B))^2]$

$= E(A^2) + 2.2E(AB) + 1.2E(B^2) - (E(A))^2 - 2.2E(A)E(B) - 1.2(E(B))^2$

$= E(A^2) - (E(A))^2 + 1.2E(B^2) - 1.2(E(B))^2 + 2.2E(AB) - 2.2E(A)E(B)$

$= Var(A) + 1.2\,Var(B) + 2.2\,Cov(A,B)$

$Var(A) = E(A^2) - (E(A))^2 = 27.4 - 25 = 2.4$

$Var(B) = E(B^2) - (E(B))^2 = 51.4 - 49 = 2.4$

$Var(A+B) = Var(A) + Var(B) + 2Cov(A,B) = 8$

$\Rightarrow Cov(A,B) = \frac{1}{2}(8 - Var(A) - Var(B)) = 1.6$

$\Rightarrow Cov(X,Y) = Var(A) + 1.2\,Var(B) + 2.2\,Cov(A,B)$

$= 2.4 + 1.2\cdot(2.4) + 2.2\cdot 1.6 = 8.8$

$Var(Y) = Var(A+1.2B) = Var(A) + 1.2^2\cdot Var(B) + 2.4\,Cov(A,B) = 2.4 + 1.2^2\cdot 2.4 - 1.2\cdot 1.6 = 9.696$

$\Rightarrow \rho(x,y) = Cov(X,Y)/\sqrt{Var(X)}\cdot\sqrt{Var(Y)} = 8.8/\sqrt{8}\cdot\sqrt{9.696} \approx 0.999$

$\because \rho(x,y) \approx 0.999 \approx 1$

$\therefore$ Y is positively proportional to X.

7.102 $\quad X' = \dfrac{X-\mu_x}{\sigma_x}$, $\quad Y' = \dfrac{y-\mu_y}{\sigma_y}$

$$LHS = Cov(X', y') = Cov\left(\frac{X-\mu_x}{\sigma_x}, \frac{y-\mu_y}{\sigma_y}\right)$$

$$= \frac{1}{\sigma_x \cdot \sigma_y} \cdot Cov(X-\mu_x, y-\mu_y)$$

$$= \frac{1}{\sigma_x \cdot \sigma_y} \cdot Cov(X, y)$$

$\because \sigma_x = \sqrt{Var(X)}$; $\sigma_y = \sqrt{Var(y)}$

$\therefore LHS = \dfrac{1}{\sqrt{Var(X)} \cdot \sqrt{Var(y)}} \cdot Cov(X, y) = RHS$, Q.E.D.