Linear Control I

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September 22, 2021

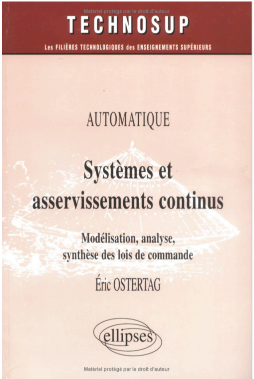
Objective

- Presentation of elementary tools for modeling, analysis and control of Linear Time-Invariant (LTI) systems.

- Modeling and analysis of LTI systems in the time and frequency domains.

- Synthesis of output feedback controllers for dynamic LTI systems with specifications in term of stability, bandwidth and accuracy.

Bibliography



Eric Ostertag, "Systèmes et asservissements continus : Modélisation, analyse, synthèse des lois de commande". ELLIPSES, ISBN-10 : 2-7298-2013-2

Content:

Chapter I: Analysis of linear time-invariant systems in the time domain

- Definitions and properties of systems
- Laplace Transform (LT)
- Transfer function, poles and zeros
- Temporal responses.

- **Chapter II**: Analysis of linear time-invariant systems in the frequency domain

- Bode diagram: reminders and definitions
- Bode diagram of elementary functions of first and second order
- Nyquist diagram

- **Chapter III**: Output feedback control of linear systems

- Introduction
- Regulation and control
- Modeling of a closed loop system
- Stability of a closed loop system
- Stability margins
- Precision and disturbance rejection
- Correction structures

Chapter IV: Feedforward Control

- Wy Feedforward?
- Disturbance feedforward
- Exemple d'application

Chapter V: Dynamic output feedback control

- Closed loop performance and specifications
- Proportional control
- Proportional Integral (PI) control
    - Ideal PI control
    - Approximate PI control
- Lead and Lag Compensators
- Design of a dynamic output feedback control based on stability, accuracy and bandwidth specifications

Chapter V: Multi-loop control

- Principle and advantages of Multi-loop control
- Tachometric feedback

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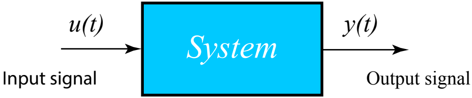
Chapter I: Analysis of linear time-invariant systems in the time domain



A system is any process or entity that has one or more well-defined inputs and one or more well-defined outputs. A dynamic system is any system whose evolution over time is governed by well identified and generally deterministic laws.

Representation by an ordinary differential equation (ODE)

$$f\left(y, \frac{dy(t)}{dt}, ..., \frac{d^n y(t)}{dt^n}, ..., u, \frac{du(t)}{dt}, ..., \frac{d^m u(t)}{dt^m}, t\right) = a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}}$$
$$+ ... + a_1 \frac{dy(t)}{dt} + a_0 y - b_m \frac{d^m u(t)}{dt^m} - b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} - ... - b_1 \frac{du(t)}{dt} - b_0 u(t) = 0$$

The coefficients a_j, b_j are constant.

It is possible to deduce the following differential equation:

$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + ... + a_1 \frac{dy(t)}{dt} + a_0 y = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}}$
$+ ... + b_1 \frac{du(t)}{dt} - b_0 u(t)$

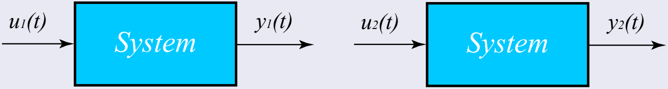
Basic example: Newton's equation (fundamental principle of dynamics):

$$m \frac{d^2 y}{dt^2} + c \frac{dy(t)}{dt} + ky = u$$

Linear Time-Invariant (LTI) systems

Linear Systems

Suppose



The system is linear if and only if:

Example of a linear system:

- System $u(t) \to y(t) : y(t) = 2u(t)$;
- System $u(t) \to y(t) : \frac{d^2y(t)}{dt^2} + ky(t) = u(y)$.

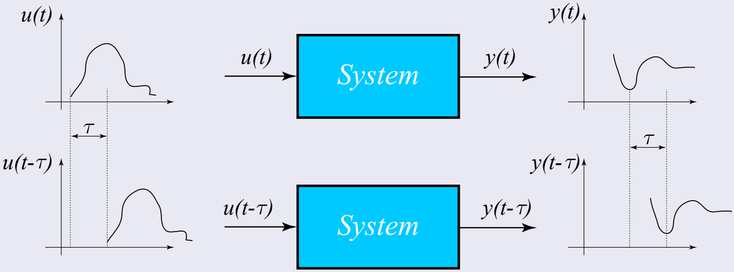
Example of a non-linear system:

- System $u(t) \to y(t) : y(t) = sin(u(t))$.

Linear Time-Invariant (LTI) systems

Time-Invariant Systems

For a time-invariant system, the relationship between the input and the output does not change when shifted in time.



$$u(t) \to y(t) \Rightarrow u(t-\tau) \to y(t-\tau)$$

Basic properties LTI systems

Output description

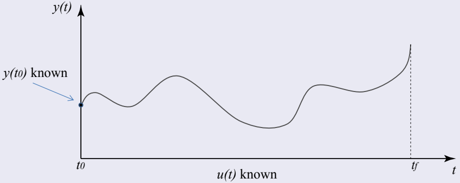
The output $y(t)$ over a time interval $[t_0, t_f]$ is completely defined if:

– the initial conditions $y(t_0)$, $\frac{dy(t)}{dt}(t_0), \ldots, \frac{d^{n-1}y(t)}{dt^{n-1}}(t_0)$;

and

– the input $u(t)$ on the time interval $[t_0, t_f]$.

are known.

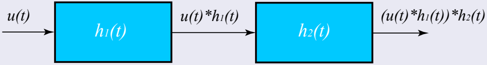


Superposition principle

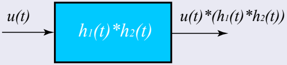
$$\left.\begin{array}{l} u_1 \to y_1 \\ u_2 \to y_2 \end{array}\right\} \Rightarrow \lambda_1 u_1 + \lambda_2 u_2 \to \lambda_1 y_1 + \lambda_2 y_2$$

Basic properties LTI systems

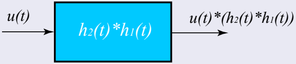
LTI systems can be cascaded in any order



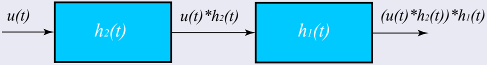
Associative:



Commutative:



Associative:



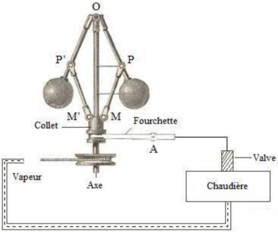
$h_i(t)$: impulse response.

System control

Purpose of control:

Ensure that the output signal resembles the user's desired signal, despite system dynamics and disturbances.

Example of a historical system driven by a feedback control



(a)

(b)

Figure 1: Simplified scheme of Watt's regulator (a) and an example of the regulator in the Georgetown PowerPlant Museum in Seattle (b).

Definition and properties

Example of modern systems driven by a feedback control

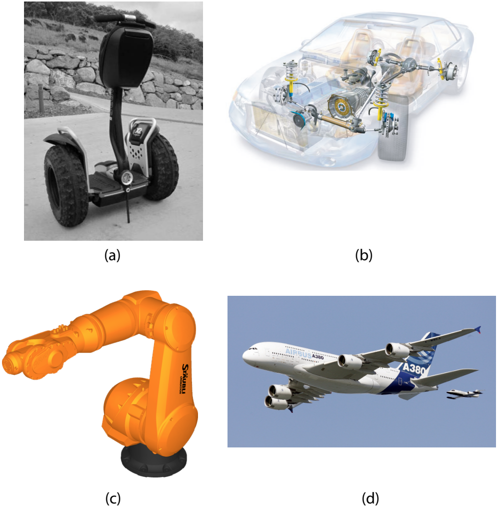


Figure 2: (a) Segway, (b) car suspension, (c) robot manipulator, (d) aircraft

System control

Feedback vs Feedforward:

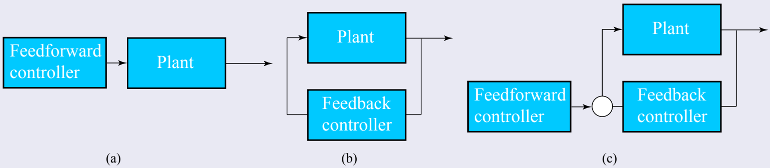


Figure 3: (a) Feedforward, (b) feedback and (c) combination of feedback and feedforward control schemes

- Feedback control is less sensitive to modeling error
- The plant model is needed for feedforward control and for model-based design of feedback control

Laplace Transform (LT)

Definition:

$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt$

$s = \sigma + j\omega$: Laplace variable

Properties:

Linearity	$af(t) + bg(t)$	$aF(s) + bG(s))$
Convolution	$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
Integration	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
Derivation	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ...$
Shift in time	$f(t-a)$	$F(s)e^{-as}$
Frequency shift	$e^{at}f(t)$	$F(s-a)$

Laplace transform of usual signals:

Impulse

$\delta(t)$	1
$\delta(t-T)$	e^{-Ts}
$\sum_{n=0}^{\infty} \delta(t-nT)$	$\frac{1}{1-e^{-Ts}}$

Step

$\delta(t)$	1
$\delta(t-T)$	e^{-Ts}
$\sum_{n=0}^{\infty} \delta(t-nT)$	$\frac{1}{1-e^{-Ts}}$

Ramp

t	$\frac{1}{p^2}$
t^m	$\frac{m!}{s^{m+1}}$

Exponential

e^{-at}	$\frac{1}{p+a}$
te^{-at}	$\frac{1}{(p+a)^2}$
$\frac{1}{(m-1)!} t^{m-1} e^{-at}$	$\frac{1}{(p+a)^m}$

Trigonometric functions

$sin(at)$	$\frac{a}{s^2+a^2}$
$cos(at)$	$\frac{s}{s^2+a^2}$

Initial value theorem:

$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$

Final value theorem:

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

From the differential equation in the time domain to the transfer function in the frequency domain

Consider a LTI system governed by the following differential equation in the time domain

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + ... + a_1 \frac{dy(t)}{dt} + a_0 y = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}}$$
$$+ ... + b_1 \frac{du(t)}{dt} - b_0 u(t)$$

By applying the Laplace Transform (LT) to the equation, the system is described in the frequency domain:

$$\mathrm{LT}[a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + ... + a_1 \frac{dy(t)}{dt} + a_0 y] = \mathrm{LT}[b_m \frac{d^m u(t)}{dt^m}$$
$$+ b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + ... + b_1 \frac{du(t)}{dt} - b_0 u(t)]$$

Let consider the following initial conditions:

$$\frac{d^m u(t)}{dt^m}(0) = 0,\ \frac{d^{m-1} u(t)}{dt^{m-1}}(0) = 0,\ ...,\ \frac{du(t)}{dt}(0) = 0,\ u(0) = 0.$$

From the differential equation in the time domain to the transfer function in the frequency domain

Therefore, the equation in the frequency domain becomes:

$$a_n[s^nY(s)-s^{n-1}y(0)-s^{n-2}\frac{dy(0)}{dt}-...]+a_{n-1}[s^{n-1}Y(s)-s^{n-2}y(0)-s^{n-3}\frac{dy(0)}{dt}-...]+...+a_1[sY(s)-y(0)]+a_0Y(s)=[b_ms^m+b_{m-1}s^{m-1}+...+b_1s+b_0]U(s)$$

Where: $Y(s)=$LT$[y(t)]$ and $U(s)=$LT$[u(t)]$

One can write

$$[a_ns^n+a_{n-1}s^{n-1}+...+a_1s+a_0]Y(s)-IC(s)=[b_ms^m+b_{m-1}s^{m-1}+...+b_1s+b_0]U(s)$$

$IC(s)$ is a polynomial function of the output initial conditions.

Therefore:

$$Y(s)=Y_1(s)+Y_2(s)$$

$Y_1(s)=\frac{b_ms^m+b_{m-1}s^{m-1}+...+b_1s+b_0}{a_ns^n+a_{n-1}s^{n-1}+...+a_1s+a_0}U(s)$ is the forced response with $IC(s)=0$.

$Y_2(s)=\frac{IC(s)}{a_ns^n+a_{n-1}s^{n-1}+...+a_1s+a_0}$ is the free response with $U(s)=0$.

Transfer function

Under the consideration that $IC(s) = 0$, one can define the dynamic transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

Some definitions:

System order: n

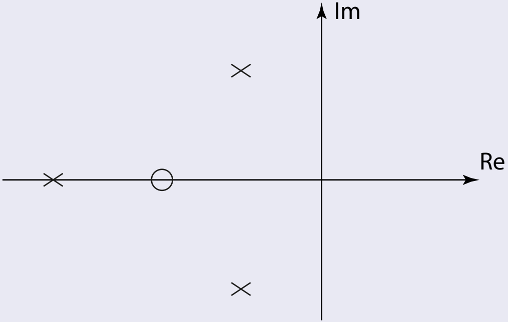
Static gain: $G(0) = \frac{b_0}{a_0}$

Pole-Zero map

$G(s) = \frac{Y(s)}{U(s)} = K\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2))\ldots(s-p_n)}$

z_i $(1 \le i \le m)$: zeros of the transfer function

p_i $(1 \le i \le n)$: poles of the transfer function



Chapter II: Analysis of linear time-invariant systems in the frequency domain

A Bode diagram is a frequency representation of a system. It allows describing the evolution of the gain and the phase of the transfer function with respect to the angular frequency ω or the frequency f.

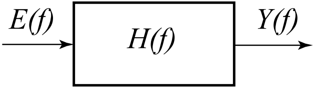


Figure 4: Système d'entrée $E(f)$ et de sortie $Y(f)$ ayant une fonction de transfert $H(f)$.

For a system with an input $e(t) = A\sin(\omega_0(t))$, the corresponding output is:

$$y(t) = A\underbrace{|H(f_0)|}_{gain}\sin\left(\omega_0 t + \underbrace{\phi_H(f_0)}_{phase}\right)$$

The transfer function can be defined as a function of ω or f by the notation $H(\omega)$ or $H(f)$ respectively.

Definitions

The gain $|H(\omega)|_{db}$ is represented in decibels.

$$|H(\omega)|_{db} = 20\log_{10}(|H(\omega)|)$$

The phase $phi[H(\omega)]$ is represented in degree or radian.

$$\phi[H(\omega)] = arg(H(\omega))$$

The horizontal axis of the Bode diagram is a logarithmic axis.

→ The intervalle $[\omega \;\; 10\omega]$ is called a decade.

→ Negative frequencies are not represented in the Bode diagram.

Gain and phase properties

$$H(\omega) = H_1(\omega).H_2(\omega)...H_n(\omega) \Rightarrow |H(\omega)| = |H_1(\omega)|.|H_2(\omega)|...|H_n(\omega)|$$

$$\begin{cases} |H(\omega)|_{db} = |H_1(\omega)|_{db} + |H_2(\omega)|_{db} + ... + |H_n(\omega)|_{db} \\ \phi[H(\omega)] = \phi[H_1(\omega)] + \phi[H_2(\omega)] + ... + \phi[H_n(\omega)] \end{cases}$$

Example

If a function $H(s)$ can be factored as a product in the form:

$$H(s) = \frac{(s-z_1).(s-z_2)...(s-z_m)}{(s-p_1).(s-p_2)...(s-p_n)}$$

With, z_i: zeros of $H(s)$ $(1 \leq i \leq m)$ and p_j: poles of $H(s)$ $(1 \leq j \leq n)$ where: $n \geq m$.

$$H(s) = (s - z_1)(s - z_2)...(s - z_m).\frac{1}{(s-p_1)}.\frac{1}{(s-p_2)}\cdots\frac{1}{(s-p_n)}.$$

Then, the gain and phase Bode diagram of $H(s)$ can be constructed by summing the Bode diagrams of the elementary functions.