Sliding Mode Control for Mismatched Uncertain Systems Using an Extended Disturbance Observer

Divyesh Ginoya, P. D. Shendge, *Member, IEEE*, and S. B. Phadke

***Abstract*—This paper extends a recent result on sliding mode control for general nth order systems with mismatched uncertainties. In this paper, a control is proposed to handle a larger class of mismatched uncertainties by extending the disturbance observer and modifying and generalizing the sliding surface. The practical stability of the overall system is proved and the results are verified by simulation of an illustrative example.**

***Index Terms*—Chatter reduction, disturbance observer (DO), sliding mode control (SMC), unmatched uncertainties.**

I. Introduction

SLIDING mode control (SMC) is an effective and popular control strategy for controlling systems affected by uncertainties and external unmeasurable disturbances and has found several applications in diverse fields [2]. The conventional SMC shows insensitivity to matched uncertainties and disturbances. While for a large class of systems, the uncertainties and disturbances are matched, there are many practical systems for which this is not true [3]. Many practical systems like the permanent magnet synchronous motors [4], [5], roll autopilots for missiles acted upon by external disturbances [6], are affected by unmatched uncertainties. In fact, all systems represented using physical state variables, in which the control is implemented by an actuator in cascade with the plant, are examples of systems with unmatched uncertainties.

Since the conventional sliding mode control loses the property of invariance in the presence of unmatched uncertainties and disturbances [7], currently considerable research activity is going on to address the issue of unmatched uncertainties in sliding mode control. A variety of control strategies like the Riccati approach [8], adaptive control [9], fuzzy and neural network based control [10]–[12], LMI-based control [13], integral sliding mode control [14], to name just a few, have been proposed in the literature to address the problem of mismatched uncertainties. Some of these methods like [8] handle mismatched uncertainties which approach zero asymptotically.

Combining SMC or other control strategies with methods that give estimates of uncertainties and disturbances is an attractive proposition. Such a combination enables a reduction in the magnitude of the discontinuous component in control and thereby offers the possibility of mitigating the chatter in control.

Manuscript received October 11, 2012; revised February 25, 2013 and June 1, 2013; accepted June 7, 2013. Date of publication June 27, 2013; date of current version September 19, 2013.

The authors are with the College of Engineering Pune, Pune 411 007, India (e-mail: dlginoya007@gmail.com; pds.instru@coep.ac.in; sbp.instru@coep.ac.in).

Digital Object Identifier 10.1109/TIE.2013.2271597

The disturbance observer (DO), originally developed in [15], is one such strategy that has been combined with SMC. The DO has been applied in conjunction with SMC [16], for systems that satisfy the so-called matching conditions.

However, recently, several interesting results using the disturbance observer approach for mismatched nonlinear systems [17], for mismatched nonlinear multi-input multi-output systems [18], for generalized Extended State Observer based control [19] and for MAGLEV suspension [20], have been reported in the literature. A novel SMC that employs DO to counteract the effect of mismatched uncertainties using a new sliding surface which includes the estimate of the unmatched disturbances is recently proposed in [1]. Apart from counteracting the effect of mismatched uncertainties, it substantially alleviates the problem of chatter in control.

All these new results are based on the assumption that the mismatched disturbances and their first derivatives are bounded and that the first derivatives go to zero in the steady state. When compared to the results which need the unmatched uncertainties themselves to be of vanishing type in the steady state, the results in [1], [17]–[20], have enlarged the class mismatched disturbances but for many systems, this assumption too can be restrictive. For example, systems affected by disturbances that are functions of powers of time t or systems affected by harmonic disturbances [21], the first derivatives do not go to zero in the steady state. In this paper, this assumption is relaxed.

In this paper, it is proposed to take the work in [1] further by extending it in several ways. The extensions are as listed as follows.

i) The class of mismatched disturbances considered here, is significantly larger.
ii) The control is proposed for a general system of order n, having mismatched uncertainties in each channel. The scheme works for uncertainties in the control input as well.
iii) The DO proposed in [1] is extended to enable the estimation of disturbances belonging to the enlarged class.
iv) The sliding surface is modified to enable improvement in the performance of the system without causing a large increase in the control, especially at $t = 0$. The novel sliding surface is extended for a general system of order n.
v) The control is changed to alleviate the chattering problem further. The control can be designed even if the bound on the estimation of uncertainty is not known exactly.
vi) Practical stability of the overall system is assured.

The rest of the paper is organized as follows: the problem and the objective is stated in Section II. The extended DO, the modified sliding surface and control are described in Section III.

Section IV gives the stability analysis. Generalization of extended disturbance observer and novel sliding surface design is derived in Section V. The performance is illustrated by simulation examples in Section VI followed by a discussion in Section VII and a conclusion in Section VIII.

II. Problem Statement

It is proposed to consider an nth order single input system affected by unmatched disturbances in all channels. However, in order to explain the basic idea of the extended disturbance observer with clarity, first a second order system is considered. The results are extended to a general nth order system in Section V.

Consider the following system as in [1]:

$$\dot{x}_1 = x_2 + d_1(t) \tag{1a}$$
$$\dot{x}_2 = a(x) + b(x)u \tag{1b}$$
$$y = x_1 \tag{1c}$$

where x_1, x_2 are the states, u is control input, $d_1(t)$ is an unmeasurable disturbance, and y is the output.

Assumption 1: The disturbance $d_1(t)$ is continuous and satisfies

$$\left|\frac{d^j d_1(t)}{dt^j}\right| \le \mu \qquad \text{for } j = 0, 1, 2, \ldots, r \tag{2}$$

where μ is a positive number.

Remark 1: The class of disturbances considered here is much larger than [1] where it is assumed that d_1 and $\dot{d}_1$ are bounded and $\dot{d}_1$ vanishes as $t \to \infty$. It may be noted that it is not required to know the bound μ.

The objective is to design the control u in such a way that the system output is unaffected by the mismatched uncertainty $d_1(t)$.

III. Control Based on DO

First the extended DO that can tackle disturbances of the type (2) is discussed.

A. Extended DO

First, a second order DO for system (1) is proposed as follows:

$$\hat{d}_1 = p_{11} + l_{11}x_1 \tag{3}$$
$$\dot{p}_{11} = -\,l_{11}(x_2 + \hat{d}_1) + \hat{\dot{d}}_1 \tag{4}$$
$$\hat{\dot{d}}_1 = p_{12} + l_{12}x_1 \tag{5}$$
$$\dot{p}_{12} = -\,l_{12}(x_2 + \hat{d}_1) \tag{6}$$

where $\hat{d}_1$ and $\hat{\dot{d}}_1$ are estimates of $d_1(t)$ and $\dot{d}_1(t)$ respectively, p_{11} and p_{12} are auxiliary variables, and l_{11}, l_{12} are user chosen constants. Let the estimation errors be defined as

$$\tilde{e}_1 = [\tilde{d}_1 \quad \tilde{\dot{d}}_1]^T \tag{7}$$
$$\tilde{d}_1 = d_1 - \hat{d}_1 \tag{8}$$
$$\tilde{\dot{d}}_1 = \dot{d}_1 - \hat{\dot{d}}_1 \tag{9}$$

where $\tilde{d}_1$ denotes the error in the estimation of $d_1(t)$ and $\tilde{\dot{d}}_1$ denotes the error in the estimation of $\dot{d}_1(t)$. From (3), (4) and (1)

$$\dot{\hat{d}}_1 = l_{11}\tilde{d}_1 + \hat{\dot{d}}_1. \tag{10}$$

Subtracting both sides of (10) from $\dot{d}_1$

$$\begin{aligned}\dot{\tilde{d}}_1 &= -\,l_{11}\tilde{d}_1 + \dot{d}_1 - \hat{\dot{d}}_1 & (11)\\ &= -\,l_{11}\tilde{d}_1 + \tilde{\dot{d}}_1. & (12)\end{aligned}$$

Working on similar lines using (5), (6) and (1)

$$\dot{\tilde{\dot{d}}}_1 = -l_{12}\tilde{d}_1 + \ddot{d}_1. \tag{13}$$

Differentiating (12) and using (13)

$$\ddot{\tilde{d}}_1 = -l_{11}\dot{\tilde{d}}_1 - l_{12}\tilde{d}_1 + \ddot{d}_1. \tag{14}$$

Since $\ddot{d}_1$ is bounded as per Assumption 1, it is necessary and sufficient to select $l_{11} > 0$ and $l_{12} > 0$ for stability of $\tilde{d}_1$. It may be noted that the extended DO described here estimates $d_1(t)$ as well as $\dot{d}_1(t)$. The observer error dynamics can be expressed in compact form as

$$\dot{\tilde{e}}_1 = D_1\tilde{e}_1 + E_1\ddot{d}_1 \tag{15}$$
$$D_1 = \begin{bmatrix} -l_{11} & 1 \\ -l_{12} & 0 \end{bmatrix};\; E_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{16}$$

The stability of the extended DO and the analysis of the accuracy of estimation is discussed in Section IV. The extended DO developed here is similar to that in [22] but there are notable differences. In [22] the form of the disturbance is assumed to be time series expansion which is very restrictive. The proposed extended DO does not assume any particular form for the disturbance. The class of disturbances in [22] is a special case of the class of disturbances that can be estimated with the DO proposed in this paper. There are also differences in the stability proofs possible for the two observers.

B. Modified Sliding Surface

The novel sliding surface given in [1] is modified as

$$\sigma^* = \sigma - \sigma(0)e^{-\alpha t} \tag{17}$$
$$\sigma = x_2 + c_1x_1 + \hat{d}_1 \tag{18}$$

where $c_1 > 0$, $\alpha > 0$ are user chosen positive constants. It may be noted that at $t = 0$, the modified sliding variable $\sigma^* = 0$. If a control is designed such that the sliding condition is satisfied with respect to the modified sliding variable, then σ^* will remain in a small neighborhood of 0 for all time $t > 0$. Implication of this modification on the magnitude of control and stability is explained in Section IV.

C. Proposed Control

The control is designed so as to ensure sliding along the surface (17). Differentiating (17) and using (1a), (1b) and (18)

$$\dot{\sigma}^* = a(x) + b(x)u + c_1x_2 + c_1d_1 + \alpha\sigma(0)e^{-\alpha t} + \dot{\hat{d}}_1. \tag{19}$$

Selecting the control u as

$$u = -\frac{1}{b(x)}\left[a(x) + c_1x_2 + c_1\hat{d}_1 + \alpha\sigma(0)e^{-\alpha t}\right] - \frac{1}{b(x)}k_l\sigma^* - \frac{1}{b(x)}k_s\,\mathrm{sgn}(\sigma^*) \quad (20)$$

where $k_l > 0, k_s > 0$ are control gains to be designed

$$\dot{\sigma}^* = -k_l\sigma^* - k_s\,\mathrm{sgn}(\sigma^*) + c_1\tilde{d}_1 + \dot{\hat{d}}_1. \quad (21)$$

The term $(-(1/b(x))k_l\sigma^*)$ in (20) helps alleviate chatter by making it possible to use a smaller k_s and assure ultimate boundedness of σ^* as will be shown in Section IV.

IV. Stability

In this section, the stability of the DO and the plant to be controlled is analyzed. In the interest of simplicity and without loss of generality, the stability of the DO is analyzed for the second order DO. From (15) and (16), it is easy to see that it is always possible to select l_{11} and l_{12} such that the eigenvalues of D_1 can be placed arbitrarily. Assuming that l_{11} and l_{12} are chosen in such a way that the eigenvalues of D_1 are in the LHP, it is always possible to find a positive definite matrix P_1 such that

$$D_1^T P_1 + P_1 D_1 = -Q_1 \quad (22)$$

for any given positive definite matrix Q_1. Let λ_m denote the smallest eigenvalue of Q_1. Defining a Lyapunov function

$$V_1(\tilde{e}_1) = \tilde{e}_1^T P_1 \tilde{e}_1 \quad (23)$$

and evaluating $\dot{V}_1(\tilde{e}_1)$ along (15)

$$\begin{aligned}
\dot{V}_1(\tilde{e}_1) &= \tilde{e}_1^T\left(D_1^T P_1 + P_1 D_1\right)\tilde{e}_1 + 2\tilde{e}_1^T P_1 E_1 \ddot{d}_1 && (24)\\
&\leq -\tilde{e}_1^T Q_1 \tilde{e}_1 + 2\|P_1E_1\|\|\tilde{e}_1\|\mu && (25)\\
&\leq -\lambda_m\|\tilde{e}_1\|^2 + 2\|P_1E_1\|\|\tilde{e}_1\|\mu && (26)\\
&\leq -\|\tilde{e}_1\|\left(\lambda_m\|\tilde{e}_1\| - 2\|P_1E_1\|\mu\right). && (27)
\end{aligned}$$

Therefore, after a sufficiently long time, the norm of the estimation error is bounded by

$$\|\tilde{e}_1\| \leq \lambda_1 \quad (28)$$

where

$$\lambda_1 = \frac{2\|P_1E_1\|\mu}{\lambda_m}. \quad (29)$$

Next the bound on $|\sigma^*|$ is found. Using (10) and (8) in (21)

$$\dot{\sigma}^* = -k_l\sigma^* - k_s\,\mathrm{sgn}(\sigma^*) + (c_1 + l_{11})\tilde{d}_1 + \dot{\hat{d}}_1. \quad (30)$$

From (9)

$$\begin{aligned}
\dot{\sigma}^* &= -k_l\sigma^* - k_s\,\mathrm{sgn}(\sigma^*) + (c_1 + l_{11})\tilde{d}_1 + \dot{d}_1 - \dot{\tilde{d}}_1 && (31)\\
\sigma^*\dot{\sigma}^* &\leq -k_l{\sigma^*}^2 - k_s|\sigma^*| + (c_1 + l_{11} + 1)\|\tilde{e}_1\||\sigma^*| + \mu|\sigma^*| && (32)
\end{aligned}$$

using the bound on $\|\tilde{e}_1\|$ found in (28)

$$\begin{aligned}
\sigma^*\dot{\sigma}^* &\leq -k_l{\sigma^*}^2 + \left[\lambda_1(c_1 + l_{11} + 1) + \mu - k_s\right]|\sigma^*| && (33)\\
&\leq -|\sigma^*|\left[k_l|\sigma^*| - (\lambda_1(c_1 + l_{11} + 1) - \mu + k_s)\right]. && (34)
\end{aligned}$$

It is easy to see that after a sufficiently long time $|\sigma^*|$ is bounded such that

$$|\sigma^*| \leq \lambda_2 \quad (35)$$

$$\lambda_2 = \frac{\lambda_1(c_1 + l_{11} + 1) + \mu - k_s}{k_l}. \quad (36)$$

It is worth noting that λ_2 can be lowered by increasing k_l. The term $k_l\sigma^*$ in control u (20) does not increase greatly on account of an increase in k_l because σ^* is held close to 0 for all time t. Specifically at $t = 0$, $\sigma^* = 0$ by the choice of sliding surface (17) and therefore at $t = 0$ the control does not increase at all.

Working on similar lines, the ultimate bound on the output $|y|$ can be calculated. It works out to

$$|y| \leq \lambda_3 \quad (37)$$

$$\lambda_3 = \frac{\lambda_1 + \lambda_2}{c_1}. \quad (38)$$

Thus from (29), (36) and (38) it can be concluded that the norm of the estimation error $\|\tilde{e}_1\|$, magnitude of sliding variable $|\sigma^*|$ and the magnitude of the output $|y|$ are ultimately bounded and the bounds can be lowered by appropriate choice of control parameters k_l, k_s, l_{11} and l_{12}. Thus the practical stability of the system is proved in the sense of [23].

V. Generalization to Systems of Order n

A. *Generalized Plant*

In this section, the results developed for a second order plant are generalized to an nth order plant given by

$$\begin{aligned}
\dot{x}_1 &= x_2 + d_1(x,t)\\
\dot{x}_2 &= x_3 + d_2(x,t)\\
&\vdots\\
\dot{x}_{n-1} &= x_n + d_{n-1}(x,t)\\
\dot{x}_n &= a(x) + b(x)u + d_n(x,u,t)\\
y &= x_1
\end{aligned} \quad (39)$$

where $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are input and output signals respectively and $a(x)$, $b(x)$ are smooth nominal functions. The system has disturbances in all channels. The disturbances $d_i(x,t)$ for $i = 1, 2, \ldots, n-1$ are unmatched, while the disturbance $d_n(x,u,t)$ is matched. The disturbances may contain external unmeasurable and/or state dependent disturbances, uncertainties and nonlinearities.

Assumption 2: The disturbances $d_i(x,t)$ are continuous and satisfies

$$\left|\frac{d^j d_i(x,t)}{dt^j}\right| \leq \mu_i \text{ for } i = 1, 2, \ldots, n; j = 0, 1, \ldots, r. \quad (40)$$

B. Generalization of Novel Sliding Surface

For a system of order greater than 2, the novel sliding surface of [1] needs to be augmented by including the estimates of the unmatched disturbances as well as their derivatives to compensate for the effects of unmatched uncertainties on the output y. It is easy to see that when $n = 3$ in system (39), with a conventional sliding surface

$$\bar{\sigma} = c_1 x_1 + c_2 x_2 + x_3. \tag{41}$$

The dynamics in sliding mode will be governed by

$$\ddot{x}_1 + c_2 \dot{x}_1 + c_1 x_1 = c_2 d_1 + \dot{d}_1 + d_2 \tag{42}$$

which suggests a sliding surface

$$\sigma = \sum_{i=1}^{3} c_i x_i + c_2 \hat{d}_1 + \hat{\dot{d}}_1 + \hat{d}_2. \tag{43}$$

Following generalization of the sliding surface can be worked out:

$$\sigma = \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} c_{j+1} \hat{d}_i^{(j-i)}, \; c_n = 1. \tag{44}$$

The modified sliding surface $\sigma^* = \sigma - \sigma(0) e^{-\alpha t}$ is as in (17).

Remark 2: The generalization clearly shows that the extended disturbance observer is not only desirable but necessary for a general higher order plant.

C. Extension of Disturbance Observer

A second order DO, developed in Section III-A, is extended in this section.

The disturbance $d_1(x, t)$ and its derivatives can be estimated by

$$\hat{d}_1^{(j-1)} = p_{1j} + l_{1j} x_1 \tag{45}$$

where auxiliary variables are defined as

$$\begin{aligned} \dot{p}_{1j} &= -l_{1j}(x_2 + \hat{d}_1) + \hat{d}_1^{(j)}, \quad j = 1, 2, \ldots, (r-1) \\ \dot{p}_{1r} &= -l_{1r}(x_2 + \hat{d}_1). \end{aligned} \tag{46}$$

To estimate the disturbance $d_i(x, t)$ and its derivatives in the ith channel, define

$$\hat{d}_i^{(j-1)} = p_{ij} + l_{ij} x_i \quad i = 1, 2, \ldots, (n-1) \tag{47}$$

where auxiliary variables are defined as

$$\begin{aligned} \dot{p}_{ij} &= -l_{ij}(x_{i+1} + \hat{d}_i) + \hat{d}_i^{(j)}, \quad j = 1, 2, \ldots, (r-1) \\ \dot{p}_{ir} &= -l_{ir}(x_{i+1} + \hat{d}_i). \end{aligned} \tag{48}$$

The disturbance $d_n(x, u, t)$ and its derivatives can be estimated as

$$\hat{d}_n^{(j-1)} = p_{nj} + l_{nj} x_n \tag{49}$$

where auxiliary variables are defined as

$$\dot{p}_{nj} = -l_{nj} \left(a(x) + b(x) u + \hat{d}_n \right) + \hat{d}_n^{(j)}$$

where

$$\begin{aligned} j &= 1, 2, \ldots, (r-1) \\ \dot{p}_{nr} &= -l_{nr} \left(a(x) + b(x) u + \hat{d}_n \right). \end{aligned} \tag{50}$$

D. Design of Control

The control is designed on lines similar to that in Section III-C. It works out to

$$u = -\frac{1}{b(x)} \left[a(x) + \hat{d}_n + \alpha \sigma(0) e^{-\alpha t} + \sum_{i=1}^{n-1} c_i (x_{i+1} + \hat{d}_i) + k_l \sigma^* + k_s \, \mathrm{sat} \sigma^* \right] \tag{51}$$

where

$$\mathrm{sat}\, \sigma^* = \begin{cases} \mathrm{sgn}\, \sigma^* & \text{if } |\sigma^*| > \epsilon \\ \frac{\sigma^*}{\epsilon} & \text{if } |\sigma^*| \le \epsilon \end{cases}$$

where ϵ is a small positive number. It may be noted, that unlike in (20) where the sgn function is used, a sat function is used in (51). A sgn function with a small amplitude alleviates the chatter but the sat function eliminates the chatter completely. Therefore, in this general case the sat function is used.

E. Stability

The gains l_{ij} of the DO for the disturbances d_i can always be chosen so that the eigenvalues of each D_i are in LHP. The observer error dynamics can be expressed in compact form on the lines of (15) and (16)

$$\dot{\tilde{e}}_i = D_i \tilde{e}_i + E_i d_i^{(r)} \tag{52}$$

$$D_i = \begin{bmatrix} -l_{i1} & 1 & 0 & \ldots & 0 \\ -l_{i2} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -l_{i(r-1)} & 0 & 0 & \ldots & 1 \\ -l_{ir} & 0 & 0 & \ldots & 0 \end{bmatrix}; E_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{53}$$

where $\tilde{e}_i = [\tilde{d}_i \; \tilde{\dot{d}}_i \; \ldots \; \tilde{d}_i^{(r)}]^T$ for $i = 1, 2, \ldots, n$.

Therefore, it is always possible to find a positive definite matrix P_i such that

$$D_i^T P_i + P_i D_i = -Q_i \tag{54}$$

for a given positive definite matrix Q_i. Let λ_{mi} denote the smallest eigenvalue of Q_i.

Defining a Lyapunov function

$$V(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n) = \sum_{i=1}^{n} \tilde{e}_i^T P_i \tilde{e}_i \tag{55}$$

and working on the lines of Section IV, it is easy to show that the bound on $\|\tilde{e}_i\|$ is

$$\|\tilde{e}_i\| \le \frac{2\|P_i E_i\|\mu_i}{\lambda_{mi}}. \tag{56}$$

Let

$$\lambda_1 = \max\left[\frac{2\|P_i E_i\|\mu_i}{\lambda_{mi}}\right] \quad \text{for all } i \tag{57}$$

therefore $\|\tilde{e}_i\| \le \lambda_1$ for all i.

Next to find the bound on σ^*, using (51) and after some simplification

$$\begin{aligned}\dot{\sigma}^* &= -k_l\sigma^* - k_s \operatorname{sat}(\sigma^*) + \sum_{i=1}^{n-1} c_i \tilde{d}_i \\ &\quad + \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\left[c_{j+1} l_{i(j-i+1)} \tilde{d}_i + c_{j+1}\hat{d}_i^{(j-i+1)}\right] \\ &= -k_l\sigma^* - k_s \operatorname{sat}\sigma^* + \sum_{i=1}^{n-1}\sum_{j=i}^{n-1} c_{j+1}\hat{d}_i^{(j-i+1)} \\ &\quad + \sum_{i=1}^{n-1}\left[c_i + \sum_{j=i}^{n-1} c_{j+1} l_{i(j-i+1)}\right]\tilde{d}_i. \end{aligned} \tag{58}$$

Now it is straightforward to find the bound on σ^*. The actual calculation is tedious and therefore omitted. The result is qualitatively similar to that obtained in (36).

VI. Simulation Example

A. Example-1

Consider the following illustrative example given in [1]:

$$\dot{x}_1 = x_2 + d_1(x,t) \tag{59a}$$

$$\dot{x}_2 = -2x_1 - x_2 + e^{x_1} + u \tag{59b}$$

$$y = x_1. \tag{59c}$$

1) Chatter Alleviation: Here, the Case 2 in [1] is compared with the control proposed in this paper. For this illustration, a constant disturbance $d = 0.5$ is applied after 2, $x(0) = [0\ 0]$ as in [1]. For the proposed control $k_l = 10$ and $k_s = 0$, for ensuring the absence of chatter. It can be seen from Fig. 1(a) and (b) that the performance is almost the same for the two controls but the proposed control is absolutely free from chatter. The results in [1] are referred to as the results with switching gain while those obtained with the proposed control are referred to as the results with linear gain.

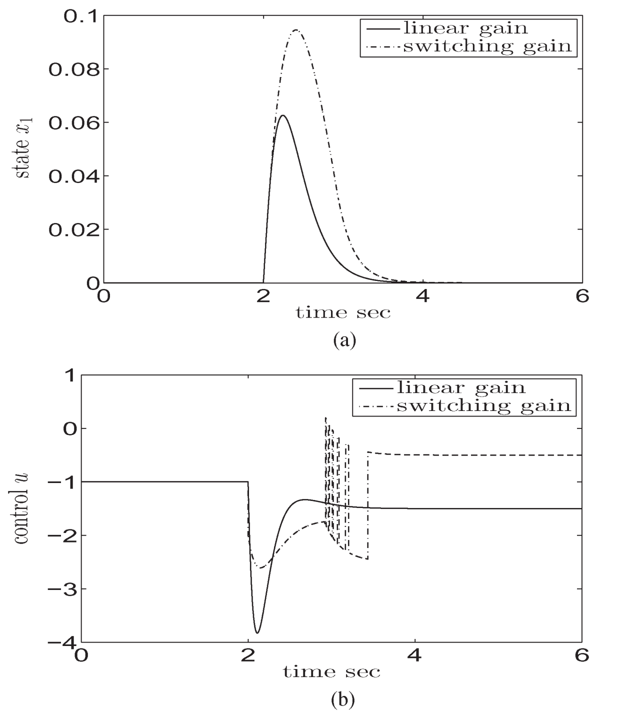

Fig. 1. Comparison between linear and switching gains for first order DO. (a) Plot of x_1; (b) plot of u.

2) Complex Disturbance: In this case, complex unmatched disturbance is considered. Unmatched disturbance

$$\begin{aligned} d_1(x,t) &= \frac{1}{2}x_1^2 + x_1\sin(2t) + 0.2x_2 + \frac{t}{6} + \sin^2(2t) \\ &\quad - \cos(2t) \quad \text{for } t < 2 \end{aligned} \tag{60}$$

$$\begin{aligned} d_1(x,t) &= \frac{1}{2}x_1^2 + x_1\sin(2t) + 0.2x_2 + \frac{t}{6} + \sin^2(2t) \\ &\quad - \cos(2t) + 1 \quad \text{for } t \ge 2 \end{aligned} \tag{61}$$

with a step change after 2 seconds, is applied. The initial condition is taken as $x(0) = [1\ 0]$. Controller parameters are set to $k_l = 50$ and $k_s = 0$, and second order DO with gains $l_{11} = 100$ and $l_{12} = 20$. The performance of the proposed controller is compared with the controller in [1] for which a switching gain of 4 and a first order DO having a pole at -10 is selected. The comparative performance is shown in Fig. 2. It can be seen that the proposed control is free from chatter and gives a better estimation of the disturbance and a better performance. It may be noted that the control in [1] is not designed for the disturbance of this type.

B. Example-2, Flexible Joint Manipulator

A single link manipulator, with the elasticity of the joint represented by a linear torsional spring with stiffness K is considered in this example. The equations of motion are given by [24]

$$\begin{aligned} I\ddot{q}_1 + MgL\sin(q_1) + K(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - K(q_1 - q_2) &= u \end{aligned} \tag{62}$$

where, I is the link inertia, J is the motor inertia, K is the spring stiffness constant, u is the input toque, M is the mass of flexible link and L is the effective length of link.

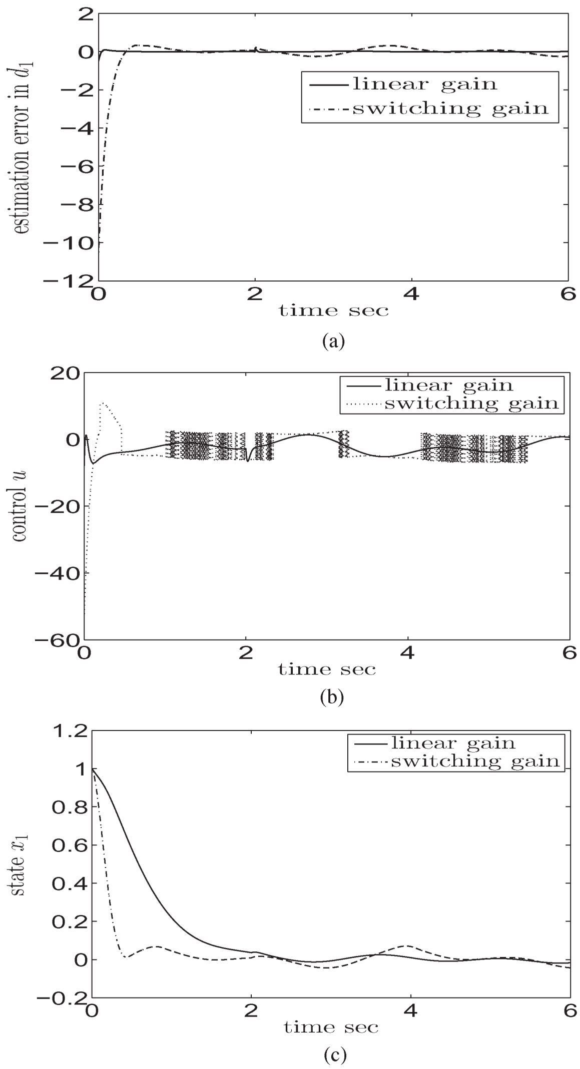

Fig. 2. Comparison of performance between linear and switching gains in the presence of complex unmatched disturbances. (a) Plot of $\tilde{d}_1$. (b) Plot of u. (c) Plot of x_1.

In state space the form of flexible joint is given as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{k_{s1}}{J_{11}} x_3' - \frac{k_{s1}}{J_{11}} x_1 - \frac{B_{11}}{J_{11}} x_2 - \sin(x_1) \pm \Delta \frac{k_{s1}}{J_{11}} x_1 \\
&\quad \pm \Delta \frac{k_{s1}}{J_{11}} x_3' \pm \Delta \frac{B_{11}}{J_{11}} x_2 \\
\dot{x}_3' &= x_4' \\
\dot{x}_4 &= \frac{k_{s1}}{J_{12}} x_1 - \frac{k_{s1}}{J_{12}} x_3' - \frac{B_{12}}{J_{11}} x_4' + \frac{K_{t1}}{J_{11}} (u+v) \pm \Delta \frac{k_{s1}}{J_{12}} x_1 \\
&\quad \pm \Delta \frac{k_{s1}}{J_{12}} x_3' \pm \Delta \frac{B_{12}}{J_{11}} x_4' \pm \Delta \frac{K_{t1}}{J_{11}} (u+v)
\end{aligned} \tag{63}$$

where system states $x = [x_1, x_2, x_3', x_4']$ are link position, link velocity, motor shaft position, motor shaft velocity respectively u is the control input to motor, v is the external unmeasurable disturbance. Table I shows the nominal parameter values used for simulation. In simulation for all cases, uncertainties up to $\pm 50\%$ over the nominal system parameters are considered.

TABLE I
NOMINAL PARAMETERS OF FLEXIBLE JOINT SYSTEM

Parameter	Description	Nominal value	Unit
K_{t1}	drive torque constant	0.119	N.m/A
J_{11}	actuated transition equivalent moment of inertia	931E-6	kg.m^2
J_{12}	load transition equivalent moment of inertia	0.230	kg.m^2
B_{11}	actuated transition equivalent viscous damping coefficient	4.5	N.m.s/rad
B_{12}	load transition equivalent viscous damping coefficient	0.07	N.m.s/rad
K_{s1}	torsional stiffness constant	9	N.m/rad

After working on (63), one can simplify the nonlinear state space form as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + d_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= a(x) + b(x)u + d_4 \\
y &= x_1
\end{aligned} \tag{64}$$

where parameters are calculated as $k_1 = k_{s1}/J_{11}$, $x_3 = k_1 x_3'$, $b_1 = B_{11}/J_{11}$, $k_2 = k_{s1}/J_{12}$, $b_2 = B_{12}/J_{11}$, $b(x) = K_{t1}/J_{11}$, $a(x) = k_2 x_1 - k_2 x_3 - b_2 x_4$, $x_4 = k_1 x_4'$. The lumped uncertainties in the second channel and the fourth channel are given by $d_2 = -k_1 x_1 - b_1 x_2 - \sin(x_1) \pm \Delta k_1 x_1 \pm \Delta k_1 x_3 \pm \Delta b_1 x_2$ and $d_4 = \pm \Delta k_2 x_1 \pm \Delta k_2 x_3 \pm \Delta b_4 x_4 \pm \Delta b(x) u + (b(x) + \Delta b(x)) v$, respectively.

The modified sliding surface σ^*, sliding surface σ and the control input u are designed using (17), (44) and (51). In all the simulations carried out following was kept unchanged: The external disturbance $v = 1 + \sin 2t$, the sliding surface coefficients $c_1 = 125$, $c_2 = 75$, $c_3 = 15$, $c_4 = 1$, $\alpha = 1$ and the initial state $x(0) = [0.1\ 0\ 0\ 0]^T$.

1) Effect of Extended DO: Here, the effect of order of DO in improving the accuracy is illustrated. The control parameters that are held constant for all the cases are $k_l = 10$, $k_s = 0$. The two cases considered are: (a) second order DO with (observer pole location is -10) $l_{21} = l_{41} = 100$, $l_{22} = l_{42} = 20$ (b) third order DO with (observer pole location is -10) $l_{21} = l_{41} = 1000$, $l_{22} = l_{42} = 300$, $l_{23} = l_{43} = 30$. It can be seen from the results shown in Fig. 3(a) that the estimation accuracy is improved with third order DO and that it can be improved further by appropriate choice of l_{21}, l_{22}, l_{23} and l_{41}, l_{42}, l_{43}. Interestingly, as seen from Fig. 3(c) the control has not changed much in magnitude. The estimation error $|\tilde{d}_2|$ is ultimately bounded by 0.01 and 0.001 for the cases (a), (b). The response of the modified sliding variable σ^* is shown in Fig. 4(a), while the plot of the output $y = x_1$ is shown in Fig. 4(b).

2) Effect of Linear Gain: Next, the effect of the linear gain k_l in bringing down the bound on σ^* is illustrated in Fig. 6(a). For the three values of $k_l = 10$, 50 and 100 considered, the bound on σ^* is seen to be 0.185, 0.049, and 0.025, respectively. Fig. 5(c) shows that the control increases only marginally as k_l goes from 10 to 100. Second order DO was employed for both the channels with gains set to $l_{21} = l_{41} = 100$, $l_{22} = l_{42} = 20$. The plot of the estimation errors $\tilde{d}_2$ and $\tilde{d}_4$ is shown in Fig. 5(a) and (b), respectively, while the plot of the output is shown in Fig. 6(b). It can be noted that the change in k_l does not affect the estimation error $\tilde{d}_2$ and $\tilde{d}_4$ at all as is to be expected.

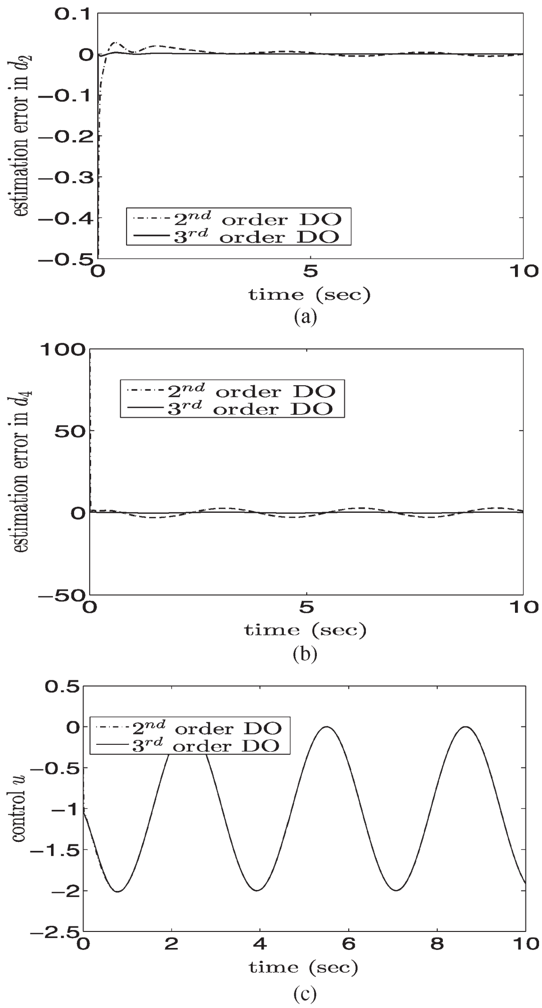

Fig. 3. Effect of extended DO on estimation accuracy and control. (a) Plot of $\tilde{d}_2$. (b) Plot of $\tilde{d}_4$. (c) Control.

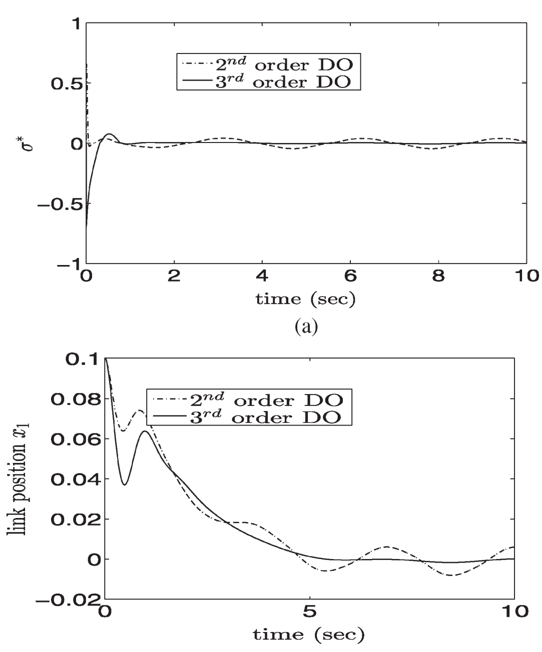

Fig. 4. Effect of order of DO on σ^* and system output. (a) Modified sliding variable. (b) Output.

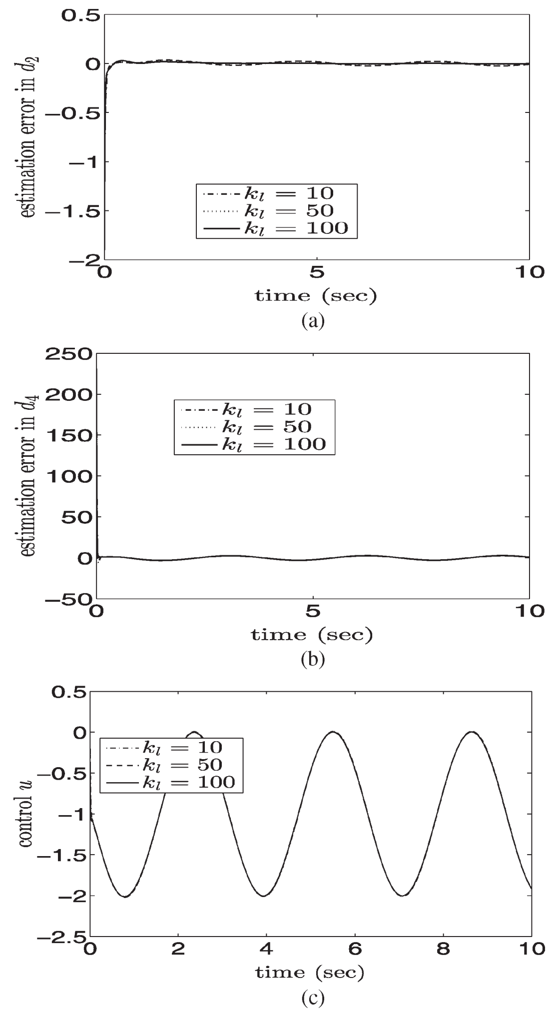

Fig. 5. Effect of linear gain k_l on controller. (a) Estimation error $\tilde{d}_2$. (b) Estimation error $\tilde{d}_4$. (c) Control.

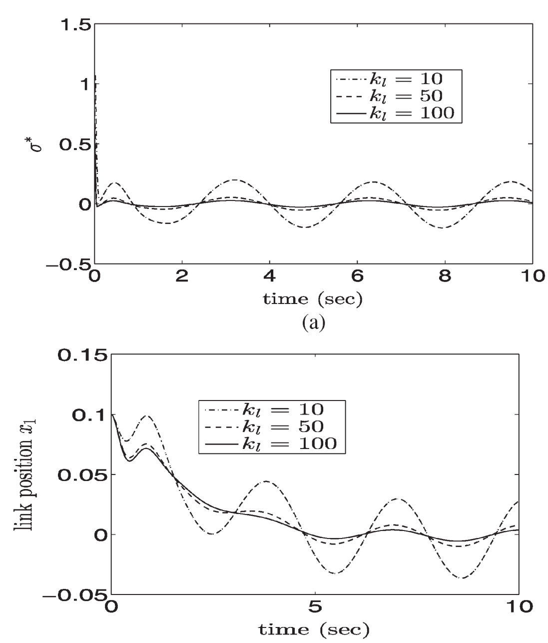

Fig. 6. Effect of linear gain k_l on system performance. (a) Modified sliding variable. (b) Output.

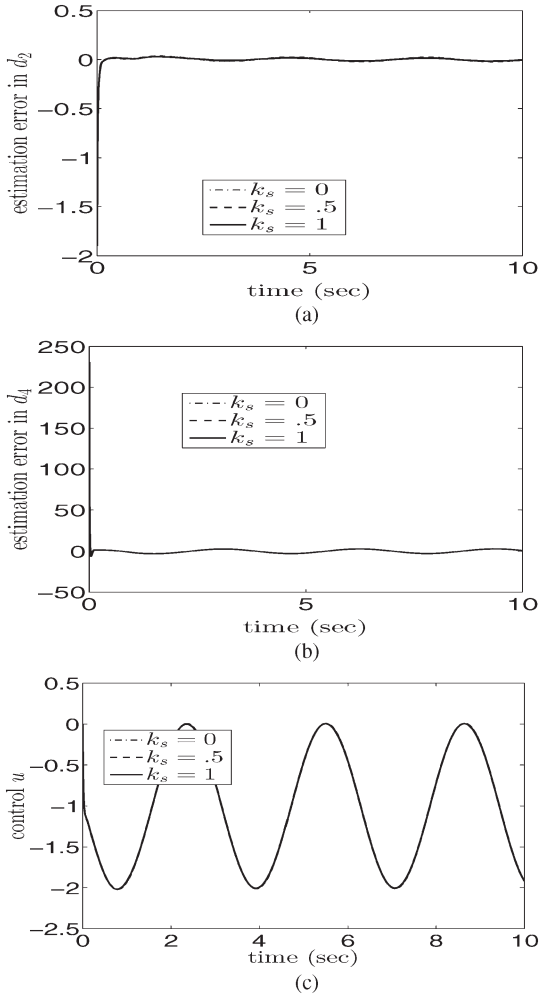

Fig. 7. Effect of switching gain k_s on controller. (a) Estimation error $\tilde{d}_2$. (b) Estimation error $\tilde{d}_4$. (c) Control.

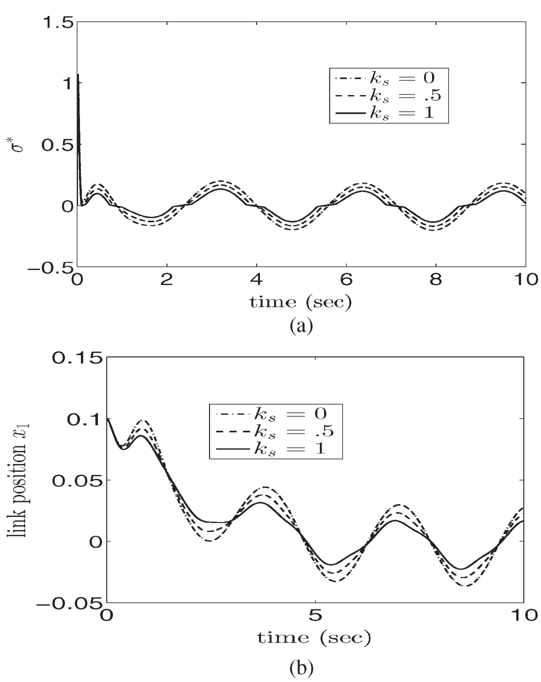

Fig. 8. Effect of switching gain k_s on system performance. (a) Modified sliding variable. (b) Output.

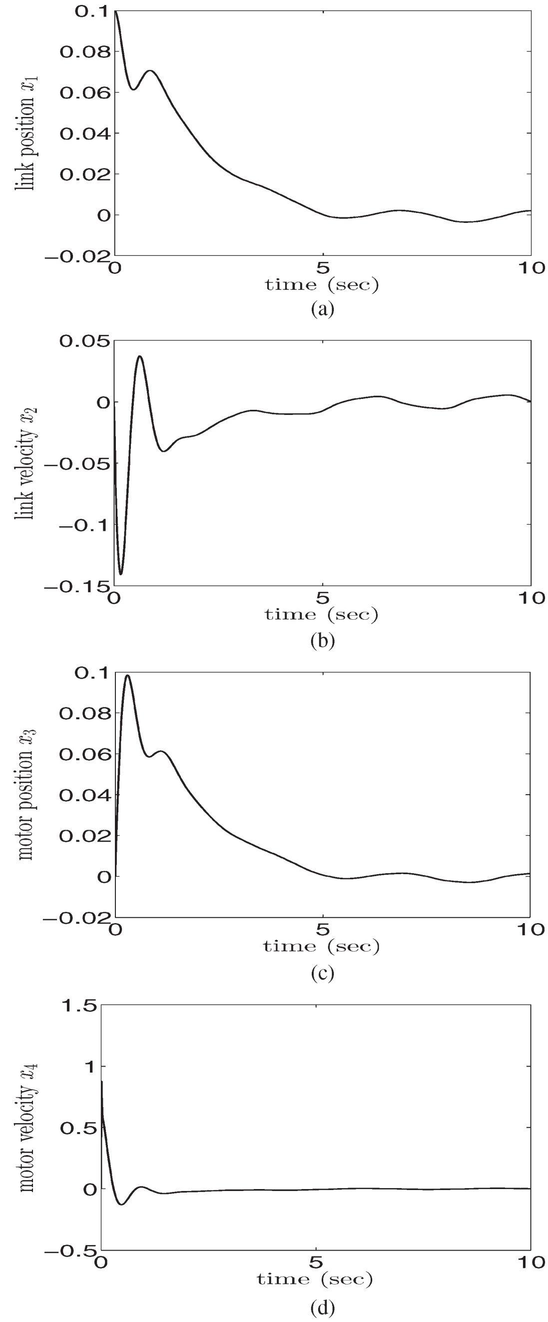

Fig. 9. States of the flexible joint. (a) Link position x_1. (b) Link velocity x_2. (c) Motor shaft position x_3. (d) Motor shaft velocity x_4.

3) Effect of Switching Gain: Next, the effect of the switching gain with usual continuous approximation inside a boundary layer is considered. The control parameter k_l was held at 10 while the switching gain k_s was taken as 0, 0.5 and 1. The other control parameters are $l_{21} = l_{41} = 100$, $l_{22} = l_{42} = 20$. The plots of the control u and σ^* in Figs. 7(c) and 8(a) show that the switching gain k_s enables further reduction in the bound on σ^*. The plot of the estimation error $\tilde{d}_1$ is shown in Fig. 7(a), while the plot of the output is shown in Fig. 8(b). It may be noted that the change in k_s does not affect the estimation error at all.

Next, taking $k_l = 50$ and $k_s = 0.5$, Fig. 9 shows all the four states of flexible joint plant. Fig. 10(a) and (b) shows the comparative plot of disturbances in second (mismatched) and

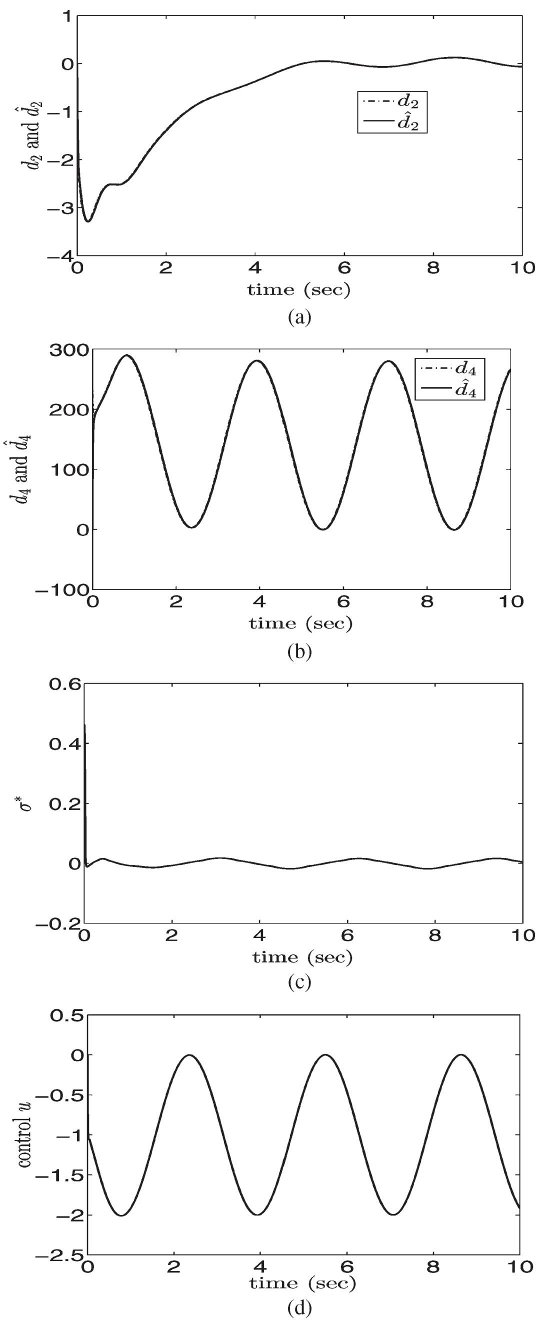

Fig. 10. Disturbance estimates, σ^* and control. (a) Comparitive plot of d_2 and $\hat{d}_2$. (b) Comparitive plot of d_4 and $\hat{d}_4$. (c) σ^*. (d) Control.

fourth channel (matched) and their estimation. Fig. 10(c) shows the plot of σ^*, while Fig. 10(d) shows the plot of control input to the plant.

VII. Discussion

The bound λ_1 on the estimation error $\tilde{d}_2$ given by (29) can be lowered by the choice of l_{11}, l_{12} and/or the order of the disturbance observer alone. Figs. 3(a), 5(a) and 7(a) illustrate this clearly. The bound λ_2 on the sliding variable σ^* can be lowered by reducing λ_1, increasing the switching gain k_s or increasing the linear gain k_l. If k_s is increased, it increases the propensity to chatter. The linear gain k_l introduced in this paper, coupled with the proposed modification of the sliding surface (17) causes only a marginal increase in control as illustrated in Fig. 5(c). The effects of k_l and k_s are complementary and a judicious combination of k_s and k_l can enable improved performance with better chatter alleviation than that is possible with k_s alone. Decreasing the effect of the unmatched disturbance on the output y requires lowering of both λ_1 and λ_2. It is not desirable to increase c_1 for lowering λ_3 because it will result in an increase in the magnitude of control.

VIII. Conclusion

This paper extends the result in [1] for a larger class of unmatched disturbances by proposing an extended disturbance observer, a modified sliding surface and a modified control and generalizing the results to systems of order n. The novel sliding surface is also generalized for higher order systems. It is proved that the ultimate boundedness of the sliding variable, disturbance estimation error and the output is guaranteed and that the bounds can be lowered by the choice of design parameters without giving rise to excessive control magnitudes. It is possible to eliminate the chatter completely or alleviate it greatly. The theoretically expected results were verified by simulations.

References

[1] J. Yang, S. Li, and X. Yu, "Sliding-mode control for systems with mismatched uncertainties via a disturbance observer," *IEEE Trans. Ind. Electron.*, vol. 60, pp. 160–169, Jan. 2013.

[2] V. I. Utkin, *Sliding Mode in Control and Optimization*. New York, NY, USA: Springer-Verlag, 1992.

[3] X. Yu and O. Kaynak, "Sliding-mode control with soft computing: A survey," *IEEE Trans. Ind. Electron.*, vol. 56, no. 9, pp. 3275–3285, Sep. 2009.

[4] W. R. Errouissi, M. Ouhrouche, and A. M. Trzynadlowski, "Robust nonlinear predictive controller for permanent-magnet synchronous motors with an optimized cost function," *IEEE Trans. Ind. Electron.*, vol. 59, no. 7, pp. 2849–2858, Jul. 2012.

[5] H. Liu and S. Li, "Speed control for PMSM servo system using predictive functional control and extended state observer," *IEEE Trans. Ind. Electron.*, vol. 59, no. 2, pp. 1171–1183, Feb. 2012.

[6] D. Chwa, J. Y. Choi, and J. H. Seo, "Compensation of actuator dynamics in nonlinear missile control," *IEEE Trans. Control Syst. Technol.*, vol. 12, no. 4, pp. 620–626, Jul. 2004.

[7] B. Drazenovich, "The invariance conditions in variable structure systems," *Automatica*, vol. 5, no. 3, pp. 287–295, May 1969.

[8] K. S. Kim, Y. Park, and S. H. Oh, "Designing robust sliding hyperplanes for parametric uncertain systems: A Riccati approach," *Automatica*, vol. 36, no. 7, pp. 1041–1048, Jul. 2000.

[9] C. C. Wen and C. C. Cheng, "Design of sliding surface for mismatched uncertain systems to achieve asymptotical stability," *J. Franklin Inst.*, vol. 345, no. 8, pp. 926–941, Nov. 2008.

[10] S. Wang, D. Yu, and D. Yu, "Compensation for unmatched uncertainty with adaptive RBF network," *Int. J. Eng. Sci. Technol.*, vol. 3, no. 6, pp. 35–43, 2011.

[11] C. Tao, M. Chan, and T. Lee, "Adaptive fuzzy sliding mode controller for linear systems with mismatched time-varying uncertainties," *IEEE Trans. Syst. Man Cybern. B, Cybern.*, vol. 33, no. 2, pp. 283–294, Apr. 2003.

[12] J. Zhang, P. Shi, and Y. Xia, "Robust adaptive sliding-mode control for fuzzy systems with mismatched uncertainties," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 4, pp. 700–711, Aug. 2010.

[13] H. Choi, "LMI-based sliding surface design for integral sliding mode control of mismatched uncertain systems," *IEEE Trans. Autom. Control*, vol. 52, no. 4, pp. 736–742, Apr. 2007.

[14] Y.-W. Liang, L.-W. Ting, and L.-G. Lin, "Study of reliable control via an integral-type sliding mode control scheme," *IEEE Trans. Ind. Electron.*, vol. 59, no. 8, pp. 3062–3068, Aug. 2012.

[15] W. H. Chen, "Nonlinear disturbance observer-enhanced dynamic inversion control of missiles," *J. Guidance, Control Dynam.*, vol. 26, no. 1, pp. 161–166, Jan./Feb. 2003.

[16] Y. S. Lu, "Sliding-mode disturbance observer with switching-gain adaptation and its application to optical disk drives," *IEEE Trans. Ind. Electron.*, vol. 56, no. 9, pp. 3743–3750, Sep. 2009.
[17] J. Yang, W. H. Chen, and S. Li, "Non-linear disturbance-observer based robust control for systems with mismatched disturbances/uncertainties," *IET Control Theory Appl.*, vol. 5, no. 18, pp. 2053–2062, Dec. 2011.
[18] J. Yang, S. Li, and W. H. Chen, "Nonlinear disturbance observer-based control for multi-input multi-output nonlinear systems subject to mismatching condition," *Int. J. Control*, vol. 85, no. 8, pp. 1071–1082, Aug. 2012.
[19] S. Li, J. Yang, and W. H. Chen, "Generalized extended state observer based control for systems with mismatched uncertainties," *IEEE Trans. Ind. Electron.*, vol. 59, no. 12, pp. 4792–4802, Dec. 2012.
[20] J. Yang, A. Zolotas, W. H. Chen, K. Michail, and S. Li, "Robust control of nonlinear MAGLEV suspension system with mismatched uncertainties via DOBC approach," *ISA Trans.*, vol. 50, no. 3, pp. 389–396, Jul. 2011.
[21] S. T. Wu, "Remote vibration control for flexible beams subject to harmonic disturbances," *ASME J. Dyn. Syst. Meas. Control*, vol. 126, no. 1, pp. 198–201, Apr. 2004.
[22] K. S. Kim, K. H. Rew, and S. Kim, "Disturbance observer for estimating higher-order disturbances in time series expansion," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1905–1911, Aug. 2010.
[23] M. Corless and G. Leitman, "Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems," *IEEE Trans. Autom. Control*, vol. AC-26, no. 5, pp. 1139–1144, Oct. 1981.
[24] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*. Hoboken, NJ, USA: Wiley, 1989.

Divyesh Ginoya received the B.E. degree in biomedical engineering from the Gujarat University, Gujarat, India, in 2008 and the M.Tech. degree in instrumentation and control engineering from the College of Engineering Pune, Pune, India, in 2011. He is currently working toward the Ph.D. degree at the College of Engineering Pune, Pune, India. His main research interests are in the fields of sliding mode control, uncertainty and disturbance observer, state and disturbance observer, nonlinear control.

P. D. Shendge (M'12) received the B.E. and M.E. degrees in instrumentation, and the Ph.D. degrees from the SGGS College of Engineering and Technology, Nanded, India in 1991, 1993, and 2007, respectively.

From 1995 to 2000, he was a Lecturer and in 2001, he became Senior Lecturer in the College of Engineering Pune, Pune, India. Currently he is Associate Professor in Instrumentation and Control Department with College of Engineering Pune from 2008. He is the author of about 38 papers, including 10 journal papers, one book chapter. His research interests include the sliding mode control, uncertainty and disturbance estimation and robust control.

S. B. Phadke was born in Pune, India, in 1949. He received the B.E. and M.E. degrees in electrical engineering from the College of Engineering, Pune, Pune, India.

He joined the Department of Aerospace Engineering, Defense Institute of Advanced Technology, Pune, where he taught control systems and missile control for over 30 years and retired as Scientist "G" in 2009. He had a brief stint in the Naval Science and Technological Laboratory, Visakhapatnam, India. He is currently Professor Emeritus at the College of Engineering Pune, Pune, India.

He has published over 40 papers in international journals and conferences and coauthored a chapter in the book Advances in Industrial Engineering and Operations Research (Springer-Verlag, 2008). He has worked extensively in sliding mode control, missile control, and missile guidance. His current interests are estimation of uncertainties, active suspension and deterministic control of uncertain systems.