Proposition 9.8 implies that all properties of (unconditional) PDFs also hold for conditional PDFs. For instance, the following conditional version of the FPF holds:

$$P(Y \in A \mid X = x) = \int_A f_{Y \mid X}(y \mid x)\, dy \tag{9.33}$$

for each subset $A \subset \mathcal{R}$.

EXAMPLE 9.12 ***Conditional Probability Density Functions***

Min and Max of a Random Sample Suppose that $X_1, \ldots, X_n$ are a random sample from a $\mathcal{U}(0, 1)$ distribution. Set $X = \min\{X_1, \ldots, X_n\}$ and $Y = \max\{X_1, \ldots, X_n\}$. Find $P(\frac{1}{4} < Y < \frac{3}{4} \mid X = \frac{1}{2})$.

Solution We first obtain a conditional PDF of Y given $X = x$. From Example 9.4(c) on page 499, a joint PDF of X and Y is $f_{X,Y}(x, y) = n(n-1)(y-x)^{n-2}$ for $0 < x < y < 1$, and $f_{X,Y}(x, y) = 0$ otherwise. And, from Example 9.9(c) on page 513, a marginal PDF of X is $f_X(x) = n(1-x)^{n-1}$ for $0 < x < 1$, and $f_X(x) = 0$ otherwise. Therefore, by Definition 9.3, for each $0 < x < 1$,

$$f_{Y \mid X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{n(n-1)(y-x)^{n-2}}{n(1-x)^{n-1}} = \frac{(n-1)(y-x)^{n-2}}{(1-x)^{n-1}},$$

for $x < y < 1$, and $f_{Y \mid X}(y \mid x) = 0$ otherwise. Hence, from Equation (9.33),

$$P(\tfrac{1}{4} < Y < \tfrac{3}{4} \mid X = \tfrac{1}{2}) = \int_{\frac{1}{4}}^{\frac{3}{4}} f_{Y \mid X}(y \mid \tfrac{1}{2})\, dy = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{(n-1)\left(y - \frac{1}{2}\right)^{n-2}}{\left(1 - \frac{1}{2}\right)^{n-1}}\, dy$$

$$= (n-1)2^{n-1} \int_0^{\frac{1}{4}} u^{n-2}\, du = \frac{1}{2^{n-1}},$$

as required. ∎

Equation (9.32) on page 515 defines the conditional PDF of Y given $X = x$. Multiplying both sides of that equation by $f_X(x)$ yields

$$f_{X,Y}(x, y) = f_X(x) f_{Y \mid X}(y \mid x). \tag{9.34}$$

Equation (9.34) holds for all real numbers x and y, not only for those where $f_X(x) > 0$. More precisely, the function f defined on $\mathcal{R}^2$ by $f(x, y) = f_X(x) f_{Y \mid X}(y \mid x)$ is a joint PDF of X and Y. This result is the **general multiplication rule** for the joint PDF of two continuous random variables, X and Y. We leave the details to you in Exercise 9.78.

EXAMPLE 9.13 ***The General Multiplication Rule***

Regression Analysis Let X have the standard normal distribution and let $-1 < \rho < 1$. Suppose that, for each $x \in \mathcal{R}$, the conditional distribution of Y given $X = x$ is a normal distribution with parameters ρx and $1 - \rho^2$; symbolically, $Y_{|X=x} \sim \mathcal{N}(\rho x, 1 - \rho^2)$.

a) Determine a joint PDF of X and Y.

b) Determine and identify a marginal PDF of Y.

Solution By assumption,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad x \in \mathcal{R},$$

and

$$f_{Y\,|\,X}(y\,|\,x) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-(y-\rho x)^2/2(1-\rho^2)}, \qquad x, y \in \mathcal{R}.$$

a) Using the general multiplication rule, Equation (9.34), we have

$$f_{X,Y}(x, y) = f_X(x) f_{Y\,|\,X}(y\,|\,x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-(y-\rho x)^2/2(1-\rho^2)},$$

or

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}. \tag{9.35}$$

b) Applying Proposition 9.7 on page 510, Equation (9.35), and the technique of completing the square, we obtain

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}\,dx \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-((x-\rho y)^2+y^2-\rho^2 y^2)/2(1-\rho^2)}\,dx \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-y^2/2} \int_{-\infty}^{\infty} e^{-(x-\rho y)^2/2(1-\rho^2)}\,dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-(x-\rho y)^2/2(1-\rho^2)}\,dx. \end{aligned}$$

The integrand in the previous expression is the PDF of an $\mathcal{N}(\rho y, 1-\rho^2)$ distribution and therefore its integral must equal 1, as is the case for any PDF. Thus

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \qquad y \in \mathcal{R}.$$

Hence $Y \sim \mathcal{N}(0, 1)$; that is, Y has the standard normal distribution. ■

Note: The joint PDF in Equation (9.35) is that of a particular *bivariate normal distribution*. We examine bivariate normal distributions in detail in Chapter 10. As you will learn there, ρ is the correlation coefficient of the bivariate normal random variables.

Multivariate Marginal and Conditional PDFs

The definitions and results that we presented for bivariate PDFs are easily extended to the general multivariate case. To begin, we explain how to obtain marginal PDFs. In the case of two continuous random variables with a joint PDF, there are two marginal PDFs.

Using the symmetry of the joint PDF of X and Y and making the substitution $u = 1 + y/5$, we obtain

$$P(|X - Y| > 5) = \iint\limits_{|x-y|>5} f_{X,Y}(x, y)\, dx\, dy$$

$$= 2\int_0^5 \left(\int_{-5}^{x-5} \frac{1}{25}\left(1 - \frac{x}{5}\right)\left(1 + \frac{y}{5}\right) dy \right) dx$$

$$= \frac{2}{5}\int_0^5 \left(1 - \frac{x}{5}\right)\left(\int_0^{x/5} u\, du \right) dx = \frac{1}{5}\int_0^5 \left(1 - \frac{x}{5}\right)\frac{x^2}{25}\, dx$$

$$= \frac{1}{625}\int_0^5 \left(5x^2 - x^3\right) dx = \frac{1}{12} = 0.083.$$

Chances are only 8.3% that the first person to arrive at the specified place will wait more than 5 minutes before the other person arrives. ■

Independence and Conditional Distributions

As Proposition 9.10 shows, independence of continuous random variables also conforms to our intuitive notion in terms of conditional distributions.

◆◆◆ **Proposition 9.10 Independence and Conditional Distributions**

Let X and Y be continuous random variables with a joint PDF. Then X and Y are independent if and only if either of the following properties holds:
a) Each conditional PDF of Y given $X = x$ is a PDF of Y.
b) Each conditional PDF of X given $Y = y$ is a PDF of X.

Proof We verify that condition (a) is equivalent to independence and leave verification of condition (b) for you. Suppose that X and Y are independent. Then the function f defined on $\mathcal{R}^2$ by $f(x, y) = f_X(x) f_Y(y)$ is a joint PDF of X and Y. Thus, if $f_X(x) > 0$,

$$f_{Y\,|\,X}(y\,|\,x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{f_X(x) f_Y(y)}{f_X(x)} = f_Y(y).$$

Hence $f_{Y\,|\,X}$ is a PDF of Y.

Conversely, suppose that condition (a) holds. To establish that X and Y are independent, we show that the function f defined on $\mathcal{R}^2$ by $f(x, y) = f_X(x) f_Y(y)$ is a joint PDF of X and Y. Applying the general multiplication rule, Equation (9.34) on page 517, we obtain

$$f_{X,Y}(x, y) = f_X(x) f_{Y\,|\,X}(y\,|\,x) = \begin{cases} f_X(x) f_Y(y), & \text{if } f_X(x) > 0; \\ 0, & \text{if } f_X(x) = 0. \end{cases} = f_X(x) f_Y(y),$$

for all x and y. Consequently, f is a joint PDF of X and Y. ◆

Note: From Proposition 9.10, if each conditional PDF of Y given $X = x$ is a PDF of Y, then each conditional PDF of X given $Y = y$ must be a PDF of X; and vice versa.

Solution By assumption, $X_j \sim \mathcal{E}(\lambda_j)$ for $1 \le j \le n$. Because a series system functions only when all the components are working, it fails at the time of the first failure among the n components. Thus, the lifetime of the series system is $X = \min\{X_1, \ldots, X_n\}$.

a) We first obtain the CDF of X. As it is easier to compute $P(X > x)$ than $P(X \le x)$, we proceed as follows. From the complementation rule, the independence of the X_js, and Proposition 8.8 on page 433, we have, for $x > 0$,

$$\begin{aligned} F_X(x) &= P(X \le x) = 1 - P(X > x) = 1 - P(\min\{X_1, \ldots, X_n\} > x) \\ &= 1 - P(X_1 > x, \ldots, X_n > x) = 1 - P(X_1 > x) \cdots P(X_n > x) \\ &= 1 - e^{-\lambda_1 x} \cdots e^{-\lambda_n x} = 1 - e^{-(\lambda_1 + \cdots + \lambda_n)x}. \end{aligned}$$

Differentiating F_X, we find that $f_X(x) = (\lambda_1 + \cdots + \lambda_n)\, e^{-(\lambda_1 + \cdots + \lambda_n)x}$ for $x > 0$, and $f_X(x) = 0$ otherwise. Thus, $X \sim \mathcal{E}(\lambda_1 + \cdots + \lambda_n)$. In other words, the lifetime of this series system has the exponential distribution with parameter $\lambda_1 + \cdots + \lambda_n$.

b) The probability that a specified component—say, component C_k—is the first to fail is $P(X = X_k)$ or, equivalently, $P\big(X_k < \min_{j \ne k}\{X_j\}\big)$. Using the same reasoning as in part (a), we conclude that $\min_{j \ne k}\{X_j\}$ has the exponential distribution with parameter $\sum_{j \ne k} \lambda_j$. Furthermore, by Proposition 6.13 on page 297, we know that X_k and $\min_{j \ne k}\{X_j\}$ are independent random variables. Referring to the result of Example 9.5(b) on page 503, we conclude that

$$P\left(X_k < \min_{j \ne k}\{X_j\}\right) = \frac{\lambda_k}{\lambda_k + \sum_{j \ne k} \lambda_j} = \frac{\lambda_k}{\sum_{j=1}^{n} \lambda_j}.$$

The probability is $\lambda_k/(\lambda_1 + \cdots + \lambda_n)$ that component C_k is the first to fail. ■

By using the same reasoning as that in Example 9.17, we get Proposition 9.11, which is especially important in the theory of stochastic processes and operations research.

◆◆◆ **Proposition 9.11 Minimum of Independent Exponentials**

Suppose that $X_1, \ldots, X_m$ are independent exponential random variables with parameters $\lambda_1, \ldots, \lambda_m$, respectively. Then the following hold.

a) We have

$$\min\{X_1, \ldots, X_m\} \sim \mathcal{E}(\lambda_1 + \cdots + \lambda_m). \tag{9.41}$$

In words, the minimum of m independent exponential random variables is also an exponential random variable with parameter equal to the sum of the parameters of the component exponential random variables.

b) For each k, with $1 \le k \le m$, we have

$$P\big(X_k = \min\{X_1, \ldots, X_m\}\big) = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_m}. \tag{9.42}$$

The probability is $\lambda_k/(\lambda_1 + \cdots + \lambda_m)$ that X_k is the minimum of $X_1, \ldots, X_m$.