Will Zong

8.94

(a) $LHS = P(|X-\mu| \le t) = P(-t \le X-\mu \le t) = P(\mu - t \le X \le \mu + t)$

$= P\left(\frac{-t}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{t}{\sigma}\right)$

$= \Phi\left(\frac{t}{\sigma}\right) - \Phi\left(\frac{-t}{\sigma}\right)$

$= \Phi\left(\frac{t}{\sigma}\right) - \left(1 - \Phi\left(\frac{t}{\sigma}\right)\right)$

$= \Phi\left(\frac{t}{\sigma}\right) + \Phi\left(\frac{t}{\sigma}\right) - 1 = RHS$

(b) $LHS = P(|X-\mu| \ge t) = P(X-\mu \le -t) + P(X-\mu \ge t)$

$P(X-\mu \le -t)$ $\qquad$ $P(X-\mu \ge t) = 1 - P(X-\mu \le t)$

$= P\left(\frac{X-\mu}{\sigma} \le \frac{-t}{\sigma}\right)$ $\qquad$ $= 1 - P\left(\frac{X-\mu}{\sigma} \le \frac{t}{\sigma}\right)$

$= \Phi\left(\frac{-t}{\sigma}\right) = 1 - \Phi\left(\frac{t}{\sigma}\right)$ $\qquad$ $= 1 - \Phi\left(\frac{t}{\sigma}\right)$

$\Rightarrow LHS = 1 - \Phi\left(\frac{t}{\sigma}\right) + \left(1 - \Phi\left(\frac{t}{\sigma}\right)\right) = 2 - 2\Phi\left(\frac{t}{\sigma}\right) = RHS.$

8.102 Let X be the R.V. representing the amount of drink dispensed.

$X \sim (\mu, \sigma = 0.25)$

$P(X \le 8) = 0.02 = P\left(\frac{X-\mu}{\sigma} \le \frac{8-\mu}{\sigma}\right) = 0.02$

$P\left(Z \le \frac{8-\mu}{\sigma}\right) = 0.02$

$\Rightarrow Z \approx -2 \Rightarrow \frac{8-\mu}{\sigma} \approx -2$

$-\mu \approx -8.5 \Rightarrow \mu \approx 8.5$

8.105 $X \sim N(0, 1/\alpha^2)$ Let $Y = \frac{1}{X^2} = X^{-2}$. $g(x) = \frac{1}{x^2} \Rightarrow g^{-1}(y) = \pm\sqrt{\frac{1}{y}}$

$x = g^{-1}(y) = \pm\sqrt{\frac{1}{y}} = \pm\, y^{-\frac{1}{2}}$

$\Rightarrow \frac{d}{dy}(x) = (g^{-1}(y))' = \pm \frac{1}{2} y^{-\frac{3}{2}} \Rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{2} y^{-\frac{3}{2}}$

$f_Y(y) = f_X\left(\sqrt{\frac{1}{y}}\right)\left|\frac{dx}{dy}\right| + f_X\left(-\sqrt{\frac{1}{y}}\right)\cdot\left|\frac{dx}{dy}\right|$

$$= \frac{1}{\sqrt{2\pi \frac{1}{\alpha^2}}} \cdot e^{\frac{-(\sqrt{\frac{1}{y}}-0)^2}{2\cdot 1/\alpha^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{y^3}} + \frac{1}{\sqrt{2\pi \frac{1}{\alpha^2}}} \cdot e^{\frac{-(-\sqrt{\frac{1}{y}}-0)^2}{2\cdot 1/\alpha^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{y^3}}$$

$$= \begin{cases} 2\cdot\left(\frac{1}{2}\frac{\alpha}{\sqrt{2\pi y^3}}\cdot e^{\frac{-\alpha^2}{2y}}\right) = \frac{\alpha}{\sqrt{2\pi y^3}}\cdot e^{\frac{-\alpha^2}{2y}}, & y>0 \\ f_Y(y)=0, & \text{otherwise} \end{cases}$$

8.116

(a) $F(x) = 1 - e^{-\lambda x}\sum_{j=0}^{r-1}\frac{(\lambda x)^j}{j!}$ $\because \forall x \geq 0,\ \lambda e^{-\lambda x} \geq 0$ and each term in the

$\frac{d}{dx}(F(x)) = \lambda e^{-\lambda x}\cdot\sum_{j=0}^{r-1}\frac{\lambda^j}{j!}\cdot j x^{j-1}$ Summation is also non-negtive

$\therefore \frac{d}{dx}(F_X(x)) \geq 0,\ \forall x \geq 0$

$\Rightarrow F_X(x)$ is non-decreasing $\forall x \geq 0$

(b) $\because \forall x \geq 0,\ e^{-\lambda x}$ is continuous; $\sum_{j=0}^{r-1}\frac{(\lambda x)^j}{j!}$ is a continuous polynomial $\therefore F_X(x)$ is continuous, $\forall x \geq 0$

$\forall x<0,\ F_X(x)=0$, which is continuous. $\therefore F_X(x)$ is continuous, $\forall x<0$

$F_X(0^+) = F_X(0) = 1 - e^{-\lambda\cdot 0}\sum_{j=0}^{r-1}\frac{(\lambda\cdot 0)^j}{j!} = 1-1\cdot 1 = 0$ $\therefore F_X(x)$ is right continuous at $X=0$.

(c) $\because F_X(x) = 0$ for $x<0$ $\therefore F_X(-\infty) = 0$ by definition

(d) $\because \lim_{x\to\infty} e^{-\lambda x} = 0$ $\therefore F(\infty) = \lim_{x\to\infty} 1 - e^{-\lambda x}\cdot\sum_{j=0}^{r-1}\frac{(\lambda x)^j}{j!} = 1-0 = 1$

$$\lim_{x\to\infty} e^{-\lambda x}\cdot\sum_{j=0}^{r-1}\frac{(\lambda x)^j}{j!} = \lim_{x\to\infty}\frac{\sum_{j=0}^{r-1}\frac{(\lambda x)^j}{j!}}{e^{\lambda x}} = \lim_{x\to\infty}\frac{\sum_{j=1}^{r-1}\frac{\lambda^j}{j!}\cdot j\cdot x^{j-1}}{\lambda e^{\lambda x}}$$

$$= \lim_{x\to\infty}\frac{\sum_{j=2}^{r-1}\frac{\lambda^j}{j!}\cdot j\cdot(j-1)\cdot x^{j-2}}{\lambda^2 e^{\lambda x}} = \underbrace{\cdots}_{\substack{J \text{ iteration} \\ \text{of using L'hospital's rule}}} = \lim_{x\to\infty}\frac{\sum_{j=r-1}^{r-1}\frac{\lambda^j}{j!}\cdot j!\cdot x^0}{\lambda^j e^{\lambda x}} = \lim_{x\to\infty}\frac{1}{e^{\lambda x}} = 0$$

$\therefore \lim_{x\to\infty} F_X(x) = 1 - 0 = 1$

As shown in (a) (b) (c) (d) above, $F_X(x)$ is the CDF of a continuous R.V.

8.120 $X \sim Beta(\alpha, \beta)$; let $Y = g(X) = X^{-1} - 1 \Rightarrow X = g^{-1}(Y) = \frac{1}{Y+1}$

$$F_Y(y) = P(Y \leq y) = P((X^{-1}-1) \leq y) = P(X \leq (y+1)^{-1}) = F_X((y+1)^{-1})$$

$$\therefore f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X((y+1)^{-1}) = f_X((y+1)^{-1}) \cdot -(y+1)^{-2}$$

$$= \frac{\left(\frac{1}{Y+1}\right)^{\alpha-1}\left(1-\frac{1}{Y+1}\right)^{\beta-1}}{B(\alpha,\beta)} \cdot (y+1)^{-2} = \frac{\left(\frac{1}{Y+1}\right)^{\alpha-1}\left(\frac{Y}{Y+1}\right)^{\beta-1}}{B(\alpha,\beta)} \cdot -(y+1)^{-2}$$

8.127 $X \sim Gamma(\alpha=5, \lambda=0.1)$

(a)

$$\therefore f_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{(0.1)^5}{\Gamma(5)} x^4 \cdot e^{-0.1x} = \frac{1}{2400,000} x^4 \cdot e^{-0.1x}, & x > 0 \end{cases}$$

$$F_X(x) = \begin{cases} \int_0^x f_X(x)\,dx = \frac{1}{12,000,000} \int_0^x x^4 \cdot e^{-0.1x} dx = 1 - e^{-0.1} = 1 - e^{-0.1x} \sum_{j=0}^{4} \frac{(0.1x)^j}{j!}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(From textbook P. 453)

(b) $P(X \leq 60) = F_X(60)$

$$= 1 - e^{-0.1\cdot 60} \cdot \left(\frac{(0.1\cdot 60)^0}{0!} + \frac{(0.1\cdot 60)^1}{1!} + \frac{(0.1\cdot 60)^2}{2!} + \frac{(0.1\cdot 60)^3}{3!} + \frac{(0.1\cdot 60)^4}{4!}\right)$$

$$= 1 - e^{-6} \cdot (1 + 6 + 18 + 36 + 54)$$

$$= 1 - e^{-6} \cdot (115) \approx 0.715$$

(c) $P(40 \leq X \leq 50) = F_X(50) - F_X(40)$

$$= \left[1 - e^{-0.1\cdot 50} \cdot \left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right)\right] - \left[1 - e^{-0.1\cdot 40} \cdot \left(\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!}\right)\right]$$

$$= [1 - e^{-5} \cdot (65.375)] - [1 - e^{-4} (34.\dot{3})]$$

$$\approx 0.5595 - 0.3712$$

$$= 0.1883$$

8.146 $X \sim Beta(\alpha, \beta)$; let $Y = g(X) = X^{-1} - 1 \Rightarrow X = g^{-1}(Y) = \frac{1}{Y+1}$

$\frac{dx}{dy} = \frac{d}{dy}((Y+1)^{-1}) = -(y+1)^{-2} \Rightarrow \left|\frac{dx}{dy}\right| = (y+1)^{-2}$

$\therefore f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f_X((Y+1)^{-1}) \cdot (y+1)^{-2}$

$$= \frac{\left(\frac{1}{Y+1}\right)^{\alpha-1} \cdot \left(1 - \frac{1}{Y+1}\right)^{\beta-1}}{B(\alpha, \beta)} \cdot (y+1)^{-2} = \frac{\left(\frac{1}{Y+1}\right)^{\alpha-1} \cdot \left(\frac{Y}{Y+1}\right)^{\beta-1}}{B(\alpha, \beta)} \cdot (y+1)^{-2}$$

8.152 $X \sim Exp(\lambda = 1)$; let $Y = g(X) = 10X^{0.8} \Rightarrow X = g^{-1}(Y) = \left(\frac{1}{10}Y\right)^{\frac{5}{4}}$

$F_Y(y) = P(Y \leq y) = P(10X^{0.8} \leq Y) = P\left(X \leq \left(\frac{1}{10}Y\right)^{\frac{5}{4}}\right) = F_X\left(\left(\frac{1}{10}Y\right)^{5/4}\right)$

$\therefore f_Y(y) = \frac{d}{dY}(F_Y(y)) = \frac{d}{dY}\left(F_X\left(\left(\frac{1}{10}Y\right)^{5/4}\right)\right) = f_X\left(\left(\frac{1}{10}Y\right)^{5/4}\right) \cdot \frac{1}{10} \cdot \frac{5}{4} \cdot \left(\frac{1}{10}Y\right)^{\frac{1}{4}}$

$$= e^{-\left(\frac{1}{10}Y\right)^{5/4}} \cdot \frac{1}{8}\left(\frac{1}{10}Y\right)^{\frac{1}{4}}$$