



# Hayashi meets Kiyotaki and Moore: A theory of capital adjustment costs<sup>☆</sup>

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## ABSTRACT

Firm-level investment is lumpy and volatile but aggregate investment is much smoother and highly serially correlated. These different patterns of investment behavior have been viewed as indicating convex adjustment costs at the aggregate level but non-convex adjustment costs at the firm level. This paper shows that financial frictions in the form of collateralized borrowing at the firm level (Kiyotaki and Moore, 1997) can give rise to convex adjustment costs at the aggregate level yet at the same time generate lumpiness in plant-level investment. In particular, our model can (i) derive aggregate capital adjustment cost functions identical to those assumed by Hayashi (1982) and (ii) explain the weak empirical relationship between Tobin's  $Q$  and plant-level investment.

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## 1. Introduction

It is well known that firm-level investment behaves quite differently from aggregate investment. In particular, firm-level investment is lumpy, whereas aggregate investment is much smoother and highly serially correlated (see, e.g., Caballero, 1999). Such a sharp difference in investment dynamics has often motivated researchers to adopt inconsistent assumptions in explaining these dynamics: assuming convex adjustment costs in aggregate models and non-convex adjustment costs in micro models. Econometric studies typically find that convex capital adjustment costs (CAC) are consistent with aggregate investment data but not with firm-level data (e.g., Bloom, 2009).

A standard neoclassical model without CAC would imply (i) the elasticity of capital supply is the same in both the short run and the long run—that is, the equilibrium capital stock can be reached instantaneously because of the possibility of an infinite investment rate; and (ii) the relative price of the investment and consumption goods is a constant independent of the relative outputs of the two goods. Such implications not only are inconsistent with data, they also create theoretical difficulties in determining the optimal rate of investment in partial equilibrium models of the firm, which motivated the

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early investment literature to adopt CAC.<sup>1</sup> In addition, investment theory requires CAC to rationalize investment decisions as a function of firm value and replacement costs of capital (Tobin, 1969; Lucas and Prescott, 1971; Abel, 1979, 1983; and Hayashi, 1982).

CAC also play an important role in contemporary dynamic stochastic general equilibrium (DSGE) models. For example, they help open economy models to explain the saving–investment correlations and the home bias puzzle (e.g., Baxter and Crucini, 1993); they are essential to explaining the equity premium puzzle in production economies with capital (e.g., Jermann, 1998; Boldrin et al., 2001); and they rationalize large welfare costs of the business cycle (e.g., Barlevy, 2004).<sup>2</sup>

However, despite the popularity and apparent “necessity” of CAC in dynamic macro models, few microfoundations have been provided in the literature to rationalize CAC, especially the properties imposed on the functional forms of CAC (Hayashi, 1982). This lack of microfoundations unavoidably invites criticisms: (i) empirical analysis based on firm-level data does not find convex adjustment costs important in explaining firm-level investment behavior<sup>3</sup>; (ii) firm-level investment is lumpy with very little serial correlation, which is inconsistent with convex adjustment costs which smooth out investment over time<sup>4</sup>; and (iii) CAC imply that Tobin's  $Q$  should be a sufficient statistic to explain firm-level investment, but firms' investments are more sensitive to cash flows than to Tobin's  $Q$ .<sup>5</sup>

The typical CAC in macro models take the following functional form (Hayashi, 1982):

$$K_{t+1} = (1 - \delta)K_t + \psi\left(\frac{I_t}{K_t}\right)K_t, \quad (1)$$

where the CAC function  $\psi(\cdot)$  is increasing, concave, and homogeneous of degree zero;  $K_t$  denotes the existing capital stock; and  $I_t$  denotes total investment expenditure as part of a firm's cash flow (CF):  $CF = F(K, N) - WN - PI$ , where  $P$  is the relative price of investment goods. In a one-good economy,  $P = 1$ . This type of CAC function implies diminishing returns to investment in capital formation—part of the investment spending is lost and does not become productive capital. Under this type of adjustment cost function, the average Tobin's  $Q$  is the same as the marginal  $Q$  (provided that the production function is constant returns to scale), which greatly facilitates empirical studies of investment behaviors (Hayashi, 1982).<sup>6</sup>

Why does aggregate capital accumulation exhibit convex adjustment costs? At least three plausible explanations are offered in the literature: (i) Installing new capital takes time and involves sunk costs, delivery lags, and learning (e.g., Cooper and Haltiwanger, 2006). (ii) Capital is firm specific, which makes investment irreversible or partially irreversible (i.e., it comes with resale costs). Irreversibility imposes costs in adjusting the capital stock downward. (iii) Firms are borrowing constrained; hence, they are not able to increase capital at an infinite speed. Borrowing constraints impose costs in adjusting capital upward.

Several questions naturally arise. Suppose these frictions are explicitly modeled in firms' optimization decisions, would they necessarily give rise to the form of CAC in Eq. (1)? If so, do they have the same policy implications as Eq. (1)? In addition, are these frictions consistent with the lumpiness of firm-level investment?

These questions are answered in this paper. First, we show that if firms' investment projects are subject to idiosyncratic risk (which affects the project's rate of returns) and financed by external credit with borrowing limit proportional to firms' collateral (capital stock), then the aggregate economy exhibits CAC that are identical in functional form to Eq. (1). Second, we show that irreversible investment—an important assumption to make when rationalizing convex adjustment costs<sup>7</sup>—is unnecessary for deriving the aggregate CAC function but imposes more structures on the CAC function. In particular, if investment is completely irreversible and the distribution of investment-specific shocks is Pareto, then the implied aggregate CAC function takes the popular Cobb–Douglas form:

$$K_{t+1} = (1 - \delta)K_t + bI_t^\theta K_t^{1-\theta}, \quad (2)$$

<sup>1</sup> See Gould (1968), Lucas and Prescott (1971), among others. For a literature survey on investment theory, see Caballero (1999).

<sup>2</sup> It is important to distinguish CAC from investment adjustment costs (IAC). This paper concerns only CAC even though IAC is just as popular in DSGE models (see, e.g., Beaudry and Portier, 2007; Christiano et al., 2005; Jaimovich and Rebelo, 2009; and Wen, 1998).

<sup>3</sup> See, e.g., Cooper and Haltiwanger (2006) and Bloom (2009).

<sup>4</sup> See, e.g., Caballero et al. (1995), Cooper et al. (1999), Doms and Dunne (1998), and Power (1994). In the data, as one moves from the plant level to more aggregated levels, such as business establishments, firms, and industries, the lumpiness of investment gradually weakens. However, even at the firm level, investment still appears to be very lumpy, much lumpier than industry-level investment (see, e.g., Doms and Dunne, 1998, p. 422). Although most empirical literature used plant-level data to document lumpy investment, in our model we assume that firms and plants are equivalent entities and use these terms synonymously (i.e., each firm has only one plant).

<sup>5</sup> See, e.g., Hassett and Hubbard (1997) and Caballero (1999).

<sup>6</sup> This form of adjustment costs in Eq. (1) is equivalent to an alternative formulation of CAC that is also popular in the investment literature (especially in empirical works). This alternative formulation maintains the neoclassical law of motion for capital,  $K_{t+1} = (1 - \delta)K_t + \tilde{I}_t$ , but redefines a firm's cash flow as  $F(K_t, N_t) - W_t N_t - P_t C(\tilde{I}_t/K_t)K_t$ , where the function  $C(\cdot)$  denotes total real costs associated with investment expenditure  $\tilde{I}_t$  measured in capital units and satisfies the properties  $C'(\cdot) > 0$  and  $C''(\cdot) > 0$  (see, e.g., Abel, 1979, 1983).

<sup>7</sup> See, e.g., Abel and Eberly (1994, 1996), Pindyck (1991), Dixit (1992), and Dixit and Pindyck (1994).

where  $\theta \in (0, 1)$  is a parameter that depends on the borrowing constraints and distribution of firm-specific shocks. Third, we also show that a microfounded CAC model with financial constraints is consistent with the following empirical facts—firm-level investment is lumpy with little serial correlation and is insensitive to Tobin's  $Q$ .<sup>8</sup>

This paper relates to the work of Carlstrom and Fuerst (1997), who show that a particular type of borrowing constraints studied by Bernanke and Gertler (1989) can imply aggregate CAC. Their specific financial frictions involve private information about investment returns and agency costs associated with costly state verification. However, this type of borrowing constraint does not imply a CAC function identical to that in Eq. (1) because the implied CAC function under agency costs is not homogeneous of degree zero and does not have the property that the marginal  $Q$  equals the average  $Q$ . Hence, our paper differs from this literature in at least two important aspects. First, the financial friction we consider is based on limited contract enforcement and collateralized borrowing as in the works of Kiyotaki and Moore (1997) and Jermann and Quadrini (2009).<sup>9</sup> Second, in an agency-cost model, investment is not lumpy because the entrepreneurs always undertake investment in equilibrium. This feature is inconsistent with data. In contrast, our setup can quantitatively match the lumpiness of firm-level investment and the weak correlation between the investment rate and Tobin's  $Q$ .

Our work also relates to that of Lorenzoni and Walentin (2007) and the associated literature that uses simulated data from theoretical models with financial frictions to investigate the quantitative relationship between Tobin's  $Q$  and investment (e.g., Gomes, 2001; and others). Lorenzoni and Walentin (2007) show that financial constraints can substantially weaken the correlation between  $Q$  and investment, relative to a frictionless benchmark (e.g., Hayashi, 1982). While our model can also explain the weak relationship between  $Q$  and investment, our approach differs from that of Lorenzoni and Walentin (2007) in one important aspect: They assume CAC in firms' investment technologies, whereas we do not need this assumption. Consequently, their model cannot explain the lumpiness of firm-level investment. Our paper also differs from theirs in the main focus of the analysis: We try to rationalize and derive CAC from microfoundations.

Thomas (2002) uses a model with non-convex adjustment costs to generate lumpy investment at the firm level and shows that such lumpiness can be unrelated to the volatility of aggregate investment in general equilibrium. Our analysis differs from hers. We show instead that borrowing constraints can simultaneously explain the lumpiness of firm-level investment and the sluggishness of aggregate investment. Consistent with her findings, however, our results suggest that there can be no causal relations between investment volatility at the firm level and that at the aggregate level. This implication holds in our model regardless of general equilibrium.

The rest of the paper is organized as follows. Section 2 presents a benchmark model with a simple form of borrowing constraints and shows how to derive Eq. (1) from the model. Section 3 studies a model with endogenous borrowing limits and their policy implications. This section also rationalizes the special forms of borrowing constraints considered in the paper using the concept of limited contract enforceability. Section 4 conducts quantitative simulations of our microfounded model and examines the model's predictions for the lumpiness of firm-level investment and its correlation with Tobin's  $Q$ . Section 5 concludes the paper.

## 2. The benchmark model

### 2.1. Firms

There is a continuum of competitive firms indexed by  $i \in [0, 1]$ . Firm  $i$ 's objective is to maximize its discounted dividends,

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} D_t(i), \quad (3)$$

where  $D_t(i)$  represents firm  $i$ 's dividend in period  $t$  and  $\Lambda_t$  the marginal utility of a representative household. The production function has constant returns to scale and is given by

$$Y_t(i) = F(K_t(i), A_t N_t(i)), \quad (4)$$

where  $A_t$  represents aggregate labor-augmenting technology which can be either deterministic or stochastic, and  $N_t(i)$  and  $K_t(i)$  are firm-level employment and capital, respectively. Each firm accumulates capital according to the law of motion,

$$K_{t+1}(i) = (1 - \delta)K_t(i) + \varepsilon_t(i)I_t(i), \quad (5)$$

where  $I_t(i)$  denotes investment expenditure and  $\varepsilon_t(i) \in \mathbb{R}^+$  is an idiosyncratic shock to the marginal efficiency of investment, which has the probability density function  $\phi(\varepsilon)$  and cumulative density function  $\Phi(\varepsilon)$ . For simplicity, assume that this shock

<sup>8</sup> The theoretical literature on lumpy investment typically assumes fixed investment costs, which are not assumed in this paper. Important examples include Veracierto (2002), Thomas (2002), Khan and Thomas (2003, 2008), Gourio and Kashyap (2007), Bachmann et al. (2008), among others.

<sup>9</sup> The literature on financial constraints and contract enforceability is vast. A selection of works closely related to those of Kiyotaki and Moore (1997) includes those of Albuquerque and Hopenhayn (2004), Cooley et al. (2004), Jermann and Quadrini (2009), and Liu and Wang (2010), among many others.

is orthogonal to any aggregate shock. A firm's dividend in period  $t$  is hence given by  $D_t(i) = Y_t(i) - P_t I_t(i) - W_t N_t(i)$ , where  $P_t = 1$  denotes the relative price of investment goods and  $W_t$  the competitive real wage.

What is the meaning of  $\varepsilon_t(i)$ ? There are at least two interpretations. First, as it is modeled here,  $\varepsilon_t(i)$  is a shock to the rate of returns to investment. A higher realization of  $\varepsilon_t(i)$  implies that the same amount of investment expenditure leads to more finished capital goods. In a world with time-to-build, it takes time and additional efforts for invested resources to become productive capital. So the efficiency shock  $\varepsilon_t(i)$  captures any idiosyncratic factor involved in the process between the time of investment spending and the time of project completion.

Second, the results are identical if we assume that the dividend is given by  $D_t(i) = Y_t(i) - \varepsilon_t(i)I_t(i) - W_t N_t(i)$  and the law of motion of capital is given by  $K_{t+1}(i) = (1 - \delta)K_t(i) + I_t(i)$ . In this alternative setting,  $\varepsilon_t(i)$  measures the cost (or its inverse) of investment. So  $\varepsilon_t(i)$  captures idiosyncratic costs associated with the ordering and installation of capital for a firm. This interpretation of  $\varepsilon_t(i)$  directly relates to the micro adjustment cost literature discussed in footnote 6.

In addition, investment-specific cost shock  $\varepsilon_t(i)$  allows us to handle the case with occasionally binding financial constraints. In the original Kiyotaki and Moore (1997) model, the constraint is assumed to be always binding even when desired investment is low. In reality, however, this may not be the case.

Denote  $n_t(i) \equiv N_t(i)/K_t(i)$  as the labor-to-capital ratio and  $f(\cdot) \equiv F(1, \cdot)$  as the output-to-capital ratio. Given the real wage, the firm's optimal labor demand is determined by the equation  $f_n(A_t n_t(i))A_t = W_t$ . Note that the labor demand function implies that all firms choose the same labor-to-capital ratio, namely,  $n_t(i) = n(w_t, A_t)$ . Firm  $i$ 's operating profits can then be expressed as  $R_t K_t(i) = \max_{N_t(i)} \{Y_t(i) - W_t N_t(i)\}$ , where

$$R_t \equiv f(\cdot) - w_t n_t \quad (6)$$

is independent of  $i$  and the capital stock. Hence, a firm's operating profit is proportional to its capital stock. The dividend is then given by  $D_t(i) = R_t K_t(i) - I_t(i)$ .

We make the following additional assumptions:

(i) A firm's investment is financed by credit and is subject to the borrowing constraint:

$$I_t(i) \leq \theta K_t(i), \quad (7)$$

where  $\theta > 0$  is a constant. This borrowing constraint specifies that total investment cannot exceed an amount proportional to the existing capital stock. We defer the justifications of such a form of borrowing constraint to a later section.

(ii) Firm-level investment may be partially irreversible:

$$K_{t+1}(i) \geq \rho(1 - \delta)K_t(i), \quad (8)$$

where the parameter  $\rho \in [0, 1]$  indicates the degree of irreversibility. For example, if  $\rho = 1$ , then investment is completely irreversible and Eq. (8) becomes  $I_t(i) \geq 0$ . At the other extreme, if  $\rho = 0$ , then investment is completely reversible and Eq. (8) becomes  $K_{t+1}(i) \geq 0$ . Hence, Eq. (8) encompasses both reversible and irreversible investments as special cases. Eq. (8) can also be rewritten as

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i), \quad (9)$$

where  $\tilde{\rho} \equiv (1 - \rho)(1 - \delta)$ . Since our general results hold for  $\rho = 0$ , irreversible investment is not essential for our analysis.

With the definition in Eq. (6), a firm's maximization problem can be rewritten as

$$\max_{\{I_t(i), K_{t+1}(i)\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} (R_t K_t(i) - I_t(i)) \quad (10)$$

subject to Eqs. (5), (7), and (9).

Denote  $\{\lambda_t(i), \mu_t(i), \pi_t(i)\}$  as the Lagrangian multipliers of constraints (5), (7), and (9), respectively. The firm's first-order conditions for  $\{I_t(i), K_{t+1}(i)\}$  are given, respectively, by

$$1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i), \quad (11)$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1 - \delta)\lambda_{t+1}(i) + \theta \mu_{t+1}(i) + \tilde{\rho} \frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} \right\}, \quad (12)$$

plus the complementarity slackness conditions,  $\pi_t(i)[I_t(i) + \rho \frac{K_t(i)}{\varepsilon_t(i)}] = 0$  and  $\mu_t(i)[\theta K_t(i) - I_t(i)] = 0$ . It is obvious that when  $\varepsilon_t(i)$  is i.i.d., the Lagrangian multipliers  $\{\lambda_t(i), \pi_t(i), \mu_t(i)\}$  depend only on aggregate states  $\mathbf{S}_t$  and  $\varepsilon_t(i)$ , which implies that the expected values of the Lagrangian multipliers are independent of  $i$ . Namely, by the law of iterated expectations, we have  $E_t \frac{\Lambda_{t+1}}{\Lambda_t} z_{t+1}(i) = E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{z}_{t+1}$ , where  $\bar{z}_{t+1} \equiv \int z_{t+1}(i) d\Phi$  denotes expected value of  $z(i)$ . Thus, Eq. (12) can be rewritten as

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1 - \delta)\bar{\lambda}_{t+1} + \theta \bar{\mu}_{t+1} + \tilde{\rho} \int \frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} d\Phi(\varepsilon) \right\}, \quad (13)$$

which shows that  $\lambda_t(i) = \lambda(\mathbf{S}_t) \equiv \lambda_t$  is also independent of  $i$ . Since the marginal cost of investment is 1 and the marginal value of newly installed capital stock is  $\lambda_t$ , the market-based measure of Tobin's  $Q$  is given by  $Q_t = \lambda_t$ , which is independent of  $i$ .

## 2.2. Investment decision rules

We use a guess-and-verify strategy to derive closed-form decision rules at the firm level. The decision rules are characterized by a cutoff strategy where the cutoff ( $\varepsilon_t^*$ ) pertains to the realization of investment-specific shocks and is defined by the opportunity cost of installing one unit of capital:

$$\lambda_t = \frac{1}{\varepsilon_t^*}. \quad (14)$$

Consider the following possible cases:

**Case A.**  $\varepsilon_t(i) > \varepsilon_t^*$ . In this case, the marginal efficiency of investment is high. Since the return to investment is high, firms opt to undertake investment up to the borrowing limit,  $I_t(i) = \theta_t K_t(i)$ , so the constraint (9) does not bind. Hence, we have  $\pi_t(i) = 0$ . Eq. (11) implies  $\mu_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*} + \pi_t(i) - 1 = \frac{\varepsilon_t(i)}{\varepsilon_t^*} - 1 > 0$ .

**Case B.**  $\varepsilon_t(i) < \varepsilon_t^*$ . In this case, the marginal efficiency of investment is low. Given this, firms opt to make minimum investment, which means  $I_t(i) = -\tilde{\rho} \frac{K_t(i)}{\varepsilon_t(i)}$ . So the constraint (7) does not bind and we have  $\mu_t(i) = 0$ . Eq. (11) implies  $\pi_t(i) = 1 + \mu_t(i) - \frac{\varepsilon_t(i)}{\varepsilon_t^*} = 1 - \frac{\varepsilon_t(i)}{\varepsilon_t^*} > 0$ .

**Case C.**  $\varepsilon_t(i) = \varepsilon_t^*$ . By Eq. (11),  $\mu_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*} + \pi_t(i) - 1 = \pi_t(i)$ . Suppose  $\{\mu_t(i), \pi_t(i)\} > 0$ ; by the slackness conditions we have  $I_t(i) = -\tilde{\rho} \frac{K_t(i)}{\varepsilon_t(i)}$  and  $I_t(i) = \theta K_t(i)$ , which is a contradiction. Hence, it must be true that  $\mu_t(i) = \pi_t(i) = 0$ . In this marginal case, Eq. (11) implies  $\lambda_t = \frac{1}{\varepsilon_t^*}$ , which confirms that the cutoff is indeed given by Eq. (14). Without loss of generality, we assume that in this marginal case a firm undertakes maximum investment.

Notice that from an individual firm's own perspective, Tobin's  $Q$  is measured by  $q_t(i) \equiv \frac{\varepsilon_t(i)}{\varepsilon_t^*}$ . A firm will undertake positive investment if  $q_t(i) \geq 1$ , otherwise the firm disinvest or remains inactive. However, because markets are incomplete and the idiosyncratic shocks are not observable (or insured) through markets, the market-based measure of Tobin's  $Q$  is  $\frac{1}{\varepsilon_t^*}$ , which is independent of  $\varepsilon_t(i)$ .

Based on the above analysis, the Lagrangian multipliers satisfy  $\mu_t(i) = \max\{q_t(i) - 1, 0\}$  and  $\pi_t(i) = \max\{1 - q_t(i), 0\}$ . The firm's decision rules for investment and capital accumulation are thus given by

$$I_t(i) = \begin{cases} \theta K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^*, \\ -\tilde{\rho} \frac{K_t(i)}{\varepsilon_t(i)} & \text{if } \varepsilon_t(i) < \varepsilon_t^*, \end{cases} \quad (15)$$

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + \frac{(1-\delta)}{\varepsilon_{t+1}^*} + O(\varepsilon_{t+1}^*) \right\}, \quad (16)$$

where the implicit function  $O(\cdot)$  in Eq. (16) is defined as

$$\begin{aligned} O(\varepsilon_{t+1}^*) &\equiv E_t \left[ \theta \mu_{t+1}(i) + \tilde{\rho} \frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} \right] \\ &= \theta \int_{\varepsilon_{t+1}(i) \geq \varepsilon_{t+1}^*} \frac{\varepsilon_{t+1}(i) - \varepsilon_{t+1}^*}{\varepsilon_{t+1}^*} d\Phi(\varepsilon) + \tilde{\rho} \int_{\varepsilon_{t+1}(i) < \varepsilon_{t+1}^*} \left( \frac{1}{\varepsilon_{t+1}(i)} - \frac{1}{\varepsilon_{t+1}^*} \right) d\Phi(\varepsilon). \end{aligned} \quad (17)$$

The investment function (15) indicates that firm-level investment is lumpy with little serial correlation. Each firm in any period has only probability  $1 - \Phi(\varepsilon_t^*)$  of undertaking positive investment and probability  $\Phi(\varepsilon_t^*)$  of remaining inactive (or disinvesting). These probabilities are determined by aggregate economic conditions that influence the cutoff ( $\varepsilon_t^*$ ) and are independent of each firm's investment history (which is highly idiosyncratic). Also, such lumpiness is independent of the value of  $\tilde{\rho}$ —i.e., the lumpiness does not hinge on irreversibility.

Notice that  $O(\cdot)$  is the option value of one unit of installed capital<sup>10</sup>. If the firm receives a favorable shock in the next period, one unit of installed capital can expand the firm's borrowing capacity by  $\theta$  units and each additional unit of installed

<sup>10</sup> Note that we used the orthogonality condition between idiosyncratic shocks and aggregate uncertainty to derive Eq. (16).

capital can bring a net profit of  $q(i) - 1$  ( $= \frac{\varepsilon(i) - \varepsilon^*}{\varepsilon^*}$ ) units. This case occurs with probability  $\int_{\varepsilon \geq \varepsilon_{t+1}^*} d\Phi(\varepsilon)$ . In the case of an unfavorable shock, the firm can withdraw  $\bar{\rho} \geq 0$  of investment units and each unit of saving can be transformed into  $\frac{1}{\varepsilon}$  units of consumption goods. By doing so, the firm can increase net profit by  $\frac{1-q}{\varepsilon}$  ( $= \frac{1}{\varepsilon(i)} - \frac{1}{\varepsilon^*}$ ) units.

Hence, Eq. (16) implies that the optimal level of investment is determined at the point where the marginal cost ( $\frac{1}{\varepsilon^*}$ ) equals the marginal benefits (= the marginal product of capital + the value of non-depreciated capital + the option value of capital). Because the optimal level of investment depends on the expected returns, which in turn depend on the probability weights of the different cases considered above (i.e., the cutoff  $\varepsilon_t^*$ ), Eq. (16) states that a firm chooses the optimal cutoff  $\varepsilon_t^*$  (as an implicit function of aggregate economic conditions) so that the marginal cost of investment equals the expected marginal gains.

Eq. (16) also shows that the optimal cutoff  $\varepsilon_t^*$  is independent of  $i$ , so it is the same across all firms. More specifically, the optimal cutoff is independent of a firm's investment rate and existing capital stock. This property allows us to characterize aggregate investment dynamics in a tractable manner without engaging numerical methods (as in the work of Krusell and Smith, 1998).

### 2.3. Properties of the aggregate investment function

Integrating the firm-level decision rules by the law of large numbers, the aggregate investment, aggregate capital stock, and the optimal cutoff are determined jointly by Eq. (16) and the following two equations:

$$\frac{I_t}{K_t} = \theta[1 - \Phi(\varepsilon_t^*)] - \bar{\rho} \int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon); \quad (18)$$

$$K_{t+1} = (1 - \delta)K_t + \theta K_t \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) - \bar{\rho} K_t \int_{\varepsilon < \varepsilon_t^*} d\Phi(\varepsilon), \quad (19)$$

where Eq. (18) is derived from Eq. (15) and Eq. (19) from Eq. (5). It can be confirmed by the eigenvalue method that this three-equation dynamic system has a unique saddle-path steady state. Hence, given the stochastic process of  $\{R_t, \Lambda_t\}$ , the equilibrium path of  $\{I_t, K_{t+1}, \varepsilon_t^*\}$  is uniquely determined.

Eq. (18) suggests that the aggregate investment rate is fully determined by  $\varepsilon_t^*$ . Given the parameters  $\theta$  and  $\bar{\rho}$ , this equation also defines the cutoff as an implicit function of the investment rate:  $\varepsilon_t^* = \varepsilon^*(\frac{I_t}{K_t})$ . Therefore, Eq. (19) can be rewritten as

$$K_{t+1} = (1 - \delta)K_t + \varphi\left(\frac{I_t}{K_t}\right)K_t, \quad (20)$$

where

$$\varphi\left(\frac{I_t}{K_t}\right) \equiv \theta \int_{\varepsilon \geq \varepsilon^*(\frac{I_t}{K_t})} \varepsilon d\Phi(\varepsilon) - \bar{\rho} \int_{\varepsilon < \varepsilon^*(\frac{I_t}{K_t})} d\Phi(\varepsilon) \quad (21)$$

is an implicit function of the aggregate investment rate.

**Proposition 1.** *The implicit function  $\varphi(\cdot)$  is increasing, strictly concave, and homogeneous of degree zero in  $\{I_t, K_t\}$ .*

**Proof.** See Appendix A.  $\square$

### 2.4. Equivalence between CAC and borrowing constraints

If we define the market value of one unit of newly installed capital (or Tobin's  $Q$ ) of a firm as

$$Q_t \equiv \lambda_t = \frac{1}{\varepsilon_t^*} \quad (22)$$

and the aggregate investment rate as  $i_t \equiv \frac{I_t}{K_t}$ , using Eq. (18), we can simplify the implicit function  $O(\varepsilon_t^*)$  in Eq. (17) to

$$\begin{aligned} O(\varepsilon_t^*) &= \frac{1}{\varepsilon_t^*} \left[ \theta \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) - \bar{\rho} \int_{\varepsilon < \varepsilon_t^*} d\Phi(\varepsilon) \right] - \left\{ \theta[1 - \Phi(\varepsilon_t^*)] - \bar{\rho} \int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon) \right\} \\ &= Q_t \varphi(i_t) - i_t. \end{aligned} \quad (23)$$

Therefore, using the defined functions for  $\{\varphi(\cdot), \varphi'(\cdot), O(\cdot), Q\}$ , the system of equations that solves for the aggregate investment rate ( $i_t$ ), the capital stock ( $K_{t+1}$ ), and Tobin's  $Q_t$  is given by

$$Q_t \varphi'(i_t) = 1, \quad (24)$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + Q_{t+1}(1 - \delta) + Q_{t+1} \varphi(i_{t+1}) - i_{t+1}\}, \quad (25)$$

$$K_{t+1} = (1 - \delta)K_t + \varphi(i_t)K_t. \quad (26)$$

Now consider a standard representative-agent macro model with CAC (e.g., Hayashi, 1982), where a representative firm solves (taking as given the marginal product of capital  $R_t$ )

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} (R_t K_t - I_t) \quad (27)$$

subject to

$$K_{t+1} = (1 - \delta)K_t + \psi\left(\frac{I_t}{K_t}\right)K_t. \quad (28)$$

Defining  $Q_t$  (Tobin's  $Q$ ) as the Lagrangian multiplier for the constraint (28) and  $i_t \equiv \frac{I_t}{K_t}$  as the investment rate, the first-order conditions for  $\{I_t, K_{t+1}\}$  are given, respectively, by

$$Q_t \psi'(i_t) = 1, \quad (29)$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + (1 - \delta)Q_{t+1} + Q_{t+1}[\psi(i_{t+1}) - \psi'(i_{t+1})i_{t+1}]\}. \quad (30)$$

Using Eq. (29), we can rewrite Eq. (30) as

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + (1 - \delta)Q_{t+1} + Q_{t+1} \psi(i_{t+1}) - i_{t+1}\}. \quad (31)$$

Notice that the system of equations (28), (29), and (31) is identical to the system of equations (24)–(26) because the two CAC functions,  $\varphi(\cdot)$  and  $\psi(\cdot)$ , have the same properties. Since our microfounded model is equivalent to the aggregate CAC model regardless of the value of  $\tilde{\rho}$ , the equivalence result is established without relying on the assumption of irreversible investment. The key assumption instead is the collateralized borrowing constraint (7).

The equivalence result holds regardless of the exogenous driving processes of  $\{R_t, W_t, \Lambda_t\}$ . That is, the two models are identical not only in the steady state but also along any transitional dynamic path. For example, the impulse responses of the two models are completely identical under either aggregate technology shocks which affect  $\{R_t, W_t\}$  or aggregate demand shocks which affect  $\Lambda_t$ .

**Example.** To further illustrate the equivalence result, suppose the distribution of  $\varepsilon$  is Pareto with support  $[1, \infty)$  and shape parameter  $\eta > 1$ —i.e.,  $\Phi(\varepsilon) = 1 - \varepsilon^{-\eta}$ ; and assume that investment is completely irreversible, i.e.,  $\tilde{\rho} = 0$ . With these assumptions, Eqs. (18) and (19) become, respectively,  $\frac{I_t}{K_t} = \theta \varepsilon_t^{*1-\eta} = \theta Q_t^\eta$  and  $K_{t+1} = (1 - \delta)K_t + \frac{\eta\theta}{\eta-1} \varepsilon_t^{*1-\eta} K_t$ , where the adjustment cost function  $\frac{\eta\theta}{\eta-1} (\frac{I_t}{K_t})^{\frac{\eta-1}{\eta}} = \varphi(\frac{I_t}{K_t})$  is homogeneous of degree zero and satisfies  $\varphi'(\frac{I_t}{K_t})Q_t = 1$ . Substituting for  $\varepsilon_t^*$ , Eq. (31) then becomes  $Q_t = E_t \beta_{t+1} \{R_{t+1} + Q_{t+1}(1 - \delta) + \frac{1}{\eta-1} \frac{I_{t+1}}{K_{t+1}}\}$ , and the law of capital accumulation becomes

$$K_{t+1} = (1 - \delta)K_t + \frac{\eta\theta^{\frac{1}{\eta}}}{\eta-1} I_t^{\frac{\eta-1}{\eta}} K_t^{\frac{1}{\eta}}, \quad (32)$$

which has the familiar Cobb–Douglas form (Eq. (2)) commonly assumed in the macro literature.

## 2.5. Intuition of the equivalence

In representative-agent models, CAC imply that the aggregate investment rate is sluggish in responding to macroeconomic environmental changes because of diminishing returns to investment in capital formation. In other words, because  $\psi(\cdot)$  is concave, aggregate investment responds to a higher future capital productivity ( $R_{t+1}$ ) less elastically than it would otherwise. As a result, the optimal capital stock can be reached only through multiple periods of investment at a finite speed instead of through a single-period investment at an infinite speed.

In our heterogeneous-agent model, firm-level investment is lumpy because a firm undertakes either a large amount of positive investment (“active”) or a large amount of negative investment (“inactive”), depending on the idiosyncratic shock to the rate of return to investment in a particular period. However, despite the lumpiness of firm-level investment, aggregate

investment is sluggish. Aggregate investment in our model has two margins: an intensive margin that depends on each firm's maximum investment level ( $\theta$ ) and an extensive margin that depends on the number of active firms ( $\varepsilon_t^*$ ) in each period. Eq. (18) shows that the aggregate investment rate depends on  $\theta$  (the intensive margin) and the proportion of active firms,  $1 - \Phi(\varepsilon^*) = \Pr[\varepsilon \geq \varepsilon^*]$  (the extensive margin, assuming  $\tilde{\rho} = 0$  for a moment). However, the extensive margin is determined by the optimal cutoff  $\varepsilon_t^*$ , which behaves sluggishly because by Eq. (16) the inverse of the cutoff  $\varepsilon_t^*$ —Tobin's  $Q$ —is a slow-moving (weighted) average of expected future marginal products ( $R_{t+j}$ ,  $j = 1, 2, \dots$ ) as well as the option values of capital ( $O_{t+j}$ ,  $j = 1, 2, \dots$ ). Hence, when  $R_{t+1}$  changes, the optimal level of aggregate capital stock cannot be reached through a single-period aggregate investment because the extensive margin (Tobin's  $Q$ ) adjusts slowly over time (since an increase in  $R_{t+1}$  has only a small impact on the cutoff).

Aggregate investment (both in our microfounded model and in the representative CAC model) depends fully and positively on Tobin's  $Q$  because Tobin's  $Q$  contains all of the information about the marginal costs and benefits of investment—a higher value of capital is required for a higher investment rate when the marginal cost of investment is increasing. However, this  $Q$ -theory of investment has not fared well empirically. Variables such as firms' cash flows are always found to be significant in explaining firm-level investment other than the average  $Q$  (see, e.g., Hassett and Hubbard, 1997).

Our approach provides an explanation for this apparent failure of the  $Q$ -theory. In our model, firm-level investment is driven by idiosyncratic cost (or efficiency) shocks  $\varepsilon(i)$  while the market-based measure of  $Q$  does not capture such information. Hence, firms' cash positions and net worth will appear to be more important than the market value of Tobin's  $Q$  in determining the rate of investment. On the other hand, without borrowing constraints, only the most productive firm—the most efficient firm with the highest draw of  $\varepsilon(i)$ —will undertake investment in each period; the model then degenerates to a one-firm (or representative-firm) model in which  $Q$  is a sufficient statistic for determining a firm's investment. Hence, both idiosyncratic shocks and borrowing constraints are important in rendering firm-level investment insensitive to  $Q$ .

At the aggregate level, however, total investment depends positively and fully on  $Q$  for the following reasons: Since a firm's investment is constrained by the firm's capital stock, only the most efficient firms will undertake positive investment and the rest will remain inactive in each period. Thus, an increase in the aggregate stock of capital requires a greater proportion of active firms. This is possible in equilibrium only if the market value of capital ( $Q$ ) increases (or the cutoff  $\varepsilon^*$  decreases) so that more firms (including the less efficient ones) will find investment profitable. In other words, the less efficient firms raise the aggregate marginal cost of investment, hence calling for a higher  $Q$  to balance it in equilibrium. Therefore, aggregate investment is closely related to  $Q$ . This explains why empirical work based on micro data tends to find firms' cash flows more important than  $Q$  in determining the rate of firm investment in the short run, but aggregate data and long-run analysis tend to find  $Q$  important and significant in determining aggregate investment (see, e.g., Caballero, 1999; Cooper and Haltiwanger, 2006).

So if we were to use both our model and the standard CAC model to generate artificial data for firms' investment rate and run regressions between this variable and Tobin's  $Q$  for the two models, the results would indicate that the regression's  $R^2 = 1$  in the aggregate CAC model and is  $< 1$  in our model.<sup>11</sup>

### 3. Endogenous borrowing constraints

In the previous benchmark model, the borrowing limit is assumed to be a fixed proportion of the existing capital stock. An implicit assumption made there is that a fair market for used capital assets does not exist, so the price of used capital is proportional (or equal) to the price of consumption goods ( $P = 1$ ). In general, with a fair market for used capital, firms' borrowing limits may depend on the asset value of the collateral (Kiyotaki and Moore, 1997). To model this, consider the following borrowing constraint with endogenous credit limits:

$$I_t(i) \leq \theta Q_t(i) K_t(i), \quad (33)$$

where  $Q_t(i)$  denotes the market value of firm  $i$ 's existing capital stock and  $\theta > 0$  is a parameter.<sup>12</sup>

**Proposition 2.** Assuming  $\tilde{\rho} = 0$  for simplicity (without loss of generality), a firm's investment decision rule is given by

$$I_t(i) = \begin{cases} \theta Q_t K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^*, \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^*, \end{cases} \quad (34)$$

the market value  $Q_t$  is determined as

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1 - \delta) Q_{t+1} + \theta Q_{t+1} \int_{\varepsilon_{t+1}(i) \geq \varepsilon_{t+1}^*} \left[ \frac{\varepsilon_{t+1}(i)}{\varepsilon_{t+1}^*} - 1 \right] d\Phi(\varepsilon) \right\}, \quad (35)$$

<sup>11</sup> For quantitative results from such regression analyses, see Table 3 in our working paper (Wang and Wen, 2010).

<sup>12</sup> For example, if the non-depreciated capital stock is fully collateralized, then  $\theta = 1 - \delta$ .



or, equivalently, as  $Q_t = \frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(i)$ ; and the firm's private value  $v_t(i)$ —the value function of a firm per unit of capital—is determined as

$$v_t(i) = \begin{cases} R_t + (1 - \delta)Q_t + \theta Q_t [Q_t \varepsilon_t(i) - 1] & \text{if } \varepsilon_t(i) \geq \varepsilon_t^*, \\ R_t + (1 - \delta)Q_t & \text{if } \varepsilon_t(i) < \varepsilon_t^*. \end{cases} \quad (36)$$

**Proof.** See Appendix B.  $\square$

The investment decision rule has the same form as that in the benchmark model (except here the parameter  $\theta$  in the benchmark model is replaced by  $\theta Q_t$ ). Note that the private value of a firm is proportional to its capital stock (i.e.,  $v_t(i)$  is independent of  $K_t(i)$ ) and the market value of a firm,  $Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(i)$ , is the same across all firms, as in the benchmark model.<sup>13</sup>

By the law of large numbers, the aggregate investment is given by

$$I_t = \theta K_t Q_t [1 - \Phi(\varepsilon_t^*)]. \quad (37)$$

Since  $Q_t = \frac{1}{\varepsilon_t^*}$ , the above equation defines the implicit function  $Q_t = Q(I_t/K_t)$ . We can use this implicit equation to rewrite Eq. (35) as

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1 - \delta)Q_{t+1} + \theta Q_{t+1}^2 \int_{\varepsilon \geq \frac{1}{Q(I_{t+1}/K_{t+1})}} \varepsilon d\Phi(\varepsilon) - \frac{I_{t+1}}{K_{t+1}} \right\}. \quad (38)$$

Similarly, the aggregate law of motion for capital accumulation is given by

$$K_{t+1} = (1 - \delta)K_t + \Psi\left(\frac{I_t}{K_t}\right)K_t, \quad (39)$$

where

$$\Psi\left(\frac{I_t}{K_t}\right) \equiv \theta Q(I_t/K_t) \int_{\varepsilon \geq \frac{1}{Q(I_t/K_t)}} \varepsilon d\Phi(\varepsilon). \quad (40)$$

**Proposition 3.** For any probability density function  $\phi(\varepsilon)$  that satisfies  $\phi'(\varepsilon) \leq 0$ , the implicit function  $\Psi(\cdot)$  in Eq. (39) is increasing, concave, and homogeneous of degree zero in  $\{I_t, K_t\}$ .

**Proof.** See Appendix C.  $\square$

**Example.** Many standard distributions, such as the Pareto distribution, the exponential distribution, and the uniform distribution, satisfy the property  $\phi'(\varepsilon) \leq 0$ . As an example, consider the Pareto distribution,  $\Phi(\varepsilon) = 1 - \varepsilon^{-\eta}$  with  $\eta > 1$ . Eq. (37) becomes  $\frac{I_t}{K_t} = \theta \varepsilon_t^{*- \eta - 1}$ , and the capital accumulation equation becomes  $K_{t+1} = (1 - \delta)K_t + \theta \frac{\eta}{\eta - 1} \varepsilon_t^{*- \eta}$ . Combining these two equations implies

$$K_{t+1} = (1 - \delta)K_t + \varphi_0 K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}, \quad (41)$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[ R_{t+1} + (1 - \delta)Q_{t+1} + \frac{1}{\eta - 1} \frac{I_{t+1}}{K_{t+1}} \right], \quad (42)$$

$$\frac{I_t}{K_t} = \theta Q_t^{\eta+1}, \quad (43)$$

where  $\varphi_0 \equiv \frac{\eta}{\eta-1} \theta^{\frac{1}{\eta+1}} > 0$ . Thus, similar to the benchmark model, we obtain a reduced-form Cobb–Douglas CAC function  $\Psi(I_t/K_t) = \varphi_0 (\frac{I_t}{K_t})^{\frac{\eta}{\eta+1}}$  with irreversible investment and the Pareto distribution of  $\varepsilon$ .

<sup>13</sup> As mentioned earlier, because markets are incomplete in the model, idiosyncratic shocks to a firm's investment return are uninsured. Hence,  $q_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}$  cannot be used by the market to determine a firm's  $Q$ . This is why the market-based measure of Tobin's  $Q$  is  $\frac{1}{\varepsilon_t^*}$  instead of  $\frac{\varepsilon(i)}{\varepsilon^*}$ .

Therefore, borrowing constraints at the firm level can fully rationalize the CAC function in Eq. (1), regardless of whether the borrowing limits are endogenous or exogenous. In other words, the specific form of aggregate CAC assumed by Hayashi (1982) and others in the existing literature can be derived from microfoundations with financial frictions which hinder firms' ability to borrow. However, there are subtle but important differences between the exogenous borrowing limit model and the endogenous borrowing limit model, as we will show next.

### 3.1. Caveats on equivalence

With endogenous borrowing limits, the equivalence between the microfounded heterogeneous-firm model and the representative-agent CAC model holds only with respect to Eq. (1). Unlike the benchmark model, however, the equilibrium in the endogenous borrowing limit model and that in the aggregate CAC model are not completely equivalent because the trajectories of investment and capital stock in the former are no longer identical to those implied by the latter. That is, even though the two models share the same law of motion for aggregate capital accumulation as in Eq. (28), the first-order conditions in Eqs. (29) and (31) (derived in the representative-firm model) no longer hold in the microfounded model with endogenous borrowing limits.

The source of the discrepancy stems from the endogeneity of the borrowing constraints in Eq. (33), where the market value of capital,  $Q(I_t/K_t)$ , is positively affected by the rate of aggregate investment. Hence, the more investment a firm makes, the higher its value, and thus the more creditworthy it becomes. However, this type of credit externality is not internalized by firms because  $Q_t$  is a market price taken as given by individual firms. As a result, the microfounded model appears to have an insufficient level of investment relative to its counterpart, the representative-agent CAC model.

The following proposition shows that the credit externality in the endogenous borrowing limit model is equivalent to a form of aggregate "investment externality" in the conventional CAC model, where the source of the aggregate investment externality is a social rate of return to the average investment rate that individual firms take as given.

**Proposition 4.** *The heterogeneous-firm model with an endogenous borrowing limit is observationally equivalent to the following representative-firm CAC model with investment externalities:*

$$K_{t+1} = (1 - \delta)K_t + \tilde{\Psi}(\bar{i}_t, i_t)K_t, \quad (44)$$

where  $\bar{i}_t \equiv \frac{\bar{I}_t}{\bar{K}_t}$  denotes the average investment-to-capital ratio in the economy that the representative firm takes as given, and the CAC function  $\tilde{\Psi}(\cdot, \cdot)$  is increasing and concave in  $\{\bar{i}_t, i_t\}$  and satisfies the decomposition:  $\tilde{\Psi}(\bar{i}_t, i_t) = \theta Q(\bar{i}_t)\varphi(i_t)$ , where the function  $\varphi(\cdot)$  satisfies

$$\varphi(i_t) = \int_{\varepsilon \geq \varepsilon^*(i_t)} \varepsilon d\Phi(\varepsilon), \quad (45)$$

which is also increasing and concave.

**Proof.** See Appendix D.  $\square$

**Example.** As an example, consider the microfounded model with Pareto distribution of  $\varepsilon$ . The model's equilibrium is characterized by Eqs. (41), (42), and (43). Now consider a representative-firm CAC model with investment externality  $(\frac{\bar{I}_t}{\bar{K}_t})^a$ :

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} [R_t K_t - I_t] \quad (46)$$

subject to

$$K_{t+1} = (1 - \delta)K_t + \varphi_0 \left( \frac{\bar{I}_t}{\bar{K}_t} \right)^a K_t^{1-b} I_t^b, \quad (47)$$

where  $\varphi_0 = \frac{\eta}{\eta-1} \theta^{\frac{1}{\eta+1}}$ ,  $a = \frac{1}{\eta(\eta+1)}$ ,  $b = \frac{\eta-1}{\eta}$ , and  $\frac{\bar{I}_t}{\bar{K}_t}$  denotes the average investment rate in the economy which the representative firm takes as given. Denoting  $Q_t$  as the Lagrangian multiplier for the constraint, the first-order conditions with respect to  $I_t$  and  $K_{t+1}$  are given, respectively, by

$$Q_t \varphi_0 b \left( \frac{\bar{I}_t}{\bar{K}_t} \right)^a K_t^{1-b} I_t^{b-1} = 1, \quad (48)$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1 - \delta)Q_{t+1} + Q_{t+1} \varphi_0 (1 - b) \left( \frac{\bar{I}_t}{\bar{K}_t} \right)^a K_{t+1}^{-b} I_{t+1}^b \right\}. \quad (49)$$

Imposing the equilibrium condition,  $\frac{\bar{I}_t}{K_t} = \frac{I_t}{K_t}$ , and plugging in the values of  $\{\varphi_0, a, b\}$ , Eq. (47) becomes

$$K_{t+1} = (1 - \delta)K_t + \frac{\eta\theta^{\frac{1}{\eta+1}}}{\eta - 1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}; \quad (50)$$

Eq. (48) becomes

$$\frac{I_t}{K_t} = \theta Q_t^{1+\eta}; \quad (51)$$

and Eq. (49) becomes

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1 - \delta)Q_{t+1} + \frac{1}{\eta - 1} \frac{I_{t+1}}{K_{t+1}} \right\}. \quad (52)$$

The three equations are identical to Eqs. (41) through (43) in the microfounded model.

### 3.2. Policy implications

The previous analysis suggests that if CAC is not a form of technology but a consequence of firm-level borrowing constraints, then the policy implications of an aggregate CAC model and those of a microfounded model may be different. As an example of the different policy implications of the two models, we have the following proposition:

**Proposition 5.** *The optimal steady state rate of capital tax in the representative-agent CAC model is zero; it is negative in the endogenous credit limit model.*

**Proof.** See Appendix E.  $\square$

The intuition behind this proposition is straightforward. The endogenous credit limit model features a positive credit externality in firms' investment. Because firms consider the borrowing limit as exogenous when in fact it is endogenously determined by the market equilibrium, the competitive equilibrium features suboptimal investment and leads to insufficient capital stock. Alternatively, since the model is equivalent to a representative-agent CAC model with positive investment externalities, the investment level determined by a representative firm in a competitive equilibrium is suboptimal. Therefore, the adoption of a negative capital tax rate to encourage more investment improves social welfare.

### 3.3. Rationalizing collateral constraints through limited contract enforceability

So far the collateral constraints have been imposed on firms in an *ad hoc* fashion. This subsection is intended to rationalize the assumptions we have made. The discussions below follow that of Jermann and Quadrini (2009).

Suppose that investment needs to be paid in advance, i.e., before production. Assuming that these payments could be financed with intra-period loans that do not incur interests, it is more convenient for the firm to finance the payments with debt than to carry cash over from the previous period. However, after taking out the intra-period loan but before making the investment, the firm could renegotiate the loan as pointed out by Kiyotaki and Moore (1997). First, suppose a fair market for used capital does not exist, so the price of used capital is proportional (or equal) to that of consumption goods ( $P = 1$ ). Then in case of default or liquidation the lender would recover a fraction of the value of the existing capital stock,  $\theta K_t(i)$ , which can be converted into consumption goods. By assuming that the firm has all the bargaining power, the lender will be willing to lend up to  $\theta K_t(i)$ . By further assuming that firms cannot raise dividends at the beginning of the period, we obtain the desired constraint (7) in the benchmark model.

On the other hand, with a fair market for used capital and assuming that the lender takes over the firm in the case of default, the value recovered by the lender would be proportional to the firm's value,

$$\theta Q_t K_t(i) = \theta \left[ \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int v_{t+1}(i) d\Phi(\varepsilon) \right] K_t(i),$$

where  $v_t(i)$  is the firm's private value as defined in Eq. (36). So the constraint (7) would be replaced by Eq. (33).

Both forms of collateral limits (or the liquidation value of capital) are standard in the literature. For example, the first form (exogenous liquidation value measured by goods-market price) is adopted by Perri and Quadrini (2011) and the second form (endogenous liquidation value measured by asset-market price) by Lorenzoni and Walentin (2007), among others.

**Table 1**  
Parameter values.

$\beta$	$\alpha$	$a_L$	$\rho$	$\sigma$	$\delta$	$\theta$	$\eta$
0.99	0.3	1.097	0.95	0.0072	0.032	0.08	2.4

#### 4. Lumpy investment at the firm level

This section solves for a general-equilibrium version of our microfounded investment model and uses simulated data from the model to investigate the lumpiness in firm-level investment. Because a firm's investment rate depends on the firm's value and other macroeconomic variables such as the real wage, a general-equilibrium model is required.

A representative consumer (i.e., the owner of firms) solves

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\log C_t - a_L N_t\} \quad (53)$$

subject to

$$C_t \leq W_t N_t + \Pi_t, \quad (54)$$

where  $\Pi_t$  denotes the lump-sum profit income from all firms. For simplicity, the household does not save. Introducing an equity market where households can buy firms' shares would give identical results (see, e.g., Wang and Wen, 2009). Denoting  $\Lambda_t$  as the Lagrangian multiplier of the household's budget constraint, the first-order conditions of the representative household are given by

$$\Lambda_t = \frac{1}{C_t}, \quad (55)$$

$$\frac{(1-\alpha)Y_t}{N_t} \frac{1}{C_t} = a_L. \quad (56)$$

The firm's problem is identical to that in the previous section with endogenous borrowing limits. The firm's decision rules are again given by Eqs. (34) through (36), and the following relationships hold:  $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$ ,  $W_t = (1-\alpha) \frac{Y_t}{N_t}$ , and  $R_t = \alpha \frac{Y_t}{K_t}$ . Under the assumption of Pareto distribution, the competitive general equilibrium of the aggregate economy is characterized by these three relationships, plus Eqs. (41)–(43), (55), and (56). This system of eight equations determines the equilibrium path of  $\{C_t, N_t, Y_t, I_t, K_{t+1}, Q_t, W_t, R_t\}$ . The equilibrium cutoff is determined as  $\varepsilon_t^* = Q_t^{-1}$ . The model has a unique saddle path near the steady state as can be easily confirmed by the eigenvalue method. We solve the model by log-linearizing around the steady state under the assumption that the aggregate productivity ( $A_t$ ) evolves according to the law of motion,

$$\log A_t = \rho \log A_{t-1} + \sigma \xi_t, \quad (57)$$

where  $\xi_t$  is i.i.d. with the standard deviation normalized to 1.

**Calibration.** We calibrate the model at quarterly frequency by setting the time discounting factor  $\beta = 0.99$ , the capital's income share  $\alpha = 0.3$ , the persistence of technology shock  $\rho = 0.95$ , and the standard deviation of innovation  $\sigma = 0.0072$  (as in the standard RBC literature). Since  $a_L$  does not enter the model's log-linear dynamic system, we choose  $a_L$  such that  $N = 1$  in the deterministic steady state. The other three parameters—the depreciation rate of capital  $\delta$ , the borrowing limit  $\theta$ , and the Pareto distribution parameter  $\eta$ —are chosen so that the model matches the distribution of firm-level investment. The parameter values are summarized in Table 1.

We follow Cooper and Haltiwanger (2006) by defining  $i_t(i) = \frac{K_{t+1}(i) - (1-\delta)K_t(i)}{K_t(i)}$  as a firm's investment rate. The annual investment rate in the model is calculated by simulation and time aggregation. We simulate 200,000 quarters of data. We first use a general-equilibrium model to obtain the cutoff  $\varepsilon_t^*$ . We then make 200,000 independent draws of  $\varepsilon_t(i)$  for a typical firm by normalizing its initial capital stock. Finally, we calculate the annual investment rate for  $\tau = 1, 2, \dots, 50,000$  as

$$i_\tau^A = \frac{K_{4\tau} - (1-\delta)^4 K_{4*(\tau-1)}}{K_{4*(\tau-1)}}. \quad (58)$$

(More details of the simulation procedure can be found in Appendix D.) The model statistics for the annualized investment rate  $i_\tau^A$  are reported in Table 2, where the empirical counterparts are based on statistics reported by Cooper and Haltiwanger (2006, p. 615, Table 1).

The table shows that our microfounded model matches the basic features of plant-level investment dynamics reported by Cooper and Haltiwanger (2006). For example, our model predicts that in any given year, about 17% of firms are inactive (making zero or negative investment), about 20% of firms undertake big investment projects (with values exceeding 20 percent of the existing capital stock), and the average investment rate is 12.7% a year. These predictions are extremely

**Table 2**

Summary statistics for annualized investment rate.

Investment rate	$i^A \leq 0$	$0 < i^A \leq 20\%$	$i^A > 20\%$	$E[i^A]$	$std[i^A]$	$\frac{E[i^A   i^A \geq 0.2]}{E[i^A]}$	$\rho(i_t^A, i_{t-1}^A)$
Model (%)	17.2	62.6	20.2	12.7	31.7	57.1	−0.0056
Data (%)	18.5	62.9	18.6	12.2	33.7	50.0	0.058

close to data. The standard deviation of the investment rate is 32% in the model, whereas it is 34% in the data. Firm-level investment is not serially correlated. The model predicts an autocorrelation of −0.0056, while this value is 0.058 in the data (the last column in the table). Therefore, our model performs quite well in explaining the lumpiness and lack of serial correlation in firms' investment behavior.<sup>14</sup>

## 5. Conclusion

This paper has addressed a long-standing inconsistency problem in investment theory: The assumption of convex adjustment costs in aggregate models and the assumption of non-convex adjustment costs in micro models. The former assumption is consistent with aggregate investment behavior but inconsistent with firm-level data. The latter assumption is consistent with micro evidence but not with aggregate data. Therefore, it is difficult to view either type of adjustment cost as a pure form of technology. This paper has shown that borrowing constraints based on limited contractual enforcement can rationalize CAC at the aggregate level and at the same time generate lumpy investment at the firm level.

The intuition is simple. In the typical CAC models, the marginal cost increases continuously as the investment increases. This assumption can rationalize the sluggishness of aggregate investment but is inconsistent with firm-level lumpy investment. In this paper, however, the marginal cost at the firm level is instead constant (at zero) until it reaches a borrowing limit, above which it goes to infinity. Thus, firm-level investment can be lumpy while aggregate investment can appear sluggish.

Our model can also explain why Tobin's  $Q$  is not a sufficient statistic to explain firm-level investment in disaggregated data. The reason is that in our model, Tobin's  $Q$  is an aggregate statistic while firm-level investment depends crucially on firm-specific shocks which are not captured by the market value of  $Q$ . In addition, we have also shown that if convex adjustment costs are no longer assumed to be part of the aggregate technology but are derived instead from market frictions and interactions, then aggregate CAC are not necessarily policy invariant.

## Appendix A. Proof of Proposition 1

**Proof.** Denoting  $i_t \equiv \frac{I_t}{K_t}$  and taking derivative of the function  $\varphi(\cdot)$  in Eq. (21) with respect to  $i_t$  gives

$$\varphi'(i_t) = [-\theta \varepsilon_t^* \phi(\varepsilon_t^*) - \tilde{\rho} \phi(\varepsilon_t^*)] \frac{\partial \varepsilon_t^*}{\partial i_t}, \quad (59)$$

where  $\phi(\varepsilon)$  denotes the PDF of  $\varepsilon$ . Differentiating Eq. (18) with respect to  $i_t \equiv \frac{I_t}{K_t}$  gives

$$\frac{\partial i_t}{\partial \varepsilon_t^*} = -\theta \phi(\varepsilon_t^*) - \tilde{\rho} \frac{1}{\varepsilon_t^*} \phi(\varepsilon_t^*). \quad (60)$$

The above two equations together imply

$$\varphi'(i_t) = \frac{\theta \varepsilon_t^* \phi(\varepsilon_t^*) + \tilde{\rho} \phi(\varepsilon_t^*)}{\theta \phi(\varepsilon_t^*) + \tilde{\rho} \frac{1}{\varepsilon_t^*} \phi(\varepsilon_t^*)} = \varepsilon_t^* > 0. \quad (61)$$

Differentiating this equation again with respect to  $i_t$  and using Eq. (60) gives

$$\varphi''(i_t) = \frac{\partial \varepsilon_t^*}{\partial i_t} = \frac{1}{-\theta \phi(\varepsilon_t^*) - \tilde{\rho} \frac{1}{\varepsilon_t^*} \phi(\varepsilon_t^*)} < 0. \quad (62)$$

Therefore, the function  $\varphi(i_t)$  is increasing and strictly concave in  $i_t$ . Since  $\varphi(i_t)$  depends only on the investment-to-capital ratio, it is homogeneous of degree zero in  $\{I_t, K_t\}$ .  $\square$

<sup>14</sup> Our working paper (2010) also runs regression analyses to quantitatively demonstrate that Tobin's  $Q$  can better explain aggregate investment than firm-level investment (see Appendix F for more details).

## Appendix B. Proof of Proposition 2

**Proof.** Denote  $V[K_t(i), \varepsilon_t(i)]$  as the value function of firm  $i$  with capital stock  $K_t(i)$ . Based on the analysis of Hayashi (1982), we conjecture that a firm's value is linearly homogeneous in its capital stock because of constant returns to scale production technology:

$$V[K_t(i), \varepsilon_t(i)] = v[\varepsilon_t(i)]K_t(i) \equiv v_t(i)K_t(i). \quad (63)$$

We verify later that this conjecture is correct. Define  $\bar{v}_t \equiv E v_t(i) = \int v_t(\varepsilon) d\Phi(\varepsilon)$  as the average value of the firm across states and  $i_t(i) \equiv \frac{I_t(i)}{K_t(i)}$  as the firm's investment rate. Firm  $i$  solves the following dynamic programming problem:

$$v_t(i)K_t(i) = \max_{K_{t+1}(i), I_t(i)} \left\{ R_t K_t(i) - I_t(i) + \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(i) K_{t+1}(i) \right] \right\} \quad (64)$$

subject to

$$K_{t+1}(i) = (1 - \delta)K_t(i) + \varepsilon_t(i)I_t(i), \quad (65)$$

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i) \quad (66)$$

and the borrowing constraint in Eq. (33). To simplify the analysis, assume  $\tilde{\rho} = 0$ .

Denote  $\{\lambda_t(i), \pi_t(i), \mu_t(i)\}$  as the Lagrangian multipliers of constraints (65), (66), and (33), respectively. The firm's first-order conditions for  $\{I_t(i), K_{t+1}(i)\}$  are given, respectively, by  $1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i)$  and

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{v}_{t+1}. \quad (67)$$

The envelope condition is given by  $v_t(i) = R_t + (1 - \delta)\lambda_t(i) + \theta Q_t(i)\mu_t(i)$ . Substituting this expression into Eq. (67) gives

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + (1 - \delta)\lambda_{t+1}(i) + \theta Q_{t+1}(i)\mu_{t+1}(i)\}. \quad (68)$$

Hence, all first-order conditions are the same as those in the benchmark model except here  $\theta = (1 - \delta)Q_t(i)$ . Therefore, following the same steps of analysis as in the benchmark model (Section 2.2) by considering all of the possible values of the Lagrangian multipliers, it can be easily shown that the Lagrangian multipliers are given by  $\mu_t(i) = \max\{q_t(i) - 1, 0\}$ ,  $\pi_t(i) = \max\{1 - q_t(i), 0\}$ , where  $q_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}$ ; the firm's optimal decision rules for investment and capital accumulation are given by Eqs. (34) and (35); and the firm's value function is given by Eq. (36). Clearly, since  $R_t$  and  $Q_t$  are independent of  $K_t(i)$ , Eq. (36) implies that the value of a firm is proportional to its capital stock:  $V[\varepsilon_t(i), K_t(i)] = v_t(i)K_t(i)$ . This confirms our initial conjecture.  $\square$

## Appendix C. Proof of Proposition 3

**Proof.** Denote  $i_t \equiv \frac{I_t}{K_t}$ , then

$$\frac{\partial \Psi}{\partial i_t} = (1 - \delta) \frac{\partial Q}{\partial i_t} \int_{\varepsilon \geq \frac{1}{Q(i_t)}} \varepsilon d\Phi(\varepsilon) + (1 - \delta)\phi(\varepsilon_t^*) Q_t^{-2} \frac{\partial Q}{\partial i_t}. \quad (69)$$

Since  $Q_t = \frac{1}{\varepsilon_t^*}$ , Eq. (37) implies

$$\frac{\partial i_t}{\partial Q_t} = (1 - \delta)[1 - \Phi(\varepsilon_t^*)] + (1 - \delta)\varepsilon_t^* \phi(\varepsilon_t^*). \quad (70)$$

The above two equations together imply

$$\Psi'(i_t) = \varepsilon_t^* \left[ \frac{\int_{\varepsilon \geq \varepsilon_t^*} \frac{\varepsilon}{\varepsilon_t^*} d\Phi(\varepsilon) + \varepsilon_t^* \phi(\varepsilon_t^*)}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^* \phi(\varepsilon_t^*)} \right] > \varepsilon_t^* > 0, \quad (71)$$

where the inequality holds because  $\int_{\varepsilon \geq \varepsilon_t^*} \frac{\varepsilon}{\varepsilon_t^*} d\Phi(\varepsilon) > 1 - \Phi(\varepsilon_t^*)$  and the support of  $\varepsilon$  is in the positive region of the real line.

Integrating by parts and rearranging, the first term in the numerator of  $\Psi'(i)$  can be written as  $\int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) = \varepsilon_t^*[1 - \Phi(\varepsilon_t^*)] + \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)] d\varepsilon$ . Thus,

$$\begin{aligned}\Psi'(i_t) &= \frac{\varepsilon_t^*[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^{*2}\phi(\varepsilon_t^*) + \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)] d\varepsilon}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^*\phi(\varepsilon_t^*)} = \varepsilon_t^* + \frac{\int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)] d\varepsilon}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^*\phi(\varepsilon_t^*)} \\ &\equiv f(\varepsilon_t^*).\end{aligned}\quad (72)$$

Notice that

$$\begin{aligned}f'(\varepsilon_t^*) &= 1 + \frac{-[1 - \Phi(\varepsilon_t^*)]\{\varepsilon_t^*\phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\} - \varepsilon_t^*\phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)] d\varepsilon}{\{\varepsilon_t^*\phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2} \\ &= 1 - \frac{[1 - \Phi(\varepsilon_t^*)]}{\varepsilon_t^*\phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]} - \frac{\varepsilon_t^*\phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)] d\varepsilon}{\{\varepsilon_t^*\phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2} \\ &= \frac{\varepsilon_t^*\phi(\varepsilon_t^*)}{\varepsilon_t^*\phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]} - \frac{\varepsilon_t^*\phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)] d\varepsilon}{\{\varepsilon_t^*\phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2}.\end{aligned}\quad (73)$$

Clearly, as long as  $\phi'(\varepsilon_t^*) \leq 0$ ,

$$f'(\varepsilon_t^*) \geq 0 \quad (74)$$

and

$$\Psi''(i_t) = f'(\varepsilon_t^*) \frac{\partial \varepsilon_t^*}{\partial i_t} \leq 0, \quad (75)$$

since  $\frac{\partial \varepsilon_t^*}{\partial i_t} < 0$  by Eq. (37). Therefore,  $\Psi(\cdot)$  is increasing and concave. In addition, it is clear that  $\Psi(\cdot)$  depends only on the investment-to-capital ratio  $i_t$ , so it is homogeneous of degree zero in  $\{I, K\}$ .  $\square$

#### Appendix D. Proof of Proposition 4

**Proof.** Consider a representative firm solving the program in Eq. (27) subject to Eq. (44), taking  $\bar{i}_t$  as given. Denoting  $Q_t$  as the Lagrangian multiplier for the constraint and imposing the equilibrium condition  $\bar{i}_t = i_t$ , the first-order conditions for  $I_t$  and  $K_{t+1}$  are given, respectively, by

$$\theta Q_t^2 \varphi'(i_t) = 1, \quad (76)$$

$$Q_t = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1} + (1 - \delta) Q_{t+1} + \theta Q_{t+1}^2 [\varphi(i_t) - \varphi'(i_{t+1}) i_{t+1}] \right\}. \quad (77)$$

Since  $\theta Q_{t+1}^2 \varphi'(i_{t+1}) = 1$ , Eq. (77) can be rewritten as

$$Q_t = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1} + (1 - \delta) Q_{t+1} + \theta Q_{t+1}^2 \int_{\varepsilon \geq \frac{1}{Q_{t+1}}} \varepsilon d\Phi(\varepsilon) - \frac{I_{t+1}}{K_{t+1}} \right\}, \quad (78)$$

which is identical to Eq. (38) in the microfounded model.  $\square$

#### Appendix E. Proof of Proposition 5

**Proof.** The first part of the proposition—that the optimal capital tax rate in the representative-agent CAC model without externalities is zero—is a standard result in the literature. Hence, we need only to prove the second part of the proposition. We add a representative household into the model so that the government's objective function is well defined. We prove the proposition in an environment without aggregate uncertainty. The household's problem is to choose consumption ( $C_t$ ) and labor supply ( $N_t$ ) in each period to solve

$$\max \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \quad (79)$$

subject to  $C_t \leq w_t N_t + \Pi_t + T_t$ , where  $\Pi_t$  denotes aggregate dividends distributed by firms and  $T_t = \int \tau_t [Y_t(i) - w_t N_t(i)] di$  is a lump-sum transferred from the government based on capital tax revenues collected from all firms, where  $\tau_t$  is the tax rate for capital income. The first-order conditions of the household can be summarized by

$$u'(C_t) w_t = v'(N_t). \quad (80)$$

On the firm side, we can show that, regardless of capital tax, the endogenous credit limit model is always equivalent to a representative-firm model with investment externality. Hence, based on the equivalence, we need only to prove that the optimal capital tax rate is negative in the representative-firm model with investment externality. For simplicity, we consider the Pareto distribution for firms' idiosyncratic shocks  $\varepsilon_t(i)$  (in the microfounded model) and the Cobb–Douglas production function,  $Y_t = AK_t^\alpha N_t^{1-\alpha}$ . Thus, the equivalent CAC function is of the Cobb–Douglas form and a representative firm in the investment externality model must solve

$$\max_{\tau} \sum_{\tau=0}^{\infty} \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} \{ (1 - \tau_t)(Y_t - w_t N_t) - I_t \} \quad (81)$$

subject to

$$K_{t+1} = (1 - \delta)K_t + \varphi_0 \left( \frac{\bar{I}_t}{\bar{K}_t} \right)^a K_t^{1-b} I_t^b, \quad (82)$$

where  $a = \frac{1}{\eta(\eta+1)}$ ,  $b = \frac{\eta-1}{\eta}$ , and  $\frac{\bar{I}_t}{\bar{K}_t}$  denotes the average investment rate in the economy which each firm takes as given. The first-order conditions for  $\{I_t, K_{t+1}\}$  in this model are given by

$$Q_t \varphi_0 b \left( \frac{\bar{I}_t}{\bar{K}_t} \right)^a K_t^{1-b} I_t^{b-1} = 1, \quad (83)$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ (1 - \tau_{t+1})R_{t+1} + (1 - \delta)Q_{t+1} + Q_{t+1}\varphi_0(1-b) \left( \frac{\bar{I}_t}{\bar{K}_t} \right)^a K_{t+1}^{-b} I_{t+1}^b \right\}, \quad (84)$$

where  $R_t = \alpha \frac{Y_t}{K_t}$  and  $w_t = \frac{(1-\alpha)Y_t}{N_t}$ . Imposing the equilibrium conditions,  $\frac{\bar{I}_t}{\bar{K}_t} = \frac{I_t}{K_t}$  and  $\Lambda_t = u'(C_t)$ , and plugging in the values of  $\{\varphi_0, a, b\}$ , and substituting out  $\{R_t, w_t, Q_t\}$ , the above two first-order conditions become

$$\frac{I_t}{K_t} = \theta Q_t^{1+\eta}, \quad (85)$$

$$\theta^{-\frac{1}{\eta+1}} \left( \frac{I_t}{K_t} \right)^{\frac{1}{\eta+1}} = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[ (1 - \tau_{t+1}) \frac{\alpha Y_{t+1}}{K_{t+1}} + (1 - \delta) \theta^{-\frac{1}{\eta+1}} \left( \frac{I_{t+1}}{K_{t+1}} \right)^{\frac{1}{\eta+1}} + \frac{1}{\eta - 1} \frac{I_{t+1}}{K_{t+1}} \right]. \quad (86)$$

The law of motion for capital accumulation becomes

$$K_{t+1} = (1 - \delta)K_t + \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta - 1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}, \quad (87)$$

and the household resource constraint becomes

$$I_t + C_t = Y_t = AK_t^\alpha N_t^{1-\alpha}. \quad (88)$$

Notice that Eqs. (80), (86)–(88), and the aggregate production function together can uniquely solve the competitive equilibrium path of  $\{C_t, I_t, Y_t, N_t, K_{t+1}\}$  as a function of the tax rate  $\tau_t$  in the externality model. The optimal tax policy is to design a sequence of tax rates  $\{\tau_t\}_{t=0}^{\infty}$  to solve

$$V(K_0) = \max_{\{\tau_t\}} \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \quad (89)$$

subject to Eqs. 80, (86)–(88), and the aggregate production function.

Instead of directly solving program (89), we first study the “first best allocation” in the externality model, which pertains to the highest possible utility that a social planner can achieve in the model when the investment externality is fully endogenized. Hence, the first best allocation also pertains to the highest possible utility that the government can achieve using tax policies in program (89).

The first best allocation solves

$$V^*(K_0) = \max_{\{C_t, N_t, I_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \quad (90)$$

subject to

$$K_{t+1} = (1 - \delta)K_t + \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta - 1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}, \quad (91)$$

$$C_t + I_t = AK_t^\alpha N_t^{1-\alpha}. \quad (92)$$



It is obvious that the lifetime utility defined in program (90) is at least as large as that defined in program (89), i.e.,  $V^*(K_0) \geq V(K_0)$ , because the former gives the first best allocation. The first-order conditions for  $\{I_t, C_t, K_{t+1}\}$  in program (90) are given, respectively, by

$$Q_t \frac{\eta}{\eta+1} \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}-1} = 1, \quad (93)$$

$$u'(C_t) \frac{(1-\alpha)Y_t}{N_t} = v'(N_t), \quad (94)$$

$$Q_t = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[ \frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta)Q_{t+1} + Q_{t+1} \frac{1}{\eta+1} \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}-1} I_t^{\frac{\eta}{\eta+1}} \right]. \quad (95)$$

Eq. (93) implies  $Q_t = \frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left( \frac{I_t}{K_t} \right)^{\frac{1}{\eta+1}}$ . Using this relationship to substitute out  $Q$ , Eq. (95) becomes

$$\frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left( \frac{I_t}{K_t} \right)^{\frac{1}{\eta+1}} = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[ \frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta) \frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left( \frac{I_{t+1}}{K_{t+1}} \right)^{\frac{1}{\eta+1}} + \frac{1}{\eta} \frac{I_{t+1}}{K_{t+1}} \right]. \quad (96)$$

Notice that Eqs. (94), (96)–(92), and the aggregate production function together uniquely solve for the first best allocation  $\{C_t, I_t, Y_t, N_t, K_{t+1}\}$  under program (90). Similarly, Eqs. (80), (86)–(88), and the aggregate production function together uniquely solve for the equilibrium path of  $\{C(\tau_t), I(\tau_t), Y(\tau_t), N(\tau_t), K(\tau_t)\}$  in a competitive equilibrium with investment externalities. Comparing these two systems of equations, except that Eq. (96) is different from Eq. (86), all other equilibrium conditions in the first best allocation are identical to those in a competitive equilibrium in terms of mathematical relationship. In particular, Eqs. (94), (91), and (92) are identical to Eqs. (80), (87), and (88), respectively.

Denote the equilibrium path of the first best allocation as  $\{C_t^*, I_t^*, Y_t^*, N_t^*, K_{t+1}^*\}$ . By comparing Eq. (96) under program (90) with Eq. (86) in the competitive equilibrium, it is obvious that the government can achieve the first best allocation in program (89) by setting the tax rate such that Eqs. (96) and (86) are identical, which implies

$$\begin{aligned} & \frac{\eta^2}{\eta^2-1} \frac{\alpha Y_{t+1}^*}{K_{t+1}^*} + (1-\delta) \theta^{-\frac{1}{\eta+1}} \left( \frac{I_{t+1}^*}{K_{t+1}^*} \right)^{\frac{1}{\eta+1}} + \frac{\eta}{\eta^2-1} \frac{I_{t+1}^*}{K_{t+1}^*} \\ &= (1-\tau_{t+1}) \frac{\alpha Y_{t+1}^*}{K_{t+1}^*} + (1-\delta) \theta^{-\frac{1}{\eta+1}} \left( \frac{I_{t+1}^*}{K_{t+1}^*} \right)^{\frac{1}{\eta+1}} + \frac{1}{\eta-1} \frac{I_{t+1}^*}{K_{t+1}^*}. \end{aligned} \quad (97)$$

Simplification gives

$$\left( \frac{1}{\eta^2-1} + \tau_{t+1} \right) \alpha Y_{t+1}^* = \frac{1}{\eta^2-1} I_{t+1}^*. \quad (98)$$

Since  $Q_t^* = \frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left( \frac{I_t^*}{K_t^*} \right)^{\frac{1}{\eta+1}}$ , we have  $Q_{t+1}^* \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta-1} K_{t+1}^{\frac{1}{\eta+1}} I_{t+1}^{\frac{\eta}{\eta+1}} = \frac{\eta+1}{\eta} I_{t+1}^*$ . So we can rewrite Eq. (96) by multiplying both sides by  $K_{t+1}^*$  as

$$Q_t^* K_{t+1}^* = \beta \frac{u'(C_{t+1}^*)}{u'(C_t^*)} [\alpha Y_{t+1}^* - I_{t+1}^* + Q_{t+1}^* K_{t+2}^*]. \quad (99)$$

This equation implies that in the steady state we must have  $\alpha Y^* > I^*$ . Then by Eq. (98), we must have  $\tau < 0$  in the steady state to achieve the first best allocation.  $\square$

## Appendix F. Model simulation

1. Simulating aggregate variables. We solve the equilibrium path of the aggregate variables by log-linear approximation around the deterministic steady state. The log-linearized variable is defined as

$$\hat{x}_t \equiv \log(X_t) - \log \bar{X}, \quad (100)$$

where  $\bar{X}$  indicates the steady-state value. We simulate the aggregate model for  $t = 200,000$  periods using the law of motion of aggregate technology in Eq. (57). Based on the simulated variables, we can use the following transformation to obtain the value of aggregate variables:

$$X_t = \bar{X} \exp(\hat{x}_t). \quad (101)$$

In this way, we obtain the sequences of capital  $K_t$ , aggregate investment  $I_t$ , Tobin's  $Q_t$ , and the cutoff  $\varepsilon_t^* = \frac{1}{Q_t}$ .

2. Generating firm data. In order to generate firm data, we need to simulate the idiosyncratic shocks,  $\varepsilon_t(i)$ . A random sample with 200,000 observations of  $\varepsilon(i)$  in each time period  $t$  can be generated using inverse transform sampling. Given a random variable  $U$  drawn from the uniform distribution on the unit interval  $(0, 1)$ , the variable

$$\varepsilon = \frac{1}{U^{\frac{1}{\eta}}} \quad (102)$$

is Pareto distributed with the distribution function

$$F(\varepsilon) = 1 - \varepsilon^{-\eta}. \quad (103)$$

Given the sequences of aggregate variables (especially the cutoff  $\varepsilon_t^*$ ), we obtain firm-level investment based on the firm's decision rule,

$$I_t(i) = \begin{cases} Q_t \theta K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^*, \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^*. \end{cases} \quad (104)$$

We normalize each firm's initial capital stock to the aggregate steady-state capital  $\bar{K}$ ; i.e.,  $K_0(i) = \bar{K}$ . We construct the firm-level capital sequence by the law of motion:

$$K_{t+1}(i) = (1 - \delta)K_t(i) + \varepsilon_t(i)I_t(i). \quad (105)$$

In each period  $t = 0, 1, \dots, 200,000$ , we track each firm  $i$ 's capital stock and positive investment level whenever  $\varepsilon_t(i) \geq \varepsilon_t^*$ .

3. Regression analysis. We run two regressions. The first is based on the aggregate time series<sup>15</sup>:

$$\frac{K_{t+1} - (1 - \delta)K_t}{K_t} = \beta_0 + \beta_1 Q_t. \quad (106)$$

The second is based on firm-level data:

$$\frac{K_{t+1}(i) - (1 - \delta)K_t(i)}{K_t(i)} = \beta_0 + \beta_1 Q_t. \quad (107)$$

The adjusted  $R^2$  is almost identical if we had used log variables for the aggregate model. For the firm-level data, since  $\frac{K_{t+1}(i) - (1 - \delta)K_t(i)}{K_t(i)}$  can be zero in some periods, we cannot use log values in the regression.

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<sup>15</sup> Tobin's  $Q$  is a sufficient statistic to determine aggregate investment in both our model and the CAC model.

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