BME2104 HW#1 **Imaging Foundation**

Zhihao Zhang, 2024291074

zhangzhh12024@shanghaitech.edu.cn

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Problem 1.

A cycle of a square wave can be expressed as following equation

$$f(x) = \begin{cases} 1, & 0 \leq x < 1/2 \\ 0, & 1/2 \leq x < 1 \end{cases}$$

Using Fourier Series, please do analytical derivation to prove that this square wave can be represented as a linear combination of sine waves of different frequencies.

Then, using a programming environment of your choice (MATLAB/Python), please build a numerical model to demonstrate that the above square wave is indeed a linear combination of sine waves. Please use plots to help explain.

Solution 1

A **Fourier series** is an expansion of a periodic function into a sum of trigonometric functions. By expressing a function as a sum of sines and cosines, many problems involving the function become easier to analyze because trigonometric functions are well understood. For a periodic function,

the Fourier series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n x) + b_n \sin(2\pi n x)]$$

Where the coefficients are:

- $a_0 = \frac{1}{T} \int_0^T f(x) dx$
- $a_n = \frac{1}{T} \int_0^T f(x) \cos(2\pi n x) dx$
- $b_n = \frac{1}{T} \int_0^T f(x) \sin(2\pi n x) dx$

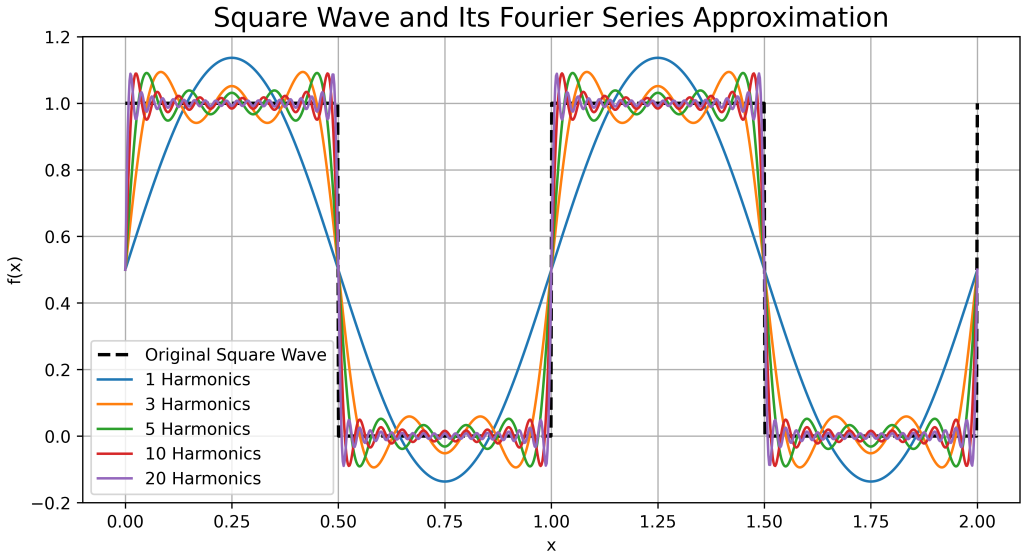
For the given square wave, the Fourier series representation and Python function can be derived as follows:

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2\pi(2k+1)x)$$

```
def fourier_square(x, harmonics):
# a0 = 1/2
f_approx = 0.5 * np.ones_like(x)
for k in range(harmonics):
   n = 2 * k + 1 # odd only
   f_approx += (2 / (np.pi * n)) * np.sin(2 * np.pi *
       n * x)
return f_approx
```

The `harmonics` parameter controls the number of sine waves used in the approximation. Increasing the number of harmonics brings the approximation closer to the original square wave. The figure below illustrates the square wave approximation using 1, 3, 5, 10, and 20 harmonics. We can see that as the number of harmonics increases, the approximation becomes more accurate, especially at the discontinuities. The Fourier series converges to

the square wave function, but it exhibits oscillations near the discontinuities, known as the Gibbs phenomenon.



Therefore, theoretically, as long as there are enough harmonics, these harmonics can closely approximate the shape of a square wave. From both the analytical derivation and numerical demonstration, we've shown that the square wave can be represented as a linear combination of sine waves of different frequencies. In addition, we also found:

- Only **odd-numbered harmonics** contribute to the series
- The amplitude of each harmonic decreases as $1/n$
- The **more terms** lead to **better approximation**, especially at discontinuities
- At discontinuities the Fourier series will produce **oscillations**, for example at $0.0, 0.5, 1.00 \ldots$ (Gibbs phenomenon)

Problem 2.

A micro-CT imaging system' s spatial resolution was characterized with a

thin tungsten wire phantom. An axial micro-CT slice image of the phantom is shown here (original image file is uploaded to Blackboard, " MTF-slice" . Note the pixel size is 7.6um).

(1). Please plot the Line Intensity Profile (LIP) of the wire in the center of the image. Please fit the LIP with a Gaussian function.

(2). Assume that the diameter of the wire is very small, therefore the LIP can be assumed to be the LSF of the micro-CT. Using the LSF, please calculate the MTF of the micro-CT. Please plot the MTF, and fit your MTF with a Gaussian function. Is your Gaussian function in (2) close to the FT (Fourier Transform) of the Gaussian function in (1)? Find your spatial resolution at 10% MTF? Explain your result.

Solution 2

(1) Line Intensity Profile

Line Intensity Profile (LIP) is defined as the intensity of the image along a line, which is useful for analyzing the spatial resolution of an imaging system. The LIP can be obtained by taking a line profile through the center of the image. As a image is in fact a 2D matrix, we can take the center row of the image and plot the intensity values along that row. The LIP is typically modeled as a Gaussian function, which is a common assumption for imaging systems. The Gaussian function is given by

$$G(x) = A \cdot \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\} + A_0$$

Where A is the amplitude,A_0 is the offset, μ is the mean (center of the Gaussian), and σ is the standard deviation (width of the Gaussian). The parameters of the Gaussian function can be estimated using curve fitting

techniques. In Python, we can use the `scipy.optimize.curve_fit` function to fit the Gaussian function to the LIP data, which uses the least squares method to minimize the difference between the observed data and the Gaussian function.

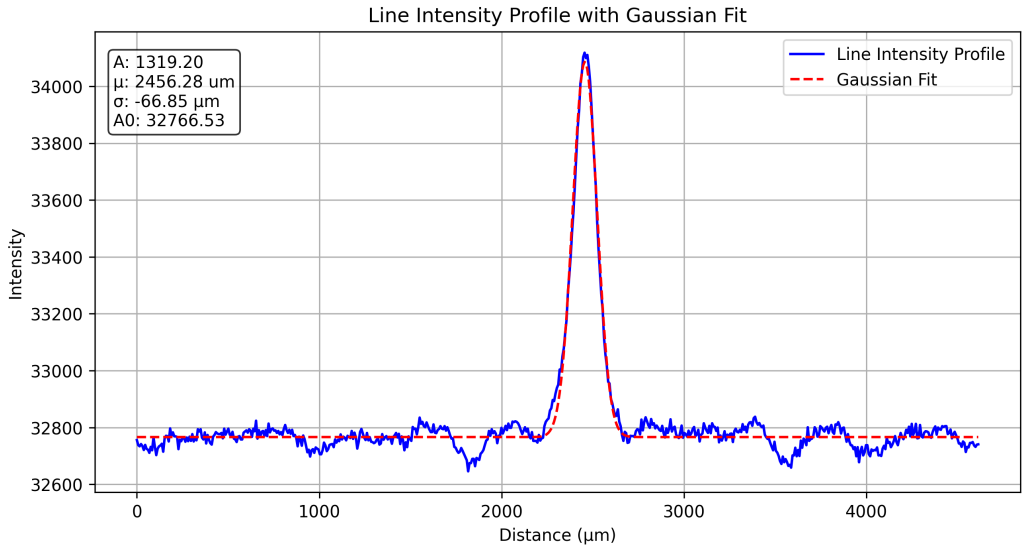
```
    # center row of the image
    line_profile = img[img.shape[0] // 2, :]
    # 7.6 um / pixel
    x = np.arange(len(line_profile)) * 7.6

    # Gaussian function
    def gaussian(x, amplitude, mean, sigma, offset):
        return amplitude * np.exp(-((x - mean) ** 2) / (2
            * sigma**2))+ offset

    # Fit Gaussian to LIP
from scipy.optimize import curve_fit
    popt, _ = curve_fit(gaussian, x,line_profile)
    y = gaussian(x, *popt)
```

The LIP of the wire in the center of the image is shown below. The blue line is the original LIP, and the orange line is the fitted Gaussian function. The parameters of the Gaussian function are also shown in the figure.

(2) Line Spread Function

Line Spread Function (LSF) is a measure of the response of an imaging system to a line source. It describes how the system blurs a line source, and is typically modeled as a Gaussian function. The LSF can be obtained from the LIP by taking the derivative of the Gaussian function. The LSF is given by:

$$LSF(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-y_0)^2}{2\sigma^2}\right)$$

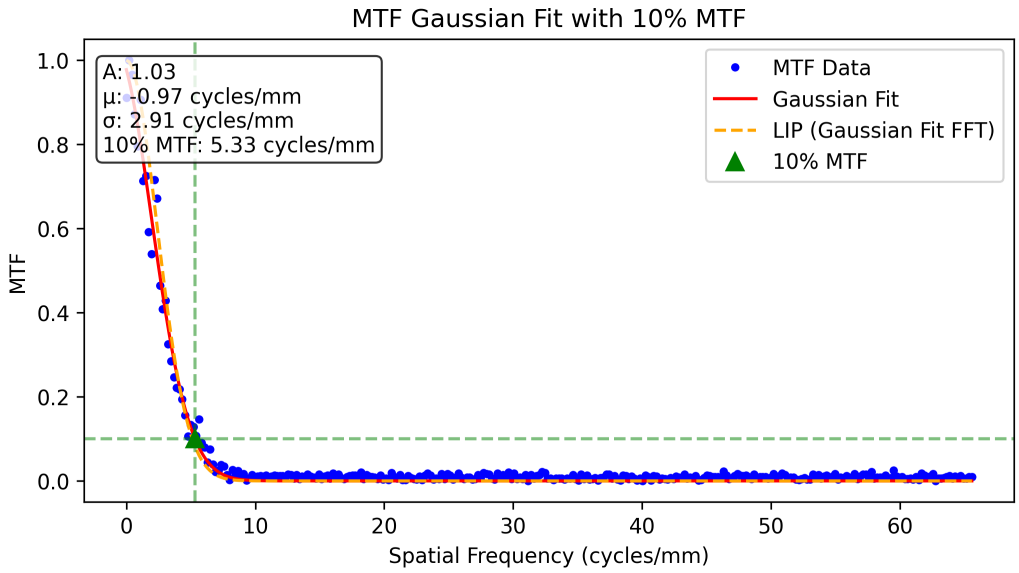


The MTF (Modulation Transfer Function) can be calculated from the LSF by taking the Fourier Transform of the LSF. The MTF describes how different spatial frequencies are transferred by the imaging system.

$$\text{MTF} = \mathcal{F}\left[\text{LSF}\right] \approx \mathcal{F}\left[\text{LSF}\right]$$

Where assumes that the diameter of the wire is very small, the LIP can be assumed to be the LSF. We can use the `numpy.fft.fft` function to calculate the Fourier Transform of the LSF. The MTF is typically normalized to the maximum value, which is 1. Thus, the MTF and Gaussian fitting is shown below. The MTF is plotted in the frequency domain, and the fitted Gaussian function is shown in red. The parameters of the Gaussian function are also shown in the figure.

Comparing the results of the fitted in (1) and (2), we can see that both LIP and MTF can be well fitted with a Gaussian function. The LIP first undergoes a Fourier transform to obtain the MTF, and then is fitted with a Gaussian function.



The FFT of gaussian function is also a Gaussian function, which is why we can still use a Gaussian function to fit MTF. The Fourier transform of Gaussian function is given by:

$$\text{MTF} = \mathcal{F}\left[\text{LSF}\right] = \exp\left(-\frac{(y-y_0)^2}{2\sigma^2}\right)$$

Therefore, there exists a definite fitting coefficient relationship between the Gaussian fitting results of LSF and MTF, and this result is consistent with obtaining the LIP through fitting with a Gaussian function followed by Fourier transformation (as seen in the graph, both have an error within an acceptable range).

Moreover, the 10% MTF is calculated as the frequency at which the MTF drops to 10% of its maximum value. 10% MTF represents the spatial resolution at which only 10% of the original contrast is preserved, reflecting the ability to resolve fine details. In other words, for structures with a contrast below 10% MTF—i.e. at spatial frequencies higher than the corresponding value—these fine details cannot be distinguished, which corre-

sponds to the area to the lower right of the 10% point in the graph.

Problem 3.

Start with any noisy digital image (it can be downloaded from the web, synthesized by yourself, or even an image in your smartphone), first convert it into 8-bit grey scale if it is not already so, then re-size it to 512 x 512 using digital down-sampling or up-sampling (combined with cropping/padding if necessary), you will end up with a noisy grey-scale image of 512 x 512. Then, do image denoising using the following two approaches:

(1). Filtering in frequency space: First find the FT version of the image, then multiply the FT version with a low-pass filter, which will give you a filtered FT version. Finally do an inverse FT to yield the denoised image. Please try with low-pass filters of different filtration levels, and show your results.

(2). Filtering in image space: design a 'smoothing/denoising' filter kernel, and convolute it with the noisy grey-scale image to receive your denoised image.

(3). For the above two denoising approaches, calculate their respective MSE, PSNR, and SSIM relative to the original image before denoising.

Solution 3

We have a photograph of the ShanghaiTech campus, which is a 3-channel RGB image. We convert it to grayscale and resize it to 512×512 pixels. The original image and resize image is shown below:

Fourier Transform (FT) is a mathematical operation that transforms a function of time (or space) into a function of frequency. The FT of a 2D image can be computed using the Fast Fourier Transform (FFT) algorithm. The FT of the image is then multiplied by a low-pass filter to remove high-frequency noise.

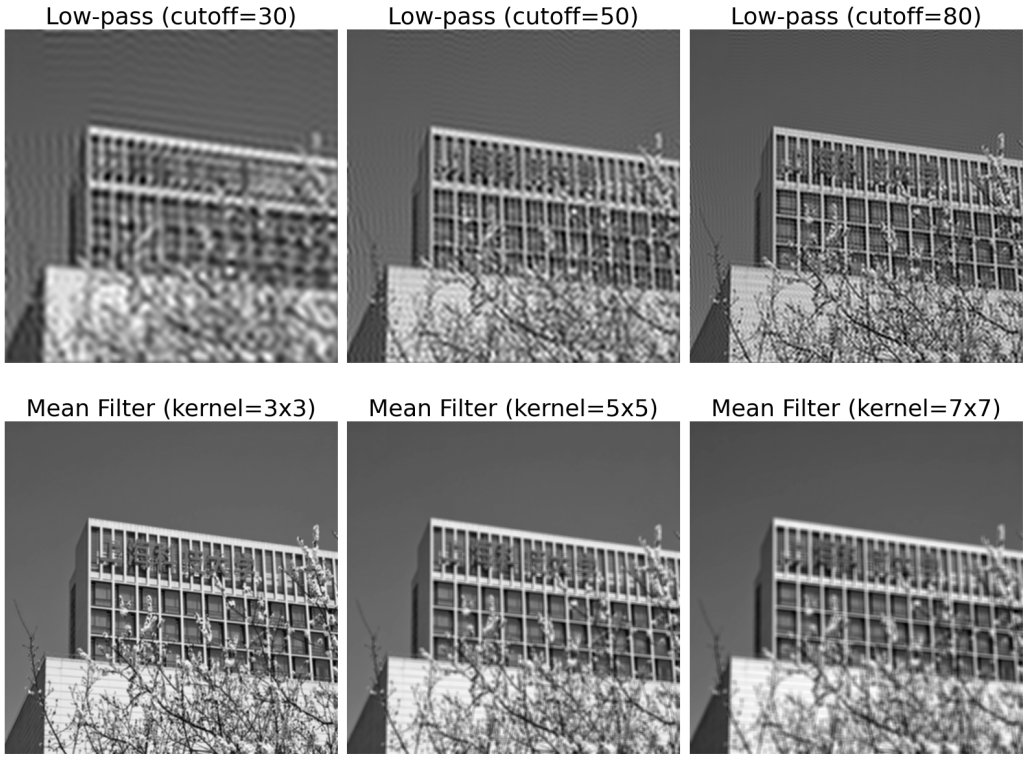

```
    # Fourier Transform
    f_transform = fftpack.fft2(image)
    f_transform_shifted = fftpack.fftshift
    (f_transform)

    # Low pass filter
    mask[crow-cutoff_frequency:crow+cutoff_frequency,
       ccol-cutoff_frequency:ccol+cutoff_frequency] = 1

    # Apply filter
    f_transform_filtered = f_transform_shifted * mask
```

Comparing to the filtering in frequency space, the filtering in image space is computed by convolving the image with a sliding filter kernel. The kernel is a small matrix that is applied to each pixel in the image, and the result is a new pixel value that is a weighted average of the surrounding pixels. To achieve noise reduction, we can use a mean filter or Gaussian filter. The mean filter replaces each pixel with the average of its neighbors, while the Gaussian filter applies a Gaussian function to weight the neighbors differently.

```
    # Mean Filter kernel
```

```
    kernel = np.ones((kernel_size, kernel_size)) / (
        kernel_size * kernel_size)

    # Apply Convolution
    filtered_image = scipy.ndimage.convolve(image, kernel
        , mode='reflect')
```

We apple the low-pass filter to the image in the frequency domain with cutoff frequency of 30, 50, and 80. And then apply the mean filter in the spatial domain with kernel size of 3, 5, and 7. To compare the two methods and different parameters, we present the results side by side as follows.



for the two denoising methods, we calculate the MSE, PSNR, and SSIM relative to the original image before denoising.

MSE (Mean Squared Error) is a measure of the average squared difference between the estimated values and the actual value. It is calculated

as:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(I(i) - \tilde{I}(i))^2$$

Where I is the original image, $\tilde{I}$ is the denoised image, and N is the number of pixels in the image. PSNR (Peak Signal-to-Noise Ratio) is a measure of the ratio between the maximum possible power of a signal and the power of corrupting noise. It is calculated as:

$$PSNR = 10 \cdot \log_{10}\left(\frac{R^2}{MSE}\right)$$

Where R is the maximum possible pixel value of the image. For an 8-bit image, $R = 255$. SSIM (Structural Similarity Index) is a measure of the similarity between two images. It is calculated as:

$$SSIM(x, y) = \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)}$$

Where x and y are the two images, μ_x and μ_y are the mean values of the images, σ_x^2 and σ_y^2 are the variances, σ_{xy} is the covariance, and C_1 and C_2 are constants to stabilize the division.

```
from skimage.metrics import structural_similarity as
    ssim
from skimage.metrics import peak_signal_noise_ratio as
    psnr

    # MSE
    mse = np.mean((original - filtered) ** 2)
    # PSNR
    psnr_value = psnr(original, filtered, data_range
        =255.0)
    # SSIM
    ssim_value = ssim(original, filtered, data_range
        =255.0)
```

Generally, a lower MSE indicates a higher PSNR and SSIM, reflecting greater similarity between the two images. The results of the two methods are presented in the table below:

Method	MSE	PSNR	SSIM
Low-pass Filter (cutoff 30)	1471.92	16.45	0.4848
Low-pass Filter (cutoff 50)	1113.05	17.67	0.5955
Low-pass Filter (cutoff 80)	714.91	19.59	0.7359
Mean Filter (3×3)	**484.64**	**21.28**	**0.8327**
Mean Filter (5×5)	915.50	18.51	0.6782
Mean Filter (7×7)	1183.50	17.40	0.5888

From the table, we can observe that the 3×3 Mean Filter achieved the best performance across all metrics with the lowest MSE (484.64), highest PSNR (21.28 dB), and highest SSIM (0.8327).Moreover, the higher cutoff frequency in the low-pass filter and the small filter size in spatial domain,the better the denoising effect.

We conclude from our analysis that, generally speaking, a large cutoff frequency and a small filter size retain more high-frequency information in the image, while both image detail and noise belong to high-frequency information. For the original input image, there is not much inherent noise; thus, excessive denoising can lead to a loss of image details and reduced quality. Therefore, in practical applications, we need to choose appropriate denoising methods and parameters according to the specific situation of the image. Specifically, if there is more noise in the original image, we can use a small cutoff frequency and a large filter size for denoising; otherwise, it may have an opposite effect.