Set 7 Exercise 7

Wiktor Hamberger
308982

April 23, 2020

We have to find distribution of $M = \frac{n}{\sigma^2}(\overline{X}-\mu)^2 = (\frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu)^2$, assuming that independent random variables $X_k \sim N(\mu, \sigma^2)$

$$M_{X_k}(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$M_{\sum\limits_{k=1}^{n} X_k}(t) = e^{n\mu t + \frac{n}{2}\sigma^2 t^2}$$

$$M_{\frac{1}{n}\sum\limits_{k=1}^{n} X_k}(t) = M_{\overline{X}}(t) = e^{\mu t + \frac{1}{2n}\sigma^2 t^2}$$

$$M_{\frac{\sqrt{n}}{\sigma}\overline{X}}(t) = e^{\mu\frac{\sqrt{n}}{\sigma}t + \frac{1}{2n}\sigma^2\frac{n}{\sigma^2}t^2} = e^{\mu\frac{\sqrt{n}}{\sigma}t + \frac{1}{2}t^2}$$

$$M_{\frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu}(t) = e^{\mu\frac{\sqrt{n}}{\sigma}t + \frac{1}{2}t^2}e^{-\frac{\sqrt{n}}{\sigma}\mu t} = e^{\frac{1}{2}t^2} \Rightarrow$$

$$\Rightarrow \frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu \sim N(0,1)$$

Let's $Z = \frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu$, then $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2(1)$ and $Z^2 = M$, so $\underline{\underline{M \sim \chi^2(1)}}$.