Pseudo-realistic simulation of the hair

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Abstract

In this brief report, we are going to outline the main aspects of our work related to hair simulation. We are mainly going to outline the mathematics invloved and then dwelve into the rendering aspects of the hair.

1 Mathematical Formalism

Here, we are going to outline the mathematics and the time discretization for a single hair strand. The simulation concerning multiple hair strands we will dealt with in the other section.

1.1 Model

We consider an inextensible rod of length L. Let $s \in [0, L]$ be the distance along the rod. The hair is described by the *centerline*, $\mathbf{r(s,t)}$, which is a curve passing through the center of mass of every cross-section. We attach a material frame $(\mathbf{n}_i(s))_{i=0,1,2}$, which are orthonormal for all s. The hair's material curvatures $(\kappa_\alpha(s,t))_{\alpha=1,2}$ are with respect to two directions of the cross-section of the hair and the twist $\kappa_0(s,t)$, is defined with respect to the tangent vector. Together, they describe the Darboux vector

$$\mathbf{\Omega}(s,t) = \kappa_0(s,t)\ \mathbf{n_0}(s,t) + \kappa_1(s,t)\ \mathbf{n_1}(s,t) + \kappa_2(s,t)\ \mathbf{n_2}(s,t) \tag{1}$$

For the hair, the degrees of freedom are its material curvatures and the twist. For purposes of simulation, we have to discretize the space. So, we divide the hair strand $s \in [0, L]$ into N segments S_Q, where $1 \leq Q \leq N$. Then, we define the material curvatures and the twist with piecewise constant functions over these segments. The explicit formulas for material curvatures and the twist reads

$$\kappa_i(s,t) = \sum_{Q=1}^{N} q_{i,Q}(t)\chi_Q(s) \tag{2}$$

where $\chi_Q(s)$ is the characteristic function of segment Q, equal to 1 if $s \in S_Q$ and 0 otherwise. Therefore, for a given segment $\kappa_i(s,t) = q_i(t)$.

Using $q's$, we can reconstruct the hair any instant of time. The explicit formulas for the centerline and the material frame are

$$\mathbf{n}_i(s) = \mathbf{n}_{i,L}^{Q\ ||} + \mathbf{n}_{i,L}^{Q\ \perp}\ cos(\Omega(s - s_L^Q)) + [\omega \times \mathbf{n}_{i,L}^{Q\ \perp}]\ sin(\Omega(s - s_L^Q)) \tag{3}$$

Here ω, equal to $\frac{\mathbf{\Omega}}{\Omega}$, is a unit vector.

$$\mathbf{r}(s) = \mathbf{r}_L^Q + \mathbf{n}_{0,L}^{Q\ ||}\ (s - s_L^Q) + \mathbf{n}_{0,L}^{Q\ \perp} \frac{sin(\Omega(s - s_L^Q))}{\Omega} + [\omega \times \mathbf{n}_{0,L}^{Q\ \perp}] \frac{1 - cos(\Omega(s - s_L^Q))}{\omega} \tag{4}$$

To write down the lagranigan for the system, we first write down the kinetic and the potential energy. The kinetic Energy (K.E.) of the hair is given by

$$T(q, \dot{q}, t) = \frac{1}{2} \int_0^L \rho(\mathbf{r}(\dot{s}, q))^2\ ds \tag{5}$$

For deriving the time integrator, we express the K.E. discrete in time and space

$$T(q^i, q^{i+1}) = \frac{1}{2} \sum_{Q=1}^{N} \sum_{j=1}^{N_Q} \rho \left[\frac{\mathbf{r}(s_L^Q + \frac{L_Q\ j}{N_Q}, q^{i+1}) - \mathbf{r}(s_L^Q + \frac{L_Q\ j}{N_Q}, q^i)}{h} \right]^2 \frac{L_Q}{N_Q} \tag{6}$$

The potential energy (U) contains two terms. One is the gravitational and the other is the internal energy. U can be written as

$$U(q,t) = \frac{1}{2}\int_0^L \sum_{j=0}^{2}(EI)_j[q_j(s,t) - q_j^o]^2 \; ds + \int_0^L \rho \; g \; z(s,t) \; ds \tag{7}$$

Here, EI's are the stiffness involved. g is the acceleration due to gravity and $z(s,t)$ is the z co-ordinate of the centerline. Now, discretizing in sapce and time we get,

$$U(q^i) = \frac{1}{2}\sum_{Q=1}^{N} \frac{L}{N}\Bigg\{\sum_{j=0}^{2}(EI)_j \; [q_j^i - q_j^o]^2\Bigg\} + \sum_{Q=1}^{N} \sum_{j=1}^{N_Q} \rho \; g \; z(s_L^Q + \frac{L_Q \; j}{N_Q}, q^i)\frac{L_Q}{N_Q} \tag{8}$$

Therefore,

$$\begin{aligned} L(q^i, q^{i+1}) = \frac{1}{2}\sum_{Q=1}^{N}\sum_{j=1}^{N_Q}\rho\Bigg[\frac{\mathbf{r}(s_L^Q + \frac{L_Q\; j}{N_Q}, q^{i+1}) - \mathbf{r}(s_L^Q + \frac{L_Q\; j}{N_Q}, q^i)}{h}\Bigg]^2 \; \frac{L_Q}{N_Q} \\ - \frac{1}{2}\sum_{Q=1}^{N} \frac{L}{N}\Bigg\{\sum_{j=0}^{2}(EI)_j \; [q_j^i - q_j^o]^2\Bigg\} \\ - \sum_{Q=1}^{N} \sum_{j=1}^{N_Q} \rho \; g \; z(s_L^Q + \frac{L_Q \; j}{N_Q}, q^i)\frac{L_Q}{N_Q} \end{aligned} \tag{9}$$

To find the time integrator we use an marching algorithm. This basically means that we solve for the equations on each segment successively. Now, performing the $d_1 L(q^i) + d_2 L(q^{i+1})$, the implicit equations of motion that we get is

$$\begin{aligned} -\sum_{j=1}^{N_Q}\rho \; \frac{L_Q}{N_Q}\Bigg[\frac{\mathbf{r}(s_L^Q + \frac{L_Q\; j}{N_Q}, q^{i+1}) - \mathbf{r}(s_L^Q + \frac{L_Q\; j}{N_Q}, q^i)}{h}\Bigg]^T \; d\mathbf{r}(q^i) \\ - (q^i - q^n)^T(EI) - \sum_{j=1}^{N_Q}\rho \; \frac{L_Q}{N_Q} \; g \; dz(q^i) \\ + \sum_{j=1}^{N_Q}\rho \; \frac{L_Q}{N_Q}\Bigg[\frac{\mathbf{r}(s_L^Q + \frac{L_Q\; j}{N_Q}, q^{i+1}) - \mathbf{r}(s_L^Q + \frac{L_Q\; j}{N_Q}, q^i)}{h}\Bigg]^T \; d\mathbf{r}(q^{i+1}) = 0 \end{aligned} \tag{10}$$

Then, we use the newton solver to find q^{i+1} given that we know q^i and q^{i-1}.

To simulate multiple hair strands we use the interpolation method.

References

[1] Bertails et.al.*Super Helices for Predicting the Dynamics of Natural Hair*. ACM Transactions on Graphics (Proceedings of the SIGGRAPH conference), 2006

[2] Bergou et.al. *Discrete Elastic Rods*. ACM Transactions on Graphics (SIGGRAPH), **28**, 2008.