Risk-Free Curve Bootstrapping Refresher

1 Purpose and Big Picture

The goal of this note is to provide a concise but rigorous refresher on how a *risk-free zero-coupon discount curve* can be constructed (bootstrapped) from a set of quoted SOFR OIS swap rates.

The intended audience is a quantitative practitioner who:

- Understands basic interest rate instruments,
- Has seen curve bootstrapping before,
- Wants a clean mental model for how SOFR swaps pin down discount factors.

We focus exclusively on the *discounting curve*, not forecasting curves.

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2 Objects of Interest

2.1 Discount Factors

We seek to construct a function

$$P(0, T)$$

where $P(0, T)$ is the time-0 price of a zero-coupon bond paying 1 unit of currency at time T.

Once $P(0, T)$ is known for all relevant maturities:

- Zero rates,
- Instantaneous forward rates,
- Present values of collateralized cashflows

follow immediately.

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2.2 Market Inputs

The market inputs are quoted **SOFR OIS par swap rates** at standard maturities:

$$T \in \{1\text{Y}, 2\text{Y}, 3\text{Y}, 5\text{Y}, 7\text{Y}, 10\text{Y}, \dots\}$$

Each quote corresponds to a fixed rate K_T such that the swap has zero value at inception.

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3 Anatomy of a SOFR OIS Swap

A SOFR OIS swap exchanges:

- A **fixed leg** paying a constant rate K_T,
- A **floating leg** paying compounded overnight SOFR.

Payments occur on a predefined schedule

$$0 < t_1 < t_2 < \cdots < t_n = T$$

with corresponding accrual factors α_i.

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3.1 Present Value of the Floating Leg

Under standard no-arbitrage assumptions and daily compounding of the overnight rate, the floating leg of an OIS swap satisfies the identity:

$$\mathrm{PV}_{\text{float}} = 1 - P(0,T).$$

Interpretation. This result mirrors the classical identity for a par floating-rate note: the floating leg effectively "returns par" at maturity, discounted back to today.

This identity is what makes OIS swaps such clean instruments for discount curve construction.

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3.2 Present Value of the Fixed Leg

The fixed leg pays:

$$K_T \alpha_i$$

at each payment date t_i. Its present value is:

$$\mathrm{PV}_{\text{fixed}} = K_T \sum_{i=1}^{n} \alpha_i P(0,t_i).$$

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4 The Par Swap Condition

Because the quoted SOFR swap rate K_T is a *par rate*, the swap has zero value at inception:

$$\mathrm{PV}_{\text{fixed}} = \mathrm{PV}_{\text{float}}.$$

Substituting the expressions above yields the core equation:

$$K_T \sum_{i=1}^{n} \alpha_i P(0,t_i) = 1 - P(0,T).$$

This single equation links:

- One market quote K_T,
- Several discount factors $P(0,t_i)$.

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5 Bootstrapping Logic

5.1 Sequential Construction

The curve is constructed *sequentially in maturity.*

- Start with the shortest maturity instrument.
- Assume all earlier discount factors are already known.
- Solve for the new, longest discount factor.

At each step:

- One new swap quote provides one equation,
- One new discount factor is introduced,
- The system remains exactly solvable.

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5.2 First Maturity

Suppose the shortest quoted swap matures at T_1.

The par condition becomes:

$$K_{T_1} \sum_{i=1}^{n_1} \alpha_i P(0, t_i) = 1 - P(0, T_1),$$

where all $t_i \le T_1$.

This equation can be solved directly for $P(0, T_1)$.

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5.3 Subsequent Maturities

For maturity T_k, the swap equation reads:

$$K_{T_k} \left(\sum_{i=1}^{n_k - 1} \alpha_i P(0, t_i) + \alpha_{n_k} P(0, T_k) \right) = 1 - P(0, T_k).$$

All discount factors except $P(0, T_k)$ are already known, allowing a direct algebraic solution.

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6 From Discount Factors to Zero Rates

Once $P(0, T)$ is known, zero rates follow immediately.

6.1 Continuously Compounded Zero Rates

$$z(T) = -\frac{1}{T} \ln P(0, T).$$

6.2 Simple or Annually Compounded Rates

Defined implicitly by:

$$P(0, T) = \frac{1}{(1 + z_T)^T}.$$

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7 Practical Implementation Details

7.1 Interpolation

Market quotes are available only at discrete maturities. Interpolation is required between bootstrapped pillars.

Common choices include:

- Log-linear interpolation in discount factors,
- Linear interpolation in zero rates,
- Spline interpolation in instantaneous forward rates.

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7.2 Day Count and Calendars

Accrual factors α_i depend on:

- Day count conventions (e.g. ACT/360),
- Business-day adjustments,
- Stub periods.

Consistency with market conventions is essential.
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7.3 Why SOFR Defines the "Risk-Free" Curve

SOFR is:

- Overnight,
- Secured by U.S. Treasuries,
- Largely free of bank credit risk.

As a result, SOFR OIS discounting is the standard framework for collateralized valuation.
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8 Summary

- SOFR OIS swaps encode the risk-free discount curve.
- The floating leg PV equals $1 - P(0, T)$.
- Each par swap quote provides exactly one equation.
- Sequential bootstrapping reveals the full zero-coupon curve.

Part II will present a fully worked numerical example using a realistic set of SOFR swap quotes.

9 Part II: Worked SOFR OIS Bootstrapping Example

We now construct a full SOFR discount curve from a realistic but hypothetical set of SOFR OIS par swap quotes.

The objective is to explicitly solve for discount factors

$$P(0,T)$$

and zero rates

$$z(T)$$

at standard market maturities.

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9.1 Market Data

Assume the following SOFR OIS par swap quotes, with annual fixed payments and ACT/360 accrual approximated as 1 for simplicity:

Maturity T	Par SOFR Swap Rate K_T
1Y	4.80%
2Y	4.60%
3Y	4.45%
5Y	4.25%
7Y	4.10%
10Y	3.95%

We assume:

- Fixed leg pays annually,
- Accrual factors $\alpha_i = 1$,
- Floating leg PV $= 1 - P(0,T)$.

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9.2 1-Year Discount Factor

For the 1Y swap, the par condition is:

$$K_{1Y} P(0,1) = 1 - P(0,1).$$

Substituting $K_{1Y} = 0.048$:

$$0.048 P(0,1) = 1 - P(0,1)$$
$$P(0,1)(1.048) = 1$$
$$P(0,1) = \frac{1}{1.048} = 0.9542.$$

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9.3 2-Year Discount Factor

The 2Y swap equation is:

$$K_{2Y}\,(P(0,1) + P(0,2)) = 1 - P(0,2).$$

Substitute values:

$$0.046(0.9542 + P(0,2)) = 1 - P(0,2).$$

Expand:

$$0.04389 + 0.046P(0,2) = 1 - P(0,2)$$
$$1.046P(0,2) = 0.95611$$
$$P(0,2) = 0.9139.$$

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9.4 3-Year Discount Factor

$$0.0445\big(P(0,1) + P(0,2) + P(0,3)\big) = 1 - P(0,3).$$

Substitute known values:

$$0.0445(0.9542 + 0.9139 + P(0,3)) = 1 - P(0,3)$$

$$0.08308 + 0.0445P(0,3) = 1 - P(0,3)$$

$$1.0445P(0,3) = 0.91692$$

$$P(0,3) = 0.8778.$$

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9.5 5-Year Discount Factor

For the 5Y swap:

$$0.0425(P_1 + P_2 + P_3 + P_4 + P_5) = 1 - P_5.$$

We interpolate $P(0,4)$ linearly between 3Y and 5Y:

$$P(0,4) \approx 0.915 \times 0.8778 + 0.5(0.820) \approx 0.848.$$

Now solve:

$$0.0425(0.9542 + 0.9139 + 0.8778 + 0.848 + P_5) = 1 - P_5$$

$$0.0425(3.5939 + P_5) = 1 - P_5$$

$$0.1527 + 0.0425P_5 = 1 - P_5$$

$$1.0425P_5 = 0.8473$$

$$P_5 = 0.8127.$$

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9.6 7-Year Discount Factor

Using interpolated 6Y:

$$P(0,6) \approx 0.785$$

$$0.041(P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7) = 1 - P_7$$

$$0.041(5.190) + 0.041P_7 = 1 - P_7$$

$$0.2128 + 0.041P_7 = 1 - P_7$$

$$1.041P_7 = 0.7872$$

$$P_7 = 0.756.$$

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9.7 10-Year Discount Factor

Using interpolated 8Y,9Y:

$$P(0,8) \approx 0.725, \quad P(0,9) \approx 0.695.$$

$$0.0395(\sum_{i=1}^{10} P(0,i)) = 1 - P(0,10).$$

$$0.0395(7.566 + P_{10}) = 1 - P_{10}$$

$$0.298 + 0.0395P_{10} = 1 - P_{10}$$

$$1.0395P_{10} = 0.702$$

$$P_{10} = 0.675.$$

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9.8 Implied Zero Rates

$$z(T) = -\frac{1}{T} \ln P(0,T).$$

T	$P(0,T)$	$z(T)$
1	0.9542	4.69%
2	0.9139	4.52%
3	0.8778	4.36%
5	0.8127	4.14%
7	0.7560	4.02%
10	0.6750	3.93%

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9.9 Interpretation

This curve shows a modestly downward-sloping term structure, typical of a market pricing declining long-term real rates.

Every number in the curve is implied by SOFR OIS swaps alone.

No LIBOR, no credit risk, no funding spreads.

This is the modern *risk-free discount curve*.