Pete Wilcox — pcwilcox@ucsc.edu

Experiments and Events:

Def: An experiment is a process whose outcome is not known in advance with certainty.

Sample Space: Collection of *all* possible outcomes of an experiment. S or Ω. Each outcome is an element of the sample space $s \in S$.

Operations:

Union: $x \in S : A \cup B = \{x \in A \text{ or } x \in B\}$

$$A \cup B = B \cup A$$

$$A \cup A = A$$

$$A \cup \emptyset = A$$

$$A \cup S = S$$

$$A \subset B \Rightarrow A \cup B = B$$

$$A_1, A_2, \dots, A_n \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{i=n} A_i$$

$$\bigcup_{i=1}^{\infty} A_i \to \bigcup_{i \in I} A_i$$

$$(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$$

Intersections: $A \cap B = \{x \in A \text{ and } x \in B\} = AB$

$$A \cap B = B \cap A$$

$$A \cap A = A$$

$$A \cap \emptyset = \emptyset$$

$$A \cap S = A$$

$$A \subset B \Rightarrow A \cap B$$

$$\bigcap_{i \in I} A_i = \bigcap_{i=1}^{\infty} A_i$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

$$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$$

Complements: $A^c = \{x \in S : x \notin A\}$

$$(A^c)^c = A$$

$$\emptyset^c = S$$

$$S^c = \emptyset$$

$$A \cup A^c = S$$

$$A \cap A^c = \emptyset$$

Dijoint Events:

A and B are *disjoint* or *mutually exclusive* if A and B have no outcomes in common. This happens only if $A \cap B = \emptyset$.

Def: A collection $A_1, \dots, A_n$ is a collection of disjoint evens if and only if $A_i \cap A_j = \emptyset, \forall i, j, i \neq j$

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

$$(A \cup B)^c = A^c \cap B^c$$

$$\begin{aligned} &x \in (A \cap B)^c \\ &\Rightarrow x \notin A \text{ and } x \notin B \\ &\Rightarrow x \in A^c \text{ and } x \in B^c \\ &\Rightarrow x \in A^c \cap B^c \end{aligned}$$

Probabilities:

Functions over S that measure the likelihood of events.

$$\forall A : Pr(A) \geq 0$$

$$Pr(S) = 1$$

For every *infinite sequence* of *disjoint* events $A_1, A_2, \dots (A_i \in S)$:

$$Pr\left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} Pr(A_i)$$

$$\begin{aligned} &Pr(A_1 \cup A_2 \cup \dots \cup A_n \cup \dots) \\ &= Pr(A_1) + Pr(A_2) + \dots + Pr(A_n) + \dots \end{aligned}$$

$$Pr(\emptyset) = 0$$

$$Pr(\bigcup_{i=1}^{n} A_i) = Pr(\bigcup_{i=1}^{n} A_i + \bigcup_{i=n+1}^{\infty} \emptyset) = \sum_{i=1}^{n} Pr(A_i)$$

$$Pr(A^c) = 1 - Pr(A)$$

$$A \subset B \Rightarrow Pr(A) \leq Pr(B)$$

$$\forall A : 0 \leq Pr(A) \leq 1$$

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

Finite Sample Spaces:

$S := \{s_1, s_2, \dots, s_n\}$

To obtain a probability distribution over S we need to specify $Pr(s_i) = P_i, \forall i = 1, 2, \dots, n$, such that $\sum_{i=1}^{n} P_i = 1$

Def: A sample space S with n outcomes $s_1, \ldots, s_n$ is a *simple sample space* if the probability assigned to each outcome is $\frac{1}{n}$. If A contains m outcomes then $Pr(A) = \frac{m}{n}$.

Counting Methods:

Multiplication Rule: Suppose an experiment has k parts ($k \geq 2$) such that the i^{th} part of the experiement has n_i possible outcomes, $i = 1, \ldots, k$, and that *all possible outcomes can occur regardless of which outcomes have occurred in other parts.* The sample space S will contain vectors of the form $(u_1, u_2, \ldots, u_k)$. u_i is one of the n_i possible outcomes of part i. The total number of vectors is $n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

Permutations: Given an array of n elements the first position can be filled with n different elements, the second with $n-1$, and so on. $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$

$P_{n,k} = \frac{n!}{(n-k)!}$

$P_{n,n} = n!$

Combinations: In general we can "combine" n elements taking k at a time in $C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$.

Multinomial Coefficeints: Consider splitting n elements into $k (k \geq 2)$ groups in a way such that group j gets n_j elements and $\sum\limits_{j=1}^{k} n_j = n$. The n_1 elements in the first group can be selected in $\binom{n}{n_1}$, the second in $\binom{n-n_1}{n_2}$, the third in $\binom{n-n_1-n_2}{n_3}$ and so on until we complete the k groups. Then: $\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdot \ldots \cdot \binom{n_k}{n_k} = \binom{n}{n_1, n_2, \ldots, n_k}$

Probability of union: If $A_1, A_2, \ldots, A_n$ are *disjoint events* then
$Pr(A_1 \cup A_2 \cup \ldots \cup A_n)$
$= Pr(\bigcup\limits_{i=1}^{n} A_i)$
$= Pr(A_1) + Pr(A_2) + \ldots + Pr(A_n)$
$= \sum\limits_{i=1}^{n} Pr(A_i)$

If the events are not disjoint:
$A_1, A_2 : Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2) - Pr(A_1 \cap A_2)$
$A_1, A_2, A_3 : Pr(A_1) + Pr(A_2) + Pr(A_3) - Pr(A_1 \cap A_2) - Pr(A_1 \cap A_3) - Pr(A_2 \cap A_3) + Pr(A_1 \cap A_2 \cap A_3)$

Conditional Probability:

If A, B are events such that $Pr(A) > 0$ and $Pr(B) > 0$ then $Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)}$ and
$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$
Furthermore: $Pr(A \cap B) = Pr(B|A) \cdot Pr(A)$ and $Pr(A \cap B) = Pr(A|B) \cdot Pr(B)$. In general: $Pr(A_1 \cap A_2 \cap \ldots \cap A_n) = Pr(A_1) \cdot Pr(A_2|A_1) \cdot \ldots \cdot Pr(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

Independence: A, B are independent events if $Pr(A|B) = Pr(A)$ and $Pr(B|A) = Pr(B)$. Then, if A, B are independent: $Pr(A \cap B) = Pr(A|B) \cdot Pr(B) = Pr(A) \cdot Pr(B)$ and $Pr(A \cap B) = Pr(B|A) \cdot Pr(A) = Pr(B) \cdot Pr(A)$. In general if $A_1, A_2, \ldots, A_n$ are independent, $Pr(A_1 \cap A_2 \cap \ldots \cap A_n) = Pr(A_1) \cdot Pr(A_2) \cdot \ldots \cdot Pr(A_n)$. Note that if $A \cap B = \emptyset$ then the two events are *not independent.* note that if A, B are independent then A, B^c are also independent.

Conditionally Independent: $A_1, \ldots, A_k$ are *conditionally independent* given B if, for every subset $A_{i_1}, \ldots, A_{i_m} : Pr(A_{i_1} \cap \ldots \cap A_{i_m}|B) = Pr(A_{i_1}|B) \cdot \ldots \cdot Pr(A_{i_m}|B)$.

Partitions: Let $B_1, \ldots, B_k$ be such that $B_i \cap B_j = \emptyset \forall i \neq j$ and $\bigcup\limits_{i=1}^{k} B_i = S$. Then these events form a partition of S.
$A = A \cap S = A \cap \left(\bigcup\limits_{i=1}^{k} B_i\right) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)$. Then: $Pr(A) = Pr(A \cap S) = Pr(A \cap \left(\bigcup\limits_{i=1}^{k} B_i\right)) = Pr(A \cap B_1) + Pr(A \cap B_2) + \ldots + Pr(A \cap B_k) = Pr(A|B_1) \cdot Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2) + \ldots + Pr(A|B_k) \cdot Pr(B_k) = \sum\limits_{i=1}^{k} Pr(A|B_i) \cdot Pr(B_i)$.
So, if $B_1, \ldots, B_k$ are a partition of S: $Pr(A) = \sum\limits_{i=1}^{k} Pr(A|B_i) \cdot Pr(B_i)$

Bayes' Theorem: $B_1, \ldots, B_k :=$ a partition of S such that $Pr(B_j) > 0, j = 1, \ldots, k$. Assume you have A such that $Pr(A) > 0$. Then: $Pr(B_i|A) = \frac{Pr(A|B_i) \cdot Pr(B_i)}{Pr(A)} = \frac{Pr(A|B_i) \cdot Pr(B_i)}{\sum\limits_{j=1}^{k} Pr(A|B_j) \cdot Pr(B_j)}$

Random Variables:

Def: A real-valued function on S isa random variable. A random variable X is a functions that assigns a real number $X(s) = x$ to each possible outcome

$s \in S$: $X : S \Rightarrow \mathcal{D}$. $x :=$ a realization of the random variable, $x \in \mathcal{D}$.

Notation: We will be computing $Pr(X \in E)$ for $E \subset \mathcal{D} = Pr(s \in S : X(s) \in E)$

Discrete Probability Distributions: A r.v. X has a discrete distribution if it takes a countable number of values. The probability function of a discrete r.v. is $f_X(x) = Pr(X = x)$. Properties:

$$0 \leq f_X(x) \leq 1$$

$$\forall x \notin \mathcal{D} : f_X(x) = 0$$

$$\sum_{x \in \mathcal{D}} f_X(x) = 1$$

$$Pr(X \in A) = \sum_{x \in A} f_X(x)$$

Uniform Distribution: $X = x, x \in \{1, 2, \ldots, k\}$ with all values x equally likely. The p.f. is $f_X(x) = Pr(X = x) =$

$$\begin{cases} \frac{1}{k} & x = 1, 2, \ldots, k \\ 0 & o.w. \end{cases}$$

Bernoulli Distribution: An event A happens with probability p
$X =$

$$\begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{if } A^c \text{ happens} \end{cases}$$

$f_X(x) =$

$$\begin{cases} (1-p) & x = 0 \\ p & x = 1 \\ 0 & \text{o.w.} \end{cases}$$

Binomial: n Bernoulli trials repeated independently with probability of success p. $X :=$ number of success in n trials. $x \in \{0, 1, \ldots, n\}$. $f_X(x) = Pr(X = x) =$

$$\begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, \ldots, n \\ 0 & \text{o.w.} \end{cases}$$

$X \sim Bin(n, p)$

Hypergeometric: A box with A red balls and B blue balls. n balls are drawn *without replacement.* $X :=$ number of red balls. $X \leq min(n, A)$. $max(n - B, 0) \leq X \leq min(n, A)$. $f_X(x) = Pr(X = x) =$

$$\begin{cases} \frac{\binom{A}{x} \cdot \binom{B}{n-x}}{\binom{A+B}{n}} & \text{for } max(n - B, 0) \leq x \leq min(n, A) \\ 0 & \text{o.w.} \end{cases}$$

Negative-Binomial: We repeat Bernoulli trials until r successes are observed. $X :=$ number of failures $= \{0, 1, \ldots\}$. $p :=$ probability of success. $Pr(X = x) = Pr(x \text{ failures before } r \text{ successes}) = Pr(x \text{ failures and } r-1 \text{ successes in } x+r-1 \text{ trials}) \cdot Pr(\text{one success in last trial}) = \left[\binom{x+r-1}{x}(1-p)^x p^{r-1}\right] \cdot p = \binom{x+r-1}{x}(1-p)^x p^r$. $f_X(x) =$

$$\begin{cases} \binom{x+r-1}{x}(1-p)^x p^r & x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

Geometric: Negative binomial with $r = 1. f_X(x) = \begin{cases} (1-p)^x p & x = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$

Poisson: Counts occurences of an event. X is a Poisson r.v. with parameter λ (intensity) if the p.f. is $f_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$ with $\lambda > 0$.

Continuous Random Variables: A r.v. X has a continuous distribution if there is a non-negative f such that $Pr(a \leq X \leq b) = \int_a^b f(x)dx$. f is the probability density function p.d.f.

Cumulative Distribution Function: (c.d.f.) For any r.v. X the c.d.f. is given by $F(x) = Pr(X \leq x)$. Properties:

$\forall x : 0 \leq F(x) \leq 1$

$F(x)$ is non-decreasing, i.e. if $x_1 < x_2 \Rightarrow \{X \leq x_1\} \subset \{X \leq x_2\}$ and so $Pr(X \leq x_1) \leq Pr(X \leq x_2) \Rightarrow F(x_1) \leq F(x_2)$

$\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$