MAT187 - Calculus 2

University of Toronto

Electrical Engineering
Year 1 Semester 2

PCE Notes by Zaeem Ahmad

Product Rule: $(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$

$$(u(x)v(x))' - u'(x)v(x) = u(x)v'(x)$$

integrate w.r.t. x

$$u(x)v'(x) = (u(x)v(x))' - u'(x)v(x)$$

$$\int u(x)v'(x)\,dx = \int (u(x)v(x))'\,dx - \int u'(x)v(x)\,dx$$

integrating second integral cancels the differentiation

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

We write $dv = v'(x)\,dx$ and $du = u'(x)\,dx$

Integration by Parts

Let u and v be differentiable functions. Then...

$$\boxed{\int u\,dv = uv - \int v\,du} \qquad \boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Example: $\int xe^{2x}\,dx$

choose $u = x$ and $dv = e^{2x}\,dx$

$du = 1\,dx \qquad v = \frac{1}{2}e^{2x}$

$$\boxed{\therefore \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

$$\int \underbrace{x}_{u}\underbrace{e^{2x}\,dx}_{dv} = \underbrace{x \cdot \frac{1}{2}e^{2x}}_{uv} - \int \underbrace{\frac{1}{2}e^{2x}}_{v}\underbrace{dx}_{du}$$

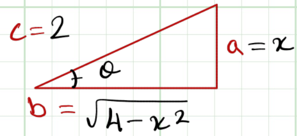
$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

What to pick for u?

Logarithm
Inverse trig function
Algebraic
Trig
Exponential

Trig identities: $\cos^2\theta + \sin^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$

Example $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$



Use pythagorean theorem

$a^2+b^2=c^2$

$b=\sqrt{c^2-a^2} \leftrightarrow \sqrt{4-x^2}$ where $c=2$, $a=x$, $b=\sqrt{4-x^2}$

→ We know that $\sin\theta = \dfrac{x}{2}$ and $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$

differentiate

$x = 2\sin\theta$ $\qquad$ $2\cos\theta = \sqrt{4-x^2}$

$dx = 2\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{2\cos\theta}\cdot 2\cos\theta\, d\theta = \int \frac{2\cos\theta}{2\cos\theta}\,d\theta$$

$$= \int 1\,d\theta = \boxed{\theta + C}$$ ← Now resubstitute to get w.r.t. x

→ We know $x = 2\sin\theta \Rightarrow \theta = \arcsin\dfrac{x}{2}$ ← plug this inside

$$\boxed{\therefore \int \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\frac{x}{2} + C}$$

Domain of Trig Functions

Look at $\sqrt{4-x^2}$ ... so $x = \pm 2 \longrightarrow -2 < x < 2$

so $x = 2\sin\theta \rightarrow -2 < x < 2$

$\theta = \arcsin\dfrac{x}{2} \rightarrow \arcsin\dfrac{-2}{2} < \theta < \arcsin\dfrac{2}{2}$

$$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

Forms of Trig Sub

1. $\int \sqrt{a^2 - x^2}\, dx$ $\quad x = a\sin\theta \quad dx = a\cos\theta\, d\theta$

2. $\int \sqrt{a^2 + x^2}\, dx$ $\quad x = a\tan\theta \quad dx = a\sec^2\theta\, d\theta$

3. $\int \sqrt{x^2 - a^2}\, dx$ $\quad x = a\sec\theta \quad dx = a\sec\theta\tan\theta\, d\theta$

Examples

1. $\int \frac{\sqrt{4-x^2}}{x^2}\, dx$

$a^2 = 4 \Rightarrow a = 2$

$x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$$= \int \frac{\sqrt{4-(2\sin\theta)^2}}{(2\sin\theta)^2} \cdot 2\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{4 - 4\sin^2\theta}}{4\sin^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{4(1-\sin^2\theta)}}{2\sin^2\theta} \cdot \cos\theta\, d\theta$$

Rational Functions

We call $f(x) = \frac{p(x)}{q(x)}$ a rational function if both $p(x)$ and $q(x)$ are polynomial functions.

→ if the degree of $p(x)$ is less than $q(x)$, $f(x)$ is a proper rational function.

eg $$g(x) = \frac{5x^2 - 10x + 3}{7x^4 + 5x - 1}$$ ← proper rational function

$$h(x) = \frac{9x^4 - 13x - 20}{3x^4 - 3x + 4}$$ ← non-proper or improper rational function

$$f(x) = \frac{e^x}{x^2 + 5x + 3}$$ ← not a rational function

Partial Fraction Decomposition

* process of splitting up a proper rational function into simpler parts that are then easier to integrate.
* undoing the process of finding a common denominator.

Eg
$$\int \frac{3x}{x^2 - x - 2} \xleftarrow{\text{common denominator}} \int \frac{1}{x+1} + \frac{2}{x-2}\,dx$$

Partial fraction decomp.

* PFD only works on proper rational functions

Example: integrate $$\frac{2x^2 + 37x + 144}{x^2 + 13x + 40}$$

Long Division

- synthetic division to get top function's degree < 2

$$\begin{array}{r} 2 \\ x^2+13x+40 \overline{)\, 2x^2+37x+144} \\ -(2x^2+26x+80) \\ \hline 0x^2+11x+64 \end{array} \Rightarrow \boxed{2 + \frac{11x+64}{x^2+13x+40}}$$

easy to integrate

focus here

Distinct Linear Factors

$$\frac{11x+64}{x^2+13x+40} = \frac{11x+64}{(x+5)(x+8)} = \frac{A}{x+5} + \frac{B}{x+8}$$

distinct factors

$$11x+64 = A(x+8) + B(x+5)$$
$$= Ax + 8A + Bx + 5B$$
$$= Ax + Bx + 8A + 5B$$
$$11x + 64 = (A+B)x + 8A + 5B$$

$$\therefore \left.\begin{array}{r} A+B=11 \\ 8A+5B=64 \end{array}\right\} \begin{array}{l} A=3 \\ B=8 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 11 \\ 8 & 5 & 64 \end{bmatrix} \sim \begin{bmatrix} 8 & 8 & 88 \\ 8 & 5 & 64 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 8 & 88 \\ 0 & -3 & -24 \end{bmatrix} \sim \begin{bmatrix} 8 & 8 & 88 \\ 0 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 0 & 24 \\ 0 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \end{bmatrix}$$

$$\boxed{\therefore \frac{11x+64}{(x+8)(x+5)} = \frac{3}{x+5} + \frac{8}{x+8}}$$

$$\int \frac{2x^2+37x+144}{x^2+13x+40}\,dx = \int 2 + \frac{11x+64}{x^2+13x+40}\,dx$$
$$= \int 2 + \frac{11x+64}{(x+8)(x+5)}\,dx$$
$$= \int 2 + \frac{3}{x+5} + \frac{8}{x+8}\,dx$$
$$\boxed{= 2x + 3\ln|x+5| + 8\ln|x+8| + C}$$

Distinct Quadratic Factors

* when we can't factor a polynomial, it's irreducible
  ↳ when $ax^2+bx+c=0$ has no real solution
* we need a polynomial of degree 1 in numerator

Example

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + x(Bx+C)$$

$$1 = Ax^2+1 + Bx^2 + Cx$$

$$0 = Ax^2+Bx^2+1+Cx \quad -1$$

$$0 = (A+B)x^2 + Cx$$

$$0 = x\left[(A+B)x + C\right]$$

$$x=0 \qquad x = \frac{C}{A+B}$$

Approximating Integrals ~ (L_n) and (R_n)

* Right and Left endpoint Riemann sum.

Eg: L_4 and R_4 for $\int_1^2 \frac{1}{x}\,dx = \ln(2) - \ln(1) = 0.6931$

(1) 4 subint of width $\Delta x = \frac{2-1}{4} = \frac{1}{4}$ $\qquad \therefore R_4 < \ln x < L_4$

(2) Right : $x^*_k = x_0 + k\Delta x = 1 + k/4$
Left : $x^*_{k-1} = x_0 + (k-1)\Delta x = 1 + (k-1)/4$

$$R_4 = \sum_{k=1}^{4} \frac{1}{x^*_k}\Delta x = \left(\frac{1}{1+1/4} + \frac{1}{1+2/4} + \frac{1}{1+3/4} + \frac{1}{1+4/4}\right)\frac{1}{4} = 0.6345$$

$$L_4 = \sum_{k=1}^{4} \frac{1}{x^*_{k-1}}\Delta x = \left(\frac{1}{1+0} + \frac{1}{1+1/4} + \frac{1}{1+2/4} + \frac{1}{1+3/4}\right)\frac{1}{4} = 0.7595$$

Midpoint Rule (M_n)

(1) choose midpoint of each subinterval
(2) evaluate function at that point
(3) compute area of resulting rectangle

Trapezoid Rule (T_n)

* approximating graph using a "connect the dots" curve and adding the areas of the resulting trapezoids.

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$\Delta x = \frac{b-a}{n}$ $\qquad T_n \approx \int_a^b f(x)\,dx$

Also, $T_n = \frac{1}{2}(L_n + R_n)$

A4. Numerical Integration

Errors

- Error $= |\text{exact value} - \text{approximation}|$
- Relative Error $= \dfrac{\text{Error}}{|\text{exact value}|}$

Statement:

If I approx the integral ___ using the ___ rule and ___ intervals, the error is less than ___

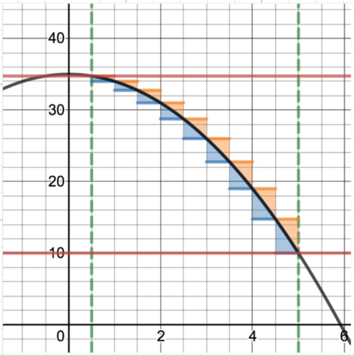
Example: Left Endpt Rule For Decreasing Functions

- if we have a decreasing function, left endpt approx is L_n

$\therefore$ Error $= |L_n - \text{exact value}|$ since L_n is overestimate for dec. functions

$$\text{Error} = L_n - \int_a^b f(x)\,dx$$

- right endpt approx is R_n, and since R_n is underestimate of decreasing functions, we have $\text{Error} = \int_a^b f(x)\,dx - R_n$



orange: error for L_n approx

blue: error for R_n approx

Add them up:

$$= (\text{orange}) + (\text{blue})$$
$$= L_n - \int_a^b f(x)\,dx + \int_a^b f(x)\,dx - R_n$$
$$= L_n - R_n$$

Since the error is represented in orange, we know

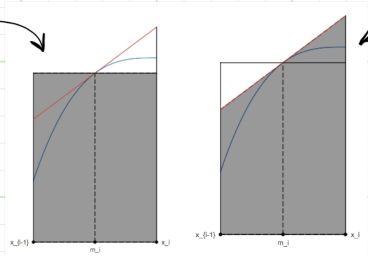
$$\boxed{\text{Error for Left endpt approx} = (\text{orange}) \le (\text{orange}) + (\text{blue}) = L_n - R_n}$$

First Rule: "For a decreasing function, if I approximate the integral with the left-endpoint-rule kand n intervals, the error I am making is less than Ln - Rn.

Different Methods, Different Errors

midpoint rule

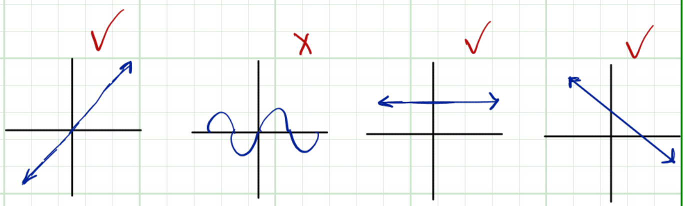
trapezoidal rule



- For concave down functions, the midpoint rule is an overestimate and the trapezoidal rule is an under estimate.
- Midpoint rule is more accurate than trapezoidal rule for concave down functions.

Question 1

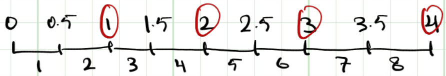
$$\int_2^5 f(x)\,dx$$



$$\text{Error} = \left| T_{2022} - \int_2^5 f(x)\,dx \right| = 0$$

Question 2

Time (s)	0	0.5	1	1.5	2	2.5	3	3.5	4
Velocity (m/s)	0	4.9	9.6	13.9	17.7	21.2	24	26.4	28.3



$$\Delta x = \frac{4}{4} = 1 \qquad M_8 = \sum_{k=1}^{8} f\left(\frac{x_k + x_{k-1}}{2}\right)\Delta x$$

$$M_4 = \frac{4}{4}(4.9 + 13.9 + 21.2 + 26.4) = 66.4$$

Left and Right Riemann Sum Error Bounds

$$\left.\begin{aligned}
&* \quad \left\| \int_a^b f(x)\,dx - L_n \right\| \leq \frac{M(b-a)^2}{2n} \\[2em]
&* \quad \left\| \int_a^b f(x)\,dx - R_n \right\| \leq \frac{M(b-a)^2}{2n}
\end{aligned}\right\} \text{ where } M = \max_{a \leq x \leq b} \| f'(x) \|$$

Midpoint and Trapezoidal Error Bounds

$$\left.\begin{aligned}
&* \quad \left\| \int_a^b f(x)\,dx - M_n \right\| \leq \frac{M(b-a)^3}{24n^2} \\[2em]
&* \quad \left\| \int_a^b f(x)\,dx - T_n \right\| \leq \frac{M(b-a)^3}{12n^2}
\end{aligned}\right\} \text{ where } M = \max_{a \leq x \leq b} \| f''(x) \|$$

Improper Integral

- Let $f(x)$ be a function cont. on $[a, \infty)$ where $a \in \mathbb{R}$

improper integral → $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

- improper integral converges if limit exists.
- improper integral diverges if limit doesn't exist.

Eg: Find $\int_0^\infty e^{-x}\,dx$

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx$$

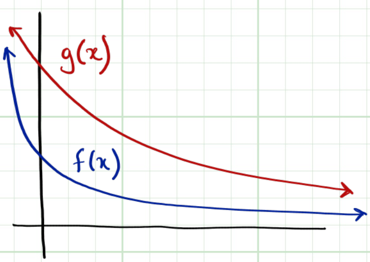
$$= \lim_{t\to\infty} -e^{-x}\Big|_0^t$$

$$= \lim_{t\to\infty} -e^{-t} - (-e^0)$$

$$= 0 + 1 = 1$$

$\therefore \int_0^\infty e^{-x}\,dx = 1$ and integral converges

Comparison Theorem



$$0 \leq \int_a^t f(x)\,dx \leq \int_a^t g(x)\,dx$$

- if $\int_0^\infty g(x)\,dx$ converges, then $\int_0^\infty f(x)\,dx$ also converges
- if $\int_0^\infty f(x)\,dx$ diverges, then $\int_0^\infty g(x)\,dx$ also diverges

Family of Comparision Functions

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{cases}$$

$$\int_0^\infty e^{-ax}\,dx \text{ is } \begin{cases} \text{convergent if } a > 0 \\ \text{divergent if } a \leq 0 \end{cases}$$

Improper Integral

* Let $f(x)$ be function that is continuous on $(a, b]$ but with an infinite discontinuity at $x = a$ for $a, b \in \mathbb{R}$

improper integral → $$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_a^b f(x)\,dx$$

* improper integral converges if limit exists.
* improper integral diverges if limit doesn't exist.

→ **Eg:** compute $\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx$

We must deal with 4 cases and break up accordingly

(1) unbounded dom as $x \to -\infty$
(2) infinite discon. as $x \to 0^-$
(3) infinite discon. as $x \to 0^+$
(4) unbounded dom as $x \to +\infty$

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \int_{-\infty}^{-1} \frac{1}{x^2}\,dx + \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx$$

$$= \lim_{a \to -\infty} \int_a^{-1} \frac{1}{x^2}\,dx + \lim_{b \to 0^-} \int_{-1}^{b} \frac{1}{x^2}\,dx + \lim_{c \to 0^+} \int_c^{1} \frac{1}{x^2}\,dx + \lim_{d \to \infty} \int_1^{d} \frac{1}{x^2}\,dx$$

Differential Equation

* Equation relating a function $y=f(t)$ to its derivative(s).

→ $t^2 \frac{d^3y}{dt^3} - 3t\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 3y = \sin t$

→ $t^2 y''' - 3ty'' + ty' - 3y = \sin t$

→ $t^2 f'''(t) - 3t f''(t) + t f'(t) - 3f(t) = \sin t$

Ordinary Differential Equations: only 1 independent variable

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 9y = \cos x$$

Partial Differential Equation: 2 or more independent variables

$$\frac{du}{dx} + \frac{du}{dy} + u = e^{x-y}$$

First Order ODE: $\frac{dy}{dx} = -\cos t + e^{5(t-5)}$

Second Order ODE: $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 9y = \cos x$

Linear ODEs: ODEs that have y only multiplied by terms that have t, not other operations (squaring, cosine, root)

$y' + 2y - 3t^3y'' = \sqrt{t-2}$ ← linear

$(y')^2 + 2y' + 2 + \sin(y) = y' + 1$ ← Not linear

Understanding ODEs

- is there a solution?
- how many are there?
- does a sol. exist for all time?
- how do I find a sol. if it exists?

Separable ODEs

A differential eq. $\frac{dy}{dt} = f(t,y)$ is called separable if it can be written as

$$\frac{dy}{dt} = \underbrace{g(t)h(y)}_{\text{product of 2 functions}}$$

depends only on t — $g(t)$; depends only on y — $h(y)$

Eg: $\frac{dy}{dt} = yt + yt^2$

$\frac{dy}{dt} = y(t+t^2)$ ← Yes seperable, since product of y stuff and t stuff

Non-Eg: $\frac{dy}{dt} = y+t$ ← Not separable since it's a sum, not a product.

Solving Separable ODE

Given IVP, find out how temp behaves over time.

$$\begin{cases} \frac{dy}{dt} = -0.02(y-20) \\ y_0 = 95 \end{cases}$$

$y(t)$ ~ temp of coffee
t ~ time in mins

$$\frac{dy}{dt} = -0.02(y-20) = g(t)h(y)$$

$g(t) = -0.02$
$h(y) = y-20$

$$\frac{1}{y-20}\frac{dy}{dt} = -0.02$$

rearrange since $y \neq 20$

$$\int \frac{1}{y-20}\frac{dy}{dt}\,dt = \int -0.02\,dt$$

$y = y(t)$

$dy = \frac{dy}{dt}\,dt$

$$\int \frac{1}{y-20}\,dy = \int -0.02\,dt$$

$e^{\ln x} = x$

$e^{x+y} = e^x e^y$

$y(0) = 95$

$$\ln|y-20| = -0.02t + C$$

$$e^{\ln|y-20|} = e^{-0.02t+C}$$

e^C is a constant, so we just replace with C

$$y-20 = (e^{-0.02t})(e^C)$$

$$y = (e^{-0.02t})(C) + 20$$

$$95 = C + 20$$

$$C = 75$$

$$\boxed{\therefore y = 75e^{-0.02} + 20}$$

First Order Linear Differential Equations

- a differential equation that can be written in the form:

standard form

$$y' + p(t)y = q(t) \qquad \mu(t) = e^{\int p(t)dt}$$

Eg: $y' = -k(y-20)$

separable

ambient temperature

what if ambient temperature given by ...

$$T_A(t) = -5\cos\left(\frac{2\pi}{24}t\right)$$

$$y' = -k(y - T_A(t))$$

$$y' = -k\left(y + 5\cos\left(\frac{2\pi}{24}t\right)\right)$$

linear first order ODE

Integrating Factor

→ Find a nice $\mu(t)$ s.t. after multiplying the ODE with $\mu(t)$, we can use product rule

Product Rule: $(\mu y)' = \mu y' + \mu' y$

Consider: $y' - 2y = 4 - t$

we can't integrate this now, but it looks like the R.H.S of product rule equation

we can integrate to get $4t - \frac{t^2}{2}$

Try integrating factor: $\mu(t) = e^{-2t}$

$$y' - 2y = 4 - t$$

× both sides with $\mu(t)$

$$e^{-2t}(y' - 2y) = (4-t)e^{-2t}$$

expand

$$e^{-2t}y' - 2e^{-2t}y = (4-t)e^{-2t}$$

$$(e^{-2t}y)' = (4-t)e^{-2t}$$

integrate both sides

$$e^{-2t}y = -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + C$$

$$\boxed{y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}}$$

Problem: Tank contains 5 kg of salt in 100 L of water. At time $t=0$, saltwater of 0.25 kg/L is entering at a rate of 4 L/min and water leaving at same rate.

a) What is concentration in tank at time t?

b) What happens to concentration in tank as $t \to \infty$?

Set up IVP

* Do I solve for $C(t)$ concentration or $Q(t)$ quantity of salt?
  ↳ $Q(t)$ is simpler
* Since no salt is being created or destroyed, we know
  a) $Q(t)$ increases if inflow > outflow
  b) $Q(t)$ decreases if inflow < outflow

$$\therefore \quad \frac{dQ}{dt} = \text{rate at which salt enters} - \text{rate at which salt leaves}$$

Step 1 rate salt enters: $0.25 \frac{\text{kg}}{\cancel{L}} \times 4 \frac{\cancel{L}}{\text{min}} = 1 \text{ kg/min}$

Step 2 rate salt leaves: $\frac{Q(t)}{100} \frac{\text{kg}}{\cancel{L}} \times \frac{4 \cancel{L}}{\text{min}} = \frac{Q(t)}{25} \text{ kg/min}$

Step 3 differential equation: $\frac{dQ}{dt} = \left(1 - \frac{Q(t)}{25}\right) \text{kg/min}$

Solve Differential Equation

Sep.

$$\frac{dQ}{dt} = 1 - \frac{Q(t)}{25}$$

$$\frac{dQ}{dt} = \frac{1}{25}(25 - Q(t))$$

$$\int \frac{1}{25-Q(t)} \, dQ = \int \frac{1}{25} \, dt$$

$$-\ln|25-Q(t)| = \frac{1}{25}t + C$$

$$\ln|25-Q(t)| = -\frac{1}{25}t + C$$

$$25 - Q(t) = Ce^{-t/25}$$

$$Q(t) = 25 - Ce^{-t/25}$$

Since $Q(0) = 5$ kg, we can solve for C.

$$5 = 25 - Ce^{-(0)/25}$$

$$5 = 25 - C$$

$$C = 20$$

$$\boxed{\therefore Q(t) = 25 - 20e^{-\frac{t}{25}}}$$

Answer Original Questions

a) Concentration at time t is just amn. of salt ÷ amn. water

$$C(t) = \frac{Q(t)}{100} = \frac{25}{100} - \frac{20}{100}e^{-t/25} = \left(0.25 - 0.2e^{-t/25}\right) \text{ kg/L}$$

b) $\lim_{t \to \infty} C(t) = 0.25$ kg/L

C1 : Complex Numbers

- complex numbers : set of all numbers of the form

$$\boxed{a+bi}$$

real numbers

$i^2 = -1$

$$\mathbb{C} := \{a+bi \mid a,b \in \mathbb{R}\}$$

- Given complex number z

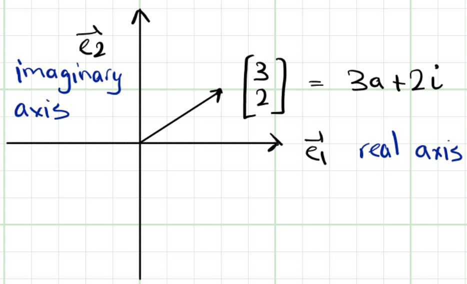
$z = a + bi$ $\qquad$ $\mathrm{Re}(z) = a$ real part

$\qquad\qquad\qquad\quad$ $\mathrm{Im}(z) = b$ imaginary part

- modulus of z : positive number $r = \sqrt{a^2 + b} = |z|$
- complex conjugate: complex number $a - bi = \bar{z}$

Visualizing Complex Numbers

- complex num is the vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$



$$2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 3y = 0$$

or

$$2y'' + 5y' - 3y = 0$$

- ordinary
- second order
- linear
- constant coefficients (no t)
- homogenous (equals zero)

Case 1: Two Distinct Real Roots

$$2y'' + 5y' - 3y = 0$$

Let $y = e^{rt}$, so $y' = re^{rt}$ and $y'' = r^2e^{rt}$ and plug back

$$2r^2e^{rt} + 5re^{rt} - 3e^{rt} = 0$$

$$e^{rt}(2r^2 + 5r - 3) = 0$$

since $e^{rt} \neq 0$ for all t, divide both sides by e^{rt}

$$\underbrace{2r^2 + 5r - 3 = 0}_{\text{characteristic equation / polynomial}}$$

"$y = e^{rt}$ is a solution if and only if $2r^2 + 5r - 3 = 0$"

solve

$$r_1 = \frac{1}{2} \quad \text{or} \quad r_2 = -3$$

$\therefore$ $y_1 = e^{\frac{1}{2}t}$, $y_2 = e^{-3t}$ are solutions to ODE

Theorem

Given homogenous 2nd order ODE of form $ay'' + by' + cy = 0$ if $y_1(t)$ and $y_2(t)$ are solutions and not constant multiples of one another, then all solutions take the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Case 2: One Real Root

$y'' - 4y' + 4y = 0$

Let $y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2e^{rt}$

$r^2e^{rt} - 4re^{rt} + 4e^{rt} = 0$

$r^2 - 4r + 4 = 0$

$\longrightarrow r_1 = r_2 = 2$, so $y_1(t) = e^{2t}$

Theorem

Given 2nd order ODE of form $ay'' + by' + cy = 0$

and char. poly. $ar^2 + br + c = 0$

↳ If $b^2 - 4ac = 0$ (only 1 real root $r = \frac{-b}{2a}$) then

$y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$

$$\underbrace{y(t) = C_1e^{rt} + C_2te^{rt}}_{\text{full sol. set}}$$

Case 3: No Real Roots

Non Homogeneous

$$ay'' + by' + cy = f(t)$$

a function of t on RHS

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

- We gotta split it by subbing $y(t) = u(t) + v(t)$

$$a\frac{d^2(u+v)}{dt} + b\frac{d(u+v)}{dt} + c(u+v) = f(t)$$

$$\underbrace{\left(a\frac{d^2u}{dt} + b\frac{du}{dt} + cu\right) + \left(a\frac{d^2v}{dt} + b\frac{dv}{dt} + cv\right)}_{\text{split it up}} = f(t)$$

Core Idea

If: $u(t)$ solves this $\longrightarrow a\frac{d^2u}{dt} + b\frac{du}{dt} + cu = 0$

And: $v(t)$ solves this $\longrightarrow a\frac{d^2v}{dt} + b\frac{dv}{dt} + cv = f(t)$

Then: $y(t) = u(t) + v(t)$ solve original ODE

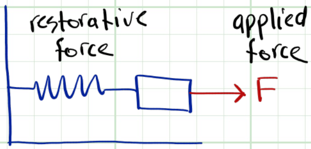
- complementary solution $y_c(t)$ solves homogeneous equation
- particular solution $y_p(t)$ solves non-homogeneous equation
- general solution given by $y(t) = y_c(t) + y_p(t)$

$r(x)$	**Initial guess for $y_p(x)$**
k (a constant)	A (a constant)
$ax+b$	$Ax+B$ (*Note*: The guess must include both terms even if $b=0$.)
ax^2+bx+c	Ax^2+Bx+C (*Note*: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x+b\sin\beta x$	$A\cos\beta x+B\sin\beta x$ (*Note*: The guess must include both terms even if either $a=0$ or $b=0$.)
$ae^{\alpha x}\cos\beta x+be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$
$\left(ax^2+bx+c\right)e^{\lambda x}$	$\left(Ax^2+Bx+C\right)e^{\lambda x}$
$\left(a_2x^2+a_1x+a_0\right)\cos\beta x$ $+\left(b_2x^2+b_1x+b_0\right)\sin\beta x$	$\left(A_2x^2+A_1x+A_0\right)\cos\beta x$ $+\left(B_2x^2+B_1x+B_0\right)\sin\beta x$
$\left(a_2x^2+a_1x+a_0\right)e^{\alpha x}\cos\beta x$ $+\left(b_2x^2+b_1x+b_0\right)e^{\alpha x}\sin\beta x$	$\left(A_2x^2+A_1x+A_0\right)e^{\alpha x}\cos\beta x$ $+\left(B_2x^2+B_1x+B_0\right)e^{\alpha x}\sin\beta x$

Table 7.2 Key Forms for the Method of Undetermined Coefficients

D3. Applications

Hooke and Newton



restorative force $= -kx$

net force $= ma$

$$\therefore \quad F = ma = m\frac{d^2x}{dt^2} = -kx$$

* Rewrite as homogeneous 2nd order ODE

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0 \qquad \text{or} \qquad mx'' + kx = 0$$

* Solve it using char. poly. $mr^2 + k = 0 \Rightarrow r = \pm\sqrt{-\frac{k}{m}}$

$$r = \pm\sqrt{\frac{k}{m}}\, i$$

$\therefore$ general solution

$$\boxed{x(t) = C_1 \cos\sqrt{\frac{k}{m}}\,t + C_2 \sin\sqrt{\frac{k}{m}}\,t}$$

simple harmonic motion

Damping Force

* friction or damping force added to make more realistic

* let the damping force $= -c\frac{dx}{dt}$ ← since it's dependent on velocity $\frac{d}{dt}x$

* total force is:

$$\text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

* ODE becomes

$$m\frac{d^2x}{dt} = -kx - c\frac{dx}{dt}$$

$$m\frac{d^2x}{dt} + c\frac{dx}{dt} + kx = 0 \quad \text{or} \quad \underbrace{mx'' + cx' + kx = 0}_{\text{damped harmonic motion}}$$

Types of Damping

- depends on relative size of damping force compared to m and k
- damping force small $\longrightarrow$ underdamped
- damping force high $\longrightarrow$ overdamped
- threshold b/t two is "critical damping"

- solving ODE with char. poly. $mr^2 + cr + k = 0$

  solutions are $r = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

$\longrightarrow$ **Case 1: $c^2 - 4mk < 0$** c is small — underdamping

$$x(t) = e^{-\frac{c}{2m}t}\left[C_1 \cos\omega t + C_2 \sin\omega t\right]$$

where $\omega = \sqrt{\dfrac{4mk - c^2}{2m}}$

$\longrightarrow$ **Case 2: $c^2 - 4mk > 0$** c large — overdamping

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

2 distinct real roots

Case 3: $c^2 - 4mk = 0$

$r_1 = r_2 = -\dfrac{c}{2m}$ $\qquad$ $x(t) = \left[C_1 + C_2 t\right] e^{-\frac{c}{2m}t}$

1 root only

E1. Intro to Taylor Polynomials

- We approximate hard functions using Taylor Polynomials

Step 1: decide a point $x=a$ where polynomial is centered

Step 2: construct polynomial $P(x) = c_0 + c_1(x-a) + c_2(x-a)^2 \ldots$

- Look for constants for which the n'th degree term of the approx at a equals the n'th derivative of the original fuction at a.

Example : $f(x) = \cos x$

Step 1: Let's pick $x=0$ as our point a.

$\Rightarrow f(0) = \cos(0) = 1$

$\Rightarrow \frac{df}{dx}(0) = -\sin(0) = 0$

$\Rightarrow \frac{d^2f}{dx^2}(0) = -\cos(0) = -1$

$\Rightarrow \frac{d^3f}{dx^3}(0) = \sin(0) = 0$

$P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

$\Rightarrow P(0) = c_0$

$1 = c_0$

$\Rightarrow P'(0) = c_1 + 2c_2(0) + 3c_3(0)^2$

$0 = c_1$

$\Rightarrow P''(0) = 2c_2 + 6c_3(0)$

$-1 = 2c_2$

$c_2 = -\frac{1}{2}$

$\Rightarrow P'''(0) = 6c_3$

$0 = 6c_3$

$c_3 = 0$

$$\boxed{\therefore P(x) = 1 - \frac{1}{2}x^2}$$

Notes

- set your constant : $\boxed{c_n = \frac{\text{desired derivative value}}{n!}}$
- Eg: $\Rightarrow \frac{d^8f}{dx^8}(0) = 4$ so $c_8 = \frac{4}{8!}$

$\Rightarrow \frac{d^8P}{dx^8}\left[c_8 x^8\right] = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\, c_8 = 8!\, c_8$

Abstraction

$f(a)$

$\frac{df}{dx}(a)$

$\frac{d^2f}{dx^2}(a)$

$\frac{d^3f}{dx^3}(a)$

$$P(x) = f(a) + \frac{df}{dx}(a)\frac{x^1}{1!} + \frac{d^2f}{dx^2}(a)\frac{x^2}{2!} + \frac{d^3f}{dx^4}(a)\frac{x^3}{3!} + \cdots$$

$\downarrow$

$$\boxed{P(x) = f(a) + \frac{df}{dx}(a)\frac{(x-a)^1}{1!} + \frac{d^2f}{dx^2}\frac{(x-a)^2}{2!} + \cdots}$$

or

$$P(x) = f(a) + \frac{f'(a)(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

Approximating Errors

* remainder $R_n(x) = f(x) - P_n(x)$
* $f(x)$ equals the Taylor approx plus the remainder

$$f(x) = p_n(x) + R_n(x)$$

* Error $= |R_n(x)| = |f(x) - P_n(x)|$

Taylor's Theorem

* error depends on
  - shape of $f(x)$
  - centre $x=a$ chosen
  - order of Taylor polynomial
* choose centre $x=a$
* If $f(x)$ has $(n+1)$ cont. derivatives on int. around $x=a$, then

$$f(x) = p_n(x) + R_n(x) \qquad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

n+1 derivative

$c \in (a, x)$

E2. Sequences & Series

- infinite series : ordered list of numbers $a_1, a_2, a_3, a_4, \cdots$

  term

Limit of a Sequence

- if the terms of a sequence $\{a_n\}$ get close to a number L, then $\{a_n\}$ is a convergent sequence and L is limit of sequence

- if $\{a_n\}$ is given by function $f(n)$ and limit of function and sequence coincide, then $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} f(n)$

Infinite Series

- infinite series : $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$

- given k, sequence of partial sums is sequence $\{s\}$ where

$$s_k = \sum_{1}^{k} a_k$$

- if sequence of partial sums converges to a real number S, then infinite series converges to S and $\sum\limits_{n=1}^{\infty} a_n = S$

Eg: Let $\{s_k\}$ be sequence of partial sums of $\sum\limits_{1}^{\infty} \frac{1}{3^{n-1}}$

$$s_3 = \sum_{k=1}^{3} \frac{1}{3^{k-1}} = \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} = 1 + \frac{1}{3} + \frac{1}{9} = \frac{13}{9}$$

E3: Power Series Convergence

- power series centred at $x=a$

$$\sum_{n=0}^{\infty} c_0 x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

power series always converge at its centre

only at centre

more than 1 value of x

all x in interval of radius R around centre a with possibly none, one, or both endpts: $x \pm R$

all real numbers x

Ratio Test

- You have series: $\sum_{n=0}^{\infty} a_n$, you let $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

  - if $\rho < 1$ — series converges
  - if $\rho > 1$ — series diverges
  - if $\rho = 1$ — inconclusive

Eg: see if $\sum_{n=0}^{\infty} (4x)^n$ converges or diverges

$$\rho = \left| \frac{(4x)^{n+1}}{(4x)^n} \right| = \frac{4^{n+1}|x|^{n+1}}{4^n|x|^n} = 4^{n+1-n}|x|^{n+1-1} = 4^1|x|^1 = 4|x|$$

We know series converges for $\rho < 1$

$4|x| < 1$

$|x| < \frac{1}{4}$

centred at $x=0$, so radius of convergence is $R = \frac{1}{4}$

Series converges $|x| < \frac{1}{4}$

diverges $|x| > 1$

E4: Taylor Series

- the larger the polynomial degree, the closer the approx. is.
  ↳ take the sum up to infinity:

Let $f(x) = e^x$ and $P_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$ is centered at $x=0$

To get exact function, we can take sum to infinity

$$\underbrace{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots}_{\text{resulting power series called Taylor Series}}$$

- By ratio test, we know series converges for every x
- To show that it converges to e^x given x, we do this

$$f(x) = P_N(x) + R_N(x) \longleftarrow$$ if this $\to 0$ as $N \to \infty$, then we have zero error

Taylor's Remainder Theorem

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{(N+1)} = \frac{e^c}{(N+1)!} x^{(N+1)}, \quad c \text{ between } 0 \text{ and } x$$

Eg: Show that $R_N(3) \to 0$ as $N \to \infty$

$$R_N(3) = \frac{e^c}{(N+1)} (3)^{(N+1)}, \quad c \text{ between } 0 \text{ and } 3$$

We know $e^c \leq e^3$, so

$$0 \leq \frac{e^c}{(N+1)} (3)^{(N+1)} \leq \frac{e^3}{(N+1)} 3^{(N+1)}$$

∴ this also $\to 0$

as $N \to \infty$, this $\to 0$

* Since we proved that error goes to zero, we say following

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

↳ Taylor Series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for all x

Integrating and Differentiating Power Series

* brute force method is hard → use existing Taylor series

→ Eg: to find Taylor Series for

$f(x) = \frac{1}{1+x^2}$, we can use the known Taylor Series

plug in something to make it $f(x)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$\therefore \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \quad \text{for } |x^2| < 1$$

$$= \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x)^{2n} \quad \text{for } |x^2| < 1$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

→ Eg: Integrating Taylor Series

use t because x is the bound of integration

We know that $\int_0^x \frac{1}{1+t^2}\,dt = \arctan(x) + C$

$$\arctan(x) + c = \int_0^x \frac{1}{1+t^2}\,dt$$

sub in Taylor $$= \int_0^x \sum_{n=0}^{\infty} (-1)^n (t)^{2n}\,dt$$

expand series $$= \int_0^x \left(1 - t^2 + t^4 - t^6 + t^8 - \cdots\right) dt$$

integrate terms $$= \left[t - \frac{t^3}{2} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \cdots \right]_0^x$$

plug in x $$= x - \frac{x^3}{2} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Since $\arctan(0) = 0 \Rightarrow c = 0$

$$\therefore \arctan(x) = x - \frac{x^3}{2} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad \text{for } |x| < 1$$

→ Eg: $$\ln(1-x) + c = \int_0^x -\frac{1}{1-t}\,dt = -\int_0^x \frac{1}{1-t}\,dt$$

$$= -\int_0^x \sum_{n=0}^{\infty} t^n\,dt$$

$$= -\int_0^x \left(1 + t + t^2 + t^3 + t^4 + \cdots\right) dt$$

$$= -\left[t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \frac{t^5}{5} + \cdots \right]_0^x$$

$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots \right)$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

$\therefore c_0 = 0 \quad c_1 = -1 \quad c_2 = -\frac{1}{2} \quad c_3 = -\frac{1}{3} \quad c_4 = -\frac{1}{4}$

Integrals with Taylor Series

We want to evaluate $\int_0^1 e^{-x^2/2}\,dx$

We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\therefore e^{-x^2/2} = e^{\left(-\frac{1}{2}x^2\right)} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^n \left(x^{2n}\right)}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}$$

$$\Rightarrow \int_0^1 e^{-\frac{x^2}{2}}\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}\,dx$$

$$= \int_0^1 \left(1 - \frac{x^2}{2^1} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \frac{x^8}{2^4 \cdot 4!} - \cdots \right) dx$$

$$= \left[x - \frac{x^3}{2^1 \cdot 3} + \frac{x^5}{2^2 \cdot 2! \cdot 5} - \frac{x^7}{2^3 \cdot 3! \cdot 7} + \frac{x^9}{2^4 \cdot 4! \cdot 9} - \cdots \right]_0^1$$

$$= 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} - \cdots$$

upper bound for error

Summing up first 4 terms, we get $\int_0^1 e^{-\frac{x^2}{2}}\,dx \approx 0.86$

Consider a convergent infinite series of the form

$$S = a_1 - a_2 + a_3 - a_4 + \cdots \qquad \text{with } a_n \geq 0$$

This series is called "alternating" because it is of the form "plus-minus-plus-minus..."

We also require that the numbers $a_1, a_2, a_3 \ldots$ are decreasing, i.e. the entries of the series are getting smaller and smaller in magnitude.

If we denote the actual value of the series by S and the sum of only the first N terms by S_N, then:

$$|\text{Error}| = |S - S_N| \leq a_{N+1}$$

The error when estimating an alternating series is bounded by the first term that we skip.

$\therefore$ We have error of at most the first term we skipped

$\Rightarrow$ error at most $\frac{1}{3456}$

Problems

a) $f(x) = \frac{1}{1-x} = (-x+1)^{-1}$

$f(0) = 1$ $\quad n=0$

$f'(0) = -1$ $\quad 1$

$f^2(0) = 2$ $\quad 2$

$f^3(0) = -6$ $\quad 3$

$f'(x) = \frac{-1}{(1-x)^2}$

$f''(x) = \frac{2}{(1-x)^3}$

$f'''(x) = \frac{-6}{(1-x)^4}$

$$P_N(x) = 1 - \frac{1x}{1} + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \cdots$$

$$= 1 - x + x^2 - x^3$$

$$= (-1)^n x^n$$

b) $f(0) = \sin 0 = 0$

$f'(0) = \cos 0 = 1$

$f^2(0) = -\sin 0 = 0$

$f^3(0) = -\cos 0 = -1$

$f^4(0) = \sin 0 = 0$

$f^5(0) = \cos 0 = 1$

$$P_4(x) = \cancel{0} + \frac{1x}{1!} + \cancel{\frac{0x^2}{2!}} - \frac{1x^3}{3!} + \cancel{\frac{0x^4}{4!}} + \cdots$$

$$= \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\therefore f(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

c) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

d) $f(0) = \ln(1+0) = 0$

$f'(0) = \frac{1}{0+1} = 1$

$f^2(0) = \frac{-1}{(0+1)^2} = -1$

$f^3(0) = \frac{2}{(0+1)^3} = 2$

$f^4(0) = \frac{-6}{(0+1)^4} = -6$

$$P_N(x) = \cancel{\frac{0}{0!}} + \frac{1x}{1!} - \frac{1x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

$$= 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

$$= \frac{(-1)^{n+1} x^n}{n}$$

→ Find $\int_0^1 \cos(x^2)\,dx$ with an 8^{th} degree Taylor Polynomial.

We know $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$$\therefore \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$$

$$\approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!}$$

$$- \frac{x^{20}}{10!} + \frac{x^{24}}{12!} - \frac{x^{28}}{14!} + \frac{x^{32}}{16!}$$

$$\therefore \int_0^1 \cos(x^2)\,dx \approx \int_0^1 \Big(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!}$$

$$- \frac{x^{20}}{10!} + \frac{x^{24}}{12!} - \frac{x^{28}}{14!} + \frac{x^{32}}{16!}\Big)\,dx$$

$$= \Big[x - \frac{x^5}{2!\cdot 5} + \frac{x^9}{4!\cdot 9} - \frac{x^{13}}{6!\cdot 13} + \frac{x^{17}}{8!\cdot 17}$$

$$- \frac{x^{21}}{10!\cdot 21} + \frac{x^{25}}{12!\cdot 25} - \frac{x^{29}}{14!\cdot 29} + \frac{x^{33}}{16!\cdot 33}\Big]_0^1$$

$$= 1 - \frac{1}{20} + \frac{1}{216} - \frac{1}{9360} + \frac{1}{685440}$$

$$- \frac{1}{76204800} + \frac{1}{11975040000} - \frac{1}{2.5282\times 10^{12}}$$

$$+ \frac{1}{6.9045\times 10^{14}}$$ $\boxed{\approx 0.95452}$ ✗ 0.90463

F1: Parametric Equations

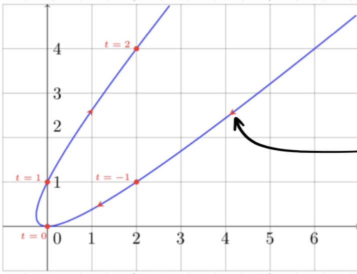
→ Given continuous functions $f(t)$ and $g(t)$ of parameter t, then

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\rangle \text{parametric equations}$$

$\longrightarrow (f(t), g(t)) \longleftarrow$ points on curve

→ Eg: given following parametric equations

$$\begin{cases} x = t^2 - t \\ y = t^2 \end{cases}$$



arrows show orientation of the curve

* Note: more than one pair of parametric equations can represent the same curve

→ orientation or starting can change, but graph remains the same

Eg: ant $= \begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ bce $= \begin{cases} x = \cos(2\pi t) \\ y = \sin(2\pi t) \end{cases}$

going ccw, 1 circle in 2π seconds

also ccw, but 1 circle in 1 sec

Derivatives

Given $= \begin{cases} x = x(t) \\ y = y(t) \end{cases} \Longrightarrow \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{y'(t)}{x'(t)}$

Problem 1

race from $(0,0) \longrightarrow (5,10)$

$$\text{Jim} = \begin{cases} x = t \\ y = 2t \end{cases} \qquad \text{Bob} = \begin{cases} x = 5t \\ y = 10t \end{cases}$$

$x:\quad 5 = t$

$y:\quad 10 = 2t$

$\quad t = 5$

$x:\quad 5 = 5t$

$\quad t = 1$

$y:\quad 10 = 10t$

$\quad t = 1$

$\therefore$ Bob wins race since only need 1 s

Problem 2

$$\text{ant} = \begin{cases} x = \cos t \\ y = \cos^2 t \end{cases}$$

$x(0) = \cos 0 = 1$

$x(\frac{\pi}{2}) = \cos \frac{\pi}{2} = 0$

$x(\pi) = \cos \pi = -1$

$x(\frac{3\pi}{2}) = \cos \frac{3\pi}{2} = 0$

$x(2\pi) = 1$

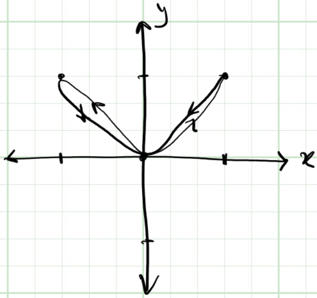
$y(0) = 1 \cdot 1 = 1$

$y(\frac{\pi}{2}) = 0 \cdot 0 = 0$

$y(\pi) = -1 \cdot -1 = 1$

$y(\frac{3\pi}{2}) = 0 \cdot 0 = 0$

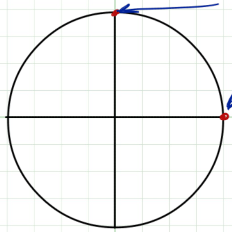
$y(2\pi) = 1 \cdot 1 = 1$



x	y
1	1
0	0
-1	1
0	0
1	1

F2. Polar Coordinates

- rectangular coordinates: move x steps right/left and y steps up/down
- polar coordinates: move r steps in direction θ



$\left(1, \frac{\pi}{2}\right)$ or $\left(1, \frac{9\pi}{2}\right)$ or $\left(-1, \frac{3\pi}{2}\right)$

$(1,0)$

Conversions

→ r-θ to x-y: $x = r\cos\theta$ $\quad y = r\sin\theta$

→ x-y to r-θ:

$r^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$ for Q1 and Q3

Curves in r-θ

- equation of circle radius 3 is: $r = 3$
- equation of horizontal line $y = 3$ is: $r\sin\theta = 3 \Rightarrow r = \frac{3}{\sin\theta}$

Problems

$(x,y) = (1, \sqrt{3}) \Longrightarrow r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$ $\quad \therefore (r,\theta) = \left(2, \frac{\pi}{3}\right)$

$= \left(-2, \frac{4\pi}{3}\right)$

$\frac{\pi}{3} + \frac{3\pi}{3} = \frac{4\pi}{3}$

Finding Tangent Line

1. Given parametric equation $x(t)$ and $y(t)$,

$$\boxed{\frac{dy}{dx} = \frac{y'(t)}{x'(t)}}$$

2. Given point on curve (r, θ), rectangular coordinates

$$\boxed{x = r(\theta)\cos\theta \qquad y = r(\theta)\sin\theta}$$

3. The point-slope equation of line $\boxed{y - b = m(x - a)}$

→ Given polar function $r(\theta)$, slope of the derivative is

product rule

$$y(\theta) = r(\theta)\sin\theta \qquad\qquad x(\theta) = r(\theta)\cos\theta$$

$$y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta \qquad\qquad x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta$$

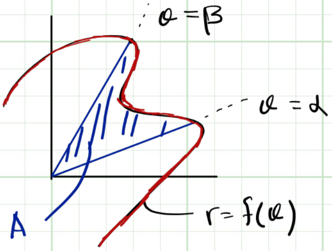
Slope of derivative is :

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta}{x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

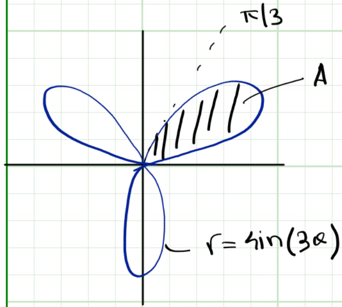
Point-slope form is :

$$y - r(\theta)\sin\theta = \frac{y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta}{x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta}\left(x - r(\theta)\cos\theta\right)$$

Integration in $r-\theta$



$$\boxed{A=\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta}$$



$$A=\frac{1}{2}\int_0^{\pi/3}\sin^2(3\theta)\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/3}\frac{1}{2}-\frac{1}{2}\cos(6\theta)\,d\theta$$

$$=\frac{1}{2}\left[\frac{1}{2}\theta\Big|_0^{\pi/3}-\frac{1}{2}\int_0^{\pi/3}\cos(6\theta)\,d\theta\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{6}-\frac{1}{2}\cdot\left(\frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/3}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{6}-\frac{1}{2}\cdot(0)\right]$$

$$=\frac{1}{2}\cdot\frac{\pi}{6}=\frac{\pi}{12}$$

$$\sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\sin^2(3\theta)=\frac{1-\cos(6\theta)}{2}$$

$$\sin\left(\frac{6\pi}{3}\right)=\sin(2\pi)=0$$

$$\boxed{\therefore\ A=\frac{\pi}{12}}$$

Problems

1. $r(\theta) = \cos(2\theta)$ at point $\theta = \frac{\pi}{4}$

$r'(\theta) = -2\sin(2\theta)$

$r\left(\frac{\pi}{4}\right) = \cos\left(2 \cdot \frac{\pi}{4}\right) = 0$ $\qquad r'\left(\frac{\pi}{4}\right) = -2\sin\left(\frac{\pi}{2}\right) = -2$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta} r(\theta)\sin\theta}{\frac{d}{d\theta} r(\theta)\cos\theta} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{-2\left(\frac{1}{\sqrt{2}}\right) + 0}{-2\left(\frac{1}{\sqrt{2}}\right)} = 1$$

$y = x - b$

$b = r(\theta)\cos\theta - r(\theta)\sin\theta$

$= 0$

$$\boxed{\therefore\ y = x}$$

2. $r(\theta) = 1 - 2\sin\theta$ on $[0, 2\pi] \longrightarrow \theta = \frac{\pi}{6}$

$r'(\theta) = -2\cos\theta$

$r\left(\frac{\pi}{6}\right) = 1 - 2\sin\frac{\pi}{6} = 0$ $\qquad r'\left(\frac{\pi}{6}\right) = -2\cos\left(\frac{\pi}{6}\right) = -\sqrt{3}$

$$\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{-\sqrt{3}\cdot\frac{1}{2}}{-\sqrt{3}\cdot\frac{\sqrt{3}}{2}} = \frac{\frac{-\sqrt{3}}{2}}{\frac{-\sqrt{3}\cdot\sqrt{3}}{2}} = \frac{-\sqrt{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{-\sqrt{3}}\cdot\sqrt{3}}$$

$= \frac{1}{\sqrt{3}}$

$$\boxed{\therefore\ y = \frac{1}{\sqrt{3}}x}$$

$y = \frac{1}{\sqrt{3}}x + b$

$b = y - \frac{1}{\sqrt{3}}x$

$= 0$

Vector Valued Funtion Definition

- 2D Space : $\vec{r}(t) = < f(t), g(t) >$
- 3D Space : $\vec{r}(t) = < f(t), g(t), h(t) >$

Calculus of Vector Valued Functions

Limits

$$\lim_{t \to \infty} \vec{r}(t) = < \lim_{t \to \infty} f(t), \lim_{t \to \infty} g(t), \lim_{t \to \infty} h(t) >$$

Derivatives

$$\vec{r}\,'(t) = < f'(t), g'(t), h'(t) >$$

Indefinite Integral

$$\int \vec{r}(t)\,dt = < \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt >$$

Definite Integral

$$\int_a^b \vec{r}(t)\,dt = < \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt >$$

Problems

1. $\vec{r}(t) = \langle a\cos t, b\sin t, t \rangle$

$\vec{r}'(t) = \langle -a\sin t, b\cos t, 1 \rangle$

$\vec{r}''(t) = \langle -a\cos t, -b\sin t, 0 \rangle$

to see if/when $\vec{r}'(t) \perp \vec{r}''(t)$, take dot product $=0$...

$$\vec{r}'(t) \cdot \vec{r}''(t) = 0 = (-a\sin t)(-a\cos t) + (b\cos t)(-b\sin t) + (1)(0)$$

$$0 = a^2 \sin t\cos t - b^2 \sin t\cos t$$

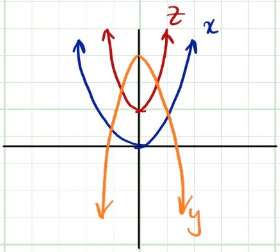
$$0 = \underbrace{(a^2-b^2)\sin t\cos t}$$

is zero when $a=b \Rightarrow b^2-b^2=0$

or when $a=-b \Rightarrow (-b)^2-b^2 = b^2-b^2=0$

2. $\vec{r}(t) = \begin{cases} x = t^2 \\ y = 3-2t^2 \\ z = 1+2t^2 \end{cases}$

$-5 \le t \le 5$



Arc Length Formula

Given vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, the arc length "L" travelled for $t_1 < t < t_2$ is

$$L = \int_{t_1}^{t_2} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \quad \text{or} \quad L = \int_{t_1}^{t_2} \|\vec{r}\,'(t)\|\, dt$$

Arc Length as Function of Time

Function $s(t)$ measures the distance travelled by particle

$$s(t) = \int_{t_0}^{t} \|\vec{r}\,'(t)\|\, dt$$

eg, $t_0 = 0$ and $s(4) = 10$ means object travelled 10 m in 4 s

Arc Length Parametrization

We say a curve $\vec{r}$ is parametrized wrt arc length if arclength between $\vec{r}(t_1)$ and $\vec{r}(t_2) = |t_2 - t_1|$

$\Rightarrow$ This is possible only if $\|\vec{r}\,'(t)\| = 1$ at all time

since $s(t) = \int_{t_0}^{t} \|\vec{r}\,'(t)\|\, dt = \int_{t_0}^{t} 1 \cdot dt = |t - t_0|$

Inverse Parametrization

$\|\vec{r}_L'(s)\| = 1 \leftarrow$ here, t is a function of s

Eg: $s(t) = \frac{1}{2}t^2 + 2t \implies r(t) = 2t\hat{i} + \frac{4}{3}t^{3/2}\hat{j} + \frac{1}{2}t^2\hat{k}$

$2s = t^2 + 4t$

$r_N(s) = 2$

$2s = t^2 + 4t + 4 - 4$

$2s = (t+2)^2 - 4$

$(t+2)^2 = 2s + 4$

$t + 2 = \pm\sqrt{2s+4}$

$t = \sqrt{2s+4} - 2$ use (+)ve root since $t \geq 0$

- $\vec{r}_N(s)$ parametrizes the same curve as $\vec{r}(t)$, but it has 1 m/s speed
  ↳ it is a normalized vector → it reparametrizes the curve.

Problems

$s(0) = (0,0,3) = (c_1, c_2, 3+c_3)$, so $c_1 = c_2 = c_3 = 0$

$\therefore s(5) = [3$

1. $\vec{r}(t) = [3\sin t, 4t, 3\cos t]$

$$\vec{r}\,'(t) = [3\cos t, 4, -3\sin t]$$

$$s(t) = \int_0^t \|\vec{r}\,'(t)\|\,dt = \int_0^t \sqrt{(3\cos t)^2 + 4^2 + (-3\sin t)^2}\,dt$$

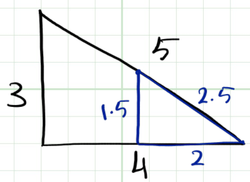
$$= \int_0^t \sqrt{9\cos^2 t + 16 + 9\sin^2 t}\,dt$$

$$= \int_0^t \sqrt{9(\sin^2 t + \cos^2 t) + 16}\,dt$$

$$= \int_0^t \sqrt{9+16}\,dt = \int_0^t \sqrt{25}\,dt = 5t$$

$$\therefore s(5) = 5(5) = \boxed{25}$$

2.



F6. Normal Vectors

- tangent vector $\longrightarrow \vec{r}\,'(t)$
- unit tangent vector $\longrightarrow \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$: only focuses on direction

Principle Unit Normal Vector

- $\vec{N}(t)$: a unit vector that points in the direction that the curve is turning
- If $\vec{T}\,'(t) \neq 0$, then $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$

Curvature

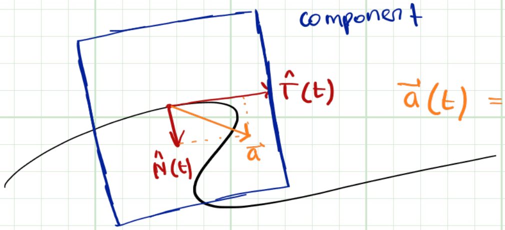
- the curvature "κ" a circle radius r is defined as

$\boxed{\kappa = \dfrac{1}{r}}$ curvature at time t : $\boxed{\kappa(t) = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}}$

- when $\vec{r}(s)$ is in arclength parametrization, $\kappa(t) = \left\| \dfrac{d\vec{T}}{ds} \right\|$

F7. Motion in Space

acceleration: $\vec{a}(t) = \underbrace{a_T \hat{T}(t)}_{\text{tangential component}} + \underbrace{a_N \hat{N}(t)}_{\text{normal component}}$



$$\vec{a}(t) = \underline{\quad}\,\hat{T}(t) + \underline{\quad}\,\hat{N}(t)$$

Tangential and Normal Acceleration

$$\vec{a}(t) = \vec{r}\,''(t) = \frac{d}{dt}\left[\vec{r}\,'(t)\right] = \frac{d}{dt}\left[\|\vec{r}\,'(t)\|\,\vec{T}(t)\right]$$

$$= \frac{d}{dt}\left[\|\vec{r}\,'(t)\|\right]\vec{T}(t) + \|\vec{r}\,'(t)\|\,\vec{T}\,'(t)$$

$$\boxed{\therefore\ \vec{a}(t) = \frac{d}{dt}\left[\|\vec{r}(t)\|\right]\hat{T}(t) + \kappa\,\|\vec{r}\,'(t)\|^2\,\hat{N}(t)}$$

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

Problems

1 $\vec{r}(t) = [5t^2, 10t^2]$

a) position after 50 km and 100 km (arc length parametrization)

$$L = \int \|\vec{r}\,'(t)\| \, dt$$

$$= \int \sqrt{(10t)^2 + (20t)^2} \, dt$$

$$= \int \sqrt{100t^2 + 400t^2} \, dt$$

$$= \int \sqrt{500} \, t \, dt$$

$$L(t) = \frac{\sqrt{500}\, t^2}{2}$$

$$2s = \sqrt{500}\, t^2$$

$$t^2 = \frac{2s}{\sqrt{500}}$$

$$t = \sqrt{\frac{2s}{\sqrt{500}}}$$

$$\vec{r}(s) = \left[5\left(\frac{2s}{\sqrt{500}}\right), 10\left(\frac{2s}{\sqrt{500}}\right)\right]$$

$$\vec{r}(s) = \left[\frac{10s}{\sqrt{500}}, \frac{20s}{\sqrt{500}}\right]$$

$$\therefore \vec{r}(50) = 22.36$$

$$\vec{r}(100) = 44.72$$

b) find average speed in that distance

$$\|\vec{r}\,'(t)\| = \sqrt{500}\, t$$