<u>Calculus 2 Student Project</u>

Zachary Handel

The spread of a rumor through a population can be modeled using logistic equations. Let y be the fraction of the population that has heard the rumor (so 0 ≤y ≤1), and thus 1 −y is the fraction of the population that has not. The rumor spreads by interactions between people who know the rumor and those who do not. The equation that models the spread of the rumor is:

$$\frac{dy}{dt} = ky(1-y)$$

where k is a positive real number.

1. <u>Solve the differential equation for y as a function of t. You must show all work to receive credit.</u>

- First, in order to solve for the differential equation for y as a function of t, we must get all elements containing *y* on one side
    - To do this, we will start by dividing both sides of the equation by (1-y):

$$\frac{\frac{dy}{dt}}{(1-y)} = \frac{ky(1-y)}{(1-y)}$$

    - This will simplify to the following equation:

$$\frac{dy}{dt(1-y)} = ky$$

    - Now, we will multiply each side by *dt* and divide each side by *y*; we will them simplify it:

$$\frac{\frac{du}{dt(1-y)}}{y} * dt = \frac{ky}{y} * dt$$

$$\to \frac{dy}{y(1-y)} = kdt$$

- Now, we can apply the Partial Fraction Rule to the left side of the equation:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

- We will them set the numerators equal to each other. To do this, we must multiply every term of the equation by the denominator on the left side of the equation:

$$y(1-y)\left[\frac{1}{y(1-y)}=\frac{A}{y}+\frac{B}{1-y}\right]$$

- This will simplify to the following equation:

$$1=A(1-y)+By$$

- We will then multiply out the A, combine like terms, and simplify:

$$1=A-Ay+By$$

$$\rightarrow 1=(B-A)y+A$$

- We will then set the terms on the right side equal to the corresponding terms on the left side with regards to the exponential value of each term:

$$B-A=0, \qquad A=1$$

- We now have our value for *A,* and in order to find B, we will plug the value of A (1) into the equation *B-A = 0* and then solve for B:

$$B-1=0\rightarrow\ B=1$$

- Now that we have found our numerator values, we can set up our integration process:

$$\int\left(\frac{1}{y}+\frac{1}{1-y}\right)dy=\int(k)dt$$

  - Since k is considered a positive real number (a constant), we can pull it to the front of the integral

$$\int\left(\frac{1}{y}+\frac{1}{1-y}\right)dy=k\int\ dt$$

  - *We must keep in mind that since we are working with logistic change, we are ultimately going to take the form of the following equation:*

$$y=Ce^{kt}$$

  - We will now integrate both sides:

$$\ln(y)-\ln|1-y|=kt+\ln\,(C)$$

  - By using *ln(C),* It will make the following process easier
  - Now, by using log rules we can simplify the equation:

$$ln\left|\frac{1-y}{y}\right| = -kt + \ln(C)$$

- Now, we will put each side of the equation as the exponent of *e* in order to remove any natural logs:

$$e^{ln\left|\frac{1-y}{y}\right|} = e^{-kt+lnC}$$

- By log rules, this simplifies to the following equation:

$$\frac{1-y}{y} = Ce^{-kt}$$

- Now, in order to solve the equation for *y*, we will do the following steps:
  - Simplify the left side to look like this:

$$\frac{1}{y} - 1 = Ce^{-kt}$$

  - Add 1 to both sides:

$$\frac{1}{y} = 1 + Ce^{-kt}$$

  - Invert both sides of the equation:

$$y = \frac{1}{1 + Ce^{-kt}}$$

- Now, the differential equation is solved for y:

$$\boldsymbol{y = \frac{1}{1 + Ce^{-kt}}}$$

2. <u>Assume UNA has 6000 students, faculty, and staff on campus. At an 8:00 am Monday faculty meeting, 15 math professors heard a rumor that Chris Hemsworth is going to be in town next month filming a movie. By 1:30 pm on Monday, 10% of the campus had heard the rumor. When had the rumor spread to 87% of the campus?</u>

- A couple of things that must be considered to start is that since 8:00am on a Monday is the starting point, this resembles t = 0.
- So if we write it in terms of a function, y(0) = *something…*
- To find this *something,* we must consider the number of people who heard the rumor at the initial time. This is 15 out of the 6000 people on campus (the professors), This means we can write the function as follows:

$$y(0) = \frac{15}{6000} = \frac{1}{400}$$

- Now, we must consider the function from question 1. We can now use the function y(0) with the equation solved for y:

$$y(0) = \frac{1}{1 + Ce^{-k(0)}} \rightarrow y(0) = \frac{1}{1 + C}$$

- Since y(0) = 1/400 and y(0) = 1/1+C, we can set them equal to each other:

$$\frac{1}{400} = \frac{1}{1 + C}$$

- With this, we can solve for the variable C for the initial value of present people who heard the 820418 rumor:
  - First, invert both fractions:

$$400 = 1 + C$$

  - Then, subtract 1 from each side to get C:

$$400 - 1 = 1 + C - 1 \rightarrow 399 = C$$

- Now that we have solved for C, we can use this to start building our equation:

$$y(t) = \frac{1}{1 + 399e^{-5.5k}}$$

- At 1:30, we can correspond this to t=5.5 since 1:30 is five and a half hours from the initial time of 8:00am.

- The function for the rumor at 1:30pm is as follows:

$$y(5.5) = \frac{1}{1 + 39e^{-5.5k}}$$

— Since at t=5.5 is 10% of the campus that heard the rumor, we can set the above equation equal to 10/100 (or 10%):

$$\frac{10}{100} = \frac{1}{1 + 399e^{-5.5k}}$$

- Our next step is to solve for k since that is our growth rate constant:
  - First, invert both fractions:

$$\frac{100}{10} = 1 + 399e^{-5.5k}$$

  - Then, simplify 100/10 and subtract 1 from both sides:

$$9 = 399e^{-5.5k}$$

- Now, divide both sides by 399:

$$\frac{9}{399} = e^{-5.5k}$$

- To get rid of the negative exponent, invert e^-5.5k:

$$\frac{1}{e^{5.5k}} = \frac{9}{399}$$

- Next, to get rid of the fraction on the left side of the equation, invert both fractions:

$$e^{5.5k} = \frac{399}{9}$$

- To get rid of the e, put both sides inside of the natural log function:

$$\ln(e^{5.5k}) = ln\left[\frac{133}{3}\right]$$

- By natural log rules, e and ln cancel bringing 5.5k down as a coefficient:

$$5.5k = ln\left[\frac{133}{3}\right]$$

- Now, divide each side by 5.5:

$$k = \frac{ln\left[\frac{133}{3}\right]}{5.5}$$

- Rounded to the 9th decimal place, the answer for k is **0.689406698**:

$$k = 0.689406698$$

- We can now substitute this back into the original equation:

$$y(t) = \frac{1}{1 + 399e^{-(0.689406698)t}}$$

- Our next step is to find what t is when y(t) = 0.87 (or when 87% of the school has heard the rumor). Since our equation is related to the spread 'amount' of the rumor, we can set 0.87 equal to said equation:

$$0.87 = \frac{1}{1 + 399e^{-(0.689406698)t}}$$

- This sets us up to find the time it took to get to 85% (or t).
  - First, turn 0.87 to fraction form and invert both fractions to remove the fraction from the right side of the equation:

$$\frac{100}{87} = 1 + 399e^{-(0.689406698)t}$$

  - Now, subtract 1 from both sides of the equation:

$$\frac{100}{87} - 1 = 399e^{-(0.689406698)t}$$

  - Now, simplify the difference on the left side and divide both sides by 399:

$$\frac{\frac{13}{87}}{399} = e^{-(0.689406698)t}$$

- Now, simplify the left side of the equation:

$$e^{-(0.689406698)t} = \frac{13}{34713}$$

- Now, put the negative exponent on the left side of the equation in the denominator and invert both sides of the equation:

$$e^{(0.689406698)t} = \frac{34713}{13}$$

- Now, set both sides of the equation inside a natural log function and through natural log rules bring the exponent of e as a coefficient:

$$t[0.689406698] = ln\left[\frac{34713}{13}\right]$$

- Now, divide each side of the equation by 0.689406698 and get the decimal answer:

$$t = 11.44450758 \approx 11.45$$

- This translates to it took 11.45 hours for the rumor to spread to 87% of the campus. This means from 8:00am, it took 11 hours and 27 minutes to get to 87% of the campus, which is **7:27pm.**

3. <u>Use calculus to determine how many of the people on campus will eventually hear this rumor.</u>

- Since we are finding the amount of people that will 'eventually' hear the rumor, there is no limit to when we should stop finding how many people have heard it. This technically means we can take the limit as it is going to infinity:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1}{1 + 399e^{-(0.689406698)t}}$$

- If we were to look at a graph of e^-x, we can see that as it goes to infinity, e^-x approaches 0. So if we were to take this limit it would look like this:

$$\lim_{t\to\infty} \frac{1}{1 + 399e^{-(0.689406698)t}} = \frac{1}{1 + 399e^{-(0.689406698)\infty}} = \frac{1}{1+0}$$

- If we simplify the right side of the equation we get 1.
- Since we are taking the calculations in terms of percentages, 1 would come out to 100% of the population.
- This means that **in due time, 100% of the population would hear the rumor.**