◆◆◆ **Proposition 9.14 Sum of Independent Normal Random Variables**
Let $X_1, \ldots, X_m$ be independent random variables with $X_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$ for $1 \leq j \leq m$, and let $a, b_1, \ldots, b_m$ be real numbers, where not all the b_js equal 0. Then

$$a + b_1X_1 + \cdots + b_mX_m \sim \mathcal{N}\left(a + b_1\mu_1 + \cdots + b_m\mu_m, b_1^2\sigma_1^2 + \cdots + b_m^2\sigma_m^2\right).$$

In particular, $X_1 + \cdots + X_m \sim \mathcal{N}\left(\mu_1 + \cdots + \mu_m, \sigma_1^2 + \cdots + \sigma_m^2\right)$.

Quotients of Continuous Random Variables

Quotients of continuous random variables occur frequently in applications, especially in statistics where they often arise as sampling distributions. Proposition 9.15 provides a general formula for a PDF of the quotient of two continuous random variables.

◆◆◆ **Proposition 9.15 PDF of the Quotient of Two Continuous Random Variables**
Let X and Y be continuous random variables with a joint PDF. Then a PDF of the random variable Y/X is given by

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} |x| f_{X,Y}(x, xz)\, dx, \qquad z \in \mathcal{R}. \tag{9.48}$$

If, in addition, X and Y are independent random variables, then Equation (9.48) takes the form

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz)\, dx, \qquad z \in \mathcal{R}. \tag{9.49}$$

Proof Applying the FPF yields

$$F_{Y/X}(z) = P(Y/X \leq z) = \iint_{y/x \leq z} f_{X,Y}(x, y)\, dx\, dy, \qquad z \in \mathcal{R}.$$

The form of the region $\{(x, y) : y/x \leq z\}$ depends on the sign of z. In each case, the shaded region in either Figure 9.13(a) or Figure 9.13(b) shows the set over which the double integral is taken. Referring to Figure 9.13, making the substitution $u = y/x$, and interchanging the order of integration, we get

$$\begin{aligned}
F_{Y/X}(z) &= \int_{-\infty}^{0}\left(\int_{zx}^{\infty} f_{X,Y}(x, y)\, dy\right) dx + \int_{0}^{\infty}\left(\int_{-\infty}^{zx} f_{X,Y}(x, y)\, dy\right) dx \\
&= \int_{-\infty}^{0}\left(\int_{z}^{-\infty} x f_{X,Y}(x, xu)\, du\right) dx + \int_{0}^{\infty}\left(\int_{-\infty}^{z} x f_{X,Y}(x, xu)\, du\right) dx \\
&= \int_{-\infty}^{0}\left(\int_{-\infty}^{z} (-x) f_{X,Y}(x, xu)\, du\right) dx + \int_{0}^{\infty}\left(\int_{-\infty}^{z} x f_{X,Y}(x, xu)\, du\right) dx \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z} |x| f_{X,Y}(x, xu)\, du\right) dx = \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} |x| f_{X,Y}(x, xu)\, dx\right) du.
\end{aligned}$$

Solution The annual payout, in millions of dollars, by the insurance company is $Y = \min\{X, 1\}$. To obtain the expected value of Y, we apply the FEF—specifically, Equation (10.8) on page 572, with $g(x) = \min\{x, 1\}$:

$$\mathcal{E}(Y) = \mathcal{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = \int_0^3 \min\{x, 1\} \frac{x(4-x)}{9}\,dx$$

$$= \frac{1}{9}\int_0^1 x \cdot x(4-x)\,dx + \frac{1}{9}\int_1^3 1 \cdot x(4-x)\,dx$$

$$= \frac{1}{9}\int_0^1 (4x^2 - x^3)\,dx + \frac{1}{9}\int_1^3 (4x - x^2)\,dx = \frac{101}{108} = 0.935.$$

On average, the insurance company pays \$0.935 million per policy year. ■

In Example 10.7, we compare an expected value determination with and without use of the fundamental expected-value formula.

EXAMPLE 10.7 ***The Fundamental Expected-Value Formula***

Speed of Gas Molecules The three velocity components of a gas molecule are independent and identically distributed normal random variables with common parameters 0 and σ^2. Determine the expected speed of a gas molecule

a) directly from the definition of expected value, Definition 10.1 on page 565.

b) by applying the FEF, Proposition 10.1.

c) Compare the methods used in parts (a) and (b).

Solution The speed of a gas molecule is $S = \sqrt{X^2 + Y^2 + Z^2}$, where X, Y, and Z denote the velocity components, which, by assumption, are independent normal random variables, each having parameters 0 and σ^2. The task is to determine the expected value of S.

a) From Example 9.18 on page 533, a PDF of S is $f_S(s) = \left(\sqrt{2/\pi}/\sigma^3\right) s^2 e^{-s^2/2\sigma^2}$ for $s > 0$, and $f_S(s) = 0$ otherwise. Applying Definition 10.1, making the substitution $u = s^2/2\sigma^2$, and using the fact that a gamma PDF integrates to 1, we get

$$\mathcal{E}(S) = \int_{-\infty}^{\infty} s f_S(s)\,ds = \int_0^{\infty} s \frac{\sqrt{2/\pi}}{\sigma^3} s^2 e^{-s^2/2\sigma^2}\,ds$$

$$= \frac{\sqrt{2/\pi}}{\sigma^3}\int_0^{\infty} s^3 e^{-s^2/2\sigma^2}\,ds = \frac{\sqrt{2/\pi}}{\sigma}\int_0^{\infty} 2\sigma^2 u e^{-u}\,du \tag{10.10}$$

$$= 2\sqrt{2/\pi}\,\sigma\,\frac{\Gamma(2)}{1^2}\int_0^{\infty} \frac{1^2}{\Gamma(2)} u^{2-1} e^{-u}\,du = 2\sqrt{2/\pi}\,\sigma.$$

The expected speed of a gas molecule is $2\sqrt{2/\pi}\,\sigma$.

b) To use the FEF to find $\mathcal{E}(S)$, we first note that a joint PDF of X, Y, and Z is

$$
\begin{aligned}
f_{X,Y,Z}(x, y, z) &= f_X(x) f_Y(y) f_Z(z) \\
&= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-z^2/2\sigma^2} \\
&= \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-(x^2+y^2+z^2)/2\sigma^2},
\end{aligned}
$$

for all $x, y, z \in \mathcal{R}$. Applying the FEF for continuous random variables and changing to spherical coordinates, we get

$$
\begin{aligned}
\mathcal{E}(S) &= \mathcal{E}\left(\sqrt{X^2 + Y^2 + Z^2}\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}\, f_{X,Y,Z}(x, y, z)\, dx\, dy\, dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}\, \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-(x^2+y^2+z^2)/2\sigma^2}\, dx\, dy\, dz \\
&= \frac{1}{(2\pi)^{3/2}\sigma^3} \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \rho e^{-\rho^2/2\sigma^2} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \\
&= \frac{1}{(2\pi)^{3/2}\sigma^3} \int_0^{\infty} \rho^3 e^{-\rho^2/2\sigma^2}\left(\int_0^{\pi} \sin\phi \left(\int_0^{2\pi} d\theta\right) d\phi\right) d\rho \\
&= \frac{2\cdot 2\pi}{(2\pi)^{3/2}\sigma^3} \int_0^{\infty} \rho^3 e^{-\rho^2/2\sigma^2}\, d\rho = \frac{\sqrt{2/\pi}}{\sigma^3} \int_0^{\infty} \rho^3 e^{-\rho^2/2\sigma^2}\, d\rho.
\end{aligned}
$$

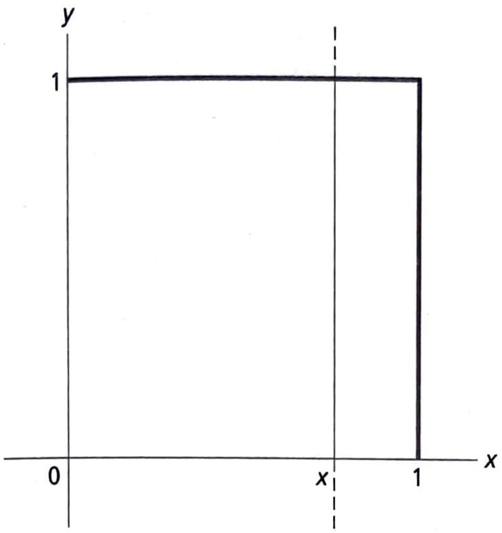
This last integral is the same as the first one in the second row of Equation (10.10) on the previous page. Therefore, $\mathcal{E}(S) = 2\sqrt{2/\pi}\,\sigma$. As in part (a), we see that the expected speed of a gas molecule is $2\sqrt{2/\pi}\,\sigma$.

c) The method used in part (a) to find $\mathcal{E}(S)$ seems easier than that in part (b). However, keep in mind the work that was required to determine a PDF of S, as shown in Example 9.18. ■

Basic Properties of Expected Value

We can now establish some of the basic and most widely used properties of expected value for continuous random variables. Actually, in Propositions 10.2–10.4, we provide only the statements of these properties. We leave the proof of Proposition 10.3 to you as Exercise 10.44. You can obtain the proofs of Propositions 10.2 and 10.4 by mimicking those in Section 7.2 for discrete random variables; simply replace sums by integrals and PMFs by PDFs. And, as we mentioned there, *these three propositions hold for all types of random variables, not just for discrete and continuous random variables.*

Figure 9.7



From Equation (9.27) and Figure 9.7,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy = \int_0^1 1\,dy = 1,$$

for $0 < x < 1$. Consequently, $f_X(x) = 1$ if $0 < x < 1$, and $f_X(x) = 0$ otherwise. So, $X \sim \mathcal{U}(0, 1)$. Similarly, $Y \sim \mathcal{U}(0, 1)$. ■

In Example 9.7, the random point (X, Y) is uniformly distributed over the unit square. We found that the marginal distributions of X and Y are also uniformly distributed. Example 9.8 shows that marginals of bivariate uniform distributions aren't always uniform.

EXAMPLE 9.8 ***Obtaining Marginal PDFs From a Joint PDF***

Bacteria on a Petri Dish A petri dish is a small, shallow dish of thin glass or plastic, used especially for cultures in bacteriology. Suppose that a petri dish of unit radius, containing nutrients upon which bacteria can multiply, is smeared with a uniform suspension of bacteria. Subsequently, spots indicating colonies of bacteria will appear. Let X and Y denote the x and y coordinates, respectively, of the center of the first spot to appear.

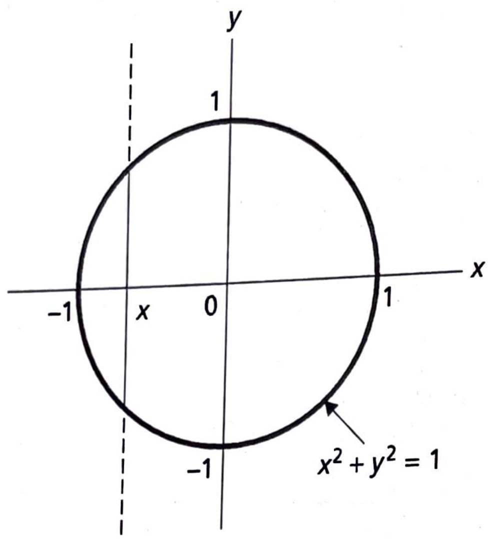
a) Determine a joint PDF of X and Y.
b) Use part (a) to find marginal PDFs of X and Y.

Solution Because the dish is smeared with a uniform suspension of bacteria, we use a geometric probability model. Specifically, we can think of the location of the center of the first spot as a point selected at random from the unit disk.

a) We see that (X, Y) is uniformly distributed over the unit disk. Hence a joint PDF of X and Y is $f_{X,Y}(x, y) = 1/\pi$ if (x, y) is in the unit disk, and $f_{X,Y}(x, y) = 0$ otherwise. (Why $1/\pi$?)

b) To obtain a marginal PDF of X, we first observe that the range of X is the interval $(-1, 1)$. Next we apply Proposition 9.7 by integrating on y the joint PDF of X and Y. Figure 9.8 shows the interval of integration for a fixed x in the range of X. The solid portion of the interval indicates where the joint PDF of X and Y is nonzero.

Figure 9.8



Noting that the equation of the boundary of the unit disk is $x^2 + y^2 = 1$, we find, in view of Equation (9.27) and Figure 9.8, that, for $-1 < x < 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2}{\pi}\sqrt{1 - x^2}.$$

Thus $f_X(x) = (2/\pi)\sqrt{1 - x^2}$ if $-1 < x < 1$, and $f_X(x) = 0$ otherwise. By symmetry, Y has that same PDF. Observe that the common distribution of X and Y isn't uniform. In fact, for each of these random variables, values near 0 are the most likely, with the likelihood decreasing as values move away from 0 in either direction. ∎

EXAMPLE 9.9 ***Obtaining Marginal PDFs***

Min and Max of a Random Sample Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with CDF F and PDF f. Set $X = \min\{X_1, \ldots, X_n\}$ and $Y = \max\{X_1, \ldots, X_n\}$.

a) Determine a PDF of X directly.
b) Determine a PDF of X by using a joint PDF of X and Y.
c) Apply the result for the PDF of X to the special case of a random sample of size n from a uniform distribution on the interval $(0, 1)$. Identify the probability distribution of X in this special case.