Dynamic Asset Allocation Using Option-Implied Distributions in an Exponentially Tempered Stable Lévy Market

Zachary Polaski

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Abstract:

This paper explores the optimal portfolio problem using option-implied distributions when the underlying price process is assumed to be driven by an exponential Lévy process. In particular, the application is carried out using an Exponentially Tempered Stable jump-diffusion process as the martingale component of the log stock price, and the investor's preferences are assumed subject to a CRRA utility function. One month risk-neutral densities are extracted from option prices by using a transform pricing method and are subsequently transformed to the risk-adjusted, or real-world density via a model preserving minimal entropy transform which importantly maintains the parameterization of the Lévy process. A stochastic optimal control result is then used to construct a portfolio consisting of a risky and risk-free asset which is rebalanced on a monthly basis. It is found that the portfolios formed using option-implied expectations under the Lévy market assumption, which are flexible enough to capture the higher moments of the implied distribution, are far more robust to left-tail market risks and offer statistically significant improvements to risk-adjusted performance when investor risk aversion is low, however this diminishes as risk aversion increases.

Keywords: Lévy Processes, Option Pricing, Stochastic Optimal Control, Portfolio Optimization

Resumo:

Este artigo explora o problema do portfólio ideal usando distribuições implícitas na opção quando o processo de preço subjacente é assumido como sendo conduzido por um processo exponencial de Lévy. Em particular, a aplicação é levada a cabo usando um processo de difusão de salto Estável Exponencialmente Temperado como o componente martingale do preço das acções de log, e as preferências do investidor são assumidas sujeitas a uma função de utilidade CRRA. Densidades de um mês neutras ao risco são extraídas dos preços das opções usando um método de precificação por transformação e são subsequentemente transformadas na densidade ajustada ao risco ou no mundo real por meio de um modelo preservando a entropia mínima que mantém a parametrização do processo Lévy. Um resultado de controle otimizado estocástico é então usado para construir um portfólio que consiste em um ativo de risco e sem risco, que é reequilibrado mensalmente. Descobriu-se que os portfólios formados usando as expectativas implícitas na opção sob a hipótese de mercado Lévy, que são flexíveis o suficiente para capturar os momentos mais altos da distribuição implícita, são muito mais robustos aos riscos de cauda esquerda e oferecem melhorias estatisticamente significativas ao desempenho ajustado ao risco quando a aversão ao risco do investidor é baixa, porém isso diminui à medida que aumenta a aversão ao risco.

Palavras-chave: Processos Lévy, Valorização co de Opções, Controle Ótimo Estocástico, Otimização de Portfólio

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1 Introduction

1.1 Motivations

This paper undertakes a study of the fundamental optimal asset allocation problem in finance and extends the current literature by assessing the (simulated) empirical performance of a dynamic strategy which relies on signals generated from extracting information from option prices under an exponential Lévy market assumption. The paper builds mainly off of the results and methodology of [37], which showed that portfolio performance could be improved by using implied forward-looking rather than historical distributions, and [32] and [58], where a number of results were derived for the optimal control strategy of exponential Lévy processes. Given that the parameterizations of Lévy processes used in financial modeling (including in particular the Exponentially Tempered Stable process used throughout this study) have economic interpretations and are supposed to offer improvements over the more simple classical models, forward-looking estimates of the parameters could reasonably be expected to improve realized portfolio performance within a framework which incorporates this information. The goal is thus to investigate whether a number of rather esoteric results in stochastic processes, option pricing, and optimal stochastic control together have any value in a quite simple and implementable investment strategy. In essence, the objective of the research was to replicate the broader result of [37], but whereas there the authors used a non-parametric approach to extract the implied distributions, here a parametric approach (i.e. an option pricing model) is used to model the distribution implied in option prices.

1.2 Literature Review

The extensive literature regarding multi-period portfolio selection was catalyzed by the seminal papers of Mossin [54], Samuelson [63], and Merton [51] in the 1960s. By applying the ideas of dynamic programming and stochastic optimal control to the portfolio selection problem, with these papers emerged an exchange of fundamental concepts in engineering to navigating the financial decision making process. But perhaps it was the widely known paper of Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities" [8], that truly kicked off the love affair of finance and mathematics. The shortcomings of the Black-Scholes model, however, became all too apparent, particularly in the aftermath of the events of October 19th, 1987, now known as "Black Monday," which saw a permanent regime change in the implied volatility structure that the Black-Scholes model simply could not handle. In [46] much of the current framework for using Lévy processes for price modeling was first developed with the Variance Gamma (VG) process, although the idea of modeling prices with infinite activity and discontinuities had already been considered years earlier in [49], [60], [59], and [9]. Since the Variance Gamma process, a number of additional Lévy processes have been proposed to more realistically capture stylized facts of the markets, perhaps the most popular being the Hyperbolic process of [22], the Normal Inverse Gaussian process of [3], the CGMY process of [12], the Kou process of [38], and the Meixner process of [67]. This study involves the merging of the optimal portfolio selection literature and the literature on Lévy processes in finance within a systematic investment strategy. The problem of optimizing a portfolio with Lévy drivers was first solved using the dynamic programming approach in [6] and [24].

In [32], the problem was also solved using the alternative martingale method approach, and this remains a seminal reference herein due to the clarity of the notation and results (the solutions are identical, as they should be, between the two approaches). As mentioned in the introduction, more recently, in [58], the optimal portfolio problem was studied in more depth numerically for exponential Lévy processes under CRRA utility. However, this paper only considered the problem cross-sectionally, that is it compared optimal portfolios under the jump assumption to the pure diffusion assumption using a single Lévy process parameterization/calibration and studied the implications. We pick up where this study left off, inspired, as mentioned above, by [37] and analyze the problem across time and investigate the performance of a periodically rebalanced strategy.

1.3 Organization of Paper

The rest of the paper is organized as follows. Section 2 is a review of foundational definitions and theorems important in formulating and understanding the results of later sections. Section 3 details the Exponentially Tempered Stable Lévy process, including the application to financial markets and an important result regarding transformations between the risk-neutral and real-world parameterization. Section 4 derives a suitable Hamilton-Jacobi-Bellman (HJB) equation for the optimal investment problem, and is solved assuming the solution for the portfolio proportions lies within the interior of a set of admissible controls and the investor's preferences are subject to the logarithmic and power HARA-type utility functions. Section 5 is a review of the Carr-Madan transform option pricing methodology. Section 6 presents the main results of the paper, where the topics of previous sections are brought together to create and backtest an investable strategy involving a simple portfolio consisting of a risky and risk-free asset rebalanced on a monthly basis.

2 Lévy Processes

2.1 Basic Definitions and Theorems

This chapter presents a number of foundational definitions and theorems regarding Lévy processes. The intent is for this text to be sufficiently self-contained that one need not consult a myriad of external sources. Nevertheless, a full treatment of the theory could never be condensed (nor is it at all the point of the paper) to a few pages, and should the reader desire a more extensive, yet largely introductory treatment, [1], [19], and [66] are recommended. What follows is mainly sourced from these texts. We begin by simply defining a Lévy process.

Definition 2.1 (Lévy Process) *A real valued and adapted stochastic process $L = \{L_t, t \geq 0\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if it possesses the following properties:*
1. $L_0 = 0$ (a.s).
2. L has independent and stationary increments, i.e., $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $s < t$ and $L_t - L_s$ has the same law as L_{t-s}.
3. L is stochastically continuous, i.e., $\forall t \in [0, T], \epsilon > 0 : \lim_{s \to t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$.

More simply, a Lévy process can be thought of as the continuous-time analog of a random walk. Every Lévy process can be associated with an infinitely divisible distribution, defined as follows.

Definition 2.2 (Infinitely Divisible Distribution) *The distribution of a random variable X is infinitely divisible if and only if for all $n \in \mathbb{N}$ there exists a random variable $X^{(1/n)}$ such that*

$$\phi_X(u) = (\phi_{X^{(1/n)}}(u))^n, \tag{2.1}$$

where $\phi_X(u)$ is the characteristic function of X and $\phi_{X^{(1/n)}}$ is the characteristic function of $X^{(1/n)}$. Equivalently, the probability distribution is infinitely divisible if there exists a convolution n-th root for each $n \in \mathbb{N}$.

A necessary and sufficient condition for an infinitely divisible distribution is the Lévy Khintchine formula, which generalizes the characteristic function for all Lévy processes as follows.

Theorem 2.3 (Lévy-Khintchine Representation) *A random variable is infinitely divisible if and only if there exists a triplet (μ, σ^2, ν) where $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_0^+$ and ν is a positive sigma-finite measure on $\mathbb{R}$, such that the characteristic function is given by*

$$\phi_X(u) = e^{\psi(u)}, \tag{2.2}$$

where $\psi(u)$ is the characteristic exponent, given by the expression

$$\psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\chi_{\{|x|\leq 1\}}(x))\nu(dx), \tag{2.3}$$

and $\chi_A(x)$ denotes the indicator function, defined as

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{2.4}$$

Since every Lévy process is associated to an infinitely divisible distribution, the characteristic function can also be expressed as

$$\phi_{L_t}(u) = e^{t\psi(u)}, \tag{2.5}$$

where $L = \{L_t, t \geq 0\}$ is the Lévy process associated with the triplet (μ, σ^2, ν). The positive measure ν is referred to as the Lévy measure of L, which must satisfy certain properties given in the following definition.

Definition 2.4 (Lévy Measure, Paths, and Moment Properties)
The Lévy measure of L, denoted ν, must satisfy the following properties:

$$\begin{cases} 1.) & \nu\{0\} = 0 \\ 2.) & \int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty, \textit{ where } A \wedge B := \min(A, B) \end{cases} . \tag{2.6}$$

The Lévy measure gives the expected number of jumps of a certain size per unit of time. The following properties describe a Lévy process and its measure on a given bounded nontrivial interval:

- If $\nu(\mathbb{R}) = \infty$ then infinitely many small jumps occur. The Lévy process is said to have infinite activity. (2.7)
- If $\nu(\mathbb{R}) < \infty$ then a.a. paths have a finite number of jumps. The Lévy process is said to have finite activity. (2.8)
- Let L be a Lévy process with Lévy triplet (μ, σ^2, ν). If $\sigma^2 = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, then a.a. paths have finite variation. If $\sigma^2 \neq 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, then a.a. paths have infinite variation. (2.9)

While the path variation properties above are seen to be only related to the small jumps and/or the Brownian motion components, the moment properties depend on the large jumps in the following way:

- L_t has finite moment of order p if and only if

$$\int_{|x|\geq 1} |x|^p \nu(dx) < \infty. \tag{2.10}$$

- L_t has finite exponential moment of order p (i.e. $\mathbb{E}[e^{pL_t}] < \infty$) if and only if

$$\int_{|x|\geq 1} e^{px} \nu(dx) < \infty. \tag{2.11}$$

The class of stable distributions is an important subclass of infinitely divisible distributions. As the name suggests, the Lévy process we will focus on later in this paper, the Exponentially Tempered Stable process, incorporates exponential decay or "tempering" terms into the Lévy measure of an α-stable random variable, presented in the following theorem (for proof, see [64]).

Theorem 2.5 (Lévy Measure of $\alpha-$Stable Random Variable) *If X is a stable random variable, that is there exists sequences $\{c_n, n \in \mathbb{N}\}, \{d_n, n \in \mathbb{N}\}, c_n > 0$ such that a linear combination of independent copies $X_1 + ... + X_n \stackrel{d}{=} c_n X + d_n$, then*

1. $When\ \alpha = 2,\ X \sim N(\mu, A).$

2. $When\ \alpha \neq 2,\ A = 0,\ and$

$$\nu(dx) = \begin{cases} \frac{c_1}{x^{1+\alpha}} dx & if\ x > 0 \\ \frac{c_2}{|x|^{1+\alpha}} dx & if\ x < 0 \end{cases},\ where\ c_1, c_2 \geq 0\ and\ c_1 + c_2 > 0. \tag{2.12}$$

It can be shown that if X is stable, then $c_n = \sigma n^{\frac{1}{\alpha}}$ with α a constant in the interval $(0, 2]$. The parameter α is called the stability index and is meaningful in characterizing the tails of the distribution. An important feature of stable laws is the existence of polynomial or "fat" tails when $\alpha \neq 2$. Thus, for $\alpha < 2$, the distribution has an infinite variance, and furthermore has an undefined mean or expected value for $\alpha < 1$. As shown above, the case when $\alpha = 2$ is the normal distribution, and thus the tails decay exponentially and do not exhibit "fatness."

An important structural property of Lévy densities is monotonicity, and we will require that the Lévy processes we work with have completely monotone jumps. Recall that a function $f:(0,\infty)\mapsto\mathbb{R}$ is said to be completely monotone if it possesses derivatives $f^{(n)}(x)$ and $(-1)^n f^{(n)}(x)\geq 0$ for all $n=0,1,2...$ and $x>0$. That is, derivatives of the same order have the same sign and are alternating in sign. A completely monotone Lévy density structurally relates arrival rates of large jump sizes to smaller jump sizes. Intuitively, we should expect that large jumps occur less frequently than small jumps. A well known necessary and sufficient condition for a function to be completely monotone is given in the following theorem.

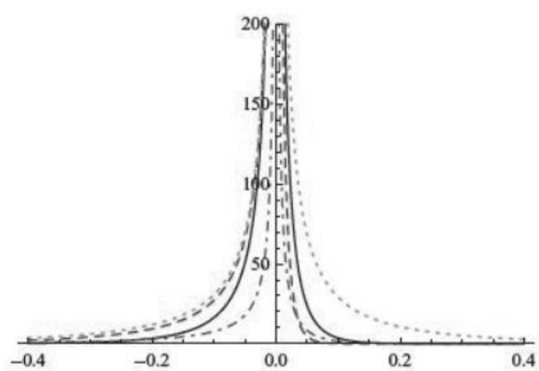
Theorem 2.6 (Bernstein's Theorem) *A function f is completely monotone if and only if it is the Laplace transform of some non-negative measure ν,*

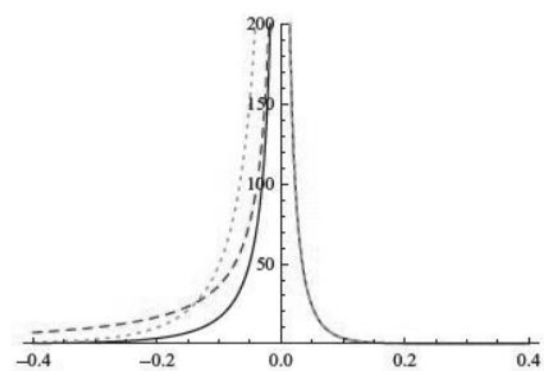
$$f(x)=\int_0^\infty e^{-ax}\nu(da). \tag{2.13}$$

Put another way, complete monotonicity asserts that for differentiable densities the derivative is negative for positive jumps and positive for negative jumps. This leads us to the following definition, taken from [29].

Definition 2.7 (Completely Monotone Jumps) *The process X has completely monotone jumps if its Lévy measure is absolutely continuous with density f decreasing exponentially when $x\to\pm\infty$ and such that $f(x)$ and $f(-x)$ are completely monotone.*

The concept of completely monotone jumps is perhaps best grasped by a visual inspection of the density of a Lévy measure. The figures below, from [31], plot the Lévy measures of two processes known as the Variance Gamma (left) and the Linear Gamma (right) processes with varying parameters (the specifics of these processes are not important). The size of the jump, x, corresponds to the horizontal axis and the number of jumps corresponds to the vertical axis. One sees that for negative jumps, we have monotonically increasing functions with a positive derivative and relatively more small jumps than large, and for positive jumps, monotonically decreasing functions with a negative derivative, but also with relatively more small jumps than large.





Finally, since many, if not most, Lévy processes do not have explicitly known probability densities, we are in general confined to work with the characteristic function, so it is helpful to recall the relationship between these functions.

Theorem 2.8 (Probability Density and Characteristic Function Bijection) *Let $\phi_X(u)$ be the characteristic function of some real-valued random variable. Then, there exists a bijection between the probability distribution and the characteristic function. That is, two distinct probability distributions never share the same characteristic function. Since the characteristic function is the Fourier transform, denoted $\mathcal{F}$, of the law f,*

$$\phi_X(u) = \mathcal{F}[f](u) = \int_{-\infty}^{\infty} e^{iux} f dx, \tag{2.14}$$

the probability density function can be recovered by a Fourier inversion of the characteristic function,

$$f(x) = \mathcal{F}^{-1}[\phi_X(u)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi_X(u)du, \tag{2.15}$$

where $\phi_X(u)$ denotes the characteristic function of x evaluated at u.

2.2 Itô Calculus for Lévy Processes

Theorem 2.9 (The 1-dimensional Itô Formula) *Suppose X_t is a Lévy-type stochastic integral of the form*

$$dX_t = \mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}} g(t,z)\bar{N}(dt,dz), \tag{2.16}$$

where

$$\bar{N}(dt,dz) = \begin{cases} N(dt,dz) - \nu(dz)dt & \text{if } |z| < R \\ N(dt,dz) & \text{if } |z| \geq R, \end{cases} \tag{2.17}$$

for some $R \in [0,\infty]$ and W_t is a standard Wiener process.
Let $f \in C^2(\mathbb{R}^2)$ and define $Y_t = f(t, X_t)$. Then Y_t is again a Lévy-type stochastic integral and

$$\begin{aligned} dY_t = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)[\mu_t dt + \sigma_t dW_t] + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial x^2}(t,X_t)dt \\ + \int_{|z|<R}\{f(t,X_{t^-}+g(t,z)) - f(t,X_{t^-}) - g(t,z)\frac{\partial f}{\partial x}(t,X_{t^-})\}\nu(dz)dt \\ + \int_{\mathbb{R}}\{f(t,X_{t^-}+g(t,z)) - f(t,X_{t^-})\}\bar{N}(dt,dz), \end{aligned} \tag{2.18}$$

and

$$\begin{cases} \text{If } R = 0, \text{ then } \bar{N} = N \text{ everywhere} \\ \text{If } R = \infty, \text{ then } \bar{N} = \tilde{N} \text{ everywhere.} \end{cases}$$

N is the so-called Poisson random measure, or a family of Poisson distributed random variables, with $\tilde{N}$ being the centered or "compensated" Poisson random measure in which the additional term is the quantity which must be subtracted from N in order to obtain a martingale. A more comprehensive treatment of these random measures can be found in [19].

2.3 Stochastic Exponential Lévy Processes

Consider a one-dimensional process $L = \{L_t, t \geq 0\}$, which is a solution of the stochastic differential equation (SDE)

$$dL_t = L_{t^-} dX_t, \tag{2.19}$$

where X is a Lévy-type stochastic integral of the form

$$dX_t = \mu_t dt + \sigma_t dW_t + \int_{|z|<1} g(t,z)\tilde{N}(dt,dz) + \int_{|z|\geq 1} w(t,z)N(dt,dz), \tag{2.20}$$

where a tilde denotes a compensated Poisson integral (importantly a martingale), which has the decomposition

$$\int_A g(t,z)\tilde{N}(dt,dz) = \int_A g(t,z)N(dt,dz) - \int_A g(t,z)\nu(dz)dt. \tag{2.21}$$

The solution of (2.19) is the "stochastic exponential" or the "Doléans-Dade exponential,"

$$L_t = \mathcal{E}_X(t) = exp\{X_t - \frac{1}{2}\sigma_t^2\} \prod_{0\leq s\leq t} (1 + \triangle X_s)e^{-\triangle X_s}. \tag{2.22}$$

For financial applications, we will require that (to avoid negative prices)

$$\inf\{\triangle X_t, t \geq 0\} \geq -1 \quad \text{a.s.} \tag{2.23}$$

An alternative form for (2.22) is

$$\mathcal{E}_X(t) = e^{S_X(t)}, \tag{2.24}$$

where

$$\begin{aligned} dS_X(t) = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t + \int_{|z|\geq 1} log(1 + w(t,z))N(dt,dz) \\ + \int_{|z|<1} log(1 + g(t,z))\tilde{N}(dt,dz) + \int_{|z|<1} (log(1 + g(t,z)) - g(t,z))\nu(dz)dt. \end{aligned} \tag{2.25}$$

2.3.1 Relation Between Ordinary and Stochastic Exponentials

An important relation exists between stochastic and ordinary exponentials. Lévy models and their associated triplets are usually presented in the literature assuming ordinary exponential form, i.e. $S_t = S_0 e^{L_t}$ for the stock price. However, as we have just seen the popular geometric type stochastic differential of equation (2.19) leads to a stochastic exponential type solution. Fortunately, there is a clear relation between these formulations, given by the following theorem from [19]:

Theorem 2.10 *If L is a Lévy process with characteristic triplet* (μ, σ^2, ν)*, then* $\mathcal{E}_L(t) = e^{L_1(t)}$ *where* L_1 *is a Lévy process with characteristic triplet* $(\mu_1, \sigma_1^2, \nu_1)$*, given by*

$\nu_1 = \nu \circ f^{-1}, \quad f(x) = \log(1 + x),$

$\mu_1 = \mu - \frac{1}{2}\sigma^2 + \int_{\mathbb{R}}[\log(1+x) - x]\nu(dx),$

$$\sigma_1 = \sigma. \tag{2.26}$$

Conversely, there exists a Lévy process L_2 with characteristic triplet $(\mu_2, \sigma_2^2, \nu_2)$ such that there exists $e^{L_t} = \mathcal{E}_{L_2}(t)$, where

$\nu_2 = \nu \circ g^{-1}, \quad g(x) = e^x - 1,$

$\mu_2 = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}}[(e^x - 1) - x]\nu(dx),$

$$\sigma_2 = \sigma. \tag{2.27}$$

2.4 Exponential Lévy Stock Price Processes, EMMs, and Esscher Transforms

As was just briefly noted, exponential Lévy process can be suitable models for stock prices, given by

$$S_t = S_0 e^{L_t}, \tag{2.28}$$

where

$$L_t = X_t + \mu t. \tag{2.29}$$

That is, the Lévy process X is taken as the martingale component of the log-stock price process. However, the Lévy process itself will have an associated drift, and therefore we must make an appropriate adjustment. Rather than evaluating each Lévy process individually and compensating for the drift, the martingale condition can be satisfied by compensating for

$$\mathbb{E}[e^{L_t}], \tag{2.30}$$

which is simply the characteristic function evaluated at $-i$. Accordingly, we add a mean-correction term to the drift of the exponential price process, given by

$$\omega = -\log[\phi(-i)], \tag{2.31}$$

where $\phi(\cdot)$ is the characteristic function of the Lévy process. The final form of the price process is thus given by

$$S_t = S_0 e^{L_t + \omega t}. \tag{2.32}$$

The mean-correcting measure can be employed such that the discounted price $\tilde{S}_t = e^{-rt}S_t$ is a martingale. Following [66], the procedure is:

1) Estimate the parameters of the process by some suitable method

2) Change the drift term of the old process, call it some μ_{old}, in such a way that

$$\mu_{new} = \mu_{old} + r - \log[\phi(-i)]. \tag{2.33}$$

Exchanging μ_{new} for μ_{old} in the Lévy process implies the discounted price $\tilde{S}_t = e^{-rt}S_t$ is a martingale. Still, it is important to note that this equivalent martingale measure is not unique in most cases involving Lévy processes. We must therefore have a criterion to reduce the class of possible measures $\mathbb{Q}$ to an appropriate subset and then obtain a unique equivalent measure $\mathbb{Q}_u$. Consider the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}_u}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{N}_u(t). \tag{2.34}$$

The measure $\mathbb{Q}_u$ is called the Esscher transform of $\mathbb{P}$ by the martingale $\mathcal{N}_u$, given by

$$\mathcal{N}_u(t) = \exp(-uX_t + t\psi(u)). \tag{2.35}$$

Significant results related to the Exponentially Tempered Stable process will be given in the next section. The important take-away here is the intuitive notion that the Esscher transform is such that the measure $\mathbb{Q}_u$ minimizes the relative entropy (or "Kullback-Leibler distance") $H(\mathbb{Q}|\mathbb{P})$ between the measures $\mathbb{Q}$ and $\mathbb{P}$. The relative entropy $H(\mathbb{Q}|\mathbb{P})$ is given by

$$H(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \tag{2.36}$$

and thus the measure $\mathbb{Q}_u$ is such that

$$H(\mathbb{Q}_u|\mathbb{P}) = \min\left\{\mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right\} = \min\left\{\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right\}. \tag{2.37}$$

A full treatment of this concept and a proof of the minimization of the relative entropy by $\mathbb{Q}_u$ can be found in [16].

3 The Exponentially Tempered Stable (ETS) Processes

The Lévy process used throughout the paper is the Exponentially Tempered Stable (hereafter ETS) process. The ETS process ([36] and [10]) is in fact just the CGMY process popularized in [12] with a generalization of one parameter based on the sign of the random variable (equivalently, the CGMY process is a special case of the ETS process). In fact, most, if not all, of the results presented below are easily understood if one is familiar first with the CGMY process, and [12] is the main reference for the mathematics presented herein. Later, in for instance [39], another special case known as Bilateral Gamma was studied. The tempered stable process has found many uses outside the field of finance, such as dynamical systems and fluid dynamics ([68], [69]). It is a popular model of turbulence in the physical sciences, where it is known more commonly as Truncated Lévy Flight ([50]). In a rather interesting application, the process has been shown, in for instance [72] and [73], to explain foraging behavior, and in [70] it was shown that great white sharks abandon a Brownian motion hunting pattern in favor of a Lévy flight as nearby prey becomes scarce. This has led to what is known as the Lévy Flight Foraging Hypothesis, which states that, as in [73], "since Lévy flights optimize random searches, biological organisms must have therefore evolved to exploit Lévy flights." But alas, we must now continue with the banalities of the financial applications.

3.1 Parameters, Properties, and Characteristic Function

The ETS process is a pure jump Lévy process with parameters λ_+, λ_-, β_+, β_- and α, with Lévy triplet (μ, σ^2, ν) given by:

$$\mu = \begin{cases} b + \int_{|x|\leq 1} x\nu(dx) & \text{if } \alpha < 1 \\ b - \int_{|x|>1} x\nu(dx) - \lambda_+\alpha\Gamma(-\alpha)(\beta_+^{\alpha-1}) + -\lambda_-\alpha\Gamma(-\alpha)(\beta_-^{\alpha-1}) & \text{if } \alpha \in (1,2), \end{cases}$$

$\sigma = 0,$

$$\nu(x) = \lambda_+ \frac{\exp(-\beta_+ x)}{x^{\alpha+1}} \chi_{x>0} + \lambda_- \frac{\exp(-\beta_-|x|)}{|x|^{\alpha+1}} \chi_{x<0}. \tag{3.1}$$

where $\lambda_+, \lambda_-, \beta_+, \beta_- > 0$ and $\alpha < 2$. The condition $\alpha < 2$ ensures that the Lévy density will integrate x^2 around 0. As the name suggests, the ETS process and a stable process with stability index $\alpha \in (0, 2)$ have similar Lévy measures, but importantly the measure of the ETS process includes the additional exponential "tempering" factors. Because of this exponential tempering within the Lévy measure, the distribution has finite moments of all orders (see Definition 2.4), and the large jumps do not require truncation. The parameters of the ETS process play an important role in capturing various desirable aspects in a stochastic process with applications to financial problems. The parameters λ_+ and λ_- can be seen as a measure of the overall level of activity of the process and, all else equal (and only considering movements greater than some arbitrarily chosen small value), the aggregate activity level may be calibrated through movements in λ_+ and λ_-. The parameters β_+ and β_-, respectively, control the rate of exponential decay on the right and left of the Lévy density, allowing for the construction of a skewed distribution. For example, if $\beta_- < \beta_+$ the exponential tempering factor for negative values of the random variable is inducing slower decay than the positive factor and thus we have a left-skewed distribution, which is consistent with the risk-neutral distribution typically implied from option prices. In the special case where $\beta_- = \beta_+$, the Lévy measure is symmetric, although non-normal distributions can still be generated through the parameters λ_+ and λ_-, which provide control over the kurtosis of the random variable. Finally, the parameter α describes the behavior of the Lévy density near zero, and is useful in characterizing the fine structure of the stochastic process. Note that the case when $\alpha = 0$ is the (generalized) Variance Gamma process of [45]. The relationship of the moments of the ETS random variable to the parameters are perhaps most easily grasped by considering the closed form expressions, and these are therefore given below in Theorem 3.1. It is clear from these equations that the variance, skewness, and kurtosis of the ETS random variable are all positively related to λ_+, λ_- and α and negatively related to β_- and β_+ (more specifically, negative skewness is inversely related to β_- and positive skewness is inversely related to β_+, and, all else equal, overall skewness is positively related to $\beta_- - \beta_+$). Additionally, the main properties of the ETS process, that is, its monotonicity (recall Theorem (2.6)), level of activity, and its variation may be defined by the value of parameter α, and this follows in Theorem 3.2 (for proof see [12]).

Theorem 3.1 (Higher Moments of the ETS Process) *The variance, skewness, and kurtosis of the ETS* *process, considering $t = 1$, are given by (for Proof see Appendix A):*

$$\textit{Variance} = \lambda_+\Gamma(2-\alpha)\left(\frac{1}{\beta_+^{2-\alpha}}\right) + \lambda_-\Gamma(2-\alpha)\left(\frac{1}{\beta_-^{2-\alpha}}\right), \tag{3.2}$$

$$\textit{Skewness} = \frac{\lambda_+\Gamma(3-\alpha)\left(\frac{1}{\beta_+^{3-\alpha}}\right)}{\textit{Variance}^{3/2}} - \frac{\lambda_-\Gamma(3-\alpha)\left(\frac{1}{\beta_-^{3-\alpha}}\right)}{\textit{Variance}^{3/2}}, \tag{3.3}$$

$$\textit{Kurtosis} = 3 + \frac{\lambda_+\Gamma(4-\alpha)\left(\frac{1}{\beta_+^{4-\alpha}}\right)}{\textit{Variance}^{2}} + \frac{\lambda_-\Gamma(4-\alpha)\left(\frac{1}{\beta_-^{4-\alpha}}\right)}{\textit{Variance}^{2}}. \tag{3.4}$$

Theorem 3.2 (Process Properties and Ranges for the Parameter α) *The ETS process*

- *has a completely monotone Lévy density for $\alpha > -1$;*
- *is a process of infinite activity for $\alpha > 0$; and*
- *is a process of infinite variation for $\alpha > 1$* (3.5)

Importantly, even though the ETS process as presented to this point is a pure jump process with no Brownian motion component, the infinite activity property afforded by setting $\alpha > 0$ can still capture small market fluctuations in the absence of a Brownian motion. Still, an orthogonal diffusion component can be added to improve the fine structure of the stock price process, resulting in a so-called "jump-diffusion" process. Although the probability density function of the ETS process is not explicitly known, the characteristic function of the ETS density admits a rather simple form, and is given in the following definition. ***For ease of notation, the vector $\overrightarrow{\mathbf{v}} = \{\lambda_+, \lambda_-, \beta_-, \beta_+, \alpha\}$ will be used to denote the ETS parameters for the rest of the paper***.

Theorem 3.3 (ETS Characteristic Function) *The characteristic function of the ETS random variable is given by (for Proof see Appendix B):*

$$\phi_{ETS}(u, t; \overrightarrow{\mathbf{v}}) = \exp(t\lambda_+\Gamma(-\alpha)[(\beta_+ - iu)^\alpha - \beta_+^\alpha] + t\lambda_-\Gamma(-\alpha)[(\beta_- + iu)^\alpha - \beta_-^\alpha]). \tag{3.6}$$

The relatively straightforward characteristic function of the ETS process lends the pricing problem well to solutions via methods which directly relate the characteristic function to the option price such as transform methods which will be elaborated in a later section.

3.1.1 Statistical (Real-World) and Risk Neutral Price Processes

The ETS model for the stock price process takes an ETS random variable X_{ETS} as the martingale component of the log stock price, making the statistical price process

$$S_t = S_0 \cdot \exp[(\mu + \omega)t + X_{ETS}(t; \overrightarrow{\mathbf{v}})], \tag{3.7}$$

where μ is the mean rate of return on the stock and ω is the mean-correction term detailed in section 2.4, given here by

$$\omega = -\log(\phi_{ETS}(-i)) = -\lambda_+ \cdot \Gamma(-\alpha)[(\beta_+ - 1)^\alpha - \beta_+^\alpha] - \lambda_- \cdot \Gamma(-\alpha)[(\beta_- + 1)^\alpha - \beta_-^\alpha]. \quad (3.8)$$

We can also consider the following "extended" or jump-diffusion model which adds an orthogonal diffusion component,

$$X_{ETS_e}(t; \overrightarrow{\mathbf{v}}, \eta) = X_{ETS}(t; \overrightarrow{\mathbf{v}}) + \eta W_t, \quad (3.9)$$

with W_t a standard Wiener process independent of X_{ETS}. The price process of this extended model is

$$S_t = S_0 \cdot \exp[(\mu + \omega - \eta^2/2)t + X_{ETS_e}(t; \overrightarrow{\mathbf{v}}, \eta)]. \quad (3.10)$$

Integral to subsequent sections, the characteristic function for the logarithm of the stock price under this extended model is given by (for Proof see Appendix C):

$$\begin{aligned} \phi_{\ln(S_t)}(u,t) &= \mathbb{E}^{\mathbb{P}}[e^{iu\ln(S_t)}] \\ &= \exp(iu\{\ln(S_0) + (\mu + \omega - \eta^2/2)t\}) \cdot \phi_{ETS}(u; \overrightarrow{\mathbf{v}}) \cdot \exp(-\eta^2 u^2/2). \end{aligned} \quad (3.11)$$

As an important corollary to the statistical price process above, the risk-neutral price process, which will later be calibrated from option market data, replaces the asset-specific drift μ with the risk-free rate r, yielding

$$S_t = S_0 \cdot \exp[(r + \tilde{\omega} - \tilde{\eta}^2/2)t + X_{ETS_e}(t; \tilde{\overrightarrow{\mathbf{v}}}, \tilde{\eta})], \quad (3.12)$$

with tildes denoting the parameters are now set under the risk-neutral measure. In this context, the characteristic function of log returns becomes, by the same logic that led to equation (3.11),

$$\begin{aligned} \tilde{\phi}_{\ln(S_t)}(u,t) &= \mathbb{E}^{\mathbb{Q}}[e^{iu\ln(S_t)}] \\ & \exp(iu\{\ln(S_0) + (r + \tilde{\omega} - \tilde{\eta}^2/2)t\}) \cdot \phi_{ETS}(u; \tilde{\overrightarrow{\mathbf{v}}}) \cdot \exp(-\tilde{\eta}^2 u^2/2), \end{aligned} \quad (3.13)$$

with

$$\tilde{\omega} = -\log(\phi_{ETS}(-i)) = -\tilde{\lambda_+} \cdot \Gamma(-\tilde{\alpha})[(\tilde{\beta_+} - 1)^{\tilde{\alpha}} - \tilde{\beta_+}^{\tilde{\alpha}}] - \tilde{\lambda_-} \cdot \Gamma(-\tilde{\alpha})[(\tilde{\beta_-} + 1)^{\tilde{\alpha}} - \tilde{\beta_-}^{\tilde{\alpha}}]. \quad (3.14)$$

3.1.2 Minimal Entropy Martingale Measure of the ETS Process

The Esscher transform concept introduced in section 2.4 can be specialized to the ETS process, affording an important result to a later part of this study. First, consider the following proposition from [35]:

Proposition 3.4 *Suppose $X_t,\ 0 \le t \le T$ is the ETS process with parameters ($\overrightarrow{\mathbf{v}}$) under $\mathbb{P}$ and the ETS process with parameters ($\tilde{\overrightarrow{\mathbf{v}}}$) under $\mathbb{Q}$. Then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent for all $t > 0$ if and only if $\alpha = \tilde{\alpha}$, $\lambda_+ = \tilde{\lambda_+}$ and $\lambda_- = \tilde{\lambda_-}$.*

This proposition is further built upon in [34] as follows. Consider again the characteristic exponent of the ETS process,

$$\psi_{ETS}(u;\overrightarrow{\mathbf{v}}) := \lambda_+\Gamma(-\alpha)[(\beta_+ - iu)^\alpha - \beta_+^\alpha] + \lambda_-\Gamma(-\alpha)[(\beta_- + iu)^\alpha - \beta_-^\alpha)]. \tag{3.15}$$

Along with Proposition 3.4, we have the following:

Theorem 3.5 (EMM Conditions for the ETS Model) *Assume $S_t,\ 0 \leq t \leq T$ is the real-world ETS stock price process with parameters $(\overrightarrow{\mathbf{v}}, \mu)$ under the measure $\mathbb{P}$, and with parameters $(\tilde{\overrightarrow{\mathbf{v}}}, (r-q))$ under the measure $\mathbb{Q}$. Then $\mathbb{Q}$ is an EMM of $\mathbb{P}$ if and only if $\alpha = \tilde{\alpha}$, $\lambda_+ = \tilde{\lambda_+}$, $\lambda_- = \tilde{\lambda_-}$, and*

$$r - q - \psi_{ETS}(-i;\lambda_+,\lambda_-,\tilde{\beta_-},\tilde{\beta_+},\alpha) = \mu - \psi_{ETS}(-i;\overrightarrow{\mathbf{v}}). \tag{3.16}$$

Choosing some $\theta,\ -\beta_- < \theta < \beta_+$, we have

$$r - q - \psi_{ETS}(-i;\lambda_+,\lambda_-,\beta_- + \theta,\beta_+ - \theta,\alpha) = \mu - \psi_{ETS}(-i;\overrightarrow{\mathbf{v}}). \tag{3.17}$$

However, this martingale measure is not yet unique, in the sense that it does not guarantee the minimal entropy. Accordingly, we recall the Esscher transform concept and consider a "model preserving minimal entropy martingale measure," so-called in [34], given by

$$H(\mathbb{Q}_u|\mathbb{P}) = H(\mathbb{Q}_{\tilde{\beta_-},\tilde{\beta_+}}|\mathbb{P}) = \min\{H(\mathbb{Q}|\mathbb{P})|\mathbb{Q} \in \text{EMM}(\mathbb{P})\}, \tag{3.18}$$

where $\mathbb{Q} \in \text{EMM}(\mathbb{P})$ denotes that the EMM condition of equation (3.17) is satisfied, and the relative entropy of the ETS process can be expressed explicitly by (for Proof see Appendix D):

$$\begin{aligned} H(\mathbb{Q}|\mathbb{P}) = t\lambda_+\Gamma(-\alpha)((\alpha-1)\tilde{\beta_+}^{\alpha} - \alpha\beta_+\tilde{\beta_+}^{\alpha-1} + \beta_+^\alpha) \\ + t\lambda_-\Gamma(-\alpha)((\alpha-1)\tilde{\beta_-}^{\alpha} - \alpha\beta_-\tilde{\beta_-}^{\alpha-1} + \beta_-^\alpha). \end{aligned} \tag{3.19}$$

4 Stochastic Optimal Control of Exponential Lévy Processes

4.1 The Dynamic Programming Principle

The dynamic programming principle was greatly advanced by Richard Bellman and his groundbreaking work in the 1950s (see [4] and [5]), although the idea had already been foreshadowed in works such as [74] and [75]. Bellman's breakthrough involved casting the value of a decision problem as an expectation of the initial value of some state variables (defined shortly) and the value of the choices to be made subsequently throughout the horizon of the decision problem. In a discrete time setting, a dynamic optimization problem is broken up into a sequence of discrete subproblems which are recursively optimized via backward induction, that is beginning with the last subsequence. The essential aspect of these problems, known as Bellman equations, is the necessary condition brought about by the Principle of Optimality, which roughly states, *Given an optimal sequence of decisions or choices, each subsequence must also be optimal.*

Considering the "decision problem" to be that of a one-dimensional portfolio strategy (i.e. a single risky asset and without consumption), the investor's goal could be thought of as maximizing terminal or end-of-period wealth,

$$\mathbb{E}^{\mathbb{P}}[U(X_T^{\pi})|X_t = x], \tag{4.1}$$

where $U(\cdot)$ denotes some applicable investor utility function and $\mathbb{P}$ denotes the problem is set under the objective or real-world probability measure. The variable of which it is necessary to know the current value, or current state, is conveniently referred to as the state variable, here denoted x. The variables chosen at any given point in time are known as the control variables, and here our control variable is π, the weight of the portfolio to be invested in the risky asset. Assuming the state process or variable x is driven by a stochastic process, equation (4.1) is known as a stochastic optimal control problem. We now move on to solving this equation via dynamic programming, importantly in *continuous time* (that is, by splitting the problem into an infinite number of subsequences), leading to a partial differential equation (PDE) known as the Hamilton-Jacobi-Bellman (HJB) equation. The control problem is then equivalent to finding the solution to the HJB equation.

4.2 Derivation of Hamilton-Jacobi-Bellman (HJB) Equation

A statement of the problem and a general HJB for Lévy processes will be derived before moving on to specializing the equation to an exponential Lévy process with a CRRA investor utility function.
Consider as above the control problem to maximize

$$\mathbb{E}_{t,x}^{\mathbb{P}}[U(X_T^{\pi})], \tag{4.2}$$

where the notation

$$\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|X_t = x].$$

We consider also the dynamics

$$dX_t^{\pi} = \mu(t, X_t^{\pi}, \pi)dt + \sigma(t, X_t^{\pi}, \pi)dW_t + \int_{\mathbb{R}} g(t, X_{t-}^{\pi}, z, \pi)\tilde{N}_t(dt, dz), \quad X_0 = x > 0, \tag{4.3}$$

with $\mu(t, x, \pi)$ and $\sigma(t, x, \pi)$ functions possibly depending on t, x, and π, and $g(t, X_t^{\pi}, z, \pi)$ a function possibly depending on t, x, π, and also the jump size z. The value function is defined by

$$J(t, x, \pi) = \mathbb{E}_{t,x}^{\mathbb{P}}[U(X_T^{\pi})]. \tag{4.4}$$

The optimal value function is defined by

$$V(t, x) = \sup_{\pi \in \Pi}\{J(t, x, \pi)\}. \tag{4.5}$$

Now assume two strategies
Strategy 1. Use the optimal control law $\hat{\pi}$

Strategy 2. Use the control law $\overset{\star}{\pi}$
with

$$\overset{\star}{\pi}(s,y) = \begin{cases} \pi(s,y), & s \in [t, t+h] \\ \hat{\pi}(s,y), & s \in (t+h, T]. \end{cases} \tag{4.6}$$

Expected utility for strategy 1. This is trivially $J(t,x,\hat{\pi}) = V(t,x)$ since $\hat{\pi}$ was defined as the optimal control law.

Expected utility for strategy 2. As presented in [7], a stylized interpretation of this strategy is that after falling asleep at time t, you wake up and realize that the state process has moved to point x. You deal as best you can with these circumstances by maximizing your utility over the remaining time, given that you now are starting at time $t+h$ in the state x. So, in the time interval $(t+h,T]$, since by definition we will use the optimal control law in this time period, we have

$$J(t,x,\overset{\star}{\pi}) = \mathbb{E}^{\mathbb{P}}_{t,x}[V(t+h, X^{\pi}_{t+h})]. \tag{4.7}$$

Now we'll compare the two strategies, and since by definition strategy 1 is optimal, we have

$$V(t,x) \geq \mathbb{E}^{\mathbb{P}}_{t,x}[V(t+h, X^{\pi}_{t+h})]. \tag{4.8}$$

From Itô's formula in Theorem 2.9, we have

$$\begin{aligned} V(t+h, X^{\pi}_{t+h}) &= V(t,x) + \int_t^{t+h} \{\frac{\partial V}{\partial s}(s, X^{\pi}_s) + \mathcal{A}^{\pi} V(s, X^{\pi}_s)\} ds + \int_t^{t+h} \frac{\partial V}{\partial x}(s, X^{\pi}_s)\sigma_s dW_s \\ &\quad + \int_t^{t+h} \int_{\mathbb{R}} \{V(s, X^{\pi}_{s^-} + g(s, X^{\pi}_{s^-}, z, \pi)) - V(s, X^{\pi}_{s^-})\} \tilde{N}(ds,dz) \\ &+ \int_t^{t+h} \int_{\mathbb{R}} \{V(s, X^{\pi}_{s^-} + g(s, X^{\pi}_{s^-}, z, \pi)) - V(s, X^{\pi}_{s^-}) - g(s, X^{\pi}_{s^-}, z, \pi)\frac{\partial V}{\partial x}(s, X^{\pi}_{s^-})\}\nu_s(dz)ds, \end{aligned} \tag{4.9}$$

with

$$\mathcal{A}^{\pi} = \mu(t, X^{\pi}_s, \pi)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X^{\pi}_s, \pi)\frac{\partial^2}{\partial x^2}. \tag{4.10}$$

Applying the expectation operator, the dW_s and $\tilde{N}(ds,dz)$ terms will drop out (as martingales by definition), and when we plug the result into equation (4.8) we obtain

$$\begin{aligned} V(t,x) &\geq \mathbb{E}^{\mathbb{P}}_{t,x}[V(t,x) + \int_t^{t+h} \{\frac{\partial V}{\partial s}(s, X^{\pi}_s) + \mathcal{A}^{\pi} V(s, X^{\pi}_s)\} ds \\ &+ \int_t^{t+h} \int_{\mathbb{R}} \{V(s, X^{\pi}_{s^-} + g(s, X^{\pi}_{s^-}, z, \pi)) - V(s, X^{\pi}_{s^-}) - g(s, X^{\pi}_{s^-}, z, \pi)\frac{\partial V}{\partial x}(s, X^{\pi}_{s^-})\}\nu_s(dz)ds] \end{aligned} \tag{4.11}$$

$\Longleftrightarrow$

$$\mathbb{E}^{\mathbb{P}}_{t,x}\Big[\int_t^{t+h}\Big[\{\frac{\partial V}{\partial s}(s,X^\pi_s)+\mathcal{A}^\pi V(s,X^\pi_s)\}ds$$
$$+\int_{\mathbb{R}}\{V(s,X^\pi_{s^-}+g(s,X^\pi_{s^-},z,\pi))-V(s,X^\pi_{s^-})-g(s,X^\pi_{s^-},z,\pi)\frac{\partial V}{\partial x}(s,X^\pi_{s^-})\}\nu_s(dz)ds\Big]\Big]\leq 0. \tag{4.12}$$

Now divide by h, move h within the expectation, and let h tend to zero (so, this is where we incorporate the infinite subproblem, or *continuous time* assumption). By the fundamental theorem of integral calculus we obtain

$$\frac{\partial V}{\partial t}(t,x)+\mathcal{A}^\pi V(t,x)+\int_{\mathbb{R}}\{V(t,x+g(t,x,z,\pi))-V(t,x)-g(t,x,z,\pi)\frac{\partial V}{\partial x}(t,x)\}\nu_t(dz)\leq 0. \tag{4.13}$$

Equality will hold at the optimum, and thus

$$\frac{\partial V}{\partial t}(t,x)+\sup_{\pi\in\Pi}\Big\{\mathcal{A}^\pi V(t,x)$$
$$+\int_{\mathbb{R}}\{V(t,x+g(t,x,z,\pi))-V(t,x)-g(t,x,z,\pi)\frac{\partial V}{\partial x}(t,x)\}\nu_t(dz)\Big\}=0. \tag{4.14}$$

Imposing the additional terminal condition, we have thus arrived at the HJB equation

$$\begin{cases} 1) \quad \frac{\partial V}{\partial t}(t,x)+\sup_{\pi\in\Pi}\Big\{\mathcal{A}^\pi V(t,x) \\ \quad +\int_{\mathbb{R}}\{V(t,x+g(t,x,z,\pi))-V(t,x)-g(t,x,z,\pi)\frac{\partial V}{\partial x}(t,x)\}\nu_t(dz)\Big\}=0 \\ 2) \quad V(T,x)=U(x) \quad \forall x\in\mathbb{R}. \end{cases} \tag{4.15}$$

4.3 Admissible Controls

Before specializing the equation and deriving the associated optimal control strategy, it is useful to discuss the details of the set of admissible controls, which we will denote Π. Recall from equation (2.23) that the random jump variable is assumed to have support on $[-1,\infty)$ to guarantee positivity of, or equivalently to preserve the limited liability structure of, the stock price. Furthermore, as discussed in [41], there are additional nuances when considering jump processes in a wealth context. It would seem reasonable to assume that the wealth process itself inherits the behavior of the underlying drivers, that is importantly if all constituent securities are driven by pure diffusions the wealth process will also be a pure diffusion process, however portfolio assets with discontinuous paths would also introduce discontinuities into the overall wealth process. In the case of pure diffusion processes without jumps, the variance of returns over some time period dt is proportional to dt, implying that as one takes $dt\to 0$, the uncertainty associated with changes in wealth also goes to zero. Thus, investors retain complete control over the portfolio via continuous optimal rebalancing, and even leveraged long and short positions can be adjusted quickly enough before wealth turns negative. However, this is not the case when discontinuities are present, as the investor may not have enough time to rebalance when a jump occurs, and as a result of this loss of control over the portfolio, negative wealth could arise due to leveraged long and short positions. This situation can be likened to illiquidity risk, since investors in illiquid assets may face large changes in the value of their portfolio before/without being able to rebalance the position. As such, the following proposition applies to the case of jump processes (see [41] or [42] for proof):

Theorem 4.1 (Bounds on Portfolio Weights) *For any $t, 0 < t \leq T$, and denoting X_{inf} and X_{sup} as the lower and upper bounds of the support of the random price jump X_t, the optimal portfolio weight $\hat{\pi}_t$ must satisfy*

$$1 + \hat{\pi}_t X_{inf} > 0 \quad and \quad 1 + \hat{\pi}_t X_{sup} > 0. \tag{4.16}$$

In particular, if $X_{inf} < 0$ and $X_{sup} > 0$,

$$-\frac{1}{X_{sup}} \leq \hat{\pi}_t \leq -\frac{1}{X_{inf}}. \tag{4.17}$$

Considering again our condition for financial applications that the support is $[-1, \infty)$, we find that

$$-\frac{1}{\infty} \leq \hat{\pi}_t \leq -\frac{1}{-1} <=> 0 \leq \hat{\pi} \leq 1. \tag{4.18}$$

Thus, the set of admissible controls $\Pi = [0, 1]$ and the investor will never take leveraged long or short positions in the risky asset.

4.4 Optimal Control of an Exponential Lévy Process Subject to CRRA Utility

Consider the case of iso-elastic or power utility, where

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{for } \gamma > 0, \gamma \neq 1 \\ \log(x) & \text{for } \gamma = 1, \end{cases} \tag{4.19}$$

in a market with two possible investments,
1) A risk-free asset with dynamics

$$dB_t = r_t B_t dt, \quad B_0 = b > 0. \tag{4.20}$$

2) A risky asset with dynamics

$$dS_t = S_{t^-}[\mu_t dt + \sigma_t dW_t + \int_{-1}^{\infty} z_t \tilde{N}(dt, dz)], \quad S_0 = s > 0, \tag{4.21}$$

with $r_t > 0$, $\mu_t > 0$, $\sigma_t > 0 \in \mathbb{R}$ variables depending on time. We additionally assume that

$$\int_{-1}^{\infty} |z| d\nu(z) < \infty, \quad \text{and} \quad \mu_t > r \; \forall t. \tag{4.22}$$

Consider now a wealth or portfolio process given by

$$dP_t = P_t\left[\pi\frac{dS_t}{S_t} + (1-\pi)\frac{dB_t}{B_t}\right], \quad P_0 = p. \tag{4.23}$$

Substituting the asset dynamics from (4.20) and (4.21), we have

$$dP_t = P_t\left[\pi\frac{S_{t^-}[\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}} z_t\tilde{N}(dt,dz)]}{S_t} + (1-\pi)\frac{r_tB_tdt}{B_t}\right], \tag{4.24}$$

which can be simplified to the following form

$$dP_t = \underbrace{P_t[r_t + (\mu_t - r_t)\pi]}_{=\mu(t,p,\pi)}dt + \int_{-1}^{\infty}\underbrace{P_t\pi z_t}_{=g(t,p,z,\pi)}\tilde{N}(dt,dz))dt + \underbrace{P_t\sigma_t\pi}_{=\sigma(t,p,\pi)}dW_t, \tag{4.25}$$

also with terminal condition

$$V(T,p) = U(p). \tag{4.26}$$

Thus, according to equation (4.15), our HJB equation is, for $\gamma \neq 1$

$$\begin{cases} 1) \quad \frac{\partial V}{\partial t}(t,p) + \sup\limits_{\pi\in\Pi}\Big\{\mathcal{A}^\pi V(t,p) \\ \quad + \int_{-1}^{\infty}\{V(t,p+p\pi z) - V(t,p) - p\pi z\frac{\partial V}{\partial x}(t,p)\}\nu_t(dz)\Big\} = 0 \\ 2) \quad V(T,p) = U(p) \quad \forall p \in \mathbb{R}. \end{cases} \tag{4.27}$$

The equation for when $\gamma = 1$ ***will be more easily handled at a later stage.*** We now search for a solution of the form

$$V(t,p) = U(e^{\delta_t}p), \tag{4.28}$$

where δ_t is a C^1 deterministic function of time such that $\delta_T = 0$, giving

$$V(t,p) = \frac{(e^{\delta_t}p)^{1-\gamma}}{1-\gamma} \text{ for } \gamma > 0, \gamma \neq 1. \tag{4.29}$$

Accordingly, note the following ancillary calculations:

$$\frac{\partial V}{\partial t} = \dot{\delta}_t(pe^{\delta_t})^{1-\gamma}, \quad \frac{\partial V}{\partial p} = e^{\delta_t(1-\gamma)}p^{-\gamma}, \quad \frac{\partial^2 V}{\partial p^2} = \gamma e^{-\delta_t(1-\gamma)}p^{-\gamma-1}. \tag{4.30}$$

To help simplify notation moving forward, let $C = \frac{\partial V}{\partial p}$, and $A = \frac{\partial^2 V}{\partial p^2}$. Realize as well that A can be rewritten as

$$A = e^{\delta_t(1-\gamma)}p^{-\gamma}p^{-1}(-\gamma) = C\cdot\frac{1}{p}\cdot(-\gamma). \tag{4.31}$$

Also note that

$$Cp = e^{\delta_t(1-\gamma)}p^{(1-\gamma)}. \tag{4.32}$$

As such, consider the full expression for equation 1) of the HJB equation (4.27),

$$\dot{\delta}_t(pe^{\delta_t})^{1-\gamma} + \sup_{\pi\in\Pi}\Big\{p[r_t + (\mu_t - r_t)\pi]C + \frac{1}{2}(p\sigma_t\pi)^2 A + \int_{-1}^{\infty}\{\frac{e^{\delta_t(1-\gamma)}(p+pz\pi)^{(1-\gamma)}}{1-\gamma} - \frac{e^{\delta_t(1-\gamma)}p^{(1-\gamma)}}{1-\gamma} - e^{\delta_t(1-\gamma)}p^{-\gamma}pz\pi\}\nu_t(dz)\Big\} = 0 \quad (4.33)$$

$\iff$

$$\dot{\delta}_t(pe^{\delta_t})^{1-\gamma} + \sup_{\pi\in\Pi}\Big\{p[r_t + (\mu_t - r_t)\pi]C + \frac{1}{2}(p\sigma_t\pi)^2 A + \int_{-1}^{\infty}\{\frac{1}{1-\gamma}\left[e^{\delta_t(1-\gamma)}(p+pz\pi)^{(1-\gamma)} - e^{\delta_t(1-\gamma)}p^{(1-\gamma)}\right] - e^{\delta_t(1-\gamma)}p^{(1-\gamma)}z\pi\}\nu_t(dz)\Big\} = 0. \quad (4.34)$$

Notice now we can factor $Cp = e^{\delta_t(1-\gamma)}p^{(1-\gamma)}$ out of each term, resulting in the final form of our HJB equation when $\gamma > 0, \gamma \neq 1$,

$$\begin{cases} 1) \quad \dot{\delta}_t(pe^{\delta_t})^{1-\gamma} + e^{\delta_t(1-\gamma)}p^{(1-\gamma)}\sup_{\pi\in\Pi}\Big\{r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \\ \quad + \int_{-1}^{\infty}\{\frac{1}{1-\gamma}\left[(1+z\pi)^{(1-\gamma)} - 1\right] - z\pi\}\nu_t(dz)\Big\} = 0 \\ 2) \quad V(T,p) = U(p) \quad \forall p \in \mathbb{R}, \end{cases} \quad (4.35)$$

but due to the strict positivity of $(pe^{\delta_t})^{(1-\gamma)}$ and the fact that $V(T,p) = U(e^{\delta_T}p) = U(p) \implies \delta_T = 0$, this is equivalent to

$$\begin{cases} 1) \quad \dot{\delta}_t + \sup_{\pi\in\Pi}\Big\{r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \\ \quad + \int_{-1}^{\infty}\{\frac{1}{1-\gamma}\left[(1+z\pi)^{(1-\gamma)} - 1\right] - z\pi\}\nu_t(dz)\Big\} = 0 \\ 2) \quad \delta_T = 0. \end{cases} \quad (4.36)$$

Now, we also revisit the limiting case when $\gamma = 1$, invoking the Cauchy/L'Hôpital theorems, and find that

$$\begin{cases} 1) \quad \dot{\delta}_t + \sup_{\pi\in\Pi}\Big\{r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \\ \quad + \int_{-1}^{\infty}\{[\log(1+z\pi) - z\pi]\,\nu_t(dz)\Big\} = 0 \\ 2) \quad \delta_T = 0. \end{cases} \quad (4.37)$$

Remark (Time-Invariance of Optimal Strategy) *Since neither equation 1) of the above HJB equations depends on the state variables P, S, or the time to maturity $T - t$, the optimal strategy will be totally myopic or time-invariant in the sense that the variable $\hat{\pi}$ controlling the proportion of wealth invested in the risky asset will not depend on the current level of wealth, but only on the variables μ_t, r_t, σ_t, and the measure ν_t.*

4.5 Existence of Solution

Consider the first equation of the general HJB given in equation (4.15) and its full form,

$$\frac{\partial V}{\partial t}(t,x) + \sup_{\pi\in\Pi}\Big\{[\mu(t,x,\pi)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x,\pi)\frac{\partial^2}{\partial x^2}]V(t,x) \\ + \int_{-1}^{\infty}\{V(t,x+g(t,x,z,\pi)) - V(t,x) - g(t,x,z,\pi)\frac{\partial V}{\partial x}(t,x)\}\nu_t(dz)\Big\} = 0. \quad (4.38)$$

Let

$$\mathcal{L}^{\pi}V(t,x) = [\mu(t,x,\pi)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x,\pi)\frac{\partial^2}{\partial x^2}]V(t,x) \\ + \int_{-1}^{\infty}\{V(t,x+g(t,x,z,\pi)) - V(t,x) - g(t,x,z,\pi)\frac{\partial V}{\partial x}(t,x)\}\nu_t(dz). \quad (4.39)$$

$\mathcal{L}^{\pi}V(t,x)$ is called the *infinitesimal generator* of the controlled process X^{π}, which are linked by the Dynkin formula,

$$\mathbb{E}^{\mathbb{P}}[f(T,X_T^{\pi})] - \mathbb{E}^{\mathbb{P}}[f(t,X_t^{\pi})] = \mathbb{E}^{\mathbb{P}}\left[\int_t^T \mathcal{L}^{\pi_u}f(u,X_u^{\pi})du\right], \quad (4.40)$$

for f good enough. We now state the verification theorem for exponential additive processes as given in [58]:

Theorem 4.2 (Verification Theorem) *Define the set $\mathcal{D}$ by*

$$\mathcal{D} := \{f \in C^{1,2}([0,T]\times\mathbb{R}) | \forall t \in [0,T],\ \textit{(4.40) holds}\ \forall \pi \in \Pi\}. \quad (4.41)$$

Let $K \in \mathcal{D}$ be a classical solution to the HJB equation (4.15). Then for all $(t,x) \in [0,T]\times\mathbb{R}$,
1) $K(t,x) \geq J(t,x,\pi)$ for every admissible control $\pi \in \Pi$;
2) there exists an admissible control $\hat{\pi} \in \Pi$ such that

$$\hat{\pi}(s,x) \in arg\, \max_{\pi} \mathcal{L}^{\pi}K(s,x)\ \forall\ (s,x),$$

then $K(t,x) = J(t,x,\hat{\pi}) = V(t,x)$.

Consider now that equations (4.36-4.37) are of the type

$$\dot{\delta}_t + \lambda_t = 0, \quad (4.42)$$

still with terminal condition $\delta_T = 0$, thus yielding $\delta_t = \int_t^T \lambda_u du$ and a candidate value function of

$$V(t,p) = U(pe^{\int_t^T \lambda_u du}) = \frac{(pe^{\int_t^T \lambda_u du})^{1-\gamma}}{1-\gamma}, \quad (4.43)$$

with a candidate optimal control $\hat{\pi}$. In order to apply Theorem 4.2, we need to prove that $V \in \mathcal{D}$. Applying Itô's formula to the candidate value function (see Theorem 2.9 and equations (4.9), (4.25) and (4.30) for guidance here), we have, for $\gamma \neq 1$

$$
\begin{aligned}
dV(t,P_t^\pi) &= \frac{\partial}{\partial t}(\int_t^T \lambda_u du)\left(P_t^\pi e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)} dt \\
&+ P_t^\pi[r_t + (\mu_t - r_t)\hat{\pi}]\left(e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)}(P_t^\pi)^{-\gamma} dt - \frac{1}{2}\gamma P_t^2 \sigma_t^2 \hat{\pi}^2 \left(e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)}(P_t^\pi)^{(-\gamma-1)} dt \\
&+ \left(e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)}(P_t^\pi)^{-\gamma} P_t^\pi \sigma_t \hat{\pi} dW_t \\
&+ \int_{-1}^{\infty}\{\frac{e^{\delta(t)(1-\gamma)}(P_t^\pi + P_t^\pi z\pi)^{(1-\gamma)}}{1-\gamma} - \frac{e^{\int_t^T \lambda_u du(1-\gamma)}(P_t^\pi)^{(1-\gamma)}}{1-\gamma}\}\tilde{N}(dt,dz) \\
&+ \int_{-1}^{\infty}\{\frac{e^{\int_t^T \lambda_u du(1-\gamma)}(P_t^\pi + P_t^\pi z\pi)^{(1-\gamma)}}{1-\gamma} - \frac{e^{\int_t^T \lambda_u du(1-\gamma)}(P_t^\pi)^{(1-\gamma)}}{1-\gamma} - e^{\int_t^T \lambda_u du(1-\gamma)}(P_t^\pi)^{-\gamma} P_t^\pi z\pi\}\nu_t(dz).
\end{aligned}
\tag{4.44}
$$

Recalling equation (4.32), note that we can factor $\left(e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)}(P_t^\pi)^{(1-\gamma)}$ out of every term. Also, recall that by the fundamental theorem of calculus, we have that $\frac{\partial}{\partial t}(\int_t^T \lambda_u du) = -\lambda_t$. Thus, we can simplify to

$$
\begin{aligned}
dV(t,P_t^\pi) &= \left(P_t^\pi e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)}\Big\{-\lambda_t + r_t + (\mu_t - r_t)\hat{\pi} - \frac{1}{2}\gamma\sigma_t^2\hat{\pi}^2 \\
&+ \int_{-1}^{\infty}\{\frac{1}{1-\gamma}[(1+z\hat{\pi})^{(1-\gamma)} - 1] - z\hat{\pi}\}\nu_t(dz)\Big\}dt \\
&+ \left(P_t^\pi e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)}\sigma_t\hat{\pi}dW_t \\
&+ \left(P_t^\pi e^{\int_t^T \lambda_u du}\right)^{(1-\gamma)}\int_{-1}^{\infty}[(1+\hat{\pi}z)^{(1-\gamma)} - 1]\tilde{N}(dt,dz).
\end{aligned}
\tag{4.45}
$$

Noting that the dt term is $\mathcal{L}^\pi V(t,P_t^\pi)$ and further denoting the remaining terms, as in [58], as dM_t, we can thus express this as

$$dV(t,P_t^\pi) = \mathcal{L}^\pi V(t,P_t^\pi)dt + dM_t. \tag{4.46}$$

Note that by analogous logic, one finds that in the case of $\gamma = 1$,

$$dM_t = \hat{\pi}\sigma_t dW_t + \int_{-1}^{\infty}\log(1+\hat{\pi}z)\tilde{N}(dt,dz). \tag{4.47}$$

For $\gamma > 0$, the Dynkin formula of equation (4.40) thus holds if

$$\mathbb{E}^{\mathbb{P}}[\int_t^T dM_t] = 0, \tag{4.48}$$

that is, if M is a martingale. As the sum of a stochastic integral and a compensated Poisson integral, this condition is clearly satisfied. Thus, we can apply the verification theorem, and we now turn the focus to the optimal strategy.

4.6 Optimal Strategy

First, it is important to note that the existence result above holds only over the interior of the compact set $\Pi = [0, 1]$. However, solutions on the boundary are possible and must be evaluated case by case. Nevertheless, we can analyze the first order condition and derive a more explicit solution by assuming an internal solution. We are thus interested in, for $\gamma \neq 1$,

$$\sup_{\pi\in\Pi}\left\{r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \quad + \int_{-1}^{\infty}\{\frac{1}{1-\gamma}\left[(1+z\pi)^{(1-\gamma)} - 1\right] - z\pi\}\nu_t(dz)\right\}, \tag{4.49}$$

and for $\gamma = 1$,

$$\sup_{\pi\in\Pi}\left\{r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \quad + \int_{-1}^{\infty}\{[\log(1+z\pi) - z\pi]\,\nu_t(dz)\right\}. \tag{4.50}$$

Working through the first order condition $\frac{\partial}{\partial\pi} = 0$ for both, recalling for the integral term the Leibniz rule $\frac{d}{dx}\left(\int_a^b f(x,t)dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)dt$, we obtain

$$(\mu_t - r_t) - \gamma\sigma_t^2\hat{\pi} + \int_{-1}^{\infty}\left(\frac{z}{(1+z\hat{\pi})^{\gamma}} - z\right)\nu_t(dz) = 0 \tag{4.51}$$

$\iff$

$$(\mu_t - r_t) - \gamma\sigma_t^2\hat{\pi} + \int_{-1}^{\infty} z((1+z\hat{\pi})^{-\gamma} - 1)\nu_t(dz) = 0, \tag{4.52}$$

for all $\gamma > 0$. It is obvious from the equation that no explicit expression exists for the control $\hat{\pi}$, however numerical methods provide a simple solution.

5 Transform Methods in Option Pricing

5.1 The Carr-Madan Method

We now move on to discussing option pricing by the Fourier transform and the efficient Fast Fourier Transform algorithm. Carr and Madan [13] lay the framework for using the Fourier transform to solve the option pricing problem. It should be noted that this method is only technically valid for options of European expiry style, though a popular related technique known as the convolution or CONV method, which can accommodate early exercise features, can be found in [43]. Furthermore, the reader will note a few points. First, this section restricts the formulations to the case of near-the-money (vanilla) call options, as this is all the later empirical study will utilize. This does not, however, mean that this is the extent of the capability of the framework. First, to obtain put prices, one can either simply use put-call parity, or utilize a closed-formulation for the put price, which is easily obtained by applying some obvious analogous calculus as will be presented for the case of call options, or referring to a multitude of easy to find sources. Furthermore, Carr and Madan also present an important extension to the case of out-of-the-money options near maturity, as some numerical difficulties arise in these cases when using the base methodology. Though very useful, since the empirical study only considers near-the-money options with at least thirty days to maturity (more on the specifics of the data used later), we will skip the details of this alternate pricing formula and the reader should refer to [13] if needed.

5.1.1 The Fourier Transform of a Near-the-Money Vanilla Call Option

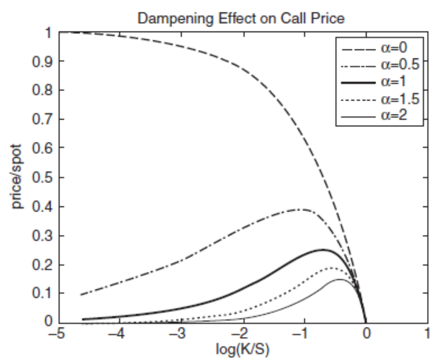
Consider the classical case of pricing a European call option. Risk neutral valuation yields

$$C_T(k) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[max(S_T - K, 0)] = e^{-rT}\int_k^{\infty}(e^s - e^k)q_T(s)ds, \tag{5.1}$$

where C_T is the time 0 price of the claim maturing at time T, $s = \log(S)$, $k = \log(K)$, and q_T is the risk-neutral density of s_T. However, one encounters an issue as $k \to -\infty$, as $C_T(k)$ approaches the underlying value, $C_T(k) \to S_0$, and is thus not square-integrable. That is, consider equation (5.1) with $k = -\infty$, resulting in

$$C_T(-\infty) = e^{-rT}\int_{-\infty}^{\infty}(e^s - e^{-\infty})q_T(s)ds = e^{-rT}\int_{-\infty}^{\infty}e^s ds \implies \int_{-\infty}^{\infty}|e^s|^2 ds = \infty. \tag{5.2}$$

Accordingly, Carr and Madan proposed using a modified call price to remedy this problem, by introducing a "dampening effect," multiplying the call price by $e^{\alpha k}$ and effectively inducing exponential decay as k becomes negative. An illustration of this effect, provided in [33], can be seen below to develop the intuition.



With the modified call price satisfying the square-integrability condition, consider now the Fourier transform of $c_T(k) = e^{\alpha k}C_T(k)$,

$$\Psi_T(\nu) = \int_{-\infty}^{\infty}e^{i\nu k}c_T(k)dk. \tag{5.3}$$

The square-integrability importantly allows for the inversion of this transform, yielding, for the call price,

$$c_T(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\nu k}\Psi_T(\nu)d\nu \tag{5.4}$$

$$= C_T(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty}e^{-i\nu k}\Psi_T(\nu)d\nu, \tag{5.5}$$

which can be expressed as

$$C_T(k) = \frac{e^{-\alpha k}}{\pi}\text{Re}[\int_0^{\infty}e^{-i\nu k}\Psi_T(\nu)d\nu], \tag{5.6}$$

due to the fact that call option prices must be real values, and thus $\Psi_T(\nu)$ is odd in its imaginary part and even in its real part. An analytical expression for $\Psi_T(\nu)$ in terms of $\phi_T(\nu)$, the characteristic function of the log return, exists and is given by (for Proof see Appendix E):

$$\Psi_T(\nu) = \frac{e^{-rT}\phi_T(\nu - (\alpha+1)i)}{\alpha^2 + \alpha - \nu^2 + i(2\alpha+1)\nu}. \tag{5.7}$$

Substituting this result for $\Psi_T(\nu)$ in the pricing formula (5.6) and evaluating the expression will yield the price of the call option as:

$$C_T(k) = \frac{e^{-\alpha k}e^{-rT}}{\pi}\mathrm{Re}[\int_0^\infty \frac{e^{-i\nu k}\phi_T(\nu - (\alpha+1)i)}{\alpha^2 + \alpha - \nu^2 + i(2\alpha+1)\nu}]. \tag{5.8}$$

The remaining problem is therefore how one should evaluate equation (5.8). As the option price has been expressed as a direct Fourier transform, the Fast Fourier Transform (hereafter FFT) algorithm is well suited.

5.1.2 The FFT Algorithm for Vanilla Option Pricing

The FFT algorithm is a method for computing sums of the form

$$\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}jk}x_j, \quad k = 0, ..., N-1. \tag{5.9}$$

We can approximate $C_T(k)$ by applying the trapezoidal integration rule and setting $\nu_j = \eta(j-1)$, resulting in

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=0}^{N-1} e^{-i\nu_j k}\Psi_T(\nu_j)\eta, \tag{5.10}$$

where $\eta > 0$ is the integration step, $\nu_j = \eta \cdot j$, and k is a vector of strikes near the money, ranging from $\theta - b$ to $\theta + b$ with $\theta = \ln(S_0)$, $b = \frac{\pi}{\eta}$ and N steps of size $\lambda = \frac{2b}{N-1}$, or $k = \theta - b + \lambda u, \quad u = 0, ..., N-1$. To obtain an accurate integration with larger values of η, Carr and Madan suggest incorporating the Simpson weighting rule, resulting in

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=0}^{N-1} e^{-i\nu_j k}\Psi_T(\nu_j)\eta \cdot S_w \tag{5.11}$$

$$\approx \frac{e^{-\alpha k}}{\pi}\sum_{j=0}^{N-1} e^{-i\nu_j k}\Psi_T(\nu_j)\frac{\eta}{3}(3 + (-1)^{j+1} - \delta_j), \tag{5.12}$$

with δ_j the Kronecker delta function. Substituting ν_j and k into the above equation yields

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=0}^{N-1} e^{-i\eta j(\theta - b + \lambda u)}\Psi_T(\eta j)\frac{\eta}{3}(3 + (-1)^{j+1} - \delta_j). \tag{5.13}$$

Simplifying a little yields the following final form of the call price formula:

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=0}^{N-1} e^{-i\eta j \lambda u} \underbrace{\Psi_T(\eta j) e^{i\eta j(b-\theta)} \frac{\eta}{3}(3 + (-1)^{j+1} - \delta_j)}_{=x_j}. \tag{5.14}$$

Recall that the FFT algorithm computes sums of the form given in equation (5.9), where we see that for our application we need $\eta\lambda = \frac{2\pi}{N}$. As long as N is chosen as a power of 2 (at least 2^{14} is a good rule of thumb), and all other parameters are set as defined above, this condition will be satisfied.

5.2 Lévy Process Parameter Calibration Using Transform Pricing

Taking market prices as a reference, the transform pricing method presented above can be used to calibrate the parameters of a given Lévy market model. That is, the option pricing problem is inverted, with market prices assumed as given, and the ETS process calibrated as to most closely replicate these given data points. To this end, an objective or cost function is constructed to achieve the best approximation to the data. A natural choice for a convex objective function is of the least squares form, given by

$$NLS = \min_{\tilde{\vec{\mathbf{v}}}, \tilde{\eta}} \frac{1}{N} \sum_{i=1}^{N} [(\text{market price})_i - (\text{calculated price})_i]^2, \tag{5.15}$$

for each i strike and where a tilde represents the parameter under the risk-neutral measure. Note that η here is from the ETS model, and this differs from the η of the FFT algorithm as presented. However, as noted in [44], this results in a bias towards more expensive in-the-money call options over the relatively cheaper out-of-the-money options. Thus, as proposed in for instance [2], we will use an average pricing error, given by

$$APE = \min_{\tilde{\vec{\mathbf{v}}}, \tilde{\eta}} \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\text{market price}_i - \text{calculated price}_i}{\text{market price}_i} \right|. \tag{5.16}$$

A significant issue remains due to this problem being ill-posed, in the sense that the calibration may be non-unique and highly sensitive to the initial value choices. There are a few ways we can deal with this. The first and perhaps most intuitive, though heuristic, is to simply run the optimization multiple times with different initial value estimates, and choose the eventual calibration which minimizes the cost function of equation (5.16). Alternatively, we can utilize the regularization framework developed in [18], [19], and [20], which draws heavily on the minimal entropy measure concepts of sections 2 and 3. This regularization framework is detailed in Appendix K. Both methods (that is, the multiple initial values approach and a regularized approach) were tested as part of the empirical study to be described in the next section, and the method of simply using multiple initial value estimates for the parameter vector and minimizing equation (5.16) was found to outperform the regularized approach. Accordingly, the reader should assume the use of the multiple initial value approach whenever calibration is discussed moving forward, and as mentioned can find a more detailed analysis of the findings in Appendix K.

6 Data, Methodology, and Results

6.1 Dataset

The dataset for the empirical analysis covers the period of January 2004 - December 2017. A set of daily option chains on the Standard and Poor's 500 index (S&P 500) was obtained from the Chicago Board Options Exchange (CBOE), known as the SPX Options Traditional product. The CBOE data has the benefit of including a quote at 3:45 PM U.S. Eastern time (15 minutes before the market close). The contract has a European exercise style and is AM-settled on the 3rd Friday of every month. Only call options were considered in the calibration of the ETS model. The average of the market bid and ask was used for the market price, and following [61] and [23], the following filters were imposed upon the data:

1) Moneyness is constrained to $-0.10 \leq (K/S_t - 1) \leq 0.10$,
2) Annualized implied volatility must be between 5% and 95%,
3) Remove options with premia below the lower bound, in particular $\max(0, S_t - q_t S_t - K e^{-r_t(T-t)})$ for call options,
4) Verify that call prices are a decreasing function of strike.

However, the option data was used in another important way, and in this case these filters proved far too restrictive, so it may be a good point to detail this consideration. In order to add context to the performance of the strategy, we of course need a suitable benchmark. In the spirit of using option-implied information to guide allocation decisions, the main benchmark will be a mean-variance version of the stochastic optimal control framework of section 4, where variance is implied from option prices using the well-known "model-free" approach (see [11],[30], and [14]). This represents an interesting benchmark because, while similarly relying on option-implied information, it also contrasts the ETS model in a number of ways. First, it is of course a mean-variance type optimization, and therefore it allows us to differentiate the performance of a portfolio that is incorporating the option market's expectations of the higher moments of the distribution of forward returns from a portfolio that does not. Secondly, as stated, it is a non-parametric or model-free approach and therefore does not rely on any option pricing model. Again, the filters listed above are far too restrictive for this methodology, and produce implied variance numbers that are nonsensically low. Accordingly, model-free implied variance is calculated in similar fashion to the widely followed VIX index (see [15] and [62]) as follows. First, the options used encompass a cross section of only *out-of-the-money* options, meaning that the options used are puts when $S_t < K$ and calls when $S_t > K$. No limit is initially placed on how "deep" out-of-the-money these options are allowed to be, however no zero bid options are included, and the data set is cut off when instances arise of two consecutive zero bid strikes. As before, option prices are taken as the average of the bid and ask price and have 30 calendar days to maturity. Model-free implied variance is then computed as

$$\sigma^2_{MFIV} = \frac{2}{t} \sum_i \frac{\triangle K_i}{K_i^2} e^{rt} Q(K_i), \tag{6.1}$$

where
K_i = strike for out-of-the-money option i.

$\triangle K_i =$ distance between successive strikes.
$Q(K_i) =$ mid-point of bid-ask spread for an option with strike K_i.

The optimal control in this framework is the the well-known result of [51]

$$\hat{\pi} = \frac{\mu_t - r_t}{\gamma \sigma^2_{MFIV}}. \tag{6.2}$$

Two benchmark portfolios will actually be formed based on this mean-variance optimal control. Recall from Theorem 4.1 that under the assumption that the risky asset (and by association the wealth process) can experience discontinuities, the investor will never take levered long or short positions in the risky asset. However, as also explained in section 4, in the case of pure diffusion processes, the investor retains a level of control over the portfolio that admits levered long and short positions. Accordingly, a non-levered version of the mean-variance portfolio, denoted "MFIV", which restricts the weights produced from equation (6.2) above to the same set of admissible controls for the ETS portfolio ($\hat{\pi} \in [0, 1]$) serves as one benchmark. Then, there is a second levered benchmark, denoted "MFIVHL", where "HL" stands for half-leverage, signifying that the leverage is tempered by the popular rule-of-thumb in practice for managing the risk of levered positions (i.e., the "half-Kelly" rule when considering log-utility). This helps temper the leverage to more realistic levels, but note this is only one such way to do so. In for instance [37], leverage was constrained by restricting the optimal control to the set $[-1, 2]$.

Additionally, the underlying spot index price along with the associated index dividend yield as well as the risk-free interest rate, chosen to be one-month LIBOR, were obtained from Bloomberg. For the investable strategy to be detailed in the next section, daily adjusted close prices for the SPDR S&P 500 ETF (ticker SPY) were also obtained from Bloomberg. Finally, a result in the next section will show that when changing from the risk-neutral to the real-world parameterization of the Lévy density, we need a number for the expected market return, $\bar{\mu}$. In order to keep with the theme of using forward looking implied information but also ensuring to always satisfy the second condition of equation (4.22), this is taken to be a long-term implied market risk-premium for the US market, available from http://www.market-risk-premia.com/us.html. For some transparency and for the reader's convenience, the calculation methodology is recreated in Appendix F.

6.2 Methodology of Investment Strategy

The first five sections of the paper lay the theoretical groundwork and supply all the tools needed for the following analysis. The goal is to take these concepts and investigate if they can be holistically employed in practice to improve the allocation choice problem and ultimately portfolio performance. As discussed in the introduction, there have already been studies, for instance [37] as mentioned but also [21], [26], and [27], which show that option-implied information can provide valuable signals in constructing portfolios. Further, in for instance [23] it was found that market information extracted under an exponential Lévy market assumption (although here only the Meixner and Normal Inverse Gaussian models were considered) showed that the risk preferences of equity investors were signaling anomalous behavior well before the U.S. subprime crisis of late 2007. These ideas are brought together and built upon in the current analysis to answer the question of whether or not these signals have any value in an investable strategy. The methodology is as follows.

Beginning in January 2004, every 30 calendar days prior to the next month's SPX option expiry, using the transform and calibration methods of section 4, the 3:45 PM price of the filtered range of option strikes is calibrated according to the risk-neutral ETS price process of section 3. These option-implied parameters effectively give a one-month forward risk-neutral density forecast for the underlying asset. The calibrated risk-neutral parameters are then transformed to real-world parameters using the model preserving minimal entropy measure detailed in section 3. Some elaboration on the numerical approach is useful here. First, the risk-neutral parameters calibrated from option prices are importantly *assumed to be under the model preserving minimal entropy martingale measure*. Thus, we adjust equation (3.17) to

$$r - q - \psi_{ETS}(-i; \tilde{\vec{\mathbf{v}}}) = \bar{\mu} - \psi_{ETS}(-i; \lambda_+, \lambda_-, \beta_- - \theta, \beta_+ + \theta, \alpha), \tag{6.3}$$

with $-\beta_+ < \theta < \beta_-$, and $\bar{\mu}$ is the implied expected market return (as mentioned previously, refer to Appendix F for the calculation methodology). An "Esscher cost function," or ECF, is thus constructed as

$$\begin{aligned} ECF = \min_{\theta} \Big\{ &[(r - q - \psi_{ETS}(-i; \tilde{\vec{\mathbf{v}}})) - (\bar{\mu} - \psi_{ETS}(-i; \lambda_+, \lambda_-, \beta_- - \theta, \beta_+ + \theta, \alpha))]^2 \\ &+ t\lambda_+ \Gamma(-\alpha)((\alpha - 1)\tilde{\beta_+}^{\alpha} - \alpha(\beta_+ + \theta)\tilde{\beta_+}^{\alpha-1} + (\beta_+ + \theta)^{\alpha}) \\ &+ t\lambda_- \Gamma(-\alpha)((\alpha - 1)\tilde{\beta_-}^{\alpha} - \alpha(\beta_- - \theta)\tilde{\beta_-}^{\alpha-1} + (\beta_- - \theta)^{\alpha}) \Big\}, \end{aligned} \tag{6.4}$$

and evaluated numerically using a search algorithm (in particular, the Nelder-Mead simplex method of [40] implemented via MatLab's *fminsearch* function). Once the real-world parameters have been calculated, they are utilized in the stochastic optimal control framework of section 4 to derive the optimal portfolio for an individual investor with some assumed level of risk aversion. Specifically, equation (4.52) is solved numerically for $\hat{\pi}$. Again a caveat is worth noting here. Recall the relation between stochastic and ordinary exponentials from Theorem 2.10. Also, realize that the optimal control of section 4 was derived starting from the geometric type stochastic differential equation for the risky asset price in equation (4.21). Accordingly, it is important to transform the triplet used as an input in equation (4.52) using the ordinary-stochastic exponential relation. In particular for the ETS process, we have (with subscript L denoting the variable under the ordinary exponential; for Proof see Appendix G):

$$\begin{aligned} &\sigma = \sigma_L, \\ &\nu(dx) = \tfrac{\lambda_+(1+x)^{-\beta_+}}{(\log(1+x))^{\alpha+1}} \chi_{\{x>0\}} dx + \tfrac{\lambda_-(1+x)^{\beta_-}}{(-\log(1+x))^{\alpha+1}} \chi_{\{-1<x<0\}} dx, \\ &\mu = \mu_L + \frac{\sigma_L^2}{2} \\ &\quad + \Gamma(-\alpha)\Big\{ -\frac{\lambda_+}{\beta_+}[\beta_+^{\alpha+1} - \alpha\beta_+^{\alpha} - \beta_+(\beta_+ - 1)^{\alpha}] - \frac{\lambda_-}{\beta_-}[\beta_-^{\alpha+1} + \alpha\beta_-^{\alpha} - \beta_-(\beta_- + 1)^{\alpha}] \Big\}. \end{aligned} \tag{6.5}$$

On the day of each calibration (recall this importantly happens at 3:45 PM U.S. Eastern time), the investor rebalances his portfolio at the market close (4:00 PM U.S. Eastern time) by purchasing the SPY ETF in proportions given by the portfolio optimization procedure, and is assumed to invest the remainder until the next rebalancing date at the prevailing one-month LIBOR rate (i.e. keep it safe in a money market account). This procedure is repeated once a month for the entire analysis period. The same methodology applies to the mean-variance (model-free implied variance) strategy introduced earlier. Additionally, the canonical buy-and-hold and 60/40 portfolios are included as benchmarks for some additional context, although these are truly not appropriate benchmarks. All calculations were carried out in MatLab. Finally, we address the importance to simulate at least some of the transaction costs involved with implementing live trades. A cursory analysis showed that the turnover of the ETS strategy (here defined as the proportion of the total portfolio required to be traded in the risky asset to bring it in line with the new optimal control) was materially higher than the other benchmarks, including even the mean-variance strategy, and thus it is imperative to investigate whether the additional costs mitigate any performance benefits. Nevertheless, building a full and realistic implementation-shortfall model (assuming such is possible) is outside the scope and purpose of this study, and so a more parsimonious path is taken. In [17], a fairly recent study by the Chicago Mercantile Exchange, it was found that the all in costs of trading the SPY ETF were about 3.25 basis points, in addition to slightly less than 0.80 basis point holding fee per month. Similar results were also found in [52], a study by Morningstar from a few years earlier (therefore the results seem to be fairly stable). Accordingly, transaction costs are modeled as 4 basis points of the trade size, and the trade size is compensated such that the optimal control is achieved *after* transaction fees.

6.3 Performance Results

We now move to the performance of the backtested portfolios. The presentation for the main part of the text is restricted to risk aversion parameters of $\gamma = 1$ (i.e. log utility) and $\gamma = 5$, as these are illustrative of the main takeaway of the results as will be discussed. However, the reader can find performance for a number of additional risk aversion parameters in Appendices H-J, which corroborate the conclusions herein. Figures 1a and 4a show the equity curves for investor risk aversion coefficients of 1 (log utility - relatively low risk aversion) and 5 (moderately high risk aversion), and the performance illustrated in these charts is quantified using a number of metrics in the preceding tables. In order to add statistical context to the risk-adjusted performance, so called "quality control" charts are included which evaluate the significance of the Sharpe ratios of the ETS portfolios. The charts are based off of the methods of [56] for assessing the statistical significance both of Sharpe ratios in isolation, and also compared with other Sharpe ratios. Figures 2a and 5a test the significance of the ETS portfolio Sharpe ratio, that is the probability of the Sharpe ratio being the result of random chance. Specifically, a null hypothesis of $SR_{ETS} = 0$ is tested against an alternative hypothesis of $SR_{ETS} \neq 0$ by plotting the cumulative annualized Sharpe ratio of the portfolio against 95% confidence bands, which are derived off of the asymptotic distribution of the Sharpe ratio,

$$\sqrt{t}(\hat{SR}) \overset{a}{\sim} N\left(0, 1 + \frac{SR^2}{4}\left[\frac{\mu_4}{\sigma^4} - 1\right] - SR\frac{\mu_3}{\sigma^3}\right), \tag{6.6}$$

making the standard error of $\hat{SR}$

$$SE(\hat{SR}) = \sqrt{\left(1 + \frac{SR^2}{4}\left[\frac{\mu_4}{\sigma^4} - 1\right] - SR\frac{\mu_3}{\sigma^3}\right)/t}, \tag{6.7}$$

and thus the confidence bands are determined with

$$\pm 1.96 \times SE(\hat{SR}). \tag{6.8}$$

Additionally, the Sharpe ratio of the ETS portfolio is compared against both the MFIV and MFIVHL portfolios in figures 2b and 2c and 5b and 5c. Here, the null hypothesis is $(SR_a - SR_b) = SR_{diff} = 0$ against an alternative hypothesis of $SR_{diff} \neq 0$, and the confidence bands are based off of the asymptotic distribution of SR_{diff}, given by

$$\sqrt{t}(\hat{SR}_{diff}) \overset{a}{\sim} N(0, Var_{diff}), \tag{6.9}$$

where

$$\begin{aligned} Var_{diff} = 2 &+ \frac{SR_a^2}{4}\left[\frac{\mu_{4a}}{\sigma_a^4} - 1\right] - SR_a\frac{\mu_{3a}}{\sigma_a^3} + \frac{SR_b^2}{4}\left[\frac{\mu_{4a}}{\sigma_b^4} - 1\right] - SR_b\frac{\mu_{3b}}{\sigma_b^3} \\ &- 2\left[\rho_{a,b} + \frac{SR_a SR_b}{4}\left[\frac{\mu_{2a,2b}}{\sigma_a^2\sigma_b^2} - 1\right] - \frac{1}{2}SR_a\frac{\mu_{1b,2a}}{\sigma_b\sigma_a^2} - \frac{1}{2}SR_b\frac{\mu_{1a,2b}}{\sigma_a\sigma_b^2}\right], \end{aligned} \tag{6.10}$$

making, analogous to the previous case, the standard error of $\hat{SR}_{diff}$

$$SE(\hat{SR}_{diff}) = \sqrt{Var_{diff}/t}, \tag{6.11}$$

and the confidence bands determined by

$$\pm 1.96 \times SE(\hat{SR}_{diff}). \tag{6.12}$$

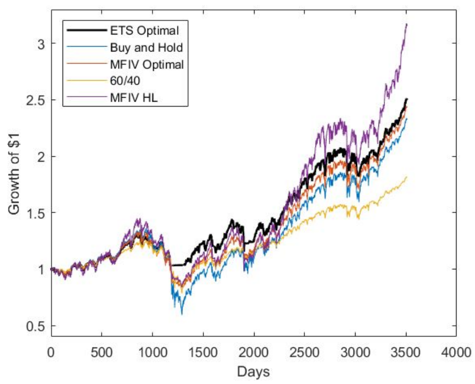
We see below that the ETS model results in statistically significant risk-adjusted performance improvements compared to the MFIV and MFIVHL portfolios when investor risk aversion is low with $\gamma = 1$, however we fail to reject the null hypothesis as risk aversion becomes moderately high with $\gamma = 5$ (again, this is corroborated in the Appendices, which show that the significance vanishes somewhere between $\gamma = 2$ and $\gamma = 3$). Still, the mean-variance framework only outperforms the Lévy framework in nominal terms for the Sortino ratio with high levels of risk aversion, and the Lévy portfolio experiences the lowest drawdowns in all cases under study. Additionally, visual inspection of the Sharpe ratio difference charts shows that most of the outperformance of the ETS model came during the financial crisis, as it did a better job of getting out of the market before the lows of the U.S. financial crisis, then relative performance of the models slowly began to converge. This is a rather interesting result, and the implications are developed further in Figures 3 and 7. These figures plot a histogram of the market's returns and overlay the coincidentally realized differential return between the ETS portfolio and the mean-variance benchmarks. If one focuses first on the left side of these charts the implication is clear - incorporating the options market's expectation for the entire forward distribution of returns, which of course includes implicitly the market's expectation of a left-tail event, *results in portfolios that are well positioned and outperform when left-tail events do occur.* The interpretation of the right-tail dependence is not as clear. As highlighted in the charts, there were two instances when the market experienced a right-tail event and the ETS portfolio underperformed the benchmarks. These were two days in October 2008 when the market experienced sharp "v-shaped" recoveries, before continuing the sell-off. Thus, the ETS portfolio was still apportioned according to the previously calibrated (and massively fat left-skewed) distribution, which turned out to be a better choice on average but simply missed out on these two days. This result is worthy of a test of the actual significance of the tail dependence, and the author hypothesizes that the left-tail dependence would be significant, but the right-tail behavior could not be distinguished from statistical noise. As an aside, note that the reason the ETS portfolio is not compared statistically to the buy-and-hold or 60/40 portfolios is due to the lack of correlation, which, as noted in [56], highly affects the power of the tests (this should be intuitive; note the correlation between the ETS and MFIV and MFIVHL portfolios is over 90%). The inclusion of these portfolios is largely conventional.

Figures 1b and 4b show the evolution of the proportion of the portfolio invested in the risky asset. The large moves in these charts signify the periodic rebalances, followed by periods of noise as the portfolio evolves over the next month. The results here are promising, and reveal the main reason for the significant outperformance of the ETS portfolio when risk aversion is low - even in these instances, the optimal strategy entails completely divesting from the risky asset at certain points, in particular prior to the lows of the financial crisis, whereas the MFIV and MFIVHL portfolios are always at least partially invested.

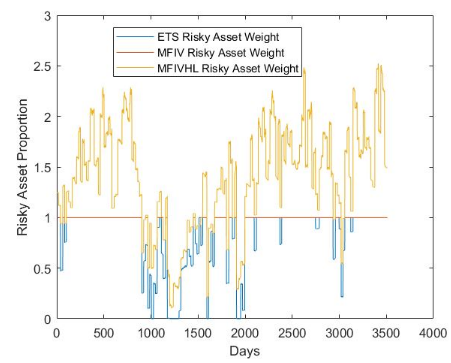
Finally, Figures 4a, 4b, and 4c, and 8a, 8b, and 8c show how the optimal control (that is, the optimal weight at each rebalancing date) evolves along with the higher moments from equations (3.2-3.4). The results here also quite thought-provoking. First, notice that the higher moments each have their own broader trends, but with significant noise about these trends. This re-motivates the consideration of a regularization framework that results in more stable parameterizations. Secondly, the counter-trends in the variance versus the skewness and kurtosis raise interesting concerns about hidden risks and complacency in the market. That is, notice that since the crisis, (calibrated) skewness has been trending lower and (calibrated) kurtosis has been trending higher, both negative developments to the utility maximizing investor. Yet, the optimal weight in the risky asset has been trending higher as well. How? The progressive trend lower in the total variance has simply overwhelmed these effects. One interpretation of this is that risk is simply being pushed into the higher moments as returns have been strong and variance has become progressively more subdued.

Table 1: Performance Metrics for $\gamma = 1$

Strategy	Ann. Excess Return	Ann. Standard Deviation	Ann. Sharpe Ratio	Ann. Sortino Ratio	Max. DD	Avg. Turnover
	$\mu - r$	σ	$\frac{\mu-r}{\sigma}$	$\frac{\mu-r}{\sigma_-}$	$\max\{\frac{Trough}{Peak} - 1\}$	$\frac{1}{N}\sum_{i=1}^{N}\lvert w_i - w_{i-1}\rvert$
ETS	5.6927%	11.704%	0.48639	0.74348	-22.53%	12.58%
MFIV	5.4818%	13.395%	0.40926	0.64058	-39.76%	6.17%
MFIV HL	7.4679%	18.468%	0.40477	0.63313	-42.17%	31.04%
B&H	5.1407%	18.414%	0.27918	0.43551	-56.47%	-
60/40	3.2812%	10.9%	0.30102	0.46734	-37.53%	0.72%

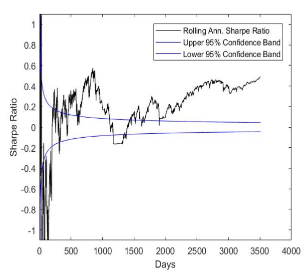


(a) Equity Curves

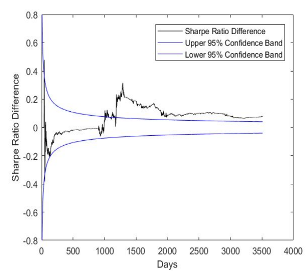


(b) ETS, MFIV, and MFIVHL Risky Asset Proportion Evolution

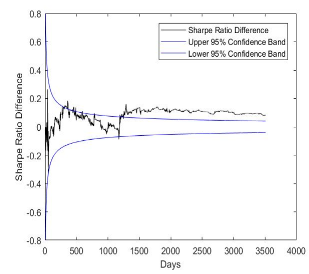
Figure 1: Equity Curves and Risky Asset Proportions for $\gamma = 1$



(a) ETS Strategy Raw SR Significance



(b) ETS-MFIV SR Difference Significance



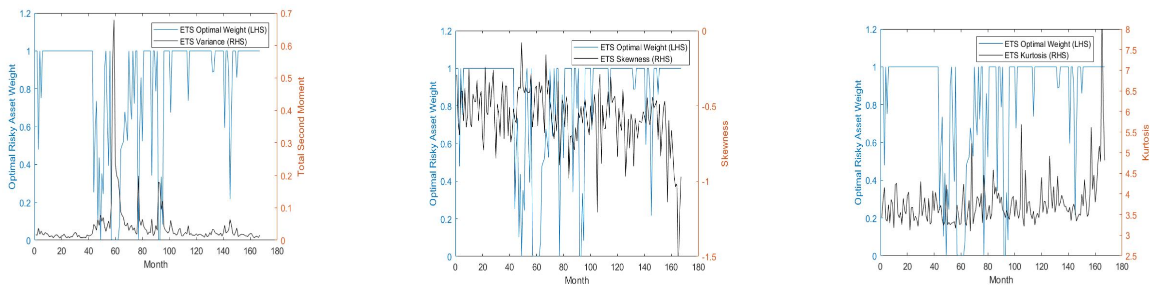
(c) ETS-MFIVHL SR Difference Significance

Figure 2: Sharpe Ratio Quality Control Charts for $\gamma = 1$

(a) ETS-MFIV Market Dependence

(b) ETS-MFIVHL Market Dependence

Figure 3: Market Dependence of Differential Returns for $\gamma = 1$



(a) Total Variance vs Optimal Control

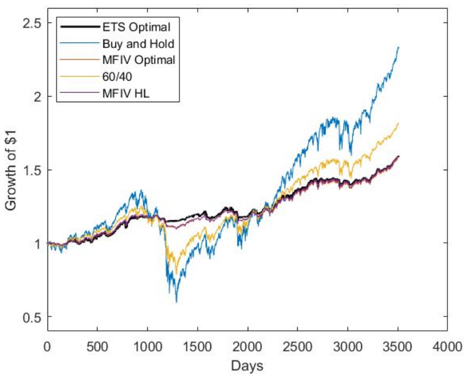
(b) Skewness vs Optimal Control

(c) Kurtosis vs. Optimal Control

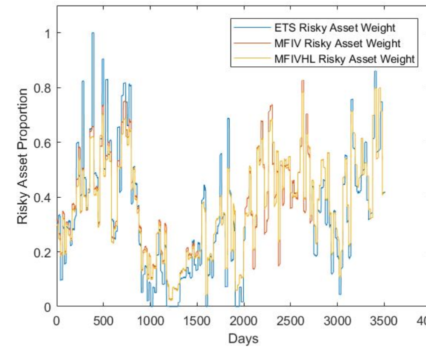
Figure 4: ETS Higher Moments vs. Optimal Control for $\gamma = 1$

Table 2: Performance Metrics for $\gamma = 5$

Strategy	**Ann. Excess Return**	**Ann. Standard Deviation**	**Ann. Sharpe Ratio**	**Ann. Sortino Ratio**	**Max. DD**	**Avg. Turnover**
	$\mu - r$	σ	$\frac{\mu-r}{\sigma}$	$\frac{\mu-r}{\sigma_-}$	$\max\{\frac{Trough}{Peak} - 1\}$	$\frac{1}{N}\sum_{i=1}^{N}\lvert w_i - w_{i-1}\rvert$
ETS	2.3021%	4.6498%	0.4951	0.75724	-7.30%	14.70%
MFIV	2.2826%	4.768%	0.47874	0.74597	-8.57%	11.40%
MFIV HL	2.3243%	4.7365%	0.49072	0.76664	-8.22%	10.55%
B&H	5.1407%	18.414%	0.27918	0.43551	-56.47%	-
60/40	3.2812%	10.9%	0.30102	0.46734	-37.53%	0.72%

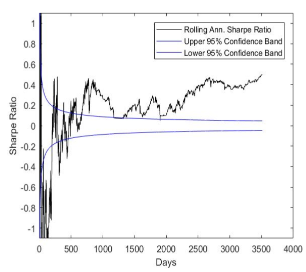


(a) Equity Curves

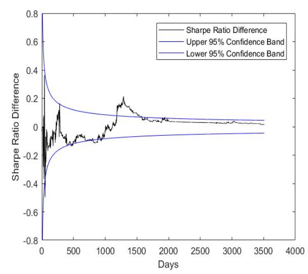


(b) ETS, MFIV, and MFIVHL Risky Asset Proportion Evolution

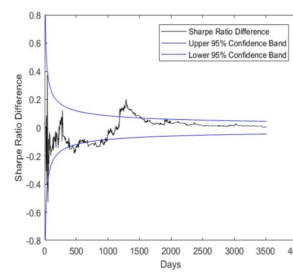
Figure 5: Equity Curves and Risky Asset Proportions for $\gamma = 5$



(a) ETS Strategy Raw SR Significance

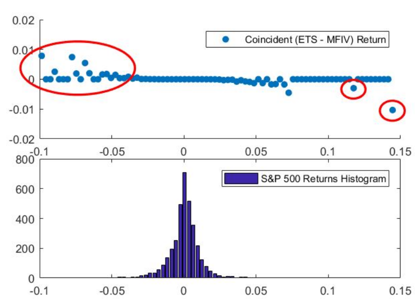


(b) ETS-MFIV SR Difference Significance

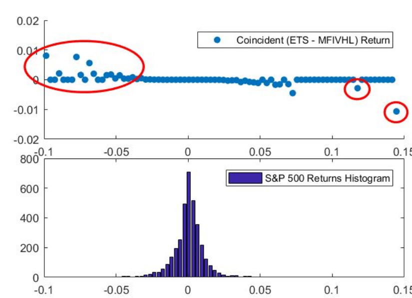


(c) ETS-MFIVHL SR Difference Significance

Figure 6: Sharpe Ratio Quality Control Charts for $\gamma = 5$

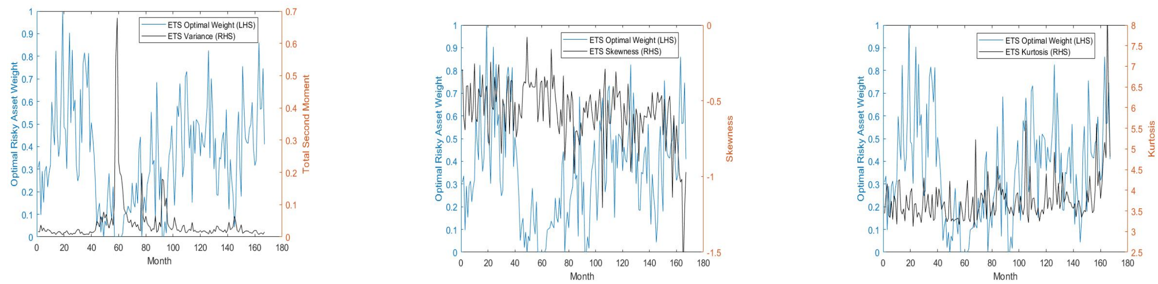


(a) ETS-MFIV Market Dependence



(b) ETS-MFIVHL Market Dependence

Figure 7: Market Dependence of Differential Returns for $\gamma = 5$



(a) Total Variance vs Optimal Control

(b) Skewness vs Optimal Control

(c) Kurtosis vs. Optimal Control

Figure 8: ETS Higher Moments vs. Optimal Control for $\gamma = 5$

6.4 Conclusion

This paper examined the performance of a dynamic investment strategy which allocates between a risky and a risk-free asset by utilizing option market information and solving the decision problem under an exponential Lévy market assumption. Using a log stock price process with an Exponentially Tempered Stable jump-diffusion process as the martingale component and a CRRA utility function as representative of the risky asset and investor preferences, respectively, the risk-neutral Lévy density was calibrated to option prices, transformed to the real-world or objective probability measure, and then the optimal portfolio was derived using the solution to a stochastic optimal control problem.

It was found that portfolios formed under the optimal strategy, when backtested and compared to mean-variance alternatives, offered statistically significant improvements in risk-adjusted performance when investor risk-aversion was low, however these gains are lost as risk aversion increases. The exact reason for this essentially linear phenomenon of convergence of the risk-adjusted performance of the strategies as risk aversion increases is an open question worthy of additional thought and research. What's more clear is that the use of option market data in calibrating the Lévy density and the flexibility of fitting in particular the skewness and kurtosis of the implied distribution does appear to have shown the ability to pick up on anomalous behavior some time prior to the financial crisis lows of 2008-2009, and the strategy offered the most value-added returns when the market experienced a left-tail event.

Broadly, the results confirm the findings of [37], and suggest a performance improvement from using both more realistic models of asset price dynamics and option implied expectations and motivates further investigation of both. Is there a Lévy measure which better models implied distributions? Can the calibration be improved to be both stable (particularly as regards the implied higher moments) and precise? Do option implied distributions, although prescient (seemingly, let's realize the sample size) of rare events, cause the investor to be systematically under-invested due to the enduring volatility skew? Are HARA type utility functions ideal in capturing investor attitudes towards non-normally distributed risks? We conclude for now.

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Appendices

A Proof of the Higher Moments of the ETS Process

For a general Lévy density $\nu(x)$ for the random variable x representing the level of the Lévy process at time 1, we have

$$\text{Variance} = \int_{-\infty}^{\infty} x^2 \nu(x) dx. \tag{A.1}$$

$$\text{Skewness} = \frac{\mathbb{E}[(x-\mathbb{E}(x))^3]}{(\mathbb{E}[(x-\mathbb{E}(x))^2])^{3/2}} = \frac{\int_{-\infty}^{\infty} x^3 \nu(x) dx}{\text{Variance}^{3/2}}. \tag{A.2}$$

$$\text{Kurtosis} = \frac{\mathbb{E}[(x-\mathbb{E}(x))^4]}{(\mathbb{E}[(x-\mathbb{E}(x))^2])^2} = 3 + \frac{\int_{-\infty}^{\infty} x^4 \nu(x) dx}{\text{Variance}^2}. \tag{A.3}$$

For the variance, we therefore have

$$\text{Variance} = \lambda_+ \int_0^{\infty} x^2 \frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx + \lambda_- \int_{-\infty}^{0} x^2 \frac{e^{-\beta_- |x|}}{|x|^{\alpha+1}} dx$$

$$= \lambda_+ \int_0^{\infty} x^{-\alpha-1} x^2 e^{-\beta_+ x} dx + \lambda_- \int_0^{\infty} x^{-\alpha-1} x^2 e^{-\beta_- x} dx. \tag{A.4}$$

It is now important to recall the gamma function, $\Gamma(\cdot)$, given by

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \tag{A.5}$$

or, equivalently

$$\Gamma(z) = r^z \int_0^{\infty} x^{z-1} e^{-rx} dx, \tag{A.6}$$

which implies that

$$r^{-z}\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-rx} dx. \tag{A.7}$$

Accordingly, we have

$$\text{Variance} = \lambda_+ \Gamma(2-\alpha)\left(\frac{1}{\beta_+^{2-\alpha}}\right) + \lambda_- \Gamma(2-\alpha)\left(\frac{1}{\beta_-^{2-\alpha}}\right), \tag{A.8}$$

and this factors clearly as in equation (3.2). For the skewness, we have

$$\text{Skewness} = \frac{\lambda_+ \int_0^{\infty} x^3 \frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx + \lambda_- \int_{-\infty}^{0} x^3 \frac{e^{-\beta_- |x|}}{|x|^{\alpha+1}} dx}{\text{Variance}^{3/2}}$$

$$= \frac{\lambda_+ \int_0^{\infty} x^{-\alpha-1} x^3 e^{-\beta_+ x} dx - \lambda_- \int_0^{\infty} x^{-\alpha-1} x^3 e^{-\beta_- x} dx}{\text{Variance}^{3/2}}$$

$$= \frac{\lambda_+\Gamma(3-\alpha)\left(\frac{1}{\beta_+^{3-\alpha}}\right) - \lambda_-\Gamma(3-\alpha)\left(\frac{1}{\beta_-^{3-\alpha}}\right)}{\text{Variance}^{3/2}}, \tag{A.9}$$

which again clearly factors as in equation (3.3). Finally, for kurtosis we have

$$\text{Kurtosis} = \frac{\lambda_+ \int_0^\infty x^4 \frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx + \lambda_- \int_{-\infty}^0 x^4 \frac{e^{-\beta_-|x|}}{|x|^{\alpha+1}} dx}{\text{Variance}^2}$$

$$= \frac{\lambda_+ \int_0^\infty x^{-\alpha-1} x^4 e^{-\beta_+ x} dx + \lambda_- \int_0^\infty x^{-\alpha-1} x^4 e^{-\beta_- x} dx}{\text{Variance}^2}$$

$$= \frac{\lambda_+\Gamma(4-\alpha)\left(\frac{1}{\beta_+^{4-\alpha}}\right) + \lambda_-\Gamma(4-\alpha)\left(\frac{1}{\beta_-^{4-\alpha}}\right)}{\text{Variance}^2}, \tag{A.10}$$

which again clearly factors to equation (3.4).

B Proof of the Characteristic Function of the ETS Process

From the Lévy Khintchine formula given in Theorem 2.3, and considering $\mu = 0, \sigma^2 = 0$, we have that

$$\psi_{ETS}(u; \overrightarrow{\mathbf{v}}) = \int_{-\infty}^{\infty} (e^{iux} - 1)\nu(dx). \tag{B.1}$$

Given the structure of the ETS Lévy measure (that is, specified for both positive and negative values of x), this can be written as the sum

$$\psi_{ETS}(u; \overrightarrow{\mathbf{v}}) = \lambda_+ \int_0^\infty (e^{iux} - 1) \frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx + \lambda_- \int_{-\infty}^0 (e^{iux} - 1) \frac{e^{-\beta_-|x|}}{|x|^{\alpha+1}} dx. \tag{B.2}$$

Due to the relation that $\int_a^b x = -\int_b^a x$, we can rewrite this as

$$\psi_{ETS}(u; \overrightarrow{\mathbf{v}}) = \lambda_+ \int_0^\infty (e^{iux} - 1) \frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx + \lambda_- \int_0^\infty (e^{-iux} - 1) \frac{e^{-\beta_- x}}{x^{\alpha+1}} dx. \tag{B.3}$$

Focusing on each term of equation (B.3) separately, we have for the first term,

$$= \lambda_+ \int_0^\infty \frac{e^{iux - \beta_+ x}}{x^{\alpha+1}} - \frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx$$

$$= \lambda_+ \int_0^\infty x^{-\alpha-1} \left\{ e^{-(\beta_+ - iu)x} - e^{-\beta_+ x} \right\} dx$$

$$= \lambda_+ \int_0^\infty x^{-\alpha-1} e^{-(\beta_+ - iu)x} dx - \lambda_- \int_0^\infty x^{-\alpha-1} e^{-\beta_+ x} dx. \tag{B.4}$$

Recalling the Gamma function of equations (A.5-A.7), the first term of equation (B.4) above yields

$$\lambda_+ \int_0^\infty x^{-\alpha-1} e^{-(\beta_+ - iu)x} dx = \lambda_+ (\beta_+ - iu)^\alpha \Gamma(-\alpha), \tag{B.5}$$

and the second term yields

$$\lambda_+ \int_0^\infty x^{-\alpha-1} e^{-\beta_+ x} dx = \lambda_+ \beta_+^\alpha \Gamma(-\alpha). \tag{B.6}$$

Thus, in total, equation (B.4) yields

$$\lambda_+ \int_0^\infty x^{-\alpha-1} e^{-(\beta_+ - iu)x} dx - \lambda_+ \int_0^\infty x^{-\alpha-1} e^{-\beta_+ x} dx = \lambda_+ \Gamma(-\alpha)[(\beta_+ - iu)^\alpha - \beta_+^\alpha]. \tag{B.7}$$

By the same logic, we evaluate now the second term of equation (B.3) by

$$\begin{aligned} \lambda_- \int_0^\infty (e^{-iux} - 1) \frac{e^{-\beta_- x}}{x^{\alpha+1}} dx &= \lambda_- \int_0^\infty x^{-\alpha-1} \left\{ e^{-(\beta_- + iu)x} - e^{-\beta_- x} \right\} dx \\ = \lambda_- \int_0^\infty x^{-\alpha-1} e^{-(\beta_- + iu)x} dx - \lambda_- \int_0^\infty x^{-\alpha-1} e^{-\beta_- x} dx &= \lambda_- \Gamma(-\alpha)[(\beta_- + iu)^\alpha - \beta_-^\alpha]. \end{aligned} \tag{B.8}$$

So, adding equations (B.7) and (B.8), we are led to the characteristic exponent of the ETS process,

$$\psi_{ETS}(u; \overrightarrow{\mathbf{v}}) = \lambda_+ \Gamma(-\alpha)[(\beta_+ - iu)^\alpha - \beta_+^\alpha] + \lambda_- \Gamma(-\alpha)[(\beta_- + iu)^\alpha - \beta_-^\alpha]. \tag{B.9}$$

Multiplying this result by t and exponentiating leads to the final form of the characteristic function ϕ_{ETS} as shown in equation (3.6).

C Proof of Characteristic Function of ETS Log Stock Price Process

The characteristic function of the logarithm of the stock price is given by

$$\phi_{\ln(S_t)}(u, t) = \mathbb{E}^{\mathbb{P}}[e^{iu\ln(S_t)}]. \tag{C.1}$$

For the ETS stock price process, we thus have

$$\begin{aligned} \phi_{\ln(S_t)}(u, t) &= \mathbb{E}^{\mathbb{P}}[e^{iu\ln(S_t)}] \\ &= \mathbb{E}^{\mathbb{P}} \left[\exp \left(iu\{\ln(S_0) \cdot \exp(\mu + \omega - \eta^2/2)t + X_{ETS_e}(t; \overrightarrow{\mathbf{v}}, \eta)\} \right) \right], \end{aligned} \tag{C.2}$$

which decomposes to:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[e^{iu\ln(S_t)}] = \mathbb{E}^{\mathbb{P}} \left[\exp\{iu \ln(S_0)\} \right] \cdot \mathbb{E}^{\mathbb{P}} \left[\exp\{iu(\mu + \omega - \eta^2/2)t + \eta W_t\} \right] \cdot ... \\ ... \cdot \mathbb{E}^{\mathbb{P}} \left[\exp\{iuX_{ETS}(t; \overrightarrow{\mathbf{v}})\} \right]. \end{aligned}$$

A few simplifications are obvious here. First, the first expectation clearly drops off as S_0 is assumed a known constant. Next, by the definition of a characteristic function the third expectation is

$$\mathbb{E}^{\mathbb{P}} \left[\exp\{iuX_{ETS_e}(t; \overrightarrow{\mathbf{v}})\} \right] = \phi_{ETS}(u; \overrightarrow{\mathbf{v}}).$$

Further, since the second expectation term is simply the characteristic function of a Gaussian random variable, we also have

$$\mathbb{E}^{\mathbb{P}} \left[\exp\{iu(\mu + \omega - \eta^2/2)t + \eta W_t\} \right] = \exp\{iu(\mu + \omega - \eta^2/2)t - \eta^2 u^2/2\},$$

and we can now bring equation (C.2) to the form shown in equation (3.11).

D Proof of the Relative Entropy of the ETS Process

From Proposition 2 in [19] (see [34]), it can be shown that

$$H(\mathbb{Q}|\mathbb{P}) = t\int_{-\infty}^{\infty}(\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1)\nu(dx), \tag{D.1}$$

where $\psi(x) = (\beta_+ - \tilde{\beta_+})x\chi_{\{x>0\}} - (\beta_- - \tilde{\beta_-})x\chi_{\{x<0\}}$. A Taylor/MacLaurin expansion of the term attached to the Lévy measure in the integrand thus yields

$$\begin{aligned}
&\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1 \\
&\quad= \left(\sum_{n=0}^{\infty}\frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1} - \sum_{n=0}^{\infty}\frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n} + 1\right)\chi_{\{x>0\}} \\
&\qquad+ \left(\sum_{n=0}^{\infty}\frac{1}{n!}(-(\beta_- - \tilde{\beta_-})x)^{n+1} - \sum_{n=0}^{\infty}\frac{1}{n!}(-(\beta_- - \tilde{\beta_-})x)^{n} + 1\right)\chi_{\{x<0\}}.
\end{aligned} \tag{D.2}$$

Now, since the first term of the second summation of each term cancels with the addition of 1, and since the first term of the first summation of each term cancels with the second term of the second summation of each term, and further since we can change the bounds of the negative part of the Lévy measure integration, the relative entropy can be written as

$$\begin{aligned}
&H(\mathbb{Q}|\mathbb{P}) \\
&\quad= t\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1}\nu(dx) - t\int_0^{\infty}\sum_{n=2}^{\infty}\frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n}\nu(dx) \\
&\qquad+ t\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}((\beta_- - \tilde{\beta_-})x)^{n+1}\nu(dx) - t\int_0^{\infty}\sum_{n=2}^{\infty}\frac{1}{n!}((\beta_- - \tilde{\beta_-})x)^{n}\nu(dx).
\end{aligned} \tag{D.3}$$

Evaluating the first integral is representative of the process for each, so let's focus on this term. To avoid confusion denote this integral as $H_1(\mathbb{Q}|\mathbb{P})$. We have

$$\begin{aligned}
&H_1(\mathbb{Q}|\mathbb{P}) = \int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1}\nu(dx) \\
&= \int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1}\left(\lambda_+\frac{e^{-\beta_+ x}}{x^{\alpha+1}}dx\right) = \int_0^{\infty}\sum_{n=2}^{\infty}\frac{1}{(n-1)!}((\beta_+ - \tilde{\beta_+})x)^{n}\left(\lambda_+\frac{e^{-\beta_+ x}}{x^{\alpha+1}}dx\right).
\end{aligned} \tag{D.4}$$

Recalling the Gamma function of equations (A.5-A.7), we have

$$\begin{aligned}
H_1(\mathbb{Q}|\mathbb{P}) &= \int_0^{\infty}\sum_{n=2}^{\infty}\frac{1}{(n-1)!}((\beta_+ - \tilde{\beta_+})x)^{n}\lambda_+ x^{-\alpha-1}e^{-\beta_+ x}dx \\
&= \sum_{n=2}^{\infty}\frac{1}{(n-1)!}(\beta_+ - \tilde{\beta_+})^{n}\beta_+^{\alpha-n}\Gamma(n-\alpha) = \sum_{n=2}^{\infty}\frac{1}{(n-1)!}\beta_+^{\alpha}\left(1 - \frac{\tilde{\beta_+}}{\beta_+}\right)^{n}\Gamma(n-\alpha).
\end{aligned} \tag{D.5}$$

Now, given that $\Gamma(n-\alpha) = (n-\alpha-1)!$, we can factor out some terms and write this as

$$
\begin{aligned}
H_1(\mathbb{Q}|\mathbb{P}) &= \lambda_+\beta_+^\alpha\Gamma(-\alpha)\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\left(\frac{\tilde{\beta_+}}{\beta_+}-1\right)^n(\alpha-n+1)! \\
&= \lambda_+\beta_+^\alpha\Gamma(-\alpha)\left(\frac{\tilde{\beta_+}}{\beta_+}-1\right)\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\left(\frac{\tilde{\beta_+}}{\beta_+}-1\right)^{n-1}\alpha(\alpha-1)\cdots(\alpha-n+1). \quad \text{(D.6)}
\end{aligned}
$$

Now, since $\frac{\partial}{\partial x}\frac{x^n}{n!} = \frac{nx^{n-1}}{n!} = \frac{x^{n-1}}{\Gamma(n)} = \frac{x^{n-1}}{(n-1)!}$, we have

$$
H_1(\mathbb{Q}|\mathbb{P}) = \lambda_+\beta_+^\alpha\Gamma(-\alpha)\left(\frac{\tilde{\beta_+}}{\beta_+}-1\right)\left[\frac{\partial}{\partial x}\sum_{n=2}^{\infty}\frac{1}{n!}x^n\alpha(\alpha-1)\cdots(\alpha-n+1)\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1}. \quad \text{(D.7)}
$$

We now need to recall a few results from the Binomial Theorem and its generalizations, specifically

$$
(1+x)^\delta = \sum_{n=0}^{\infty}\binom{\delta}{n}x^n, \quad \text{(D.8)}
$$

where

$$
\binom{\delta}{n} = \frac{\delta(\delta-1)\cdots(\delta-n+1)}{n!} = \frac{\delta!}{n!\,(\delta-n)!}. \quad \text{(D.9)}
$$

Expanded, equation (D.8) is written as

$$
(1+x)^\delta = \binom{\delta}{0}x^0 + \binom{\delta}{1}x^1 + \binom{\delta}{2}x^2 + \cdots + \binom{\delta}{\delta-1}x^{\delta-1} + \binom{\delta}{\delta}x^\delta. \quad \text{(D.10)}
$$

Accordingly, note that if we start the summand at an integer other than zero, in particular from 2 as in equation (D.7), we have

$$
\sum_{n=2}^{\infty}\binom{\delta}{n}x^n = (1+x)^\delta - \binom{\delta}{0}x^0 - \binom{\delta}{1}x^1. \quad \text{(D.11)}
$$

Considering this equation, note that

$$
\binom{\delta}{0}\frac{\delta!}{0!\,(\delta-0)!} = \frac{\delta!}{\delta!} = 1, \quad \binom{\delta}{1} = \frac{\delta!}{1!\,(\delta-1)!} = \delta, \quad \text{(D.12)}
$$

and thus

$$
\sum_{n=2}^{\infty}\binom{\delta}{n}x^n = (1+x)^\delta - 1 - \delta x. \quad \text{(D.13)}
$$

Using these results, we revisit equation (D.7), where we find we have

$$
\begin{aligned}
H_1(\mathbb{Q}|\mathbb{P}) &= \lambda_+\beta_+^\alpha\Gamma(-\alpha)\left(\frac{\tilde{\beta_+}}{\beta_+}-1\right)\left[\frac{\partial}{\partial x}\sum_{n=2}^{\infty}\binom{\alpha}{n}x^n\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1} \\
&= \lambda_+\beta_+^\alpha\Gamma(-\alpha)\left(\frac{\tilde{\beta_+}}{\beta_+}-1\right)\left[\frac{\partial}{\partial x}((1+x)^\alpha-1-\alpha x)\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1} \\
&= \lambda_+\beta_+^\alpha\Gamma(-\alpha)\left(\frac{\tilde{\beta_+}}{\beta_+}-1\right)\left[\alpha(1+x)^{\alpha-1}-\alpha\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1}. \quad \text{(D.14)}
\end{aligned}
$$

Finally, a few more simplifications are made, yielding

$$H_1(\mathbb{Q}|\mathbb{P}) = \lambda_+\beta_+^\alpha\Gamma(-\alpha)\left[\alpha(\frac{\tilde{\beta_+}}{\beta_+})^\alpha - \alpha(\frac{\tilde{\beta_+}}{\beta_+})^{\alpha-1} - \alpha(\frac{\tilde{\beta_+}}{\beta_+}) + \alpha\right]$$
$$= \lambda_+\Gamma(-\alpha)(\alpha\tilde{\beta_+}^{\alpha} - \alpha\beta_+\tilde{\beta_+}^{\alpha-1} - \alpha\tilde{\beta_+}\beta_+^{\alpha-1} + \alpha\beta_+^\alpha). \quad \text{(D.15)}$$

Using analgous arguments the other three integrals of equation (D.3) are evaluated and can be simplified together as in equation (3.19).

E Proof of Fourier Transform of European Vanilla Call Option

Expressing the option price as in the risk-neutral valuation form of equation (5.1) and inverting equation (5.4) in terms of $\Psi_T(\nu)$, we have

$$\Psi_T(\nu) = \int_{-\infty}^{\infty} e^{i\nu k}\int_k^\infty e^{\alpha k}e^{-rT}(e^s - e^k)q_T(s)dsdk. \quad \text{(E.1)}$$

We now change the order of integration, giving

$$\Psi_T(\nu) = \int_{-\infty}^{\infty} e^{-rT}q_T(s)\int_{-\infty}^{s}(e^{s+\alpha k} - e^{(1+\alpha)k})e^{i\nu k}dkds \quad \text{(E.2)}$$

$$= \int_{-\infty}^{\infty} e^{-rT}q_T(s)(\frac{e^{(\alpha+1+i\nu)s}}{\alpha+i\nu} - \frac{e^{(\alpha+1+i\nu)s}}{\alpha+1+i\nu})ds. \quad \text{(E.3)}$$

Moving e^{-rT} and a common denominator for the term in parenthesis out of the integral (since neither depends on s), we have

$$\Psi_T(\nu) = \frac{e^{-rT}}{(\alpha+i\nu)(\alpha+1+i\nu)}\int_{-\infty}^{\infty} e^{(\alpha+1+i\nu)s}(\alpha+1+i\nu-\alpha-i\nu)q_T(s)ds \quad \text{(E.4)}$$

$$= \frac{e^{-rT}}{(\alpha+i\nu)(\alpha+1+i\nu)}\int_{-\infty}^{\infty} e^{(\alpha+1+i\nu)s}q_T(s)ds. \quad \text{(E.5)}$$

We can now factor the exponential in the integrand to produce

$$\Psi_T(\nu) = \frac{e^{-rT}}{(\alpha+i\nu)(\alpha+1+i\nu)}\int_{-\infty}^{\infty} e^{(\nu-(\alpha+1)i)is}q_T(s)ds, \quad \text{(E.6)}$$

and see that the integral is in fact the characteristic function evaluated at $(\nu - (\alpha+1)i)$, that is,

$$\Psi_T(\nu) = \frac{e^{-rT}\phi_T(\nu-(\alpha+1)i)}{(\alpha+i\nu)(\alpha+1+i\nu)}. \quad \text{(E.7)}$$

Noting that the denominator can be expanded by

$$(\alpha+i\nu)(\alpha+1+i\nu) = \alpha^2 + \alpha + \underbrace{\alpha i\nu + \alpha i\nu}_{=2\alpha i\nu} + i\nu + \underbrace{(i\nu)^2}_{=-\nu^2}$$
$$= \alpha^2 + \alpha + \underbrace{2\alpha i\nu + i\nu}_{=i\nu(2\alpha+1)} - \nu^2, \quad \text{(E.8)}$$

we can thus write $\Psi_T(\nu)$ as in equation (5.7).

F Calculation of Implied Expected Market Return

The idea that the value of a financial asset is the discounted value of expected future cash flows is perhaps the main tenet of financial theory. When you consider "expected future cash flows" to be in the form of dividends, you are left with the well-known Dividend Discount Model. If one considers a "two-stage" version of this model, where the first three years are forecasted explicitly followed by a terminal growth period, the market value of the subject asset is given by:

$$MV_0 = \frac{D_1}{(1+r_e)} + \frac{D_2}{(1+r_e)^2} + \frac{D_3}{(1+r_e)^3} + \frac{(1+g)\cdot D_3}{(1+r_e)^3\cdot(r_e-g)}, \tag{F.1}$$

where
- r_e is the cost of equity capital.
- MV_0 is the current market value of the asset.
- D_1, D_2, D_3 are the 1,2, and 3 year-ahead dividend forecasts.
- g is the long-term growth rate of dividends.

Recalling that we are inferring r_e, one can quickly gather than the long-term growth rate g is a major input and the resulting value for the implied cost of equity capital is highly sensitive to this estimate. In this regard, a simple assumption is made: Payout ratios and growth rates from year three on must be consistent. Furthermore, assuming that earnings and dividends cannot grow faster than book values over the long-run (that is, assuming that return on equity will stay at the level it reached just before the terminal period starts), the long-term growth rate can be expressed as:

$$g = \frac{E_3 - D_3}{BV_2}. \tag{F.2}$$

Inserting this into equation (F.1), the formula for MV_0 simplifies as follows:

$$\begin{aligned} MV_0 &= \frac{D_1}{(1+r_e)} + \frac{D_2}{(1+r_e)^2} + \frac{D_3}{(1+r_e)^3} + \frac{(1+\frac{E_3-D_3}{BV_2})D_3}{(r_e-\frac{E_3-D_3}{BV_2})(1+r_e)^3} \\ &= \frac{D_1}{(1+r_e)} + \frac{D_2}{(1+r_e)^2} + \frac{(r-\frac{E_3-D_3}{BV_2})D_3}{(r-\frac{E_3-D_3}{BV_2})(1+r_e)^3} + \frac{(1+\frac{E_3-D_3}{BV_2})D_3}{(r_e-\frac{E_3-D_3}{BV_2})(1+r_e)^3} \\ &= \frac{D_1}{(1+r_e)} + \frac{D_2}{(1+r_e)^2} + \frac{r_eD_3+D_3}{(r_e-\frac{E_3-D_3}{BV_2})(1+r_e)^3} \\ &= \frac{D_1}{(1+r_e)} + \frac{D_2}{(1+r_e)^2} + \frac{D_3}{(r_e-\frac{E_3-D_3}{BV_2})(1+r_e)^2}. \end{aligned} \tag{F.3}$$

Substituting g for equation (F.2) and using a rearrangement of this equation also for D_3, we have finally

$$MV_0 = \frac{D_1}{(1+r_e)} + \frac{D_2}{(1+r_e)^2} + \frac{E_3 - g\cdot BV_2}{(r_e-g)(1+r_e)^2}. \tag{F.4}$$

To calculate the implied cost of equity capital, r_e, for the aggregate market, the remaining input parameters are aggregated across all companies, i.e. the total dollar-amount of expected dividends (years 1 & 2), book-value (year 2), earnings (year 3), and market capitalization (current).

G Proof of Stochastic Exponential Triplet of the ETS Process

Recalling Thereom 2.10, the result for $\nu(dx)$ follows directly from evaluating the ordinary exponential Lévy measure at $log(1+x)$, which is the anti-function of $e^x - 1$, i.e. $log(1 + e^x - 1) = x$. As such,

$$\nu(dx) = \lambda_+ \frac{\exp(-\beta_+(\log(1+x)))}{\log(1+x)^{\alpha+1}} \chi_{x>0} + \lambda_- \frac{\exp(\beta_- \log(1+x))}{-\log(1+x)^{\alpha+1}} \chi_{-1<x<0}, \tag{G.1}$$

and the simplification to equation (6.5) is obvious. For the drift μ, we need to evaluate the integral

$$\int_{\mathbb{R}} (e^x - 1 - x) \left\{ \lambda_+ \frac{\exp(-\beta_+ x)}{x^{\alpha+1}} \chi_{x>0} + \lambda_- \frac{\exp(-\beta_- |x|)}{|x|^{\alpha+1}} \chi_{x<0} \right\}. \tag{G.2}$$

Splitting the integral into the positive and negative parts and beginning with the former, we have

$$\lambda_+ \int_0^\infty x^{-\alpha-1} e^x e^{-\beta_+ x} dx - \lambda_+ \int_0^\infty x^{-\alpha-1} e^{-\beta_+ x} dx - \lambda_+ \int_0^\infty x^{-\alpha-1} x e^{-\beta_+ x} dx. \tag{G.3}$$

Recalling again the Gamma function from equations (A.5-A.7), this evaluates as

$$\begin{aligned} \lambda_+ \int_0^\infty x^{-\alpha-1} e^{-x(\beta_+-1)} dx - \lambda_+ \int_0^\infty x^{-\alpha-1} e^{-\beta_+ x} dx - \lambda_+ \int_0^\infty x^{-\alpha-1} x e^{-\beta_+ x} dx \\ = \lambda_+ (\beta_+ - 1)^\alpha \Gamma(-\alpha) - \lambda_+ \beta_+^\alpha \Gamma(-\alpha) - \lambda_+ \beta_+^{\alpha-1} \Gamma(1-\alpha) \\ = -\frac{\lambda_+}{\beta_+} \Gamma(-\alpha) [\beta_+^{\alpha+1} - \alpha \beta_+^\alpha - \beta_+ (\beta_+ - 1)^\alpha]. \end{aligned} \tag{G.4}$$

Now, for the negative part of the integral we have

$$\lambda_- \int_{-\infty}^0 |x|^{-\alpha-1} e^x e^{\beta_- x} dx - \lambda_- \int_{-\infty}^0 |x|^{-\alpha-1} e^{\beta_- x} dx - \lambda_- \int_{-\infty}^0 |x|^{-\alpha-1} x e^{\beta_- x} dx. \tag{G.5}$$

Changing the limits of each integral and evaluating yields

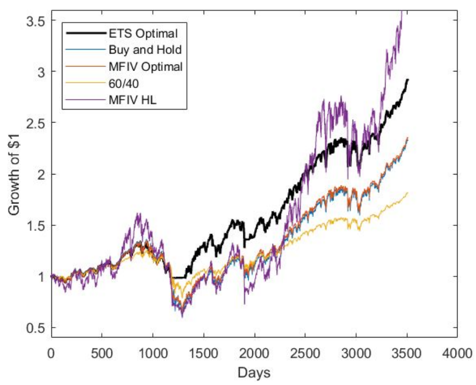
$$\begin{aligned} \lambda_- \int_0^\infty x^{-\alpha-1} e^{-x(\beta_-+1)} dx - \lambda_- \int_0^\infty x^{-\alpha-1} e^{-\beta_- x} dx + \lambda_- \int_0^\infty x^{-\alpha-1} x e^{-\beta_- x} dx \\ = \lambda_- (\beta_- + 1)^\alpha \Gamma(-\alpha) - \lambda_- \beta_-^\alpha \Gamma(-\alpha) + \lambda_- \beta_-^{\alpha-1} \Gamma(1-\alpha) \\ = -\frac{\lambda_-}{\beta_-} \Gamma(-\alpha) [\beta_-^{\alpha+1} + \alpha \beta_-^\alpha - \beta_- (\beta_- + 1)^\alpha]. \end{aligned} \tag{G.6}$$

Adding equations (G.4) and (G.6) thus yields the final term of equation (6.5).

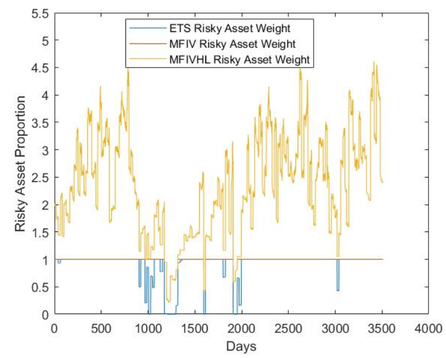
H Performance Results for $\gamma = 0.5$

Table 3: Performance Metrics for $\gamma = 0.5$

Strategy	Ann. Excess Return	Ann. Standard Deviation	Ann. Sharpe Ratio	Ann. Sortino Ratio	Max. DD	Avg. Turnover
	$\mu - r$	σ	$\frac{\mu - r}{\sigma}$	$\frac{\mu - r}{\sigma_-}$	$\max\{\frac{Trough}{Peak} - 1\}$	$\frac{1}{N}\sum_{i=1}^{N}\lvert w_i - w_{i-1}\rvert$
ETS	6.8585%	13.024%	0.52662	0.81558	-27.15%	8.41%
MFIV	5.2115%	16.111%	0.32347	0.50728	-51.55%	2.15%
MFIV HL	9.7678%	31.905%	0.30615	0.48075	-63.07%	60.00%
B&H	5.1407%	18.414%	0.27918	0.43551	-56.47%	-
60/40	3.2812%	10.9%	0.30102	0.46734	-37.53%	0.72%

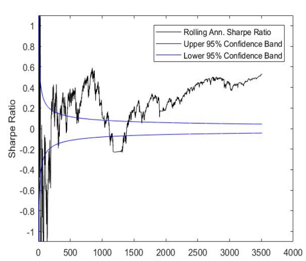


(a) Equity Curves

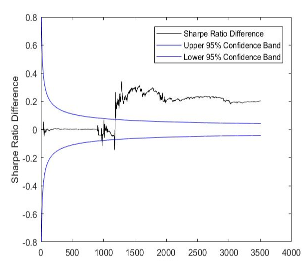


(b) ETS, MFIV, and MFIVHL Risky Asset Proportion Evolution

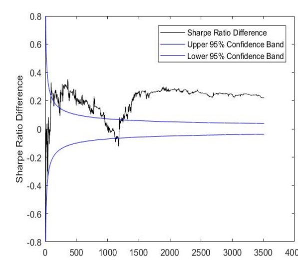
Figure 9: Equity Curves and Risky Asset Proportions for $\gamma = 0.5$



(a) ETS Strategy Raw SR Significance

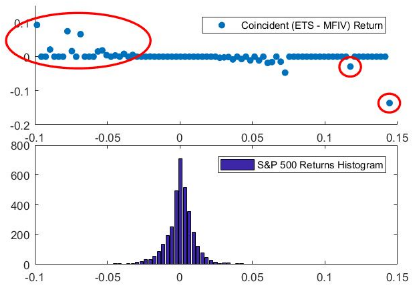


(b) ETS-MFIV SR Difference Significance

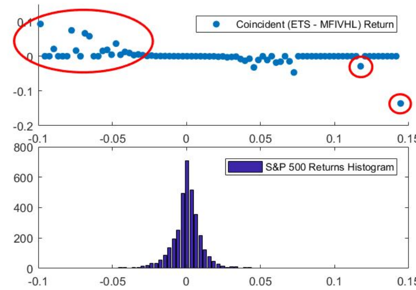


(c) ETS-MFIVHL SR Difference Significance

Figure 10: Sharpe Ratio Quality Control Charts for $\gamma = 0.5$

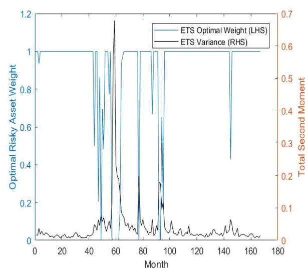


(a) ETS-MFIV Market Dependence

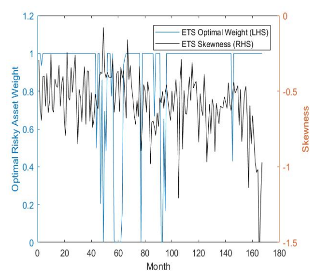


(b) ETS-MFIVHL Market Dependence

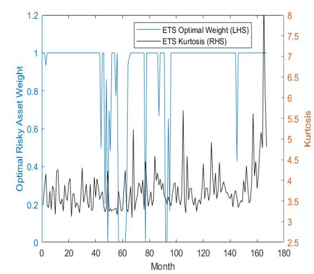
Figure 11: Market Dependence of Differential Returns for $\gamma = 0.5$



(a) Total Variance vs Optimal Control



(b) Skewness vs Optimal Control



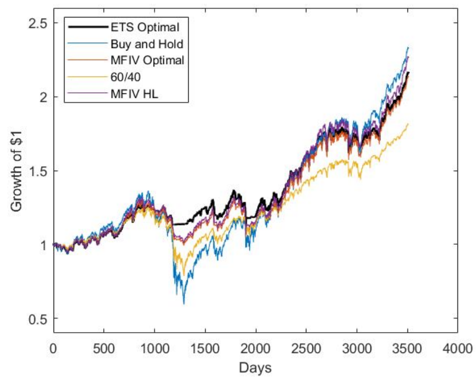
(c) Kurtosis vs. Optimal Control

Figure 12: ETS Higher Moments vs. Optimal Control for $\gamma = 0.5$

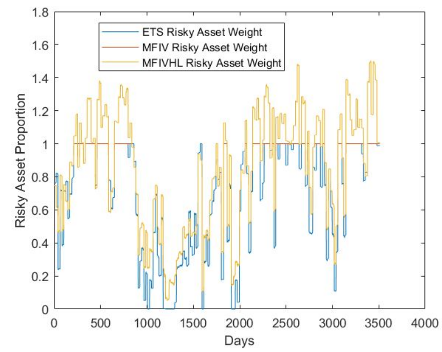
I Performance Results for $\gamma = 2$

Table 4: Performance Metrics for $\gamma = 2$

Strategy	**Ann. Excess Return**	**Ann. Standard Deviation**	**Ann. Sharpe Ratio**	**Ann. Sortino Ratio**	**Max. DD**	**Avg. Turnover**
	$\mu - r$	σ	$\frac{\mu-r}{\sigma}$	$\frac{\mu-r}{\sigma_-}$	$\max\{\frac{Trough}{Peak} - 1\}$	$\frac{1}{N}\sum_{i=1}^{N}\lvert w_i - w_{i-1}\rvert$
ETS	4.5798%	9.3441%	0.49013	0.75427	-17.86%	16.67%
MFIV	4.4803%	10.345%	0.43307	0.67332	-23.52%	12.48%
MFIV HL	4.9336%	11.096%	0.44463	0.69257	-23.52%	19.76%
B&H	5.1407%	18.414%	0.27918	0.43551	-56.47%	-
60/40	3.2812%	10.9%	0.30102	0.46734	-37.53%	0.72%

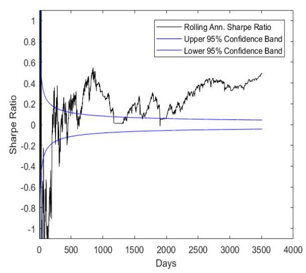


(a) Equity Curves

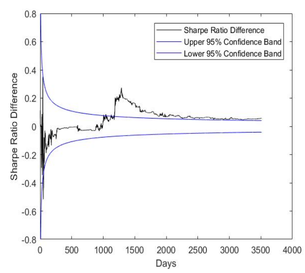


(b) ETS, MFIV, and MFIVHL Risky Asset Proportion Evolution

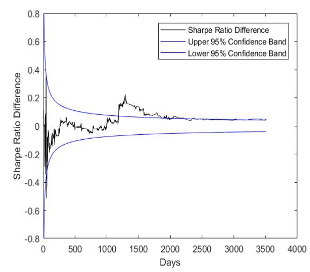
Figure 13: Equity Curves and Risky Asset Proportions for $\gamma = 2$



(a) ETS Strategy Raw SR Significance

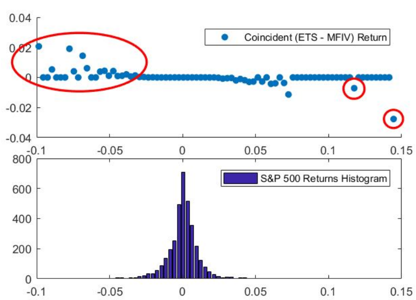


(b) ETS-MFIV SR Difference Significance

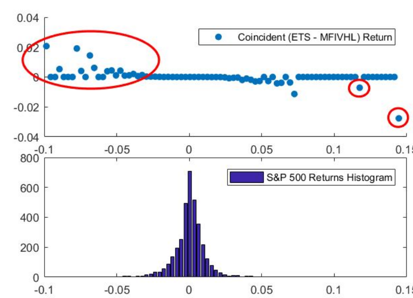


(c) ETS-MFIVHL SR Difference Significance

Figure 14: Sharpe Ratio Quality Control Charts for $\gamma = 2$

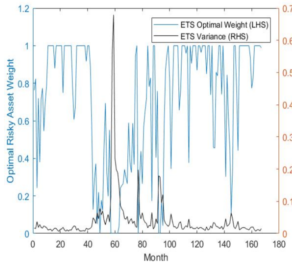


(a) ETS-MFIV Market Dependence

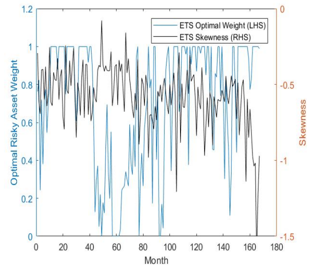


(b) ETS-MFIVHL Market Dependence

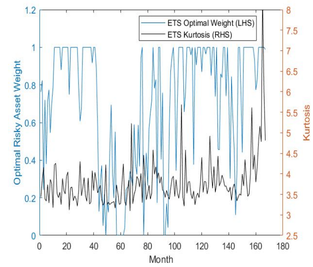
Figure 15: Market Dependence of Differential Returns for $\gamma = 2$



(a) Total Variance vs Optimal Control



(b) Skewness vs Optimal Control



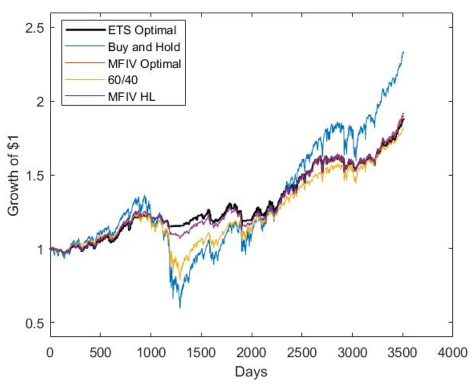
(c) Kurtosis vs. Optimal Control

Figure 16: ETS Higher Moments vs. Optimal Control for $\gamma = 2$

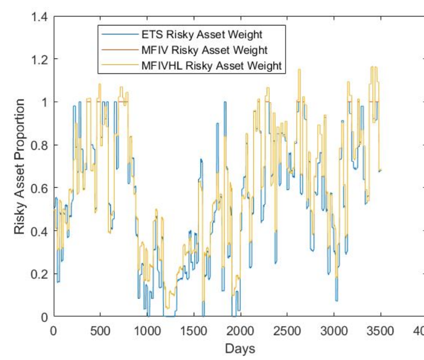
J Performance Results for $\gamma = 3$

Table 5: Performance Metrics for $\gamma = 3$

Strategy	Ann. Excess Return	Ann. Standard Deviation	Ann. Sharpe Ratio	Ann. Sortino Ratio	Max. DD	Avg. Turnover
	$\mu - r$	σ	$\frac{\mu-r}{\sigma}$	$\frac{\mu-r}{\sigma_-}$	$\max\{\frac{Trough}{Peak} - 1\}$	$\frac{1}{N}\sum_{i=1}^{N}\lvert w_i - w_{i-1}\rvert$
ETS	3.5297%	7.2488%	0.48693	0.74823	-12.06%	19.12%
MFIV	3.5982%	7.8005%	0.46129	0.7194	-15.25%	15.62%
MFIV HL	3.679%	7.8606%	0.46804	0.7291	-15.25%	16.70%
B&H	5.1407%	18.414%	0.27918	0.43551	-56.47%	-
60/40	3.2812%	10.9%	0.30102	0.46734	-37.53%	0.72%

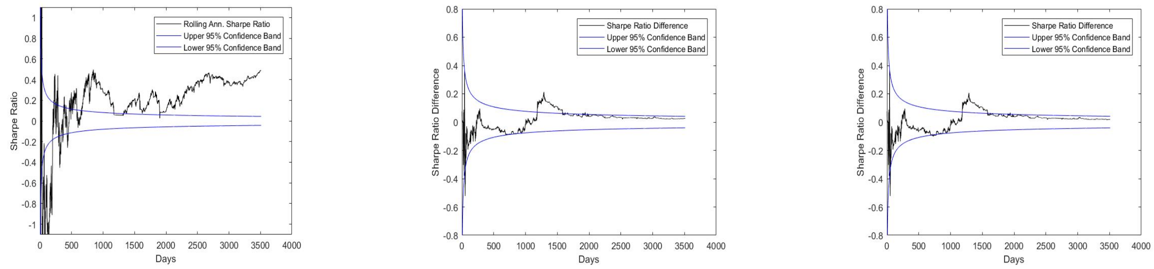


(a) Equity Curves



(b) ETS, MFIV, and MFIVHL Risky Asset Proportion Evolution

Figure 17: Equity Curves and Risky Asset Proportions for $\gamma = 3$



(a) ETS Strategy Raw SR Significance

(b) ETS-MFIV SR Difference Significance

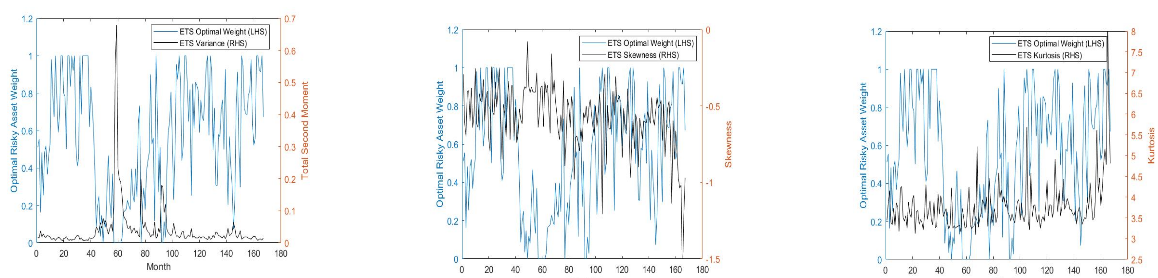
(c) ETS-MFIVHL SR Difference Significance

Figure 18: Sharpe Ratio Quality Control Charts for $\gamma = 3$

(a) ETS-MFIV Market Dependence

(b) ETS-MFIVHL Market Dependence

Figure 19: Market Dependence of Differential Returns for $\gamma = 3$



(a) Total Variance vs Optimal Control

(b) Skewness vs Optimal Control

(c) Kurtosis vs. Optimal Control

Figure 20: ETS Higher Moments vs. Optimal Control for $\gamma = 3$

K Analysis of Calibration Methodologies

As introduced in section 5, there are a few ways we can deal with the ill-posedness of the calibration problem. The first and perhaps most intuitive, though heuristic, is to simply run the optimization multiple times with different initial value estimates, and choose the eventual calibration which minimizes the cost function of equation (5.16). Alternatively, we can utilize the regularization framework developed in [18], [19], and [20], which relates heavily to the minimal entropy measure concepts of sections 2 and 3. Due to the convexity of the relative entropy (see equations (2.36-2.37)), we can augment the objective function of equation (5.16) with the relative entropy as a regularization term, giving

$$AAPE = \min_{\vec{\tilde{\mathbf{v}}},\tilde{\eta}} \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\text{market price}_i - \text{calculated price}_i}{\text{market price}_i} \right| + \lambda H(\mathbb{Q}|\mathbb{Q}_0), \tag{K.1}$$

where $\mathbb{Q}_0$ is a chosen prior and λ is the regularization scalar, or the weight assigned to the relative entropy term. When λ is large, we place more weight on the prior information, and when λ is small, we place more weight on the new information extracted from the current market. Of course, when $\lambda = 0$, the augmented objective function reduces to the non-augmented function of equation (5.16). We have still to make choices for two variables, the regularization scalar λ and the prior measure $\mathbb{Q}_0$. Importantly, λ cannot be chosen *a priori* since it depends on the noise present in the current data, and so is normally chosen by the method of [53] for its simplicity. Given the convexity of the relative entropy $H(\cdot)$, we expect that

$$AAPE \geq APE. \tag{K.2}$$

We should thus be willing to trade off the precision of the calibration and the uniqueness of the calibration by a scaled *a priori* pricing error,

$$AAPE = \varrho \cdot APE. \tag{K.3}$$

While there is no "optimal" value for ϱ, in [19] and [20] it is recommended to be in the range of $1.1 \leq \varrho \leq 1.5$, and 1.1 was chosen throughout this study. Accordingly, λ is calibrated such that

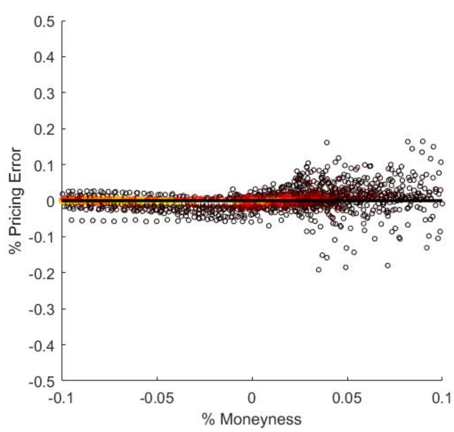
$$AAPE \approx 1.1 \cdot APE \tag{K.4}$$

$\iff$

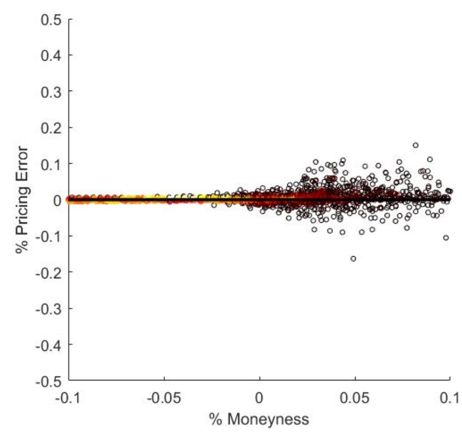
$$\min_{\vec{\tilde{\mathbf{v}}},\tilde{\eta}} \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\text{market price}_i - \text{calculated price}_i}{\text{market price}_i} \right| + \lambda H(\mathbb{Q}|\mathbb{Q}_0)$$
$$\approx 1.1 \cdot \min_{\vec{\tilde{\mathbf{v}}},\tilde{\eta}} \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\text{market price}_i - \text{calculated price}_i}{\text{market price}_i} \right|. \tag{K.5}$$

Finally, we must address the choice of the prior measure, $\mathbb{Q}_0$. In [19] and [20], a number of methods are suggested in this regard, including using a historical prior estimated using a time series of the underlying, using the updated risk-neutral unregularized calibrated parameters, and using a long-term average of the unregularized risk-neutral calibrated parameters. For the current study, the second method will be utilized. While the third alternative may seem preferable to the second due to the increased stability in the prior by using a long-term average, the investment methodology of the paper is based on the idea that *timely* updates to market sentiment calibrated via the ETS process are useful in allocating capital, and thus historical calibrations should offer fleeting value.

One of the first steps of the empirical analysis involved testing and comparing these calibration methods. Figure a below shows the results for the calibration of the ETS model using the relative entropy approach (that is, minimizing the objective function of equation (K.1)), where the prior $\mathbb{Q}$ is chosen as an initial unregularized calibration (however, importantly only once and always with the same initial guess). Figure b below shows the results for the approach involving using multiple initial values and choosing the calibration which achieves the lowest cost function (that is, the objective function used is equation (5.16) rather than equation (K.1)). The figures are presented as a scatter heat plot, with brighter colors (i.e. yellow) representing a higher density of points clustered tightly together, and each point corresponds to the calibration of a single call option price, where the vertical axis is the percentage pricing error and the horizontal axis is a moneyness measure of strike relative to underlying price (for instance, -0.1 denotes an option which is 10% in-the-money and 0.1 denotes an option that is 10% out-of-the-money). As shown, the multiple initial value approach performs materially better than the relative entropy approach. Accordingly, the parameter calibrations used for the eventual portfolio constructions used calibrations produced from the multiple initial value approach. It is also important to note that the gradually larger dispersion of error in the out-of-the-money options is not due to a bias in the cost function, as was mentioned in section 5, but due to the fact that the implied volatility curve for these options can become more "wiggly," and the Lévy model (at least in the form used with only a single decay parameter for each side of the distribution) fits a smooth curve.



(a) Relative Entropy Approach



(b) Multiple Initial Values Approach

Figure 21: Calibration Methodology Results