MEAM 520
Lecture 14: Velocity Kinematics

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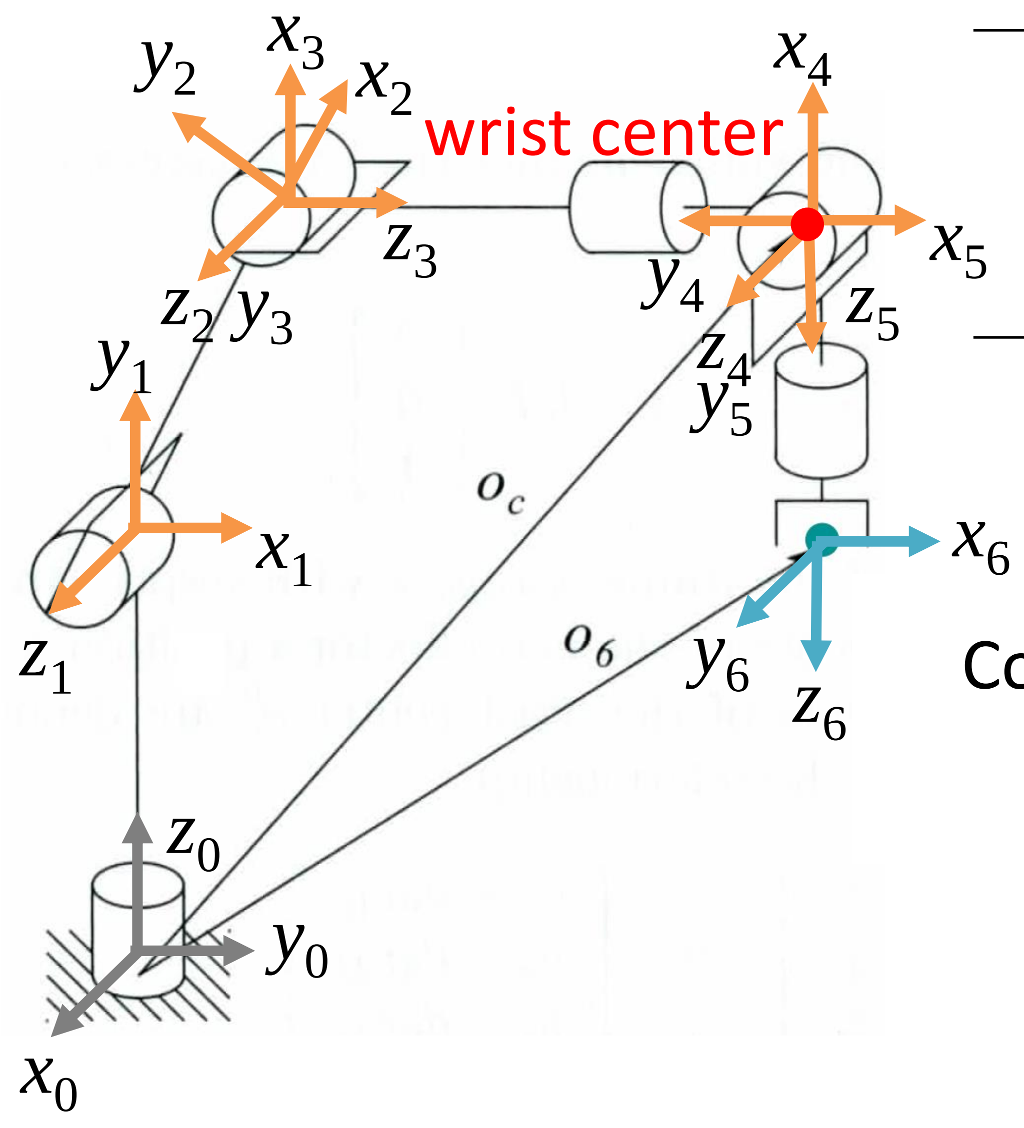
Recap of the semester so far:



Link	x step		z step	
	a_i	α_i	d_i	θ_i
1	0	$-\pi/2$	d_1	θ_1
2	a_2	0	0	θ_2
3	0	$\pi/2$	0	θ_3
4	0	$-\pi/2$	d_4	θ_4
5	0	$\pi/2$	0	θ_5
6	0	0	d_6	θ_6

DH convention

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

FK

$\boldsymbol{q}$ Configuration Space

$\mathbf{T}_n^0$ Task Space

IK

Kinematic Decoupling

position

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

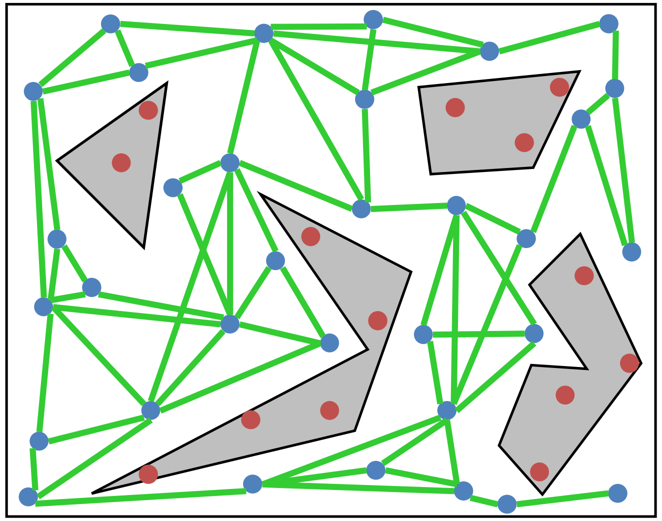
orientation

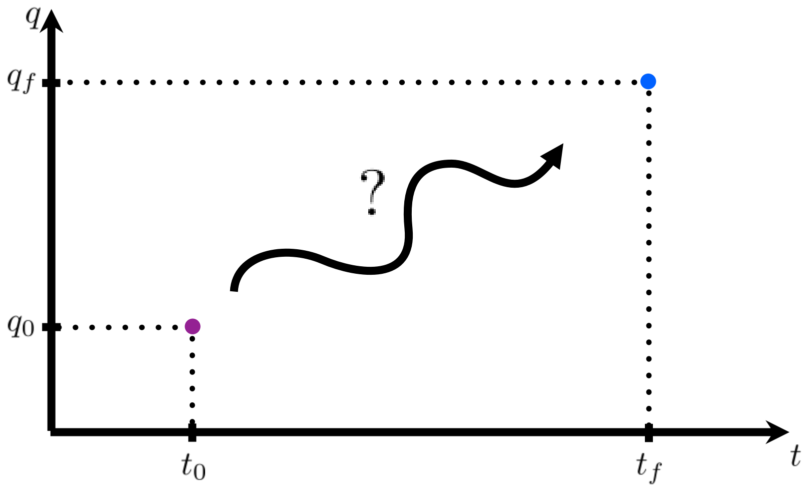
$$\mathbf{R}_6^3 = (\mathbf{R}_3^0)^{-1}\mathbf{R} = (\mathbf{R}_3^0)^{\mathrm{T}}\mathbf{R}$$

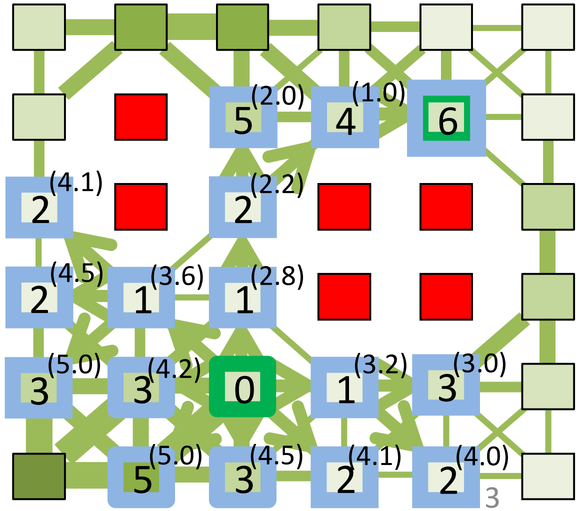
Recap of the semester so far:

Planning strategy:

1. Convert your free C-space into a graph/roadmap
2. Find a path from q_{start} to a node q_a that is in the roadmap
3. Find a path from q_{goal} to a node q_b that is in the roadmap
4. Search the roadmap for a path from q_a to q_b

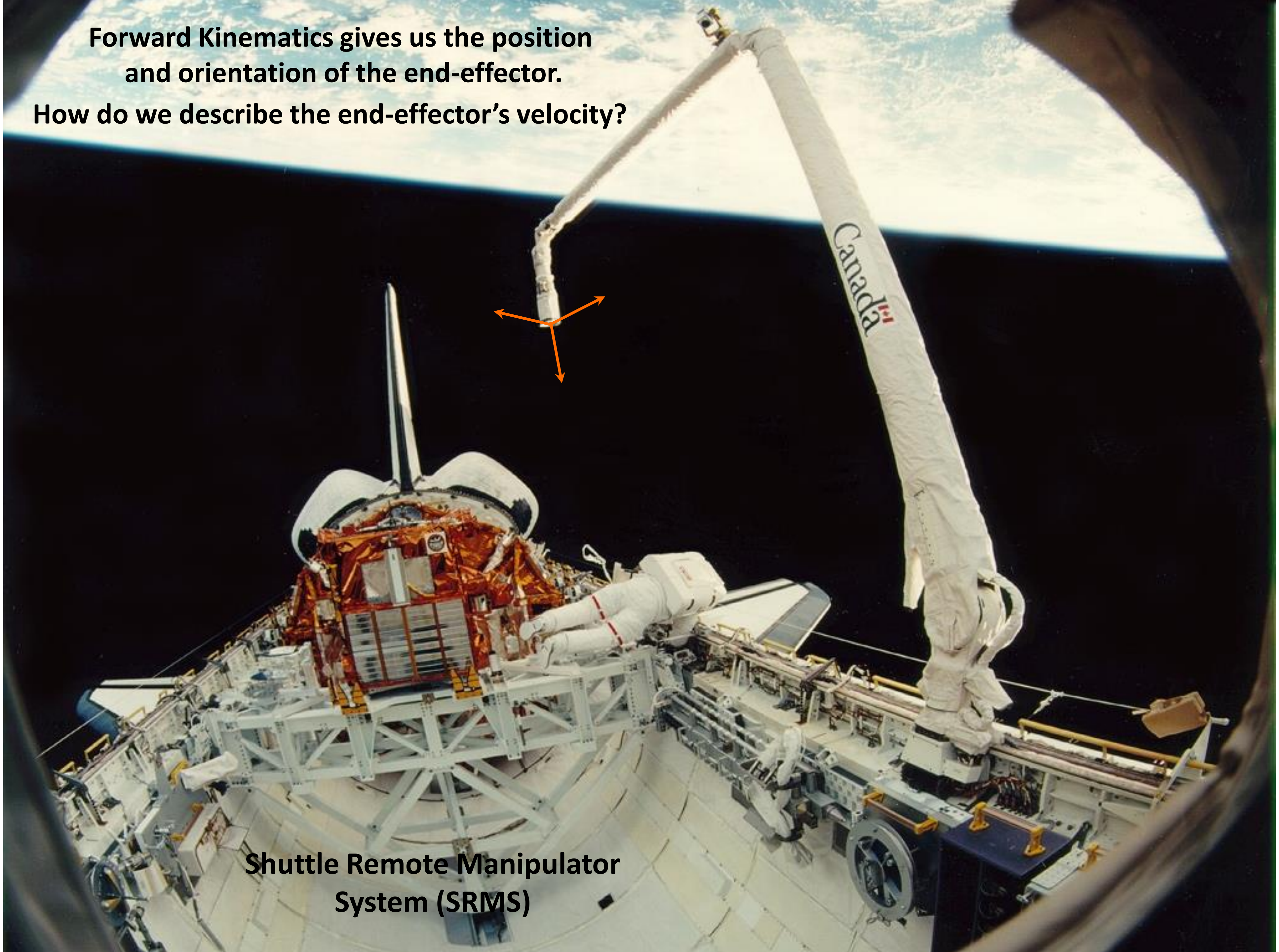






Forward Kinematics gives us the position and orientation of the end-effector.

How do we describe the end-effector's velocity?



Shuttle Remote Manipulator System (SRMS)

Next few weeks (Jacobians)

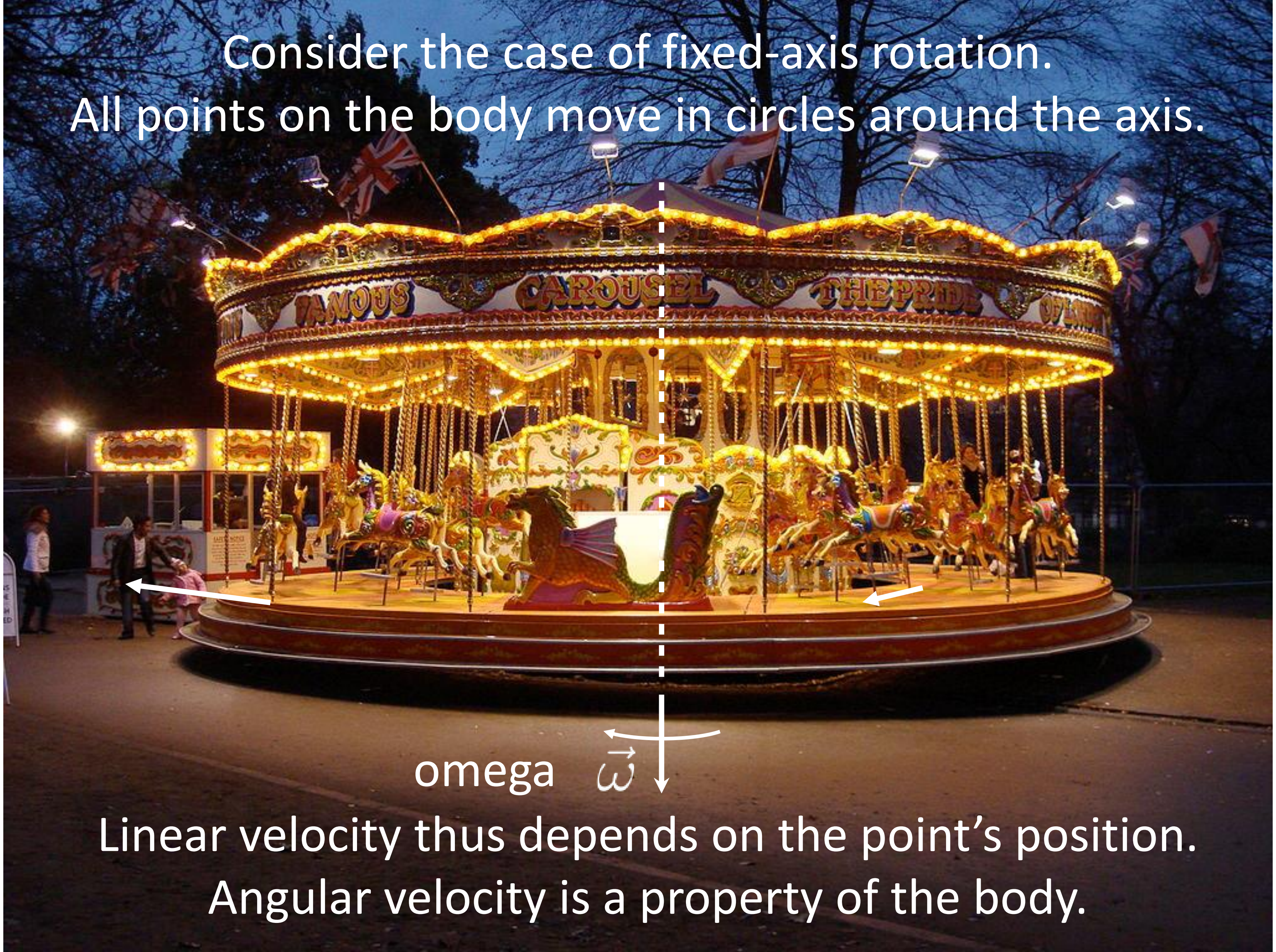
- Today: Angular velocities and derivatives of rotation matrices
- 10/20: Forward velocity kinematics for a manipulator arm
- 10/22: Inverse velocity kinematics and singularities
- 11/6: Lab 4 due = velocity FK/IK

- 10/27: Forces using Jacobians
- 10/29: Potential fields: using Jacobians for reactive path planning
- 11/20: Lab 5 due = potential fields

- Quick update on final project: multi-robot competition, rules to be announced

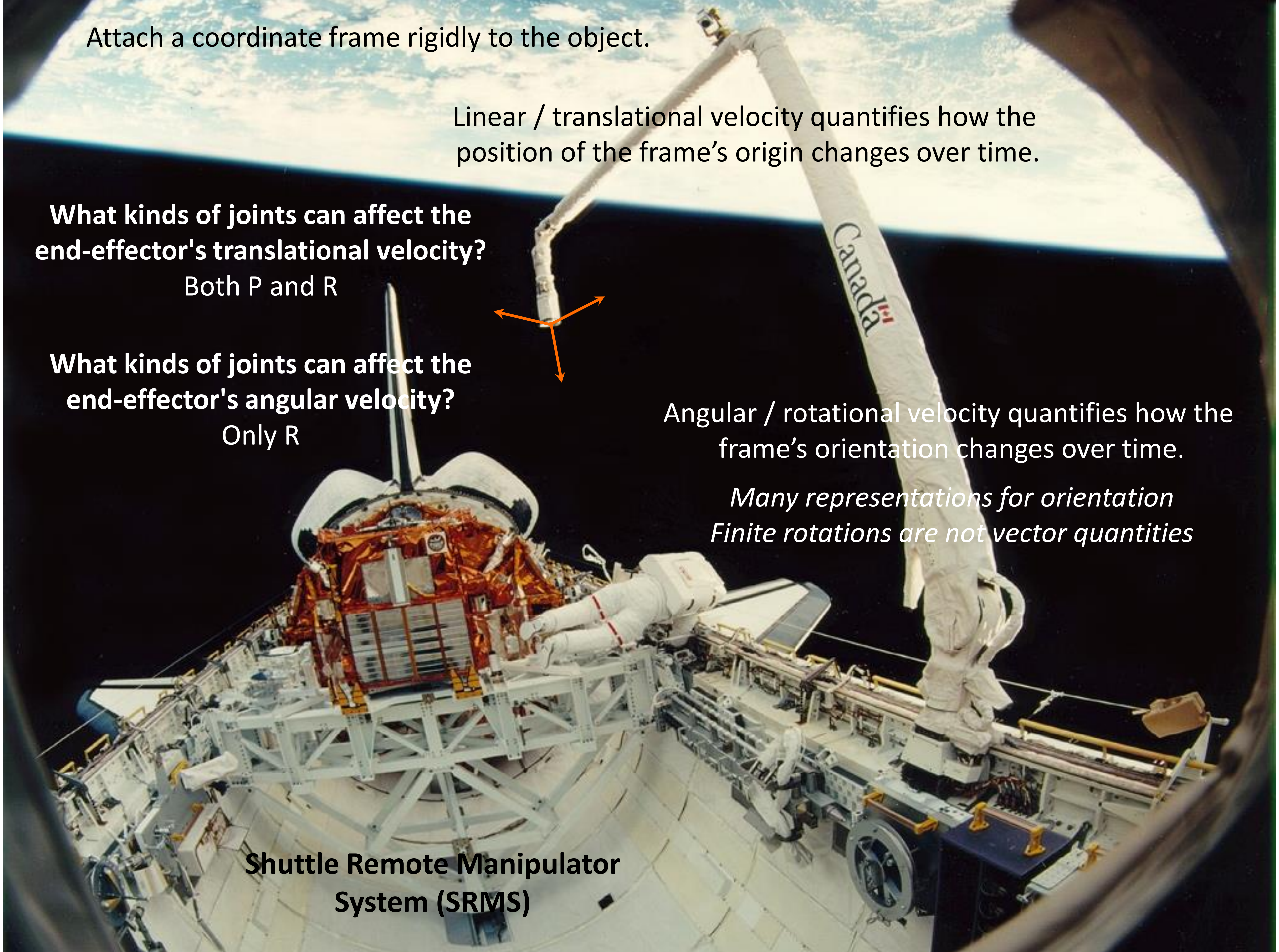
Consider the case of fixed-axis rotation.

All points on the body move in circles around the axis.



Linear velocity thus depends on the point’s position.

Angular velocity is a property of the body.


Attach a coordinate frame rigidly to the object.
Linear / translational velocity quantifies how the position of the frame's origin changes over time.
What kinds of joints can affect the end-effector's translational velocity?
Both P and R
What kinds of joints can affect the end-effector's angular velocity?
Only R
Angular / rotational velocity quantifies how the frame's orientation changes over time.
Many representations for orientation
Finite rotations are not vector quantities
Canada
Shuttle Remote Manipulator System (SRMS)

What is the time derivative of a rotation matrix?

$$\dot{R} = \frac{dR}{dt} = ?$$

To start, consider a rotation matrix that is a function of only one variable:

$$R = R(\theta) \in SO(3)$$

$$\text{e.g., } R(\theta) = R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Angle/Axis: $\mathbf{R}_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$

$$\dot{R} = \frac{dR}{dt} = \overset{?}{\boxed{\frac{dR}{d\theta}}} \overset{\checkmark}{\boxed{\frac{d\theta}{dt}}}$$

What is the time derivative of a rotation matrix?

$$\frac{dR}{d\theta} = ?$$

What do we know about rotation matrices?

orthogonality

$$R\,R^T = I$$

$$\frac{d}{d\theta}\left(R\,R^T\right) = \frac{d}{d\theta}(I)$$

product rule

$$\frac{dR}{d\theta}R^T + R\frac{dR^T}{d\theta} = 0$$

Sum of two matrices equals zero.

$$\text{define } S = \frac{dR}{d\theta}R^T \qquad S^T = \left(\frac{dR}{d\theta}R^T\right)^T = R\frac{dR^T}{d\theta}$$

$$S + S^T = 0$$

Sum of a matrix and its transpose equals zero.

Skew-Symmetric Matrices

$$S + S^T = 0$$

$$S = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

What do you know about the elements of S?

Skew-Symmetric Matrices

$$S + S^T = 0$$

$$S = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

What do you know about the elements of S?

Zeros along the diagonal.
Positive and negative values across the diagonal.

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Skew-Symmetric Matrices

$$S + S^T = 0 \qquad S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Define the operator S

$$\vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

The operator S is linear

$$S(\alpha\vec{a} + \beta\vec{b}) = \alpha S(\vec{a}) + \beta S(\vec{b})$$

But what does S do?

$$S(\vec{a})\,\vec{p} = ?$$

What ideas do you have?

Skew-Symmetric Matrices

$$S(\vec{a})\,\vec{p} = ? = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

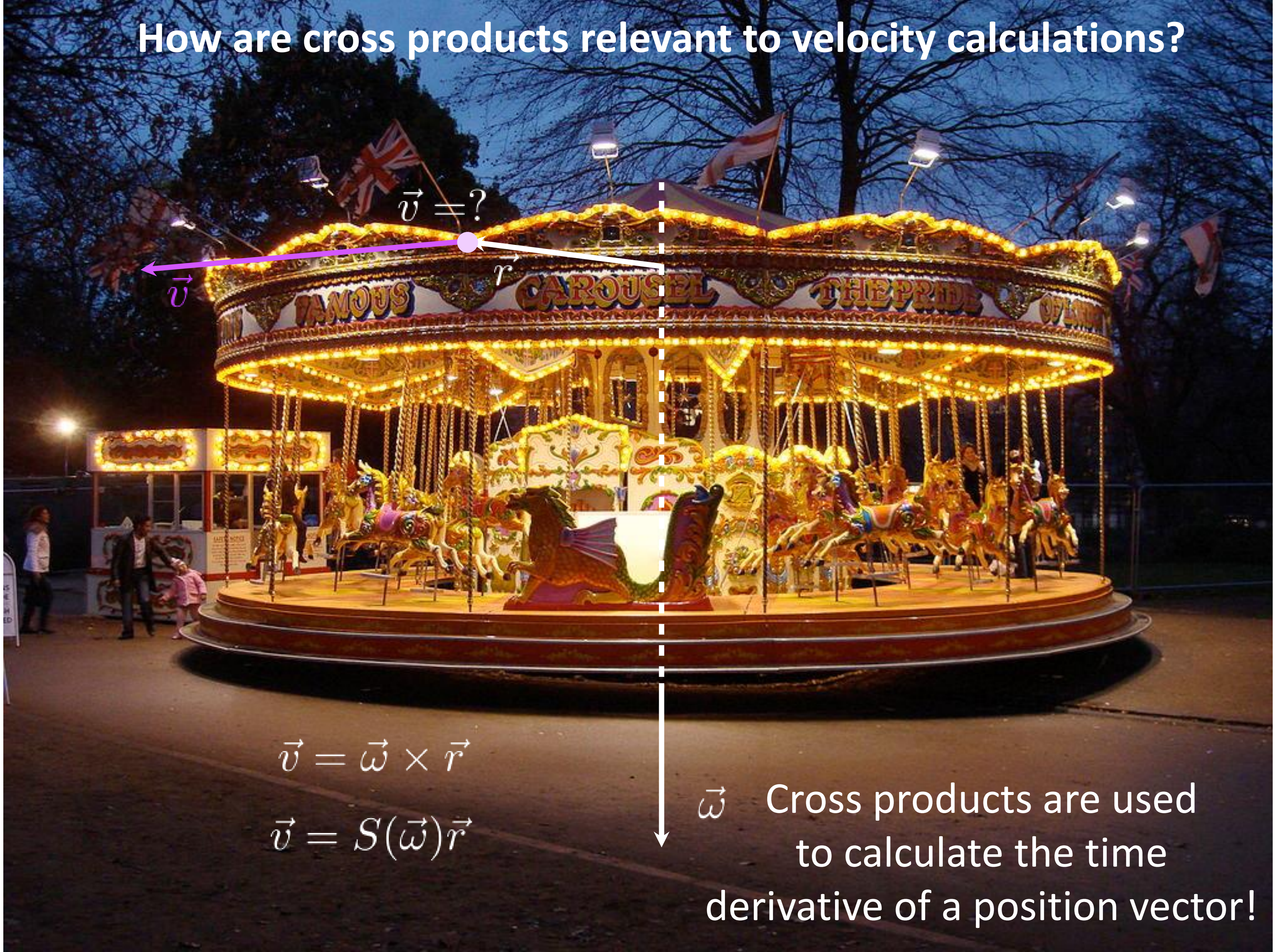
$$= \begin{bmatrix} -a_z p_y + a_y p_z \\ a_z p_x - a_x p_z \\ -a_y p_x + a_x p_y \end{bmatrix}$$

$$= \begin{bmatrix} a_y p_z - a_z p_y \\ a_z p_x - a_x p_z \\ a_x p_y - a_y p_x \end{bmatrix}$$

$$S(\vec{a})\vec{p} = \vec{a} \times \vec{p}$$

Skew-symmetric matrices are a matrix-based way to represent a cross-product between vectors.

How are cross products relevant to velocity calculations?



$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{v} = S(\vec{\omega})\vec{r}$$

Cross products are used to calculate the time derivative of a position vector!

What is the time derivative of a rotation matrix?

$$\frac{dR}{d\theta} = ? \qquad \text{define } S = \frac{dR}{d\theta} R^T \qquad S + S^T = 0$$

This matrix is skew-symmetric.

It also contains the quantity we are seeking.

Multiply both sides on the right by R.

$$S\,R = \frac{dR}{d\theta} R^T R \qquad R^T R = I$$

$$\boxed{\frac{dR}{d\theta} = S\,R}$$

What do you get when you multiply S into R?

This crosses the vector in S into each column of R.

Computing the derivative of a rotation matrix R is equivalent to multiplying that matrix R by a skew-symmetric matrix S.

But we don't yet know how to calculate that matrix S from R!

$$\boxed{\begin{aligned} S &= \frac{dR}{d\theta} R^T \\ \frac{dR}{d\theta} &= S\,R \end{aligned}}$$

Example

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\dot{R}_{x,\theta} = ?$$

Let's solve by direct calculation to discover what S must be.

$$\dot{R}_{x,\theta} = \frac{dR_{x,\theta}}{dt} = \frac{dR_{x,\theta}}{d\theta}\frac{d\theta}{dt} = SR_{x,\theta}\frac{d\theta}{dt}$$

$$S = ? = \frac{dR_{x,\theta}}{d\theta} R^T_{x,\theta}$$

$$= \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(\hat{i})$$

$$\vec{a} = ? = \hat{i}$$

$$S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

S is a skew-symmetric matrix of the axis of rotation!

$$\boxed{\begin{gathered} S = \frac{dR}{d\theta} R^T \\ \frac{dR}{d\theta} = S\,R \end{gathered}}$$

Example

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\dot{R}_{x,\theta} = ?$$

$$\dot{R}_{x,\theta} = \frac{dR_{x,\theta}}{dt} = \frac{dR_{x,\theta}}{d\theta}\frac{d\theta}{dt} = SR_{x,\theta}\frac{d\theta}{dt}$$

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(\hat{i})$$

The skew-symmetric matrix S defines the axis about which rotation is occurring.

Exactly what you would get by differentiating each element w.r.t. time.

$$\dot{R}_{x,\theta} = S(\hat{i})R_{x,\theta}\dot{\theta}$$

$$\dot{R}_{x,\theta} = S(\dot{\theta}\hat{i})R_{x,\theta}$$

$$\vec{\omega} = \dot{\theta}\hat{i}$$

$$\boxed{\dot{R}_{x,\theta} = S(\vec{\omega})R_{x,\theta}}$$

Crossing omega into each column of R...

$$\boxed{\dot{R}_{x,\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\theta}\sin\theta & -\dot{\theta}\cos\theta \\ 0 & \dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{bmatrix}}$$

In general, you simply get S from the angular velocity vector, and you don't need to differentiate the matrix.

a skew-symmetric matrix
formed from omega

The time derivative of a
rotation matrix is...

$$\dot{R}(t) = S(\vec{\omega}(t))R(t)$$

times the rotation
matrix itself

angular velocity of rotating frame
w.r.t. the fixed frame at time t

Another Example:

Frame 1 is instantaneously aligned with frame 0, and their origins are always coincident. Frame 1 has the following angular velocity vector relative to frame 0, expressed in frame 0:

$$\vec{\omega}_{0,1}^{0} = \begin{bmatrix} 0 \text{ rad/s} \\ 2 \text{ rad/s} \\ 2 \text{ rad/s} \end{bmatrix}$$

$$S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$R_1^0 = ?$$

$$\dot{R}_1^0 = ?$$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{\omega}_{0,1}^0 = \begin{bmatrix} 0 \text{ rad/s} \\ 2 \text{ rad/s} \\ 2 \text{ rad/s} \end{bmatrix}$$

a skew-symmetric matrix formed from omega

$$\dot{R}_1^0 = ? = S(\vec{\omega})R_1^0$$

times the rotation matrix itself

$$S(\vec{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \text{ rad/s} & 2 \text{ rad/s} \\ 2 \text{ rad/s} & 0 & 0 \\ -2 \text{ rad/s} & 0 & 0 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{\omega}_{0,1}^0 = \begin{bmatrix} 0 \text{ rad/s} \\ 2 \text{ rad/s} \\ 2 \text{ rad/s} \end{bmatrix}$$

a skew-symmetric matrix
formed from omega

$$\dot{R}_1^0 = ? = S(\vec{\omega})R_1^0$$

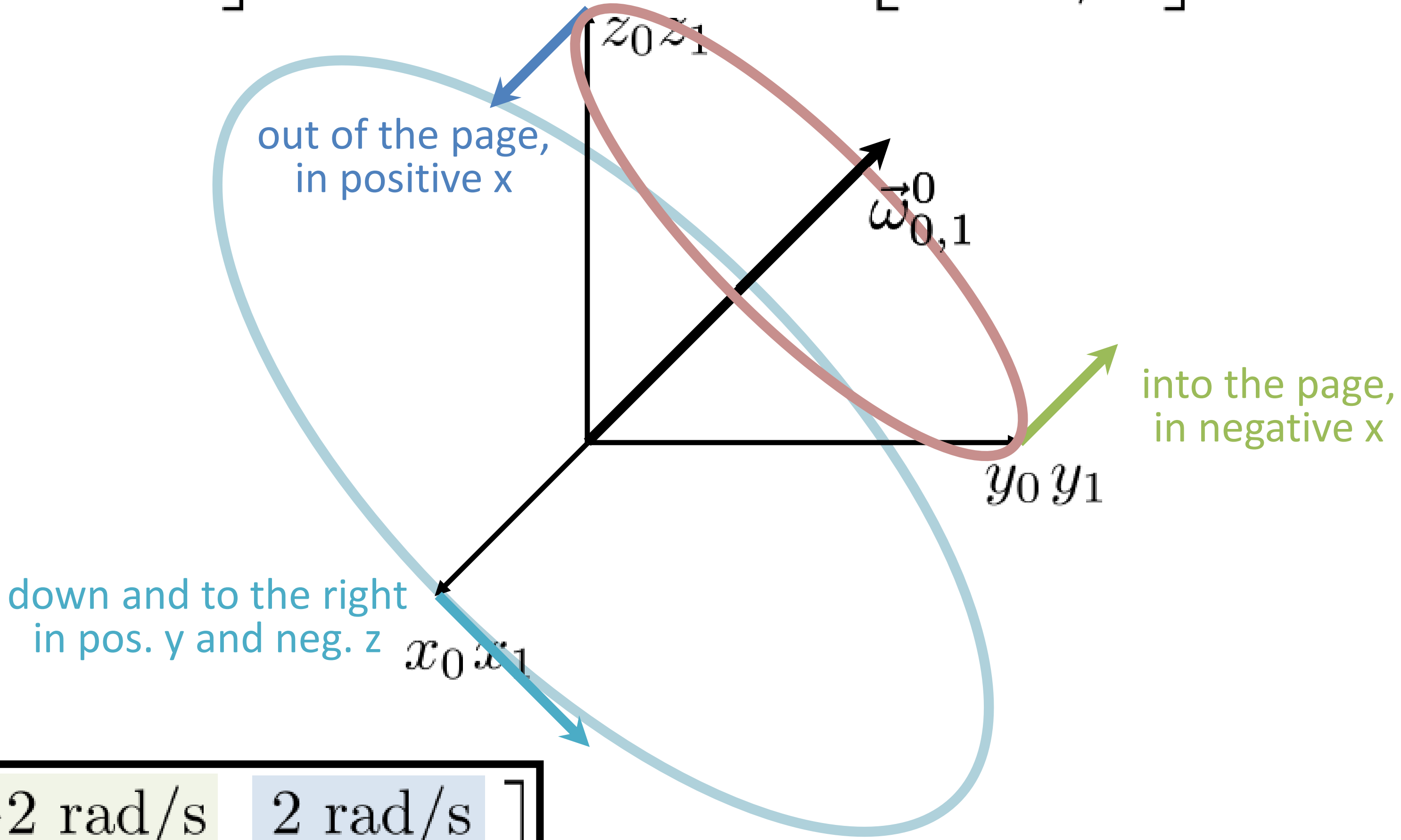
times the rotation
matrix itself

$$\dot{R}_1^0 = \begin{bmatrix} 0 & -2 \text{ rad/s} & 2 \text{ rad/s} \\ 2 \text{ rad/s} & 0 & 0 \\ -2 \text{ rad/s} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\dot{R}_1^0 = \begin{bmatrix} 0 & -2 \text{ rad/s} & 2 \text{ rad/s} \\ 2 \text{ rad/s} & 0 & 0 \\ -2 \text{ rad/s} & 0 & 0 \end{bmatrix}}$$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\omega}_{0,1}^0 = \begin{bmatrix} 0 \text{ rad/s} \\ 2 \text{ rad/s} \\ 2 \text{ rad/s} \end{bmatrix}$$



$$\boxed{\dot{R}_1^0 = \begin{bmatrix} 0 & -2 \text{ rad/s} & 2 \text{ rad/s} \\ 2 \text{ rad/s} & 0 & 0 \\ -2 \text{ rad/s} & 0 & 0 \end{bmatrix}}$$

What questions do you have?

Why is this useful?

Calculating the velocity of a point in a rotating frame.

Calculating the linear velocity of a point on a rigid body (e.g., the end effector) that is both translating and rotating.

Understanding how angular velocities combine on a robotic manipulator.

Calculating the velocity of a point in a rotating frame.

See SHV 4.3: Angular Velocity: The General Case

$$p^0 = R^0_1 p^1$$

A vector to a point that is fixed to frame 1, expressed in frame 0.

$$\frac{d}{dt}p^0 = ? = \dot{R}^0_1 p^1$$

$$\dot{R}(t) = S(\vec{\omega}(t))R(t)$$

$$= S(\vec{\omega})R^0_1 p^1$$

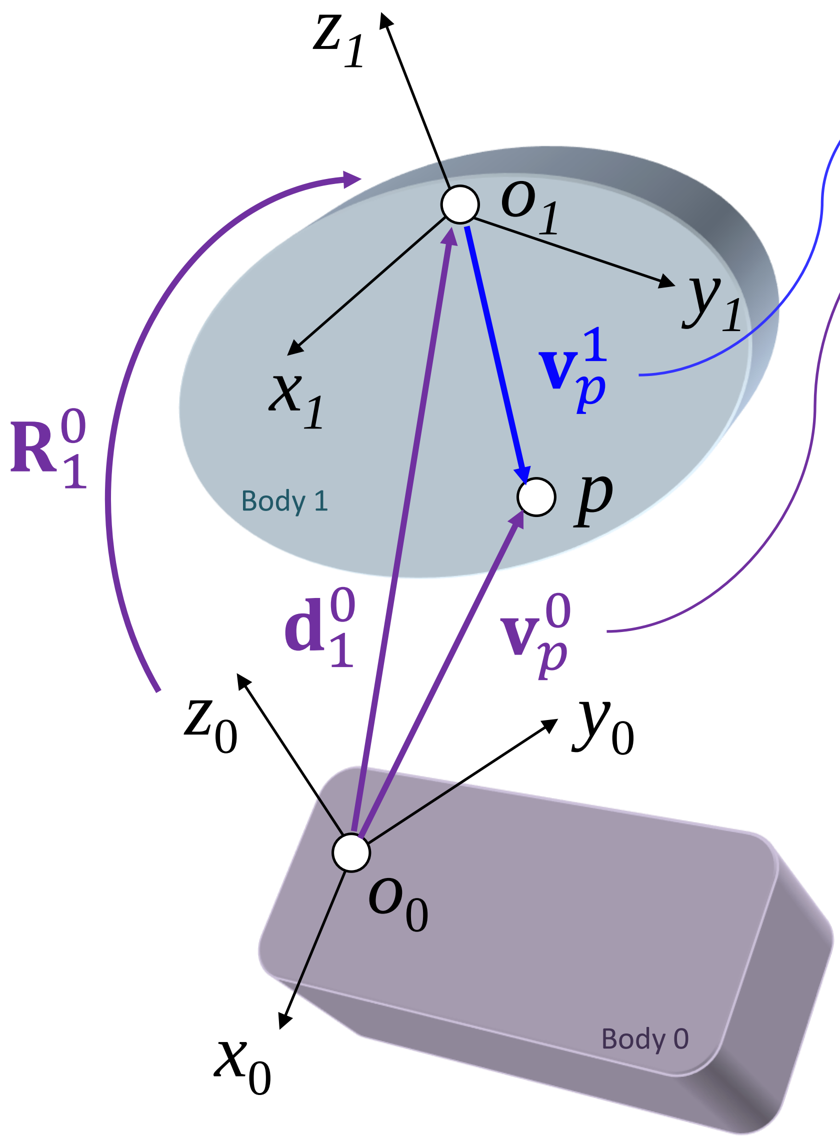
$$= \vec{\omega} \times R^0_1 p^1$$

$$= \vec{\omega} \times p^0$$

$$\boxed{\dot{p}^0 = S(\vec{\omega}(t))R^0_1 p^1}$$

Calculating the linear velocity of the end-effector of a robot

See SHV 4.5: Linear Velocity of a Point Attached to a Moving Frame



$$p^0 = R_1^0(t)p^1 + o_1^0(t)$$

point p is rigidly fixed in frame 1

$$\dot{p}^0 = \dot{R}_1^0 p^1 + \dot{o}_1^0$$

$$\dot{p}^0 = S(\omega^0)R_1^0 p^1 + \dot{o}_1^0$$

$$\dot{R}(t) = S(\vec{\omega}(t))R(t)$$

angular velocity of body 1 in frame 0

$$\dot{p}^0 = \omega^0 \times p^0 + \dot{o}_1^0$$

You can calculate the linear velocity of the end-effector from the angular velocity of its frame, its position relative to its frame's origin, and the linear velocity of its frame's origin.

Combining angular velocities on a robotic manipulator

See SHV 4.4: Addition of Angular Velocities

$$R_2^0(t) = R_1^0(t) R_2^1(t)$$

Differentiate both sides with respect to time.

$$\dot{R}_2^0 = S(\omega_{0,2}^0) R_2^0$$

$$\frac{d}{dt}(R_1^0 R_2^1) = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

$$\dot{R}(t) = S(\vec{\omega}(t)) R(t)$$

$$\frac{d}{dt}(R_1^0 R_2^1) = S(\omega_{0,1}^0) R_1^0 R_2^1 + \underbrace{R_1^0}_{\text{express in frame 0}} \underbrace{S(\omega_{1,2}^1) R_2^1}_{\dot{R}_2^1 \text{ written in frame 1}}$$

You can add angular velocity vectors!

$$\omega_{0,2}^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1$$

$$\frac{d}{dt}(R_1^0 R_2^1) = S(\omega_{0,1}^0) R_2^0 + \underbrace{S(R_1^0 \omega_{1,2}^1) R_2^0}_{\dot{R}_2^1 \text{ written in frame 0}}$$

The angular velocity of frame 2 relative to frame 0 is equal to the angular velocity of frame 1 relative to frame 0, expressed in frame 0, plus the angular velocity of frame 2 relative to frame 1, expressed in frame 0

Uses for Skew-Symmetric Matrices

What questions do you have?

You can derive the fact that you can add angular velocity vectors by expressing them in the same frame.

$$\omega^0_{0,2} = \omega^0_{0,1} + R^0_1 \omega^1_{1,2}$$

The angular velocity of frame 2 relative to frame 0 is equal to the angular velocity of frame 1 relative to frame 0, expressed in frame 0, plus the angular velocity of frame 2 relative to frame 1, expressed in frame 0

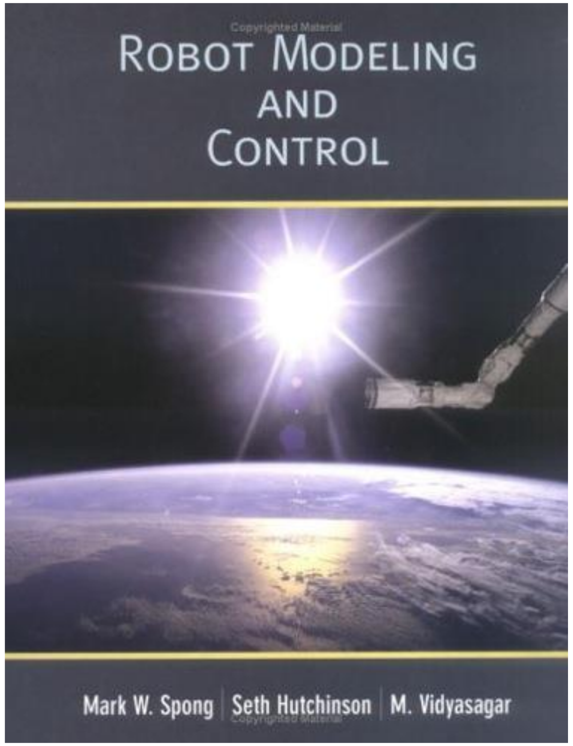
You can calculate the velocity of a point that is fixed to a rotating (but not translating) frame.

$$\begin{aligned} p^0 &= R^0_1 p^1 \\ \frac{d}{dt} p^0 &= ? = \dot{R}^0_1 p^1 \\ &= S(\vec{\omega}) R^0_1 p^1 \\ &= \vec{\omega} \times R^0_1 p^1 \\ &= \vec{\omega} \times p^0 \end{aligned}$$

You can calculate the velocity of a point that is fixed to a rotating and translating frame.

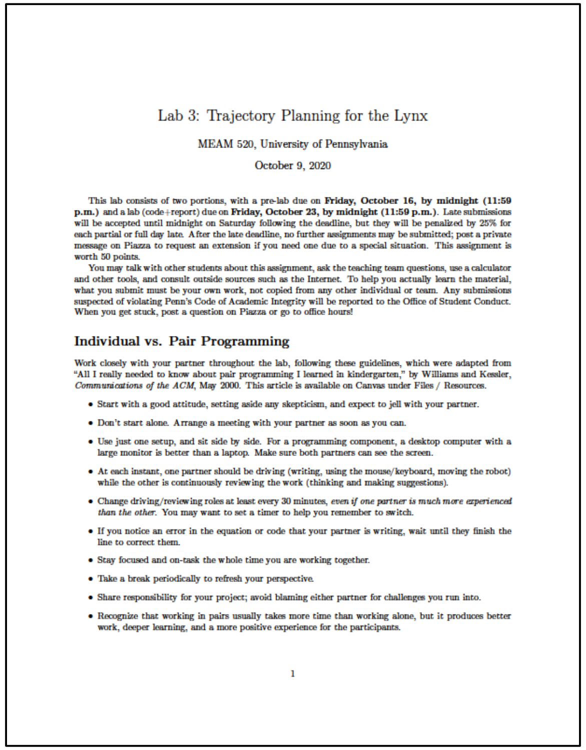
$$\begin{aligned} p^0 &= R^0_1(t) p^1 + o^0_1(t) \\ \dot{p}^0 &= \dot{R}^0_1 p^1 + \dot{o}^0_1 \\ \dot{p}^0 &= S(\omega^0) R^0_1 p^1 + \dot{o}^0_1 \\ \dot{p}^0 &= \omega^0 \times p^0 + \dot{o}^0_1 \end{aligned}$$

Next time: More Velocity Kinematics



Chapter 4: Velocity Kinematics

- Read 4.5-4.7



Lab 3: Trajectory Planning for the Lynx

MEAM 520, University of Pennsylvania

October 9, 2020

This lab consists of two portions, with a pre-lab due on **Friday, October 16, by midnight (11:59 p.m.)** and a lab (code+report) due on **Friday, October 23, by midnight (11:59 p.m.)**. Late submissions will be accepted until midnight on Saturday following the deadline, but they will be penalized by 25% for each partial or full day late. After the late deadline, no further assignments may be submitted; post a private message on Piazza to request an extension if you need one due to a special situation. This assignment is worth 50 points.

You may talk with other students about this assignment, ask the teaching team questions, use a calculator and other tools, and consult outside sources such as the Internet. To help you actually learn the material, what you submit must be your own work, not copied from any other individual or team. Any submissions suspected of violating Penn's Code of Academic Integrity will be reported to the Office of Student Conduct. When you get stuck, post a question on Piazza or go to office hours!

Individual vs. Pair Programming

Work closely with your partner throughout the lab, following these guidelines, which were adapted from "All I really needed to know about pair programming I learned in kindergarten," by Williams and Kessler, *Communications of the ACM*, May 2000. This article is available on Canvas under Files / Resources.

- Start with a good attitude, setting aside any skepticism, and expect to jell with your partner.
- Don't start alone. Arrange a meeting with your partner as soon as you can.
- Use just one setup, and sit side by side. For a programming component, a desktop computer with a large monitor is better than a laptop. Make sure both partners can see the screen.
- At each instant, one partner should be driving (writing, using the mouse/keyboard, moving the robot) while the other is continuously reviewing the work (thinking and making suggestions).
- Change driving/reviewing roles at least every 30 minutes, *even if one partner is much more experienced than the other*. You may want to set a timer to help you remember to switch.
- If you notice an error in the equation or code that your partner is writing, wait until they finish the line to correct them.
- Stay focused and on-task the whole time you are working together.
- Take a break periodically to refresh your perspective.
- Share responsibility for your project; avoid blaming either partner for challenges you run into.
- Recognize that working in pairs usually takes more time than working alone, but it produces better work, deeper learning, and a more positive experience for the participants.

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Lab 3: Trajectory Planning due 10/23