Exercise 1 set 4

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Given is function $f(x,y) = C(x+y)exp\{-(x+y)\}$, where $x > 0, y > 0$.

1. Compute the value of C such that f(x, y) is the density of 2-dimensional r.v. (X, Y).

$$\int_0^\infty \int_0^\infty f(x,y)dxdy = 1$$

$$= \int_0^\infty \int_0^\infty C(x+y)e^{-(x+y)}dxdy = C\int_0^\infty \int_0^\infty (x+y)e^{-x}e{-y}dxdy =$$
$$= C\int_0^\infty e^{-y} \int_0^\infty xe^{-x}+ye^{-x}dxdy = C\int_0^\infty e^{-y}(\int_0^\infty xe^{-x}dx+\int_0^\infty ye^{-x}dx)dy =$$
$$= C\int_0^\infty e^{-y}(0-0+0-(-1)+y(0-(-1)))dy = C\int_0^\infty e^{-y}(1+y)dy =$$
$$= C(\int_0^\infty e^{-y}dy+\int_0^\infty e^{-y}ydy) = C(0-(-1)+0-0-0-(-1)) = 2C = 1 \implies C = \frac{1}{2}$$

2. Check if variables X, Y are independent.
Variables X, Y are independent if $\forall x, y \in \mathbf{R}\ f(x,y) = f_1(x) * f_2(y)$

$$f_1(x) = \int_0^\infty \frac{1}{2}(x+y)e^{-x}e^{-y}dy = \frac{1}{2}e^{-x}\int_0^\infty (x+y)e^{-y}dy = \frac{1}{2}e^{-x}(x+1)$$

$$f_2(y) = \int_0^\infty \frac{1}{2}(x+y)e^{-x}e^{-y}dx = \frac{1}{2}e^{-y}(y+1)$$

For $x, y = 2$, $f_1(2) = \frac{3}{2e^2}$, $f_2(2) = \frac{3}{2e^2}$, $f(2,2) = \frac{2}{e^4} \implies$ they are not independent.
Also f(x, y) is not defined for $x, y \leq 0$, so it's another proof that X and Y are not independent.

3. Find moments m_{10}, m_{01}.

$$m_{pq} = \int_{\mathbf{R}}\int_{\mathbf{R}} x^p y^q f(x,y)dydx$$

$$m_{10} = \int_0^\infty \int_0^\infty x^1 y^0 \frac{1}{2}(x+y)e^{-x}e^{-y}dydx = \int_0^\infty \frac{1}{2}xe^{-x}(x\int_0^\infty e^{-y}dy+\int_0^\infty ye^{-y}dy)dx =$$
$$= \int_0^\infty \frac{1}{2}xe^{-x}(x+1)dx = \frac{1}{2}(\int_0^\infty x^2e^{-x}dx+\int_0^\infty xe^{-x}dx) = \frac{1}{2}[(-e^{-x}x^2)\Big|_0^\infty +3\int_0^\infty xe^{-x}dx] =$$
$$= \frac{1}{2}(0+3*1) = 1.5$$

$m_{10}\ and\ m_{01}$ are symmetric so $m_{10} = m_{01} = 1.5$