

A note of Tom Leinster, Higher Operads, Higher Categories.
 Siyuan Chen, Aug 23, 2023

$$\begin{array}{lll} \text{Lax} & \phi_{A,B} & FA \otimes FB \rightarrow F(A \otimes B) \\ & \phi & I \rightarrow FI \end{array}$$

$$\text{Colax} \quad \leftarrow$$

$$\text{Weak} \quad \overset{\text{iso.}}{\longleftrightarrow}$$

$$\text{Strong} \quad \overset{\text{id.}}{\longleftrightarrow}$$

$$\text{Strict } n\text{-cat:} \quad \text{Str-0-cat} = \text{Set}$$

$$\text{Str-(n+1)-Cat} = (\text{Str-n-Cat})\text{-Cat}$$

$$\text{Bicat: } B_0, B(A,B), B(B,C) \times B(A,B) \rightarrow B(A,C).$$

$$1_A \in B(A,A), \quad (h \circ g) \circ f \quad \downarrow \alpha_{h,g,f} \quad \alpha \text{ iso.}$$

$$\begin{array}{ccc} 1 \circ f & f \circ 1 & h \circ (g \circ f) \\ \downarrow \lambda_f & \downarrow \rho_f & \downarrow \lambda_{h,g,f} \\ f & f & \lambda \cdot \rho \text{ iso.} \end{array}$$

$$\begin{array}{ccc} \text{Lax transf.} & FA & \xrightarrow{ff} FB \\ & \downarrow \sigma_A & \Rightarrow \sigma_f \downarrow \sigma_B \\ & GA & \xrightarrow{Gf} GB \end{array}$$

$$\text{Multicat.} \quad \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \quad \triangle \quad a + \text{comp.} + \text{id.}$$

1 object Multicat = Operad.

without comp., id. = Multigraph.

PRD: (S, A) . S set A strict monoidal category.

monoid of $\text{Ob } A = \text{free monoid on } S$

$$\coprod_{a_1^{k_1}, \dots, a_n^{k_n}} (A(a_1^{k_1}, \dots, a_1^{k_1}), [a_1]) \times \dots \times A([a_n^{k_n}, \dots, a_n^{k_n}], a_n)$$

bije.

$$\longrightarrow A([b_1, \dots, b_m], [a_1, \dots, a_n])$$

$$(a_1^{k_1}, \dots, a_1^{k_1}, \dots, a_n^{k_n}, \dots, a_n^{k_n}) = [b_1, \dots, b_n].$$

tree. $\text{tr}(1)$ has "1", $\text{tr}(k_1) + \dots + \text{tr}(k_n) \rightarrow \text{tr}(k_1, \dots, k_n)$
 $T_1 \quad T_n \mapsto (T_1, \dots, T_n)$

ctr: $\text{ctr}(p)$ has "1", $\text{ctr}(1)$ has "1" $\text{ctr}(k_1) + \text{ctr}(k_2) \rightarrow \text{ctr}(k_1 + k_2)$
 $T_1 \quad T_2 \mapsto (T_1, T_2)$

(0 or 2). $\triangleright, \Rightarrow$

Lax MonCat $\xrightarrow{\text{lax (weak, strong)}}$ $P: A \rightarrow A' \quad \prod_{a_1, \dots, a_n} (p_{a_1} \otimes \dots \otimes p_{a_n})$
 $\downarrow \text{(unbiased, strong)}$ $\rightarrow p(a_1 \otimes \dots \otimes a_n)$ is lax. (weak, strong).

$\gamma((a_1^{k_1}, \dots, a_1^{k_1}), \dots, (a_n^{k_n}, \dots, a_n^{k_n})):$

$$[(a_1^{k_1} \otimes \dots \otimes a_1^{k_1}) \otimes \dots \otimes (a_n^{k_n} \otimes \dots \otimes a_n^{k_n})] \rightarrow$$

$$(a_1^{k_1} \otimes \dots \otimes a_n^{k_n}) \text{ is lax.}$$

$\text{La}: a \rightarrow (a)$ is lax. (unbiased, strong)

strict cover: $(P, \pi) \cdot \text{st}(A) \rightarrow A$. P full, faithful, essentially surjective

Cartesian = Preserve pullback.

$$\Sigma(T) \quad d \swarrow M \searrow c \quad \text{for 1 cell.}$$

$$\begin{array}{ccc} & M & \\ d \swarrow & & \searrow c \\ TE & & E' \end{array}$$

$$\begin{array}{ccc} & M & \\ \swarrow & \downarrow & \searrow \\ TE & N & E' \end{array} \quad \text{for 2 cell.}$$

$$\begin{array}{c} M' \circ M \\ \swarrow \quad \searrow \\ TM \quad M \\ \swarrow \quad \searrow \\ Td \quad Tc \quad d' \quad c' \\ \swarrow \quad \searrow \quad \searrow \\ T^2E \quad TE' \quad E'' \\ \swarrow \quad \searrow \\ ME \quad TE \end{array} \quad \text{for 1-comp.}$$

$$\begin{array}{ccc} & E & \\ \eta_E \swarrow & & \searrow \\ TE & & E \end{array} \quad \text{for 1-id.}$$

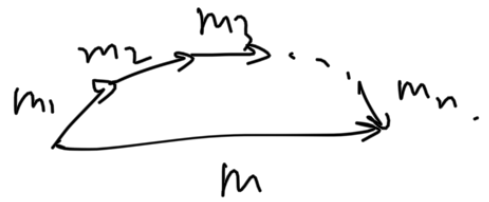
T -multicat is a monad in bicat. $\Sigma(T)$.

$$\begin{array}{ccc} \text{diag} \swarrow C_1 \searrow \text{cod} & & C_1 \circ C_1 = C_1 \times_{TC_0} TC_1 \xrightarrow{\text{comp}} C_1 \\ \swarrow \quad \searrow & & \downarrow \text{id} \\ TC_0 \quad C_0 & & C_0 \xrightarrow{\text{id}} C_1 \end{array}$$

T -Operad. T -graph. (similarly defined).

$$\begin{array}{ccccc} a_0 & \xrightarrow{m_1} & a_1 & \xrightarrow{m_2} & \dots & \xrightarrow{m_n} & a_n \\ \downarrow f & & & \downarrow \theta & & & \downarrow f' \\ a & \xrightarrow{m} & & & & & a' \end{array} \quad \begin{array}{l} + \text{comp.} \\ + \text{id.} \end{array}$$

plain-multi : $f, f' = \text{id.}$



weak double cat.

$$D_1^{(n)} = \lim \begin{array}{c} D_1 \\ \swarrow \text{dom} \searrow \text{cod} \\ D_0 \end{array} \begin{array}{c} D_1 \\ \swarrow \text{dom} \searrow \text{cod} \\ D_0 \end{array} \begin{array}{c} D_1 \\ \swarrow \text{dom} \searrow \text{cod} \\ D_0 \end{array} \dots \begin{array}{c} D_1 \\ \swarrow \text{dom} \searrow \text{cod} \\ D_0 \end{array}$$

$$\text{comp}^{(n)}: \begin{array}{ccc} & D_1^{(n)} & \\ \swarrow & \downarrow \text{comp}^{(n)} & \searrow \\ D_0 & & D_0 \end{array}$$

$$\begin{array}{ccc} T\text{-Multicat}' & \varepsilon/E \xrightarrow{S} \varepsilon/E & \\ & \downarrow e_1 \quad \Downarrow \phi \quad \downarrow e_1 & (\text{colax}). \\ & \varepsilon/\tilde{E} \xrightarrow{S'} \varepsilon/\tilde{E} & \end{array}$$

$$\text{Opetope} \quad \Sigma_0 = \text{Set} \quad T_0 = \text{id.} \quad \Sigma_{n+1} = \Sigma_n / T_n$$

$$T_{n+1} = (\text{free } T_n\text{-operad}).$$

n	\mathcal{O}_n	Σ_n	T_n
0	1	Set	id.
1	1	Set	free monoid
2	\mathbb{N}	Set/ \mathbb{N}	free plain operad.
3	{trees}	Set/{trees}	

\mathcal{O}_0 \bullet

$\mathcal{O}_1 \rightarrow$

$\mathcal{O}_2 \left\{ \begin{array}{c} \text{diagram (0)} \\ (0) \end{array} , \begin{array}{c} \text{diagram (1)} \\ (1) \end{array} , \begin{array}{c} \text{diagram (2)} \\ (2) \end{array} , \dots \right\}$ (1-pasting diagram)

$\mathcal{O}_3 \left\{ \dots , \begin{array}{c} \text{diagram} \\ \parallel \\ \text{diagram} \end{array} , \dots \right\}$ (2-pd)

Strict tree $stTr(1)$ has "1".
 $stTr(k_1) + stTr(k_2) + \dots + stTr(k_n) \rightarrow stTr(k_1 + \dots + k_n)$
 $T_1 \quad T_2 \quad T_n \quad \mapsto (T_1, \dots, T_n)$

$(stTr(0) = \emptyset, stTr(2) = \{ \text{diagram} \})$

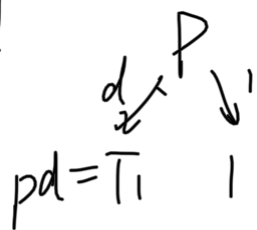
opetopic set: $\mathcal{O} \rightarrow \text{Set}$

Globular pd.

$\begin{array}{c} \text{diagram} \\ \circ_1 \end{array} = \begin{array}{c} \text{diagram} \end{array} \quad pd(2) \times_{pd(1)} pd(2) \rightarrow pd(2)$

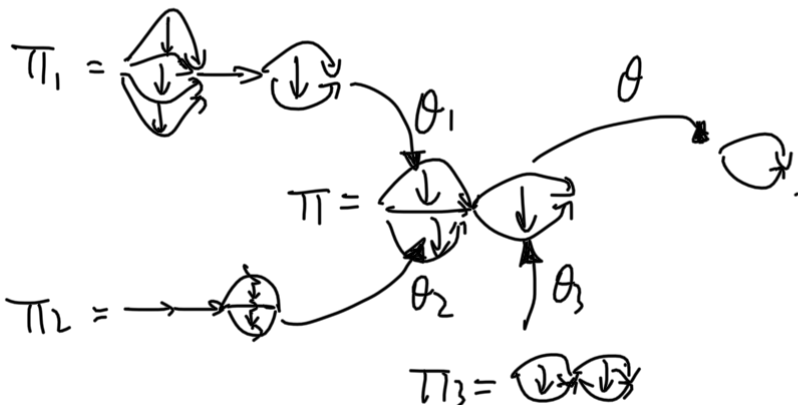
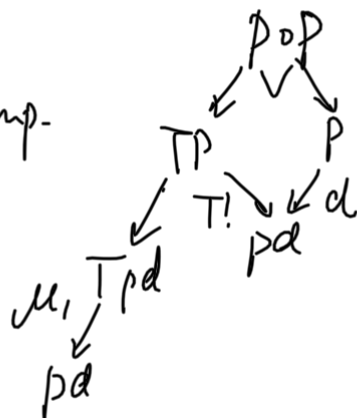
$\begin{array}{c} \text{diagram} \\ \circ_0 \end{array} = \begin{array}{c} \text{diagram} \end{array} \quad pd(3) \times_{pd(1)} pd(3) \rightarrow pd(3)$

Global operad

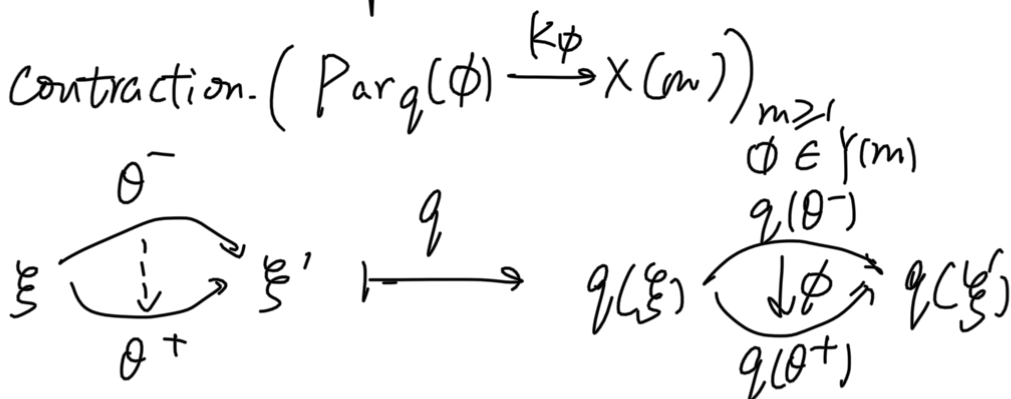


$$P(\pi) \xrightarrow{s} P(\partial\pi)$$

Comp.



weak n -cat. parallel: $\alpha^-, \alpha^+ \in X(m)$, $m=0$ or $m \geq 1$,
 $s(\alpha^-) = s(\alpha^+)$
 $t(\alpha^-) = t(\alpha^+)$



operads with contraction (P, κ) . P globular operad.

κ contraction. $(P, \kappa) \xrightarrow{f} (P', \kappa')$ preserve contraction

$$f(\kappa_\pi(\theta^-, \theta^+)) = \kappa'_\pi(f\theta^-, f\theta^+), \forall m \geq 1, \pi \in \text{pd}(m), (\theta^-, \theta^+) \in \text{Par}_P(\pi)$$

OC has an initial obj. (L, λ) .

A weak ω -cat is an L -alg.

