A note of Vladimir Hinich, Lectures on Infinity Categories

Siyuan Chen, Aug 28, 2023.

Yoneda embedding

$Y: \mathcal{C} \to P(\mathcal{C})$ $\quad Y(x)(y) = \mathrm{Hom}_{\mathcal{C}}(y,x)$. fully faithful

Yoneda lemma

$$F \in P(\mathcal{C}) \quad \underset{a}{\mathrm{Hom}_{P(\mathcal{C})}(Y(x), F)} \to \underset{a(\mathrm{id}_x)}{F(x)} \quad \text{bij.}$$

$X \in$ Kan sSet. $\Rightarrow \pi_n(X,x) =$ heq. class of $(\Delta^n, \partial\Delta^n) \to (X,x)$.

j^n_k $(k \neq 0,n)$, $X_n = \mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$. bij iff $X=$nerve.
sur iff $X=$quasicat.

$$\left(\mathrm{Hom}^R(x,y)\right)_n = (n+1)\text{-simplices } h \text{ s.t. } \begin{cases} d_{n+1} h = x. \\ d_0 \cdots d_n h = y. \end{cases}$$

$\mathrm{Hom}_{\mathrm{Ho}(C)}(x,y) = \pi_0(\mathrm{Hom}^R_C(x,y))$.

$h : s\mathrm{Set} \rightleftarrows \mathrm{Cat} : N$ (nerve functor).

For a quasicat X, $h(X)$ canonically iso. to $\mathrm{Ho}(X)$.

$\mathcal{C}$ quasicat $\Rightarrow \mathrm{Hom}^R_C(x,y)$ Kan. $\forall x,y$.

$\mathbb{C}^\vee$: $\mathbb{C}^0 = *$. $\mathbb{C}^1 = 0 \to 1$ $\quad \mathbb{C}^2 =$
$$\begin{tikzcd} & 1 \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & {} & 2 \end{tikzcd}$$

$\mathbb{C} : s\mathrm{Set} \rightleftarrows s\mathrm{Cat} : \mathfrak{N}$. $\mathfrak{N}(\mathcal{C})_n = \mathrm{Hom}_{s\mathrm{Cat}}(\mathbb{C}^n, C)$.

DK eq. 1. $\forall x,y \in C$, $\mathrm{Hom}^R_C(x,y) \to \mathrm{Hom}^R_D(fx, fy)$ is w.e. (ff).

2. $\forall z \in D$ $\exists$ eq. : $f(x) \to z$ for some $x \in C$. (es).

$\mathfrak{N}$: Kan mapped $\to$ quasicat.

$\mathbb{C} \circ \mathfrak{N}(\mathcal{C}) \to C$ is DK eq.

$\mathcal{C}$ fibrant mapped. X quasicat. eq. $\mathfrak{C}(X) \to \mathcal{C} \Rightarrow$ eq. $X \to \mathfrak{N}(\mathcal{C})$.

Y Kan $\Rightarrow \mathrm{Fun}(X, Y)$ Kan $\quad Y$ quasicat. $\Rightarrow \mathrm{Fun}(X, Y)$ quasicat.

homotopy coherent func. $\quad I \to (N(I) \to \mathcal{S})$.

$\mathcal{S}$: ho. coh. nerve of sim. cat. of Kan.

Cat_∞ obj = quasicat. $\mathrm{Map}(X, Y)$ = maximal Kan of $\mathrm{Fun}(X, Y)$.

Quillen eq. cofibrant $X \in \mathcal{C}$. fibrant $Y \in \mathcal{D}$. $a: X \to G(Y)$ is w.e iff $a': F(X) \to Y$ is w.e. OR induced adjunction of ho. cat. is eq.

Top mod. str. $\quad$ w.e. = ho. eq.

f. = Serre f.

c. = LLP of trivial f.

sSet mod. str. $\quad$ w.e. : $|f|$ is w.e.

$f: X \to Y$

I(g. c.) = $\partial\Delta^n \hookrightarrow \Delta^n$

J(g. t. c.) = $\Lambda^n_k \to \Delta^n$.

sCat mod. str. (Bergner)

w. e. (DK) = $\begin{cases} \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(fx, fy) \text{ w.e.} \\ \text{e.s.} \end{cases}$

f. = $\begin{cases} \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(fx, fy) \text{ Kan fi} \\ \text{eq. } \alpha: f(c) \to d \text{ in } \mathcal{D} \text{ can be lifted} \\ \text{to eq } a: c \to c' \text{ in } \mathcal{C} \text{ s.t. } \alpha = f(a) \end{cases}$

$(\mathbb{1}_S : \mathrm{Map}(0,0) = \{id\} = \mathrm{Map}(1,1).\ \mathrm{Map}(1,0) = \emptyset,\ \mathrm{Map}(0,1) = S)$

$$g.c. = \begin{cases} \coprod_{\partial\Delta^n} \to \coprod_{\Delta^n} \\ \emptyset \to * \end{cases}$$

$$g.t.c. = \begin{cases} \coprod_{\Lambda^n_k} \to \coprod_{\Delta^n} , \quad 0 \le l \le n \\ * \to \mathcal{J} \end{cases}$$

($\mathcal{J}$ obj. $= \{0,1\}$ countable sim. in each Map w. contractible

sSet mod. str. w.e. = Map carried to DK eq. by $\mathfrak{C}$.

(Joyal) fibrant obj. = quasicat.

c. = inj. maps

$(\mathfrak{C}, \mathfrak{N})$ is Quillen eq.

Hammock

$$\underbrace{\begin{array}{ccccccccccc} x & - & \cdot & - & \cdots & - & \cdot & - & y \\ \| & & \downarrow & & & & \downarrow & & \| \\ x & - & \cdot & - & \cdots & - & \cdot & - & y \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ x & - & \cdot & - & \cdots & - & \cdot & - & y \\ \| & & \downarrow & & & & \downarrow & & \| \\ x & - & \cdot & - & \cdots & - & \cdot & - & y \end{array}}_{n \text{ maps}}$$

same direction for a row.
different direction for adjacent rows.
$\leftarrow$ in W

weak path functor $\forall S \in sSet$, $X \in \mathcal{G}$, $Y \to Hom_{sSet}(S, Map_{\mathcal{C}}(Y, X))$

is representable. (by X^S)

ssSet mod. str. w.e. $= f: X \to Y$ s.t. $f_n : X_n \to Y_n$ is w.e. of sSet

(Reedy) $$(t.)f. = \begin{cases} X_0 \to Y_0 \ (t.) \text{ Kan } f. \\ X_n \to Y_n \times_{Y(\partial\Delta^n)} X(\partial\Delta^n) \ (t.) \text{ Kan } f. \end{cases}$$

c. $= X_n \to Y_n$ c. (inj.)

Rezk nerve $n \mapsto (C^{[n]})^{iso}$

Segal space: Reedy fibrant. $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ w.e.

ssSet mod. str. w.e. = f: $\mathrm{Map}(f, X)$ is a w.e. of sSet,
(CSS) $\forall X$ complete Segal
fibrant obj. = complete Segal spaces
c. = monomor.

Reedy w.e. is a w.e. in CSS. converse holds for a map between two CSS.

Bousfield localization $\mathrm{id}: M \rightleftarrows M_{loc} : \mathrm{id}$.

$i: A \to B$ $j: C \to D$ c. $\Rightarrow (A \times D) \coprod_{A \times C} (B \times C) \to B \times D$ c.
one of is t.c. $\Rightarrow$ t.c.
X CSS $\Rightarrow \forall K \in$ ssSet X^K CSS.

CSS completion: X Segal. $i: X \to \hat{X}$. $\hat{X}$ CSS. i $\begin{cases} \text{w.e. in CSS} \\ \text{DK eq.} \end{cases}$

Left fibration RLP $\{0\} \to \Delta^1$, $\Lambda^2_0 \to \Delta^2$.
Grothendieck construction $f: C \to D$. $F: D \to \mathcal{G}rp$.
$F(d) = f^{-1}(d)$. the fibre of a left fibration is a groupoid.
base change of l.f. is l.f. $X \to Y$ l.f. $\Rightarrow X^Z \to Y^Z$ is l.f.

Stop by P72