(note: Latex for some reason throws a fit when I use the minus sign, so I switched all minus signs with the words 'minus sign')

Part 1: Compare to known click rate

(a) Derive expressions for $E[\overline{X}]$ and $\text{var}[\overline{X}]$ under the null hypothesis in terms of p_a. You will need to use the properties of expectations and variances described below. Here, I give you the derivation for $E[\overline{X}]$, you need to do the same for $\text{var}[\overline{X}]$.

$E(\overline{x}) = p_a$ $V(\overline{x}) = V(\frac{1}{n}\sum_{i=1}^{n} x_i) = \frac{1}{n^2}V(\sum_{i=1}^{n} x_i) = \frac{1}{n^2}\sum_{i=1}^{n} V(x_i) = \frac{1}{n^2}\sum_{i=1}^{n} p_a(1-p_a) = \frac{p_a(1-p_a)}{n}$

(b) Based on your derivation, compute values for $E[\overline{X}]$ and $\text{var}[\overline{X}]$ based on $p_a = 0.5$ and n = 50. Use R or python to do this.

```
expected <- pa
variance <- (pa * (1-pa)) / n

cat("Variance: ", variance, "\n")
```

```
## Variance:  0.005
```

```
cat("Expected Value: ", expected, "\n")
```

```
## Expected Value:  0.5
```

(c) Using the result above, you can now use the CLT by approximating the distribution of X as $N(E[\overline{X}], \sqrt{var(X)})$. Based on this approximation, compute $P(X > pB)$. Use the R function pnorm, or norm.cdf in scipy.stats to compute this.

```
pb <- 0.6
N <- pnorm(pb, expected, sqrt(variance))
cat("P(mean(x) > 0.6) =", 1 - N)
```

```
## P(mean(x) > 0.6) = 0.0786496
```

(d) Should you reject the null hypothesis pB ≤ pA? Why?

We can only reject the null hypothesis if $P(\overline{x} > 0.6) < \alpha$. In this case, $\alpha = 0.05 < P(\overline{x} > 0.6)$, so we cannot reject the null hypothesis.

(e) What if you had observed the same pB = 0.6 but with n = 100 samples. Should you reject the null hypothesis in this case? Why?

```
variance_100 <- (pb * (1-pb)) / 100

N <- pnorm(pb, expected, sqrt(variance_100))
cat("P(mean(x) > 0.6) =", 1 - N)
```

```
## P(mean(x) > 0.6) = 0.02061342
```

Yes. If n = 100, $P(\overline{x} > 0.6) < \alpha$, so we should reject the null hypothesis in this scenario.

(f) What is the smallest value pB you would reject the null hypothesis with n = 100. Use the qnorm function in R or norm.ppf in scipy.stats for this. Denote this smallest value as qB.

```
variance_100 <- (pb * (1-pb)) / 100


qb <- qnorm(0.95, expected, sqrt(variance_100))
cat("smallest value =", qb)
```

```
## smallest value = 0.580581
```

(g) Based on (f), the smallest detectable improvement for pA = 0.5 with n = 100 is then qB (minus sign) pA. What is the smallest detectable improvement in your experiment (that is, with n = 50)?

```
qb <- qnorm(0.95, expected, sqrt(variance))

improvement <- qb - pa
cat('Smallest detectable improvement =', improvement)
```

```
## Smallest detectable improvement = 0.1163087
```

Part 2: Compare to known click rate

(a) What are the values of $E[\overline{X}]$ and $var(\overline{X})$ under the null hypothesis in this case.

```
expected <- pa
variance <- (pa*(1-pa))/n

cat("E(mean(x)) = ", expected, "\n")
```

```
## E(mean(x)) =  0.75
```

```
cat("var(mean(X)) = ", variance, "\n")
```

```
## var(mean(X)) =  0.00375
```

(b) Based on the CLT approximation, compute $P(\overline{X} > p_B)$ under the null hypothesis.

```
approx <- 1 - pnorm(pb, expected, sqrt(variance))
cat("P(mean(x) > pb) =", approx)
```

```
## P(mean(x) > pb) = 0.9928471
```

(c) Should you reject the null hypothesis pB ≤ 0.75? Why?

No. 0.9928471 is significantly greater than 0.05.

(d) What if you had observed the same pB = 0.6 but with n = 100 samples. Should you reject the null hypothesis in this case? Why?

```
n <- 100

variance <- (pa*(1-pa))/n

approx <- 1 - pnorm(pb, expected, sqrt(variance))
cat("P(mean(x) > pb) =", approx)
```

```
## P(mean(x) > pb) = 0.999734
```

No. The odds actually go up.

(e) What is the smallest value p_B you should reject the null hypothesis with n = 100. Denote this smallest value as q_B.

```
variance_100 <- (pa * (1-pa)) / 100


qb <- qnorm(.95, expected, sqrt(variance_100))
cat("smallest value =", qb)
```

```
## smallest value = 0.8212243
```

(f) Based on (e), the smallest detectable improvement for pA = 0.75 with n = 100 is then qB (minus sign) pA. What is the smallest detectable improvement in your experiment (n = 50)?

```
n <- 50

variance <- (pa * (1-pa)) / n

qb <- qnorm(0.95, expected, sqrt(variance))

improvement <- qb - pa
cat('Smallest detectable improvement = ', improvement)
```

```
## Smallest detectable improvement =  0.1007263
```

Part 3

Consider your answers for parts (1g) and (2f). Is the smallest detectable improvement in Question (1g) larger or smaller than in Question (2f)? Explain why this makes sense mathematically.

The smallest detectable improvement is larger in 1g than in 2f. This makes sense because we've decreased the variance in part 2 by raising the threshold (p_A). We can't improve as much when there's less variance.

Part 4: Comparing to estimated click rate pA.

(a) Derive expressions for E[Y] and var(Y) under the null hypothesis in terms of pA = pB = p. You will need to use the properties of expectations and variances described below. Here, I give you the derivation for E[Y], you need to do the same for var(Y).

$$E[Y] = 0$$

$$var[Y] = var[\overline{X_B} - \overline{X_A}] = var[\overline{X_B}] - var[\overline{X_A}] = \frac{p_B(1-p_B)}{n} - \frac{p_A(1-p_A)}{n} = 0$$

(b) Based on your derivation, compute values for $E[\overline{X}]$ and $var[\overline{X}]$ based on pA = 0.5 and n = 50. Use R or python to do this.

```
pa <- .5
n <- 50

na <- 55
sa <- 35
nb <- 45
sb <- 35

estimate_mean <- ((sa + sb) / (nb + na))

cat('mean =', estimate_mean)
```

```
## mean = 0.7
```

(c) Now that you have an estimate of p, compute a value for var(Y).

```
variance <- (.7) * (1-.7)

cat('variance =', variance)
```

```
## variance = 0.21
```

(d) What is your estimate y of pB (minus sign) pA based on the data your recorded for this experiment?

```
pb <- sb / nb
pa <- sa / na

p <- pb - pa
cat('y_hat =', p)
```

```
## y_hat = 0.1414141
```

(e) Using the CLT approximation, what is $P(Y > y)$

```
variance <- p * (1 - p)
expected <- p

probability = 1 - pnorm(expected, 0, sqrt(variance))

cat("Our fantastic probability is", probability)
```

```
## Our fantastic probability is 0.3424302
```

(f) Can you reject the null hypothesis of no improvement in this case? Why? Remember, we are using $\alpha = 0.05$.

No. We cannot reject the null hypothesis becuase our fantastic probability is larger than 0.05.