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9.148 152

Let $Z = Y/X$, $u = X$

Set $g(x,y) = Y/X$, $h(x,y) = X$

then $J(X,Y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} -YX^{-2} & X^{-1} \\ 1 & 0 \end{vmatrix} = -X^{-1}$

$\therefore f_{Z,u} = (Z,u) = |-X| \cdot f_{X,Y}(x,y)$

$= |-u| \cdot f_{X,Y}(u, Zu)$

$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} |-u| \cdot f_{X,Y}(u, zu)\, du$

9.152 Let $X, Y \sim N(0, \sigma^2)$; Set $S = X^2 + Y^2$, $T = \frac{Y}{X}$

By independence, $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$$f_{X,Y}(x,y) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \cdot \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2}$$

$$= \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)}$$

Let $g(x,y) = S = X^2 + Y^2$, $h(x,y) = T = \frac{Y}{X}$

$$J(X,Y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -yx^{-2} & x^{-1} \end{vmatrix} = 2 + 2y^2x^{-2} = 2 + \frac{2y^2}{x^2}$$

$\because$ S is not one-to-one $\therefore$ divide $J(X,Y)$ by 2.

$$\Rightarrow f_{S,T}(s,t) = \frac{1}{|J(X,Y)|} f_{X,Y}(x,y)$$

$$= \frac{1}{1 + 1\left(\frac{y}{x}\right)^2} \cdot \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)}$$

$$= \frac{1}{2\pi\sigma^2 \cdot (1+T^2)} \cdot e^{-\frac{S}{2\sigma^2}}$$

$$f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2(1+T^2)} e^{-\frac{s}{2\sigma^2}} \cdot dT$$

$$= \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{s}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{1+T^2}\, dT = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{s}{2\sigma^2}} \cdot \left[\tan^{-1} T\right]_{-\infty}^{+\infty}$$

$$= \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{s}{2\sigma^2}} \cdot \pi = \frac{1}{2\sigma^2} \cdot e^{-\frac{s}{2\sigma^2}}$$

$$f_T(t) = \int_0^{\infty} \frac{1}{2\pi\sigma^2 \cdot (1+T^2)} \cdot e^{-s/2\sigma^2}\, ds \qquad \text{let } u = -\frac{s}{2\sigma^2}, \quad \frac{du}{ds} = -\frac{1}{2\sigma^2}$$

$$= \frac{1}{2\pi\sigma^2 \cdot (1+T^2)} \cdot \int_0^{\infty} e^{-s/2\sigma^2} ds \qquad du = -\frac{1}{2\sigma^2} \cdot ds \Rightarrow ds = -2\sigma^2 du$$

$$= \frac{1}{2\pi\sigma^2(1+T^2)} \cdot (-2\sigma^2) \int e^u\, du$$

$$= \frac{1}{2\pi\sigma^2 (1+T^2)} \cdot (-2\sigma^2) \cdot \left[e^{-s/2\sigma^2}\right]_0^{\infty}$$

$$= \frac{1}{2\pi\sigma^2 (1+T^2)} \cdot 2\sigma^2 = \frac{1}{\pi(1+T^2)}$$

Test independence: S, T are independent if $f_{S,T}(s,t) = f_S(s) \cdot f_T(t)$

$$f_S(s) \cdot f_T(t) = \frac{1}{2\sigma^2} e^{-\frac{s}{2\sigma^2}} \cdot \frac{1}{1\pi\ (1+T^2)}$$

$$= \frac{1}{2\pi\sigma^2(1+T^2)} e^{-\frac{s}{2\sigma^2}} = f_{S,T}(s,t) \Rightarrow S, T \text{ are independent.}$$

10.9 $E(X) = \alpha / (\alpha + \beta)$

Let $X \sim B(\alpha, \beta) \Rightarrow f_X(x) = \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$

Where $B(\alpha,\beta) = \dfrac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$E(X) = \int_0^1 x \cdot f_X(x)\, dx \qquad \text{let } u-1 = \alpha, \quad u = \alpha + 1$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \int_0^1 x \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \int_0^1 x^{\alpha}(1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \int_0^1 x^{u-1}(1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \frac{\Gamma(u) \cdot \Gamma(\beta)}{\Gamma(u+\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{(\alpha+\beta-1)!}{(\alpha-1)! \cdot (\beta-1)!} \cdot \frac{\alpha(\alpha-1)! \cdot (\beta-1)!}{(\alpha+\beta)(\alpha+\beta-1)!} = \frac{\alpha}{\alpha+\beta}$$

10.13

$\because f_X(x) \propto (1+x)^{-4} \therefore f_X(x) = k\cdot(1+x)^{-4}, \ \forall x>0, \ k\in\mathbb{R}$

$P_{tot} = \int_0^\infty f_X(x)\,dx = 1$

$\Rightarrow k\int_0^\infty \frac{1}{(1+x)^4}\,dx = 1$ Let $u=1+x$, $\frac{du}{dx}=1 \Rightarrow du=dx$

$k\cdot\int u^{-4}\,du = 1$

$k\cdot(-\frac{1}{3})\cdot\left[(1+x)^{-3}\right]_0^\infty = 1$

$\frac{1}{3}k = 1 \Rightarrow k=3 \Rightarrow f_X(x) = 3\cdot(1+x)^{-4}$

$\Rightarrow E(x) = \int_0^\infty x\cdot 3\cdot(1+x)^{-4}\,dx$

$= 3\int_0^\infty \frac{x}{(1+x)^{-4}}\,dx$ Let $u=x+1 \Rightarrow du=dx$

$= 3\cdot\int \frac{u-1}{u^{-4}}\,du$ $\Rightarrow x = u-1$

$= 3\cdot\int u^{-3}\,du - 3\int u^{-4}du = 3\cdot(-\frac{1}{2})\cdot\left[(x+1)^{-2}\right]_0^\infty + 3\cdot\frac{1}{3}\left[(x+1)^{-3}\right]_0^\infty$

$= \frac{3}{2} - 1 = \boxed{\frac{1}{2}}$

10.23

(a) Since the square of a normal distribution is a chi-squared distribution, and that W is a linear combination of 3 R.V.s — X^2, Y^2, and Z^2, by the linearity principle, W is also a chi-squared distribution with 3 degrees of freedom

(b) $W \sim \chi^2(3) \Rightarrow f_W(w) = \frac{1}{2^{3/2}\cdot\Gamma(3/2)}\cdot w^{\frac{1}{2}}e^{-w/2}$

$E[S] = E[(X^2+Y^2+Z^2)^{\frac{1}{2}}]$

$= E[(W^{\frac{1}{2}}\cdot\sigma)]$

$= \sigma E[W^{\frac{1}{2}}]$

$= \sigma\int_0^\infty w^{\frac{1}{2}}\cdot f_W(w)\,dw$

$= \sigma\cdot\frac{1}{2^{3/2}\cdot\Gamma(\frac{3}{2})}\int_0^\infty w^{\frac{1}{2}}\cdot w^{\frac{1}{2}}\cdot e^{-w/2}\,dw$

$= \sigma\cdot\frac{1}{2^{3/2}\cdot 2^{-1}\cdot\Gamma(\frac{1}{2})}\cdot\int_0^\infty w\cdot e^{-w/2}\,dw$ Let $u=w$, $du = 1dw$ $V=-2e^{-w/2}$, $dV/dt = e^{-w/2}$

$= \sigma\cdot\frac{1}{2^{\frac{1}{2}}\cdot\sqrt{\pi}}\cdot\left(-2\left[w\cdot e^{-w/2}\right]_0^\infty + 2\int_0^\infty e^{-w/2}\,dw\right)$

$$= \frac{1}{\sqrt{\pi}\cdot\sqrt{2}} \cdot \sigma \cdot (-4) \cdot \left[e^{-w/2} \right]_0^{\infty}$$

$$\Rightarrow \boxed{E[S] = \frac{4}{\sqrt{\pi}\cdot\sqrt{2}} \cdot \sigma}$$

10.29 Let Y be a R.V. representing the lifetime of the equipment.

$$\frac{1}{\lambda} = 10 \Rightarrow \lambda = \frac{1}{10} \Rightarrow Y \sim Exp\left(\lambda = \frac{1}{10}\right) \Rightarrow F_Y = 1 - e^{-\frac{1}{10}y}$$

Let Z be the insurance payment $\Rightarrow Z = \begin{cases} X & 0 < Y \le 1 \\ \frac{X}{2} & 1 < Y \le 3 \\ 0 & Y > 3 \end{cases}$

$$E(Z) = X \cdot P(Y \le 1) + \frac{1}{2}X\, P(1 < Y \le 3) + 0 \cdot P(Y > 3)$$

$$= X \cdot \left(1 - e^{-\frac{1}{10}\cdot 1}\right) + \frac{1}{2}X \cdot \left[\left(1 - e^{-\frac{1}{10}\cdot 3}\right) - \left(1 - e^{-\frac{1}{10}\cdot 1}\right) \right]$$

$$= X \cdot \left(1 - e^{-\frac{1}{10}}\right) + \frac{1}{2}X \left(1 - e^{-\frac{3}{10}} - 1 + e^{-\frac{1}{10}}\right)$$

$$= X - X \cdot e^{-\frac{1}{10}} - \frac{1}{2}X \cdot e^{-\frac{3}{10}} + \frac{1}{2}X e^{-\frac{1}{10}}$$

$$= X - \frac{1}{2}Xe^{-\frac{1}{10}} - \frac{1}{2}X \cdot e^{-\frac{3}{10}} = 1000$$

$$X\left(1 - \frac{1}{2}e^{-\frac{1}{10}} - \frac{1}{2}e^{-\frac{3}{10}}\right) = 1000 \Rightarrow \boxed{X \approx 5644.23}$$

10.52

(a) According to E.g. 9.8 (b) on P.511, $f_X(x) = (2/\pi) \cdot \sqrt{1 - x^2}$, $(-1 < X < 1)$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \int_{-1}^{1} x^2 \cdot f_X(x)\, dx - \left[\int_{-1}^{1} x \cdot f_X(x)\, dx \right]^2$$

$$= (2/\pi) \int_{-1}^{1} x^2 \cdot (1 - x^2)^{\frac{1}{2}} dx - \left[(2/\pi) \int_{-1}^{1} x \cdot (1 - x^2)^{\frac{1}{2}} dx \right]^2$$

$$= \frac{2}{\pi} \cdot \left(\frac{\pi}{8}\right) - \left(\frac{4}{\pi^2}\right) \cdot 0^2 = \frac{1}{4}$$

(b) $Var(Y) = Var(X) = \frac{1}{4}$ because X, Y are identically distributed

(c) $f_{X,Y}(x,y) = \frac{1}{\pi} \Rightarrow E(XY) = \int_{-1}^{1}\int_{-1}^{1} xy \cdot \frac{1}{\pi}\, dx\, dy$

$$E(XY) = \int_{-1}^{1} \frac{y}{\pi} \left[\frac{1}{2}x^2 \right]_{-1}^{1} dy = 0 \Rightarrow cov(X,Y) = E(XY) - E(X) \cdot E(Y) = 0$$

$$\therefore \rho(X,Y) = cov(X,Y) / \sqrt{Var(X) \cdot Var(Y)} = 0$$

(d) No, because having $\rho(x,y)$ only implies $cov(x,y)=0$

$Cov(x,y)=0$ does not imply independence (i.e., $f_{x,y}(x,y)=f_x(x)\cdot f_y(y)$)

(e) No, X and Y are dependent

Because: $f_X(x)\cdot f_Y(y)=\left(\frac{2}{\pi}\sqrt{1-x^2}\right)\left(\frac{2}{\pi}\sqrt{1-y^2}\right)\neq f_{xy}(x,y)$

10.61

(a) Let $Z=X+Y=1 \Rightarrow Y=1-X$; Given that $Var(Z)=Var(1)=0$

$\therefore Var(X+Y)=0 \Rightarrow Var(X)+Var(Y)+2cov(X,Y)=0$

$Var(X)+Var(1-X)+2cov(X,1-X)=0$

$Var(X)+(-1)^2\cdot Var(X)-2cov(X,X)=0$

$Var(X)+Var(X)-2Var(X)=0$

$2Var(X)=2Var(X)$

$\Rightarrow Var(X)=Var(1-Y)=(-1)^2Var(Y)=Var(Y)$, Q.E.D.

(b) Given that $(Var(X)+Var(Y)+2cov(X,Y)=0)\wedge(Var(Y)=Var(X))$

$Var(X)+Var(X)+2cov(X,Y)=0$

$2Var(X)=-2cov(X,Y)$

$\Rightarrow -Var(X)=cov(X,Y)$, Q.E.D.

(c) $LHS=cov(X,Y)=cov(X,1-X)$

$=cov(X,-X)+cov(X,1)$

$=-cov(X,X)+0=-Var(X)=RHS$, Q.E.D.

(d) $\rho(X,Y)=cov(X,Y)/\sqrt{Var(X)\cdot Var(Y)}$

$=-Var(X)/\sqrt{(Var(X))^2}$

$=-Var(X)/Var(X)=-1$, Q.E.D.

(e) Given that $X+Y=1$, we deduce that $Y=1-X$

By proposition 7.16 (d) on P. 372, $\rho(X,Y)=-1$ since $-1<0$.

10.64 $Var(X) = E(X^2) - (E(X))^2$

Let X be an R.V. representing the insurance payment, Y be an R.V. represent repair cost

$\therefore X = \begin{cases} 0 & Y \leq 250 \\ Y-250 & 250 < Y \leq 1500 \end{cases}, \quad f_X(x) = f_Y(y) = \frac{1}{1500-0} = \frac{1}{1500}$

$\Rightarrow E(X) = \int_{250}^{1500} (Y-250)\cdot f_Y(y)\,dy$

$= \frac{1}{1500}\int_{250}^{1500} Y - 250\, dy = \frac{1}{1500}\left[\frac{1}{2}Y^2 - 250Y\right]_{250}^{1500} \approx 520.83$

$\Rightarrow E(X^2) = \int_{250}^{1500} (Y-250)^2 \cdot \frac{1}{1500}\, dy = \frac{1}{1500}\int_{250}^{1500} Y^2 - 500Y + 62500\, dy$

$= \frac{1}{1500}\left[\frac{1}{3}Y^3 - 250Y^2 + 62500Y\right]_{250}^{1500} \approx 434027.78$

$\Rightarrow Var(X) = E(X^2) - (E(X))^2 \approx 162763.89$

$\Rightarrow \sigma = \sqrt{Var(X)} \approx \boxed{403.44}$

10.78 Given that $f_{X,Y}(x,y) = \begin{cases} 2x, & x<y<x+1 \\ 0, & \text{otherwise} \end{cases}$

$f_{Y|X}(Y|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

$f_X(x) = \int_x^{x+1} 2x\, dy = 2x\,[y]_x^{x+1} = 2x\cdot[(x+1)-x] = 2x$

$\Rightarrow f_{Y|X}(Y|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2x}{2x} = 1, \quad (x<y<x+1)$

$\Rightarrow E(Y|X=x) = \int_x^{x+1} y\cdot 1\, dy = \left[\frac{1}{2}y^2\right]_x^{x+1} = \frac{1}{2}\left[(x+1)^2 - x^2\right]$

$= \frac{1}{2}(x^2+2x+1-x^2) = \boxed{x+\frac{1}{2}}$

$E(Y^2|X=x) = \int_x^{x+1} y^2\cdot 1\, dy = \frac{1}{3}[y^3]_x^{x+1} = \frac{1}{3}[(x+1)^3 - x^3] = \frac{1}{3}(x^3+1+3x^2+3x-x^3) = x^2+x+\frac{1}{3}$

$\Rightarrow Var(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2 = x^2+x+\frac{1}{3} - (x+\frac{1}{2})^2 = x^2+x+\frac{1}{3} - x^2 - x - \frac{1}{4} = \boxed{\frac{1}{12}}$