# MAT290 Important Formulas

Zaeem Ahmad

## Complex Numbers

→ $z = x + iy$ , $Re\{z\} = x$ , $Im\{z\} = y$

→ $z = |z| \angle\theta = r\cos\theta + ir\sin\theta$

→ $|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$

→ $Arg(z) = \theta_0 = \tan^{-1}(y/x)$ , $-\pi < \theta_0 \leq \pi$

→ $arg(z) = \theta_0 + 2\pi k$ , $k = 0, \pm 1, \pm 2, \cdots$

→ $\bar{z} = x - iy$

→ $i^2 = -1$ , $z\bar{z} = |z|^2$ , $|z_1 + z_2| \leq |z_1| + |z_2|$

→ **Euler's Formula** : $e^{i\theta} = \cos\theta + i\sin\theta$

→ **$n$'th power of $z$** : $z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n \angle n\theta$

→ **$n$ roots of $z$** : $w_k = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta_0 + 2\pi k}{n}\right) + i\sin\left(\frac{\theta_0 + 2\pi k}{n}\right)\right]$

all lie on circle radius $|z|^{1/n} = r^{1/n}$ and are equally spaced on this circle by $\frac{2\pi}{n}$ starting at $\frac{\theta_0}{n}$

→ **$\cos z$** $= \frac{e^{iz} + e^{-iz}}{2} = \cos x \cosh y + i \sin x \sinh y$    **$\sin z$** $= \frac{e^{iz} - e^{-iz}}{2i} = \sin x \cosh y + i \cos x \sinh y$

→ **$\cosh z$** $= \frac{e^{iz} + e^{-iz}}{2} = \cos(iz)$    **$\sinh z$** $= \frac{e^{iz} - e^{-iz}}{2} = -i\sin(iz)$

→ **$\ln(z)$** $= \log_e|z| + i[Arg(z) + 2n\pi]$, $n = 0, \pm 1, \pm 2, \cdots$

regular log laws hold for $\ln(z)$, but not always for $Ln(z)$

→ **$Ln(z)$** $= \log_e|z| + i\,Arg(z)$

→ **$z^\alpha = e^{\alpha \ln z}$** , principal value of $z^\alpha = e^{\alpha Ln z}$

# Ordinary Differential Equations

→ **separable method :**

$$\frac{dy}{dx} = f(y)\,g(x) \Rightarrow \frac{1}{f(y)}\frac{dy}{dx} = g(x) \Rightarrow \int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

→ **integrating factor method :**

$$y' + P(x)y = f(x) \Rightarrow \text{integrating factor } \rho(x) = e^{\int P(x)dx}$$

$$\boxed{\begin{aligned} &\Rightarrow \rho(x)y = \int f(x)\rho(x)\,dx \\ &\Rightarrow y(x) = \frac{1}{\rho(x)}\left[\int \rho(x) f(x)\,dx + C\right] \end{aligned}}$$

→ **reduction of order :**

$y'' + P(x)y' + Q(x)y = 0$ and $y_1(x)$ s.t. $y_2(x) = u(x)y_1(x)$

$$\Rightarrow \boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx} \Rightarrow y(x) = C_1y_1(x) + C_2y_2(x)$$

→ **Variation of parameters :**

$y'' + P(x)y' + Q(x)y = f(x)$

$y_c$ is $sol^{n}$ to homogenous $equ^{n}$
$y_p$ is $sol^{n}$ to nonhomogenous $equ^{n}$

$\Rightarrow y_c = C_1y_1 + C_2y_2$

$$W(y_1,y_2) = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$\Rightarrow y_p = u_1y_1 + u_2y_2$

$$\Rightarrow \boxed{u_1 = -\int \frac{y_2 f}{W(y_1,y_2)}\,dx} \Rightarrow \boxed{u_2 = \int \frac{y_1 f}{W(y_1,y_2)}\,dx}$$

→ **conditions for existence and uniqueness :**

1) all $a_n(x)$'s and $g(x)$ are continuous
2) $a_n(x) \neq 0$

→ **Lin. Indep. $sol^{n}s$ :**

$y_1 \quad y_2$ → $W \neq 0$ Lin. Indep.
→ $W = 0$ we dunno

$y_1 \quad y_2 \quad y_3$ → $W \neq 0$ Lin. Indep
→ $W = 0$ Lin. Dep.

## Laplace Transforms

→ $F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$ ⟶ 1) $f(t)$ is piecewise continuous
2) $f(t)$ is of expontial order
3) If 1) and 2), then $\lim_{s \to +\infty} F(s) = 0$

→ property quick notes:

- $s$-shift: func$^n$ in $s$-dom shifted by $a$ → multiply by $e^{at}$ in $t$-dom
- $t$-shift: func$^n$ in $t$-dom shifted by $a$ → multiply by $e^{-as}$ in $s$-dom
- time differentiation ⟶ multiplication by $s$
- time integration ⟶ division by $s$
- $s$ differentiation ⟶ multiplication by $t$

## Complex Analysis

→ a domain is an open connected set

→ a function is differentiable at a point $z_0$ if:

1) Cauchy Riemann equations satisfied

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

2) $u(x,y)$ and $v(x,y)$ and all their partials are continuous

→ analytic function: not only differentiable at $z_0$, but in neighborhood of $z_0$

→ harmonic functions: $u(x,y)$ and $v(x,y)$ of analytitic fun$^n$s are harmonic

$$\nabla^2 f = 0 = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f = 0$$

→ <u>contour integration:</u>

1) Parametrize curve $C$: $z(t) = x(t) + i\,y(t)$ $\quad a \leq t \leq b$

2) Integral: $\boxed{\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt}$

→ <u>ML bounding:</u> $\left\| \int_C f(z)\,dz \right\| \leq ML$, $M$ is max $f(z)$ on $C$, $L$ is length of $C$

→ <u>very important integral:</u> $\boxed{\int_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{if } n \neq 1 \end{cases}}$

☆ to do: add the circulation + flux notes

MAT290 Final

## 1. Complex Numbers

$e^{i\theta} = \cos\theta + i\sin\theta$, $e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y)$

$z^{n} = |z|^{n}(\cos n\theta + i\sin n\theta)$ ; $z\bar{z} = |z|^{2}$

$w_k = z^{1/n} = r^{1/n} \angle \frac{\theta_0 + 2\pi k}{n}$ • all lie on circle radius $|z|^{1/n}$ and spaced by $\frac{2\pi}{n}$ starting at $\frac{\theta_0}{n}$

$\ln(z) = \log_e(|z|) + i(\theta_0 + 2\pi k)$

$\mathrm{Ln}(z) = \log_e(|z|) + i\,\mathrm{Arg}(z) = \ln|z| + i\theta_0$ $\quad z^{\alpha} = e^{\alpha \mathrm{Ln} z}$

$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$ $\quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$

$\cosh z = \frac{e^{z} + e^{-z}}{2}$ $\quad \sinh z = \frac{e^{z} - e^{-z}}{2}$

$\cos(n\theta) = \frac{z^{n} + z^{-n}}{2}$ $\quad \sin(n\theta) = \frac{z^{n} - z^{-n}}{2i}$

## 2. Differential Equations

→ integrating factor $\mu(x) = e^{\int P(x)dx}$

→ reduction of order $y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$

→ variation of parameters

$W(y_1 y_2) = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \Rightarrow u_1 = -\int \frac{y_2 f}{W(y_1, y_2)}dx \quad u_2 = \int \frac{y_1 f}{W(y_1, y_2)}dx$

## 3. Laplace Trasform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad , \quad s = \sigma + i\omega$$

* F(s) exists if
  1) f(t) piecewise cont. on $[0,\infty)$
  2) f(t) is of exponential order

$$\left.\begin{array}{l}\text{1) } f(t) \text{ piecewise cont. on } [0,\infty) \\ \text{2) } f(t) \text{ is of exponential order}\end{array}\right\} \lim_{s\to\infty} F(s) = 0$$

## 4. Complex Analysis

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

Cauchy Riemann Equ$^{r}$s:

$$\left.\begin{array}{l}\text{1) } \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \text{2) } \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}\end{array}\right\} \begin{array}{l}\text{if these are satisfied and} \\ u, v \text{ partials cont; analytic}\end{array}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Contour Integral: $\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$

Circulation and Flux:

$$\oint_C \overline{f(z)}\,dz \begin{cases} \to \text{real: circulation} \\ \to \text{im: flux} \end{cases}$$

Cauchy Goursat: $\oint_C f(z)\,dz = 0$ if $f(z)$ analytic on and inside C

FTC, Path Indep: $\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$

Cauchy Integral formula:

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Cauchy Residue Thm: $\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$

Residues: "$a_{-1}$ term of Laurent Series"

simple: $\operatorname{Res}(f(z), z_0) = \lim_{z\to z_0} \left[f(z)(z-z_0)\right]$

order n: $\operatorname{Res}(f(z), z_0) = \dfrac{1}{(n-1)!} \lim_{z\to z_0} \dfrac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z)\right]$