Mutually Exclusive Versus Independent Events

The terms *mutually exclusive* and *independent* refer to different concepts. Mutually exclusive events are those that can't occur simultaneously. Independent events are those for which the occurrence of some doesn't affect the probabilities of the others occurring. If two or more events are mutually exclusive, the occurrence of one precludes the occurrence of the others. Hence, two or more events with positive probabilities can't be both mutually exclusive and independent. See Exercise 4.55 for more on this issue.

EXERCISES 4.3

Basic Exercises

4.43 Verify the following statements made on page 148.

a) If event B is independent of event A in the sense of Definition 4.3, then the two events are independent in the sense of Definition 4.4.

b) If $P(A) > 0$ and events A and B are independent in the sense of Definition 4.4, then event B is independent of event A in the sense of Definition 4.3.

4.44 The U.S. National Center for Health Statistics compiles data on injuries and publishes the information in *Vital and Health Statistics.* A contingency table for injuries in the United States, by circumstance and sex, is as follows. Frequencies are in millions.

Circumstance

Sex		Work C_1	Home C_2	Other C_3	Total
	Male S_1	8.0	9.8	17.8	35.6
	Female S_2	1.3	11.6	12.9	25.8
	Total	9.3	21.4	30.7	61.4

a) Are events C_1 and S_2 independent? Explain.

b) Is the event that an injured person is male independent of the event that an injured person was hurt at home? Explain.

4.45 Refer to the joint probability distribution in Exercise 4.8 on page 135 for living arrangement and age of U.S. citizens 15 years of age and older. Are events A_2 and L_1 independent? Interpret your answer.

4.46 In the game of Yahtzee, five balanced dice are rolled. What is the probability

a) of rolling all 2s?

b) that all the dice come up the same number?

c) of getting a full house—three of one number and two of another?

4.47 Let A be an event of a sample space. Verify the following statements.

a) If $P(A) = 0$ or $P(A) = 1$, then, for each event B of the sample space, A and B are independent events.

b) If A and A are independent events, then $P(A) = 0$ or $P(A) = 1$.

4.48 Suppose that a number is chosen at random from the interval (0, 1). For the number obtained, let A be the event that the first decimal digit is 6 and let B be the event that the second decimal digit is 4. Determine whether events A and B are independent.

4.49 Respond *true* or *false* and explain your answer: If events A and B are independent and events B and C are independent, then events A and C are independent.

4.50 According to *Accident Facts*, published by the National Safety Council, a probability distribution of age group for drivers at fault in fatal crashes is as follows.

Age (yrs)	Probability
16–24	0.255
25–34	0.238
35–64	0.393
65 & over	0.114

Of three fatal automobile crashes, find the probability that

a) the drivers at fault in the first, second, and third crashes are in the age groups 16–24, 25–34, and 35–64, respectively.

b) two of the drivers at fault are between 16 and 24 years old and one of the drivers at fault is 65 years old or older.

4.51 Determine the number of equations for checking mutual independence of n events.

4.52 Refer to Example 4.15 on page 152 and also to Example 2.18 on page 44, where we provided six potential probability assignments (Table 2.4) and discovered that only assignments #1–#4 are legitimate.

a) Of the assignments #1–#4, which correspond to independent tosses of the coin? Explain your reasoning.

b) For those assignments that correspond to independent tosses of the coin, identify the probability of a head on any given toss.

4.53 Refer to Example 4.16 on page 153 regarding component analysis.

a) Solve part (b) by using the inclusion–exclusion principle.

b) Provide the details for part (c).

4.54 Referring to Example 4.18 on page 155, determine the probability of winning a game of craps.

4.55 In this exercise, you are to further examine the concepts of independent events and mutually exclusive events.

a) If two events are mutually exclusive, determine their joint probability.

b) If two events with positive probability are independent, explain why their joint probability is not 0. Conclude that the two events can't be mutually exclusive.

c) Give an example of two events that are neither mutually exclusive nor independent.

4.56 Suppose that you have two coins, one balanced and the other with probability p of a head. You select one of the coins at random and toss it twice. Determine the probability that

a) the first toss is a head. **b)** the second toss is a head. **c)** both tosses are heads.

d) Show that the events "first toss a head" and "second toss a head" are independent if and only if the second coin is also balanced.

4.57 Suppose that, for children born to a certain couple, the events "first is a boy," "second is a boy," "third is a boy," and "fourth is a boy" are mutually independent. Further suppose

Now suppose that the person selected is a smoker. On the basis of this additional information, we can revise the probability that the person has lung disease. We do so by determining the conditional probability that a randomly selected person has lung disease, given that the person selected is a smoker: $P(L \mid S) = 0.211$ (from Example 4.21). This revised probability is called a **posterior probability** because it represents the probability that the person selected has lung disease *after* we know that the person is a smoker.

EXERCISES 4.4 Basic Exercises

4.66 The National Sporting Goods Association collects and publishes data on participation in selected sports activities. For Americans 7 years old or older, 17.4% of males and 4.5% of females play golf. And, according to the U.S. Census Bureau's *Current Population Reports*, of Americans 7 years old or older, 48.6% are male and 51.4% are female. From among those Americans who are 7 years old or older, one is selected at random. Find the probability that the person selected

a) plays golf, given that the person is a female.
b) is a female, given that the person plays golf.
c) Interpret your answers in parts (a) and (b) in terms of percentages.

4.67 A survey conducted by TELENATION/Market Facts, Inc., combined with information from the U.S. Census Bureau's *Current Population Reports,* yielded the table given in Exercise 4.30 on page 145. What percentage of adult moviegoers are between 25 and 34 years old?

4.68 In a certain population of registered voters, 40% are Democrats, 32% are Republicans, and 28% are Independents. Sixty percent of the Democrats, 80% of the Republicans, and 30% of the Independents favor increased spending to combat terrorism. If a person chosen at random from this population favors increased spending to combat terrorism, what is the probability that the person is a Democrat?

4.69 An insurance company classifies people as *normal* or *accident prone*. Suppose that the probability that a normal person has an accident in a specified year is 0.2 and that for an accident prone person this probability is 0.6. Further suppose that 18% of the policyholders are accident prone. A policyholder had no accidents in a specified year. What is the probability that he or she is accident prone?

4.70 Refer to Example 4.18 on page 155. If you win a game of craps, what is the probability that your first toss resulted in a sum of 4?

4.71 An urn contains one red marble and nine green marbles; a second urn contains two red marbles and eight green marbles; and a third urn contains three red marbles and seven green marbles. An urn is chosen at random, and then one marble is randomly selected from the chosen urn.

a) Given that the marble obtained is red, what is the probability that the chosen urn was the first one; the second one; the third one?
b) Modify the problem by adding one million additional green marbles to each urn. Given the unlikely result that the marble selected is red, what are the posterior probabilities of the three urns? How is the result affected by the addition of the extra green marbles?
c) In part (a), find the posterior probabilities of the three urns, given that the marble selected is green. Then do the same after the additional green marbles of part (b) have been introduced. How is this result affected by the addition of the extra green marbles?

4.72 Suppose that you have four chests, each with two drawers, and that each drawer contains either a gold coin or a silver coin. Chests A and B each have one gold and one silver coin, chest C has two silver coins, and chest D has two gold coins. You select a chest at random, open one of the drawers, and find a silver coin within. What is the probability that the other drawer also contains a silver coin?

4.73 At a grocery store, eggs come in cartons that hold a dozen eggs. Experience indicates that 78.5% of the cartons have no broken eggs, 19.2% have one broken egg, 2.2% have two broken eggs, 0.1% have three broken eggs, and that the percentage of cartons with four or more broken eggs is negligible. An egg selected at random from a carton is found to be broken. What is the probability that this egg is the only broken one in the carton?

4.74 If you toss a balanced coin, in the long run, you get a head half the time. If you toss a certain unbalanced coin, in the long run, you get a head two-thirds of the time. First you choose one of these two coins at random, then you toss the chosen coin twice. Find the conditional probability that the balanced coin is chosen, given that

a) the first toss is a head and the second toss is a tail.
b) the second toss is a head and the first toss is a tail.
c) exactly one of the two tosses is a head.

4.75 Refer to Exercise 4.74. Find the conditional probability that the second toss is a head, given that the first toss is a head.

Advanced Exercises

4.76 Medical tests are frequently used to decide whether a person has a particular disease. The *sensitivity* of a test is the probability that a person having the disease will test positive; the *specificity* of a test is the probability that a person not having the disease will test negative. A test for a certain disease has been used for many years. Experience with the test indicates that its sensitivity is 0.96 and that its specificity is 0.98. Furthermore, it is known that roughly 1 in 1000 people has the disease.

a) Interpret the sensitivity and specificity of this test in terms of percentages.
b) Determine the probability that a person testing positive actually has the disease.
c) Interpret your answer from part (b) in terms of percentages.
d) Your naive colleague objects to the result of part (b), saying, "A positive test should be evidence of illness, not evidence of wellness. But, if there is only a 4.6% chance of illness—and therefore a 95.4% chance of wellness—after the patient tests positive, then a positive test would be evidence against illness." Set your colleague straight by comparing and contrasting the prior and posterior probabilities of illness, given a positive test.
e) The application of Bayes's rule in this exercise can be abbreviated as

$$(0.001,\ 0.999) \cdot (0.96,\ 0.02) \mapsto (1,\ 999) \cdot (96,\ 2) \mapsto (96,\ 1998)$$

$$\mapsto \left(\frac{96}{96+1998},\ \frac{1998}{96+1998}\right) = (0.046,\ 0.954).$$

Explain why this formulation is really the same thing as Bayes's rule, as stated in Proposition 4.8 on page 162.

4.77 Stratification: For purposes of statistically analyzing a specified attribute (e.g., female), statisticians often divide the population under consideration into subpopulations called *strata*. A finite population is divided into m strata such that, for $1 \le k \le m$, stratum k has N_k members of which $100p_k\%$ have the specified attribute. What percentage of members having the specified attribute belong to stratum j $(1 \le j \le m)$?

Identify in words and as a subset of the sample space each of the following events.
b) $\{X \geq 4\}$ **c)** $\{X = 0\}$ **d)** $\{X = 5\}$ **e)** $\{X \geq 1\}$
Use random variable terminology to represent each of the following events. The number of working components is
f) two. **g)** at least two. **h)** at most two. **i)** between two and four, inclusive.

5.4 Suppose that a family is selected at random from among those in a population that have three children.
a) Obtain a sample space for this random experiment that takes into account gender and order of birth.
b) Let Y denote the number of female children in the family obtained. Construct a table showing the value of Y for each outcome in the sample space.
c) Is Y a discrete random variable? Explain your answer.

5.5 An archer shoots an arrow into a square target 6 feet on a side whose center we call the origin. The outcome of this random experiment is the point in the target hit by the arrow. The archer scores 10 points if she hits the bull's eye—a disk of radius 1 foot centered at the origin; she scores 5 points if she hits the ring with inner radius 1 foot and outer radius 2 feet centered at the origin; and she scores 0 points otherwise. For one arrow shot, let S be the score.
a) Determine the value of S for each possible outcome of the random experiment.
b) Is S a discrete random variable? Explain your answer.
Identify in words and as a subset of the sample space each of the following events.
c) $\{S = 5\}$ **d)** $\{S > 0\}$ **e)** $\{S \leq 7\}$
f) $\{5 < S \leq 15\}$ **g)** $\{S < 15\}$ **h)** $\{S < 0\}$

5.6 A factory received a shipment of 100 parts that are vital to the construction of its product. The shipment contains an unknown number (possibly 0) of defective parts. The quality-control inspector decides to take a random sample of 5 parts without replacement and to accept the shipment if and only if at most 1 of the parts in the sample is defective. Let X denote the number of defective parts in the sample and set $Y = 0$ if the shipment is rejected and $Y = 1$ if the shipment is accepted.
a) Describe the sample space of this random experiment.
b) Express the random variable Y in terms of the random variable X.

5.7 Urn I contains four red balls and one white ball; Urn II contains one red ball and four white balls. An urn is chosen at random and then two balls are drawn without replacement from the chosen urn. Let X be the number of red balls drawn.
a) Construct a sample space for this random experiment whose outcomes specify which urn is chosen, the color of the first ball drawn, and the color of the second ball drawn.
b) Construct a table showing the value of X for each possible outcome.

5.8 Six men and five women apply for a job at Alpha, Inc. Three of the applicants are selected for interviews. Let X denote the number of women in the interview pool.
a) What are the possible values of the random variable X?
b) Obtain the number of interview pools corresponding to each possible value of X; that is, for each possible value x of X, find the number of ways that event $\{X = x\}$ can occur.
c) Describe the event $\{X \leq 1\}$ in words and find the number of ways in which it can occur.

5.9 Refer to the coin tossing illustration in Example 5.2 on page 179. Let Y denote the difference between the number of heads obtained and the number of tails obtained.
a) Construct a table showing the value of Y for each possible outcome.
b) Identify in words and as a subset of the sample space the event $\{Y = 0\}$.

5.10 The cafeteria at a school has two salads that cost \$1.00 and \$1.50; three entrees that cost \$2.50, \$3.00, and \$3.50; and two desserts that cost \$1.50 each. A person chooses a meal consisting of at most one selection from each of the three categories. Let X be the cost of the meal, in dollars.
a) Determine the value of X for each possible meal.
b) Identify in words and as a subset of the sample space the event $\{X \leq 3\}$.

5.11 Suppose that you play a certain lottery by buying one ticket per week. Let W be the number of weeks until you win a prize.
a) Is W a random variable? Explain.
b) Is W a discrete random variable? Explain.
Identify in words each of the following events.
c) $\{W > 1\}$ **d)** $\{W \leq 10\}$ **e)** $\{15 \leq W < 20\}$

5.12 Suppose that a number is selected at random from the interval $(0, 1)$, a task that a basic random number generator aims to accomplish. Let X denote the number obtained. For integers m and n with $m < n$, set $Y = \lfloor m + (n - m + 1)X \rfloor$, where $\lfloor x \rfloor$ denotes the *floor function*—the greatest integer smaller than or equal to x. (The floor function is identical to the greatest-integer function.)
a) What are the possible values of Y?
b) Is Y a discrete random variable? Explain.
c) Let y be a possible value of Y. Express the event $\{Y = y\}$ in terms of X.

Theory Exercises

5.13 Prove that a random variable with countable range is a discrete random variable in the sense of Definition 5.2 on page 178. That is, if X is a random variable whose range is countable, there is a countable set K of real numbers such that $P(X \in K) = 1$.

5.14 Prove that, for any random variable X, the set $\{\, x \in \mathcal{R} : P(X = x) \neq 0 \,\}$ is countable. Use the following steps.
a) Show that, for each $n \in \mathcal{N}$, no more than $n - 1$ values of $x \in \mathcal{R}$ can satisfy the inequality $P(X = x) > 1/n$.
b) Use the result of part (a) to deduce that $\{\, x \in \mathcal{R} : P(X = x) \neq 0 \,\}$ is countable. *Hint:* A countable union of countable sets is countable.
c) Interpret the result in part (b).

Advanced Exercises

Note: In the remaining exercises for this section, adhere strictly to Definition 5.2 on page 178 of a discrete random variable.

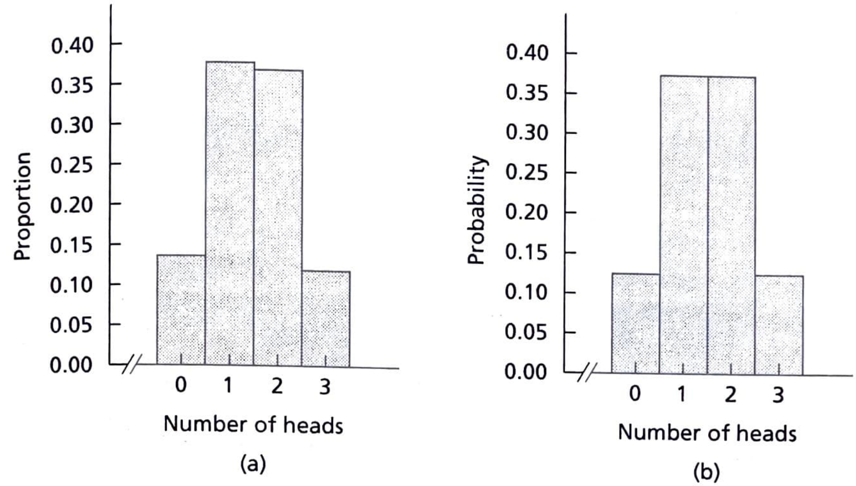
5.15 Construct a discrete random variable with an uncountable range. *Hint:* Use the fact that there is an uncountable subset C of $[0, 1]$ that has length 0 (in the extended sense).

5.16 Prove, for any random variable X, that $0 \leq \sum_x P(X = x) \leq 1$, where the notation $\sum_x$ means the sum over all $x \in \mathcal{R}$. *Hint:* Refer to Exercise 5.14.

5.17 Prove that a random variable X is discrete if and only if $\sum_x P(X = x) = 1$.

5.18 Prove that X is a discrete random variable if and only if there is a random variable X_0 with a countable range such that $P(X_0 = x) = P(X = x)$ for all $x \in \mathcal{R}$.

Figure 5.2 (a) Histogram of proportions for the number of heads obtained in three tosses of a balanced coin for 1000 observations; (b) probability histogram for the number of heads obtained in three tosses of a balanced coin



EXERCISES 5.2 Basic Exercises

Note: Many of the exercises in this section continue those presented in Section 5.1.

5.24 Suppose that two balanced dice are rolled. Let X be the sum of the two faces showing.
a) Obtain the PMF of the random variable X.
b) Construct a probability histogram for X.
In the game of craps, a first roll of a sum of 7 or 11 wins, whereas a first roll of a sum of 2, 3, or 12 loses. To win with any other first sum, that sum must be repeated before a sum of 7 is rolled. Use part (b) and the FPF to determine the probability of
c) a win on the first roll. **d)** a loss on the first roll.
e) For each of parts (c) and (d), identify the set A appearing in Equation (5.4) on page 190.

5.25 Refer to the component-analysis illustration in Example 2.4 on page 27. Assume that the components act independently and that, for $1 \leq j \leq 5$, the probability is 0.8 that, at the specified time, component j is working. Let X denote the number of components that are working at the specified time.
a) Obtain the PMF of the random variable X.
b) Use part (a) and the FPF to obtain the probability that between 1 and 3 inclusive of the components are working at the specified time. In this case, identify the set A appearing in Equation (5.4) on page 190.
c) Suppose that the unit is a series system—that is, it functions only when all five components are working. Express the event that the unit is functioning at the specified time in terms of the random variable X. Then use part (a) to find the probability of that event.
d) Suppose that the unit is a parallel system—that is, it functions when at least one of the five components is working. Express the event that the unit is functioning at the specified time in terms of the random variable X. Then use part (a) to find the probability of that event.
e) Repeat parts (a)–(d) if the probability is p that, at the specified time, any particular component is working.