

MAT 290 Important Formulas

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Complex Numbers

$$\rightarrow z = x + iy, \operatorname{Re}\{z\} = x, \operatorname{Im}\{z\} = y$$

$$\rightarrow z = |z| |\theta| = r \cos \theta + i r \sin \theta$$

$$\rightarrow |z| = r = \sqrt{x^2 + y^2} = \sqrt{zz^*}$$

$$\rightarrow \operatorname{Arg}(z) = \theta_0 = \tan^{-1}(y/x), -\pi < \theta_0 \leq \pi$$

$$\rightarrow \arg(z) = \theta_0 + 2\pi k, k=0, \pm 1, \pm 2, \dots$$

$$\rightarrow \bar{z} = x - iy$$

$$\rightarrow i^2 = -1, z\bar{z} = |z|^2, |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\rightarrow \text{Euler's Formula: } e^{i\theta} = \cos \theta + i \sin \theta$$

$$\rightarrow n^{\text{th}} \text{ power of } z: z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n [\cos(n\theta) + i \sin(n\theta)] = r^n \ln z$$

$$\rightarrow n \text{ roots of } z: w_k = z^{1/n} = r^{1/n} \left[\cos\left(\frac{\theta_0 + 2\pi k}{n}\right) + i \sin\left(\frac{\theta_0 + 2\pi k}{n}\right) \right]$$

all lie on circle radius $|z|^{1/n} = r^{1/n}$ and are equally spaced on this circle by $\frac{2\pi}{n}$ starting at $\frac{\theta_0}{n}$

$$\rightarrow \cosh z = \frac{e^{iz} + e^{-iz}}{2} = \cos x \cosh y + i \sin x \sinh y \quad \sinh z = \frac{e^{iz} - e^{-iz}}{2i} = i \sin x \cosh y + i \cos x \sinh y$$

$$\rightarrow \cosh z = \frac{e^{iz} + e^{-iz}}{2} = \cos(i z)$$

$$\sinh z = \frac{e^{iz} - e^{-iz}}{2i} = -i \sin(i z)$$

$$\rightarrow \ln(z) = \log|z| + i \{\operatorname{Arg}(z) + 2n\pi\}, n=0, \pm 1, \pm 2, \dots$$

regular log laws hold for $\ln(z)$, but not always for $\ln(z)$

$$\rightarrow \ln(z) = \log|z| + i \operatorname{Arg}(z)$$

$$\rightarrow z^\alpha = e^{\alpha \ln z}, \text{ principal value of } z^\alpha = e^{\alpha \ln z}$$

Ordinary Differential Equations

→ Separable method :

$$\frac{dy}{dx} = f(y) g(x) \Rightarrow \frac{1}{f(y)} \frac{dy}{dx} = g(x) \Rightarrow \int \frac{1}{f(y)} dy = \int g(x) dx$$

→ integrating factor method :

$$y' + P(x)y = f(x) \Rightarrow \text{integrating factor } \varphi(x) = e^{\int P(x) dx}$$

$$\begin{aligned} &\Rightarrow \varphi(x)y = \int f(x) \varphi(x) dx \\ &\Rightarrow y(x) = \frac{1}{\varphi(x)} \left[\int \varphi(x) f(x) dx + C \right] \end{aligned}$$

→ reduction of order :

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{and} \quad y_1(x) \quad \text{s.t.} \quad y_2(x) = u(x)y_1(x)$$

$$\Rightarrow y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \quad \Rightarrow y(x) = C_1 y_1(x) + C_2 y_2(x)$$

→ Variation of parameters :

$$y'' + P(x)y' + Q(x)y = f(x)$$

y_c is sol^r to homogenous equ^r
 y_p is solⁿ to nonhomogenous equ^r

$$\Rightarrow y_c = C_1 y_1 + C_2 y_2$$

$$\omega(y_1, y_2) = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2$$

$$\Rightarrow u_1 = - \int \frac{y_2 f}{\omega(y_1, y_2)} dx$$

$$\Rightarrow u_2 = \int \frac{y_1 f}{\omega(y_1, y_2)} dx$$

→ conditions for existence and uniqueness :

- 1) all $a_n(x)$'s and $g(x)$ are continuous
- 2) $a_n(x) \neq 0$

→ Lin. Indep. Sol^rs :

$$\begin{vmatrix} y_1 & y_2 \\ y_2 & y_2 \end{vmatrix} \begin{cases} \rightarrow \omega \neq 0 \\ \rightarrow \omega = 0 \end{cases}$$

Lin.
Indep.
we dunno

$$\begin{cases} \begin{array}{c} \overbrace{y_1} \\ \overbrace{y_2} \\ \overbrace{y_3} \end{array} \end{cases} \begin{cases} \rightarrow \omega \neq 0 \\ \rightarrow \omega = 0 \end{cases}$$

Lin. Indep.
Lin. Dep.

Laplace Transforms

$$\rightarrow F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad \begin{aligned} &1) f(t) \text{ is piecewise continuous} \\ &2) f(t) \text{ is of exponential order} \\ &3) \text{ If 1) and 2), then } \lim_{s \rightarrow \infty} F(s) = 0 \end{aligned}$$

\rightarrow property quick notes:

- ζ -shift: func^r in s -dom shifted by $a \rightarrow$ multiply by e^{at} in t -dom
- t -shift: func^r in t -dom shifted by $a \rightarrow$ multiply by e^{-as} in s -dom
- time differentiation \rightarrow multiplication by s
- time integration \rightarrow division by s
- s differentiation \rightarrow multiplication by t

Complex Analysis

\rightarrow a domain is an open connected set

\rightarrow a function is differentiable at a point z_0 if:

1) Cauchy Riemann equations satisfied

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

2) $u(x,y)$ and $v(x,y)$ and all their partials are continuous

\rightarrow analytic function: not only differentiable at z_0 , but in neighborhood of z_0

\rightarrow harmonic functions: $u(x,y)$ and $v(x,y)$ of analytic fun^rs are harmonic

$$\nabla^2 f = 0 = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f = 0$$

→ contour integration:

1) Parametrize curve $C: z(t) = x(t) + iy(t)$ $a \leq t \leq b$

2) Integral:

$$\int_C f(z) dz = \int_a^b f[z(t)] z'(t) dt$$

→ ML bounding: $\left\| \int_C f(z) dz \right\| \leq ML$, M is max $|f(z)|$ on C , L is length of C

→ very important integral:

$$\int_C \frac{1}{(z-z_0)^n} dz = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

★ to do: add the circulation + flux notes