A note of Tom Leinster, Higher Operads, Higher Categories.

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| | | | | |
|---|---|---|---|---|
| Lax | $\phi_{A,B}$ | $FA \otimes FB$ | $\longrightarrow$ | $F(A \otimes B)$ |
| | $\phi$ | $I$ | $\longrightarrow$ | $FI$ |
| Colax | | | $\longleftarrow$ | |
| Weak | | | $\xleftrightarrow{\text{iso.}}$ | |
| Strong | | | $\xleftrightarrow{\text{id.}}$ | |

Strict $n$-cat: $\quad \text{Str-}0\text{-cat} = \text{Set}$

$$\text{Str-}(n+1)\text{-Cat} = (\text{Str-}n\text{-Cat})\text{-Cat}$$

Bicat: $B_0$. $B(A,B)$. $B(B,C) \times B(A,B) \to B(A,C)$.

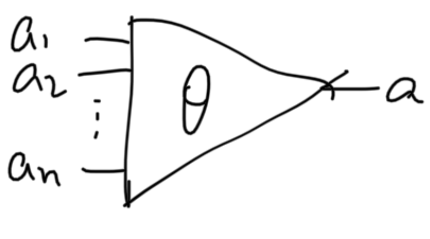
$1_A \in B(A,A)$.

$$\begin{tikzcd} \cdot \arrow[r, bend left=50, "(h\circ g)\circ f"] \arrow[r, bend right=50, "h\circ(g\circ f)"'] \arrow[r, phantom, "\Downarrow\alpha_{h,g,f}"] & \cdot \end{tikzcd} \qquad \alpha \text{ iso.}$$

$$\begin{tikzcd} \cdot \arrow[r, bend left=50, "1\circ f"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Downarrow\lambda_f"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, bend left=50, "f\circ 1"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Downarrow\rho_f"] & \cdot \end{tikzcd} \qquad \lambda, \rho \text{ iso.}$$

Lax transf.

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\sigma_A"'] \arrow[dr, phantom, "\Rightarrow\sigma_f"] & FB \arrow[d, "\sigma_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

Multicat.  + comp. + id.

1 object Multicat = Operad.

without comp., id. = Multigraph.

PRO: $(S, A)$. $S$ set $A$ strict monoidal category.

monoid of Ob $A$ = free monoid on $S$

$$\coprod_{a_1^1,\dots,a_n^{k_n}} \left(A((a_1^1,\dots,a_1^{k_1}),(a_1)) \times \dots \times A((a_n^1,\dots,a_n^{k_n}),(a_n))\right) \xrightarrow{bij} A((b_1,\dots,b_m),(a_1,\dots,a_n))$$

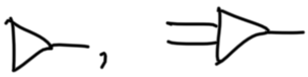
$$(a_1^1,\dots,a_1^{k_1};\dots,a_n^1,\dots,a_n^{k_n}) = (b_1,\dots,b_n).$$

tree. $tr(1)$ has "|", $tr(k_1)+\dots+tr(k_n) \to tr(k_1,\dots,k_n)$

$$T_1 \qquad T_n \mapsto (T_1,\dots,T_n).$$

ctr: $ctr(0)$ has "•", $ctr(1)$ has "|" $ctr(k_1)+ctr(k_2) \to ctr(k_1+k_2)$

$$T_1 \qquad T_2 \mapsto (T_1 \cdot T_2)$$

(0 or 2). 

LaxMonCat $\xrightarrow[\text{lax}]{\text{(weak, strong)}}$ $P: A \to A'$ $\pi_{a_1,\dots,a_n}: (Pa_1 \otimes \dots \otimes Pa_n) \to P(a_1 \otimes \dots \otimes a_n)$ is lax (weak, strong).

$\downarrow$ (Unbiased, strong)

$\gamma((a_1^1,\dots,a_1^{k_1}),\dots,(a_n^1,\dots,a_n^{k_n}))$:

$$((a_1^1 \otimes \dots \otimes a_1^{k_1}) \otimes \dots \otimes (a_n^1,\dots,a_n^{k_n})) \to$$

$$(a_1^1 \otimes \dots \otimes a_n^{k_n}) \text{ is lax.}$$

(unbiased, strong)

$\iota_a: \quad a \to (a)$ is lax.

Strict cover: $(P, \pi)$ $St(A) \to A$. $P$ full, faithful, essentially surjective

Cartesian = Preserve pullback.

$\mathcal{E}(T)$
$$\begin{tikzcd} & M \arrow[dl, "d"'] \arrow[dr, "c"] & \\ TE & & E' \end{tikzcd}$$
for 1 cell.

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] \arrow[d] & \\ TE & N \arrow[l] \arrow[r] & E' \end{tikzcd}$$
for 2 cell.

$$\begin{tikzcd} & & M' \circ M \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\ & TM \arrow[dl, "Td"'] \arrow[dr, "Tc"] & & M' \arrow[dl, "d'"'] \arrow[dr, "c'"] & \\ T^2E \arrow[d, "\mu_E"'] & & TE' & & E'' \\ TE & & & & \end{tikzcd}$$
for 1-comp.

$$\begin{tikzcd} & E \arrow[dl, "\eta_E"'] \arrow[dr, "1"] & \\ TE & & E \end{tikzcd}$$
for 1-id.

T-multicat is a monad in bicat. $\mathcal{E}(T)$.

$$\begin{tikzcd} & C_1 \arrow[dl, "dom"'] \arrow[dr, "cod"] & \\ TC_0 & & C_0 \end{tikzcd}$$

$C_1 \circ C_1 = C_1 \times_{TC_0} TC_1 \xrightarrow{comp} C_1$

$C_0 \xrightarrow{ids} C_1$

T-Operad. T-graph. (similarly defined).

fc-multicat.
$$\begin{tikzcd} a_0 \arrow[r, "m_1"] \arrow[d, "f"] & a_1 \arrow[r, "m_2"] \arrow[d, phantom, "\Downarrow\theta"] & \cdots \arrow[r, "m_n"] & a_n \arrow[d, "f'"] \\ a \arrow[rrr, "m"] & & & a' \end{tikzcd}$$
+ comp.

+ id.

plain-multi : $f, f' = \mathrm{id}$.

$$
\begin{tikzcd}
\bullet \arrow[r, "m_1"] \arrow[rrrr, bend right, "m"'] & \bullet \arrow[r, "m_2"] & \bullet \arrow[r, "m_3"] & \cdots \arrow[r, "m_n"] & \bullet
\end{tikzcd}
$$

week double cat.

$$
D_1^{(n)} = \lim
\begin{tikzcd}[column sep=small]
 & D_1 \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & & D_1 \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & & D_1 \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
D_0 & & D_0 & & \cdots & & D_0
\end{tikzcd}
$$

$\mathrm{comp}^{(n)}$:

$$
\begin{tikzcd}
 & D_1^{(n)} \arrow[dl] \arrow[dr] \arrow[d, "\mathrm{comp}^{(n)}"] & \\
D_0 & D_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & D_0
\end{tikzcd}
$$

T-Multicat'

$$
\begin{tikzcd}
\mathcal{E}/E \arrow[r, "S"] \arrow[d, "e_!"'] \arrow[dr, phantom, "\Swarrow\phi"] & \mathcal{E}/E \arrow[d, "e_!"] \\
\mathcal{E}/\tilde{E} \arrow[r, "S'"'] & \mathcal{E}/\tilde{E}
\end{tikzcd}
\quad \text{(colax).}
$$

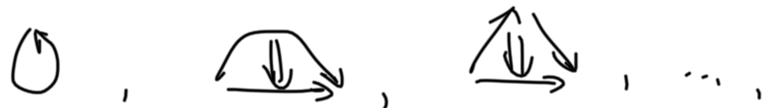
Opetope $\mathcal{E}_0 = \mathrm{Set}$ $T_0 = \mathrm{id}$. $\mathcal{E}_{n+1} = \mathcal{E}_n / T_n 1$

$T_{n+1}$ = (free $T_n$-operad).

| $n$ | $O_n$ | $\mathcal{E}_n$ | $T_n$ |
|---|---|---|---|
| 0 | 1 | Set | id. |
| 1 | 1 | Set | free monoid |
| 2 | $\mathbb{N}$ | Set/$\mathbb{N}$ | free plain operad. |
| 3 | {trees} | Set/{trees} | |

$O_0$ $\bullet$

$O_1$ $\longrightarrow$

$O_2$ $\{$  $\}$ (1-pasting diagram)

(0) (1) (2)

$O_3$ $\{ \cdots ,$  $, \cdots \}$ (2-pd).

Strict tree $\quad StTr(1)$ has "|".

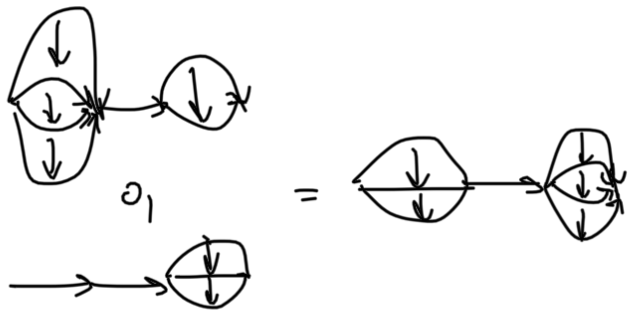
$StTr(k_1) + StTr(k_2) + \dots + StTr(k_n) \to StTr(k_1 + \dots + k_n)$

$T_1 \quad T_2 \quad T_n \qquad \mapsto (T_1, \dots, T_n)$

$(StTr(0) = \emptyset. \quad StTr(2) = \{ Y \}.$

opetopic set: $\mathbb{O} \to Set.$

Globular pd.

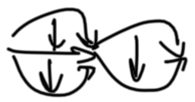


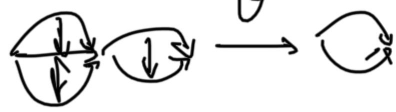
$pd(2) \times_{pd(1)} pd(2) \to pd(2)$

$pd(3) \times_{pd(0)} pd(3) \to pd(3).$

Globular operad

$$\begin{tikzcd} & P \arrow[dl, "d"'] \arrow[dr, "!"] & \\ pd = T1 & & 1 \end{tikzcd}$$

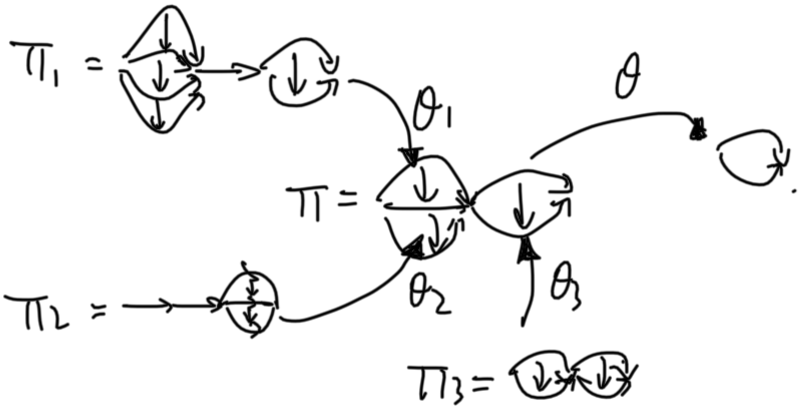
$\pi =$ 

$\theta \in P(\pi)$: 

$$P(\pi) \overset{s}{\underset{t}{\rightrightarrows}} P(\partial\pi).$$

Comp.

$$\begin{tikzcd} & & P \circ P \arrow[dl] \arrow[dr] & \\ & TP \arrow[dl] \arrow[dr, "T!"] & & P \arrow[dl, "d"] \\ \mu, Tpd \arrow[d] & & pd & \\ pd & & & \end{tikzcd}$$




weak n-cat. parallel: $\alpha^-, \alpha^+ \in X(m)$. $m=0$ or $m \geq 1$, $s(\alpha^-) = s(\alpha^+)$, $t(\alpha^-) = t(\alpha^+)$.

Contraction. $\left( Parq(\phi) \xrightarrow{K\phi} X(m) \right)_{m \geq 1}$

$\phi \in Y(m)$

$$\begin{tikzcd} \xi \arrow[r, bend left, "\theta^-"] \arrow[r, bend right, "\theta^+"'] & \xi' \end{tikzcd} \xmapsto{\quad q \quad} \begin{tikzcd} q(\xi) \arrow[r, bend left, "q(\theta^-)"] \arrow[r, bend right, "q(\theta^+)"'] \arrow[r, phantom, "\Downarrow\phi"] & q(\xi') \end{tikzcd}$$

operads with contraction $(P, K)$. $P$ globular operad.

$K$ contraction. $(P, K) \xrightarrow{f} (P', K')$ preserve contraction

$$f(K_\pi(\theta^-, \theta^+)) = K'_\pi(f\theta^-, f\theta^+). \ \forall m \geq 1. \ \pi \in pd(m). \ (\theta^-, \theta^+) \in Parp(\pi)$$

OC has an initial obj. $(L, \lambda)$.

A weak $\omega$-cat is an $L$-alg.