Geometric Camera Models

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Objective

“ This chapter introduces the analytical machinery necessary to establish quantitative constraints between image measurements and the position and orientation of geometric figures measured in some arbitrary external coordinate system.”

Forsyth

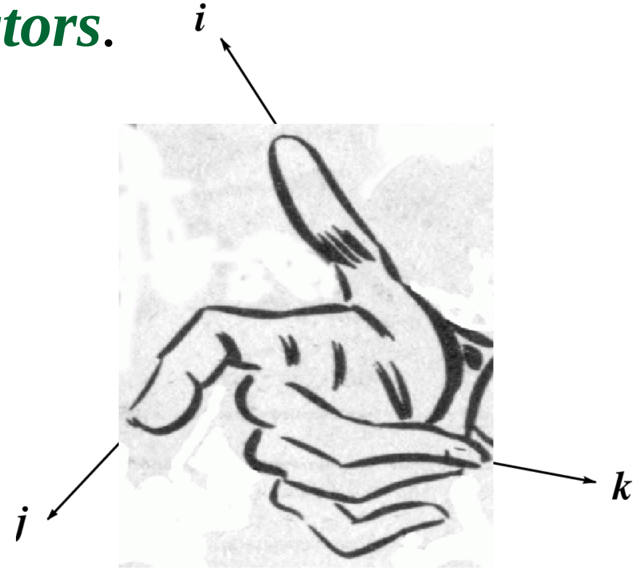
Content:

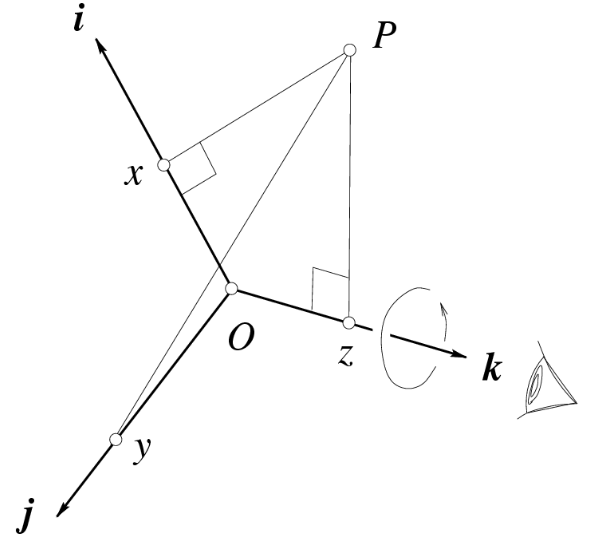
- Elements of Analytical Euclidean Geometry
- Camera Parameters and the Perspective Projection
- Affine Cameras and Affine Projection Equations

Coordinate Systems

An ***orthonormal coordinate frame*** (F) is defined by

- a point O in the Euclidean space $\mathbb{E}^3$ called ***origin***, and
- three unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ orthogonal to each other, called ***base vectors***.



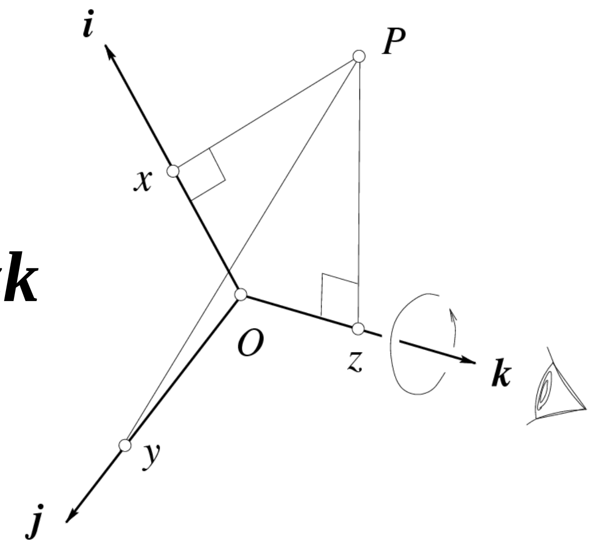


right-handed coordinate system

Coordinate Systems

The ***coordinates*** x,y,z ***of a point*** P in the coordinate frame F is the signed lengths of the orthogonal projections of vector $\overrightarrow{OP}$ onto the vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

$$\begin{cases} x = \overrightarrow{OP} \cdot \boldsymbol{i} \\ y = \overrightarrow{OP} \cdot \boldsymbol{j} \\ z = \overrightarrow{OP} \cdot \boldsymbol{k} \end{cases} \Leftrightarrow \quad \overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$



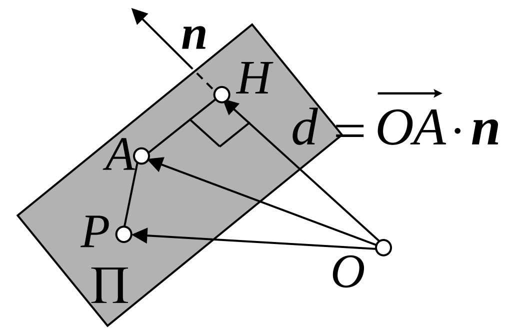
Coordinate Systems

- $\boldsymbol{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ is the ***coordinate vector of the point*** P in the frame (F).

The ***coordinate vector associated with a free vector*** $\boldsymbol{v}$ are the lengths of its projections onto the basis vector of (F), which is are (clearly) independent on the choice of the origin O.

Homogeneous Coordinates

Let Π be a plane, A a point in Π and $\boldsymbol{n}$ a perpendicular unitary vector to Π. Points lying on Π are characterized by



$$\overrightarrow{AP}\cdot\boldsymbol{n}=0$$

If $\mathbf{P}=(x,y,z)^T$ e $\boldsymbol{n}=(a,b,c)^T$, then

$$\overrightarrow{OP}\cdot\boldsymbol{n}-\overrightarrow{OA}\cdot\boldsymbol{n}=0 \quad \text{or} \quad ax+by+cz-d=0.$$

This can be rewritten as

$$\Pi\cdot\boldsymbol{P}=0, \quad \text{where}$$

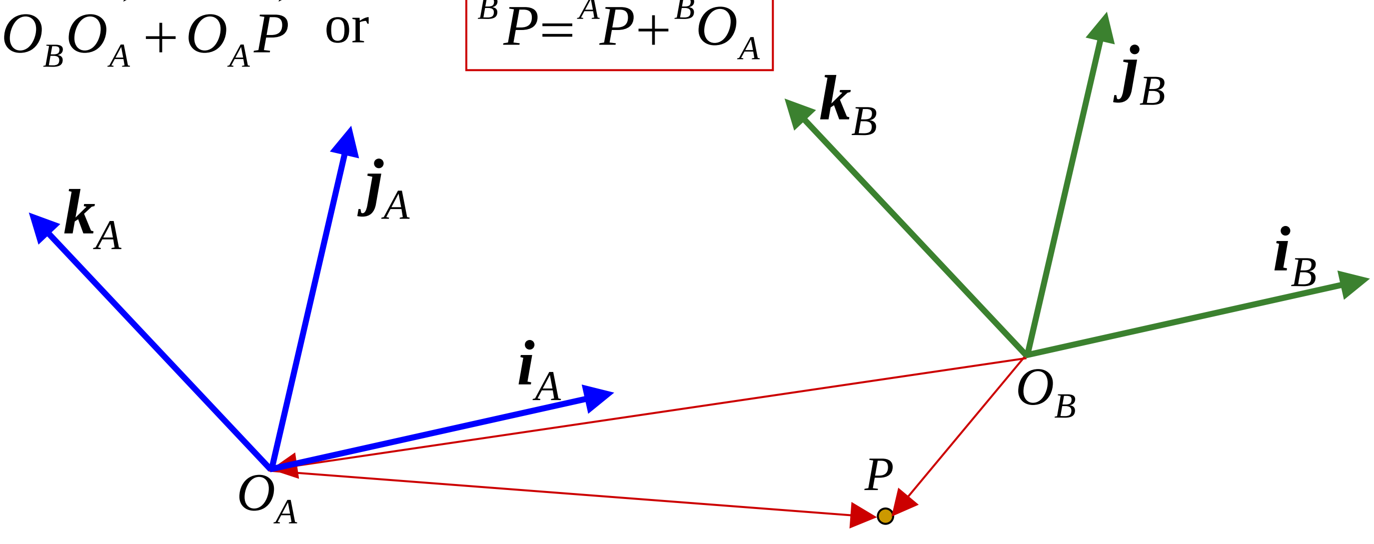
$\Pi=\begin{pmatrix} a \\ b \\ c \\ -d \end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ are the ***homogeneous coordinates*** of Π and P in (F).

Rigid Transformations
Pure Translation

Let ${}^{A}P$ and ${}^{B}P$ be the coordinate vectors of the point P respectively in coordinate systems $A=(O_A,\boldsymbol{i}_A,\boldsymbol{j}_A,\boldsymbol{k}_A)$ and $B=(O_B,\boldsymbol{i}_B,\boldsymbol{j}_B,\boldsymbol{k}_B)$.

If the base vectors are parallel to each other, both coordinate systems are separated by a ***pure translation***, and we have

$$\overrightarrow{O_B P} = \overrightarrow{O_B O_A} + \overrightarrow{O_A P} \quad \text{or} \quad \boxed{{}^{B}P = {}^{A}P + {}^{B}O_A}$$



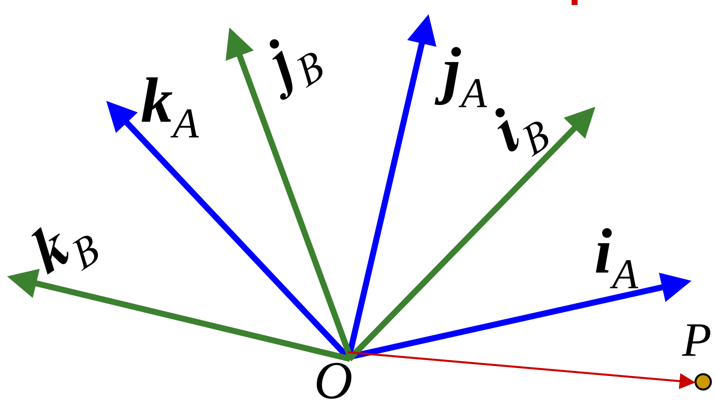
Rigid Transformations
Pure Rotation

If $O_A = O_B = O$ the coordinate systems are separated by a ***pure rotation***

projections of the A base onto the B base

$$ {}^{B}P = {}^{B}_{A}\mathcal{R}\,{}^{A}P \text{ where } {}^{B}_{A}\mathcal{R} = \begin{pmatrix} \boldsymbol{i}_A\boldsymbol{i}_B & \boldsymbol{j}_A\boldsymbol{i}_B & \boldsymbol{k}_A\boldsymbol{i}_B \\ \boldsymbol{i}_A\boldsymbol{j}_B & \boldsymbol{j}_A\boldsymbol{j}_B & \boldsymbol{k}_A\boldsymbol{j}_B \\ \boldsymbol{i}_A\boldsymbol{k}_B & \boldsymbol{j}_A\boldsymbol{k}_B & \boldsymbol{k}_A\boldsymbol{k}_B \end{pmatrix} = \begin{pmatrix} {}^{B}\boldsymbol{i}_A & {}^{B}\boldsymbol{j}_A & {}^{B}\boldsymbol{k}_A \end{pmatrix} = \begin{pmatrix} {}^{A}\boldsymbol{i}_B^T \\ {}^{A}\boldsymbol{j}_B^T \\ {}^{A}\boldsymbol{k}_B^T \end{pmatrix} $$

projections of the B base onto the A base



Clearly $\quad {}^{A}_{B}\mathcal{R} = {}^{B}_{A}\mathcal{R}^T$

Rigid Transformations
Pure Rotation

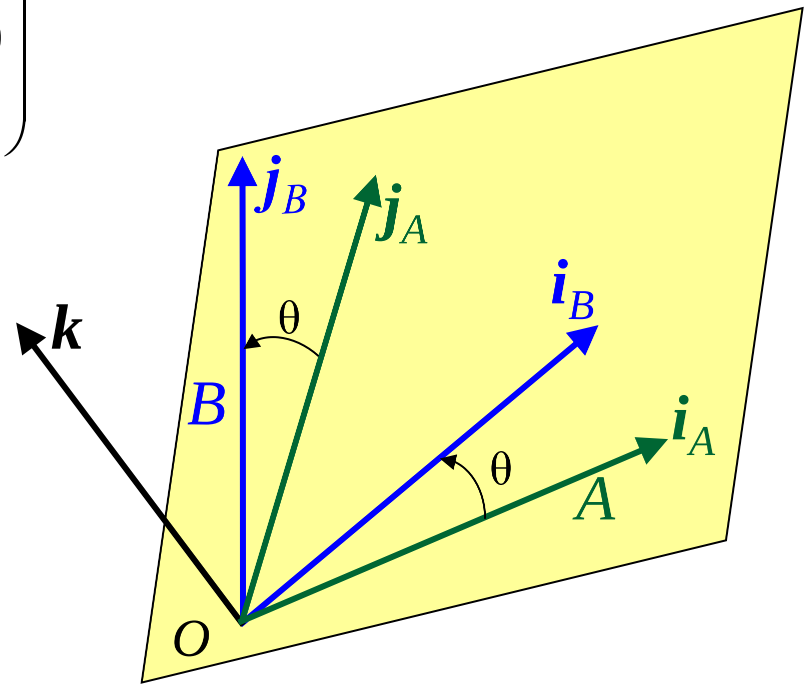
Let $\boldsymbol{k}_A=\boldsymbol{k}_B=\boldsymbol{k}$ and θ_k the rotation angle

around $\boldsymbol{k}$ $\quad {}^B_A\mathcal{R}(\theta_k)=\begin{pmatrix}\cos\theta_k & \sin\theta_k & 0\\ -\sin\theta_k & \cos\theta_k & 0\\ 0 & 0 & 1\end{pmatrix}$

Similarly rotation

around $\boldsymbol{j}$ $\quad {}^B_A\mathcal{R}(\theta_j)=\begin{pmatrix}\cos\theta_j & 0 & \sin\theta_j\\ 0 & 1 & 0\\ -\sin\theta_j & 0 & \cos\theta_j\end{pmatrix}$

around $\boldsymbol{i}$ $\quad {}^B_A\mathcal{R}(\theta_i)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_i & \sin\theta_i\\ 0 & -\sin\theta_i & \cos\theta_i\end{pmatrix}$



Rigid Transformations
Properties of Rotation

1. Rotation matrices admit an inverse
$$\mathcal{R}^{-1}=\mathcal{R}^{T}$$
2. Determinant is equal to 1
$$||\mathcal{R}|| = 1$$
3. Rows and columns form an orthonormal base
4. Rotation matrices equipped with the matrix product form a *group*, that is, if $\mathcal{R}$, $\mathcal{R}_1$ an $\mathcal{R}_2$ are rotation matrices, then
   - The product of two rotation matrices ($\mathcal{R}_1\,\mathcal{R}_2$) is also a rotation matrix
   - Matrix product is associative $(\mathcal{R}_1\,\mathcal{R}_2)\,\mathcal{R}_3 = \mathcal{R}_1\,(\mathcal{R}_2\,\mathcal{R}_3)$
   - there is an identity matrix Id such that $\mathcal{R}\,\mathrm{Id} = \mathrm{Id}\,\mathcal{R} = \mathcal{R}$
   - $\mathcal{R}^{-1}=\mathcal{R}^{T}$, such that $\mathcal{R}^{-1}\mathcal{R}= \mathcal{R}\mathcal{R}^{-1}=\mathrm{Id}$

Rigid Transformations

- Frames with distinct origins and bases are separated by a rigid transformation if

$$ {}^{B}P = {}^{B}_{A}\mathcal{R}\,{}^{A}P + {}^{B}O_A \quad \text{alternatively,} \quad \begin{pmatrix} {}^{B}P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{B}_{A}\mathcal{R} & {}^{B}O_A \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^{A}P \\ 1 \end{pmatrix} $$

  So, if ${}^{A}P$ and ${}^{B}P$ are in homogeneous coordinates :

$$ {}^{B}P = {}^{B}_{A}\mathcal{T}\;{}^{A}P \qquad \text{where} \qquad {}^{B}_{A}\mathcal{T} = \begin{pmatrix} {}^{B}_{A}\mathcal{R} & {}^{B}O_A \\ \mathbf{0}^T & 1 \end{pmatrix} $$

- Rigid transformations map a coordinate system onto another one

$$ {}^{F}P' = \mathcal{R}\,{}^{F}P + \boldsymbol{t} \Leftrightarrow \begin{pmatrix} {}^{F}P' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{R} & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^{F}P \\ 1 \end{pmatrix} $$

  where $\mathcal{R}$ is a rotation matrix and $\boldsymbol{t}$ is an element of $\mathbb{R}^{3.}$

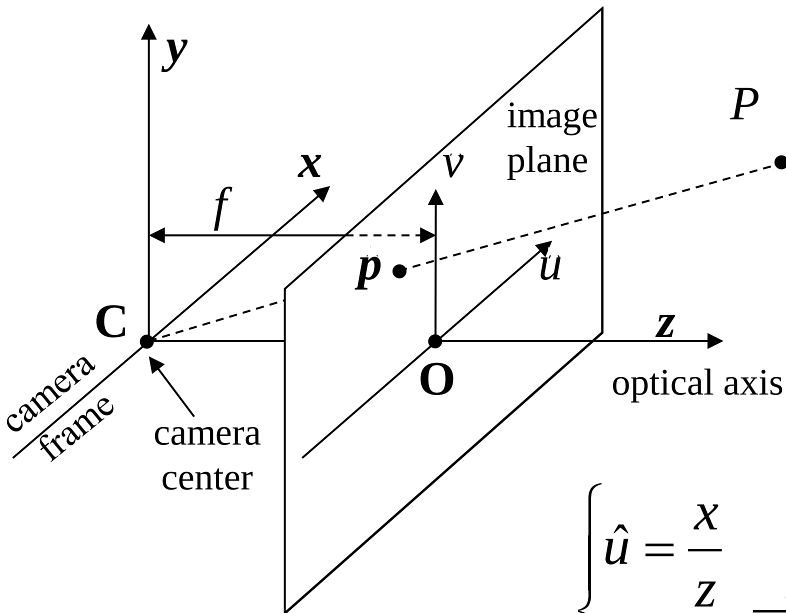
- Rigid transformations preserve distances and angles.

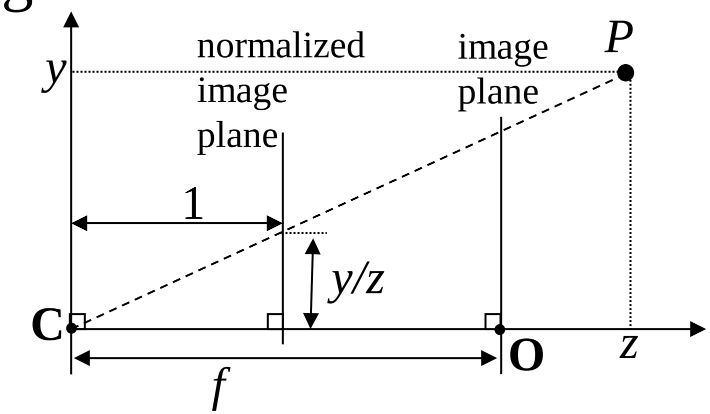
Content:

- Elements of Analytical Euclidean Geometry
- Camera Parameters and the Perspective Projection
- Affine Cameras and Affine Projection Equations

Intrinsic Camera Parameters

Let ${}^{C}P = (x,y,z,1)^T$ be the homogeneous coordinates of a point P in the <u>camera frame</u> and $\hat{p} = (\hat{u},\hat{v},1)$ the coordinates of its projection onto the normalized image frame.

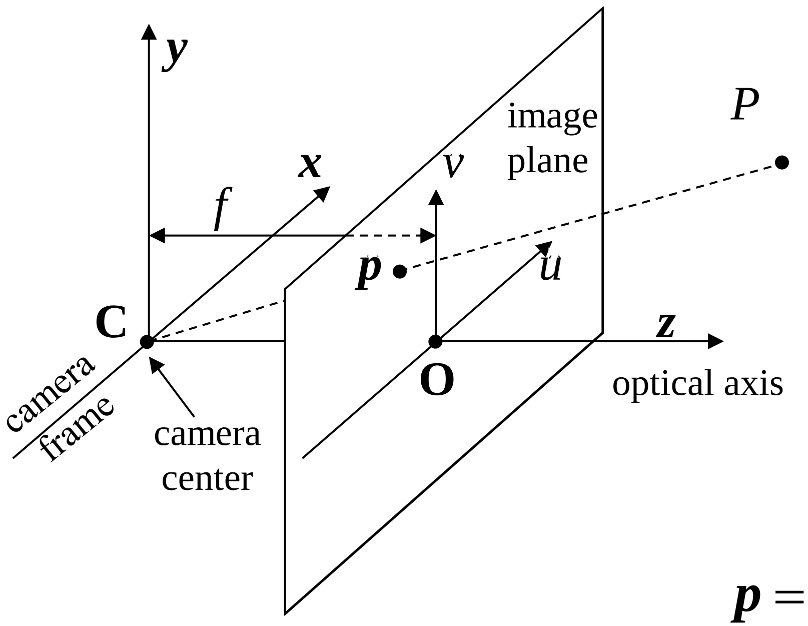


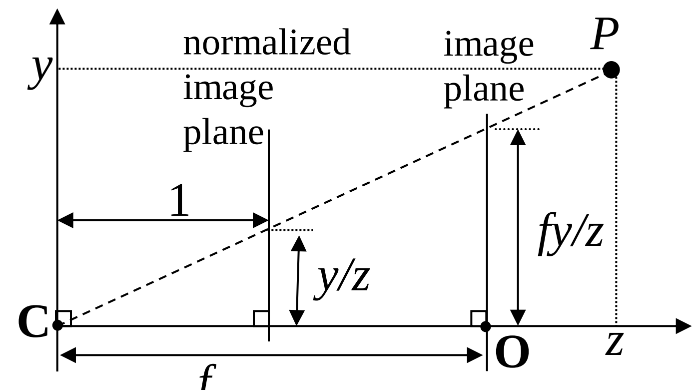


$$\begin{cases} \hat{u} = \dfrac{x}{z} \\[2ex] \hat{v} = \dfrac{y}{z} \end{cases} \rightarrow \hat{\mathbf{p}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix} = \frac{1}{z}\left(\mathrm{I} \,|\, \mathbf{0}\right) \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \frac{1}{z}\left(\mathrm{I} \,|\, \mathbf{0}\right) {}^{C}P$$

Intrinsic Camera Parameters

The coordinates $\boldsymbol{p}=(u,v,1)$ of the projection of point P in the image plane is





$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} f & & \\ & f & \\ & & 1 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix} = \begin{pmatrix} f & & \\ & f & \\ & & 1 \end{pmatrix} \hat{\boldsymbol{p}}$$

Intrinsic Camera Parameters

For $p=(u,v,1)$ to be expressed in (*row*,*column*) coordinates, we have to consider the horizontal and vertical pixel dimensions, say $1/k$ and $1/l$ be, (in meter, for instance)

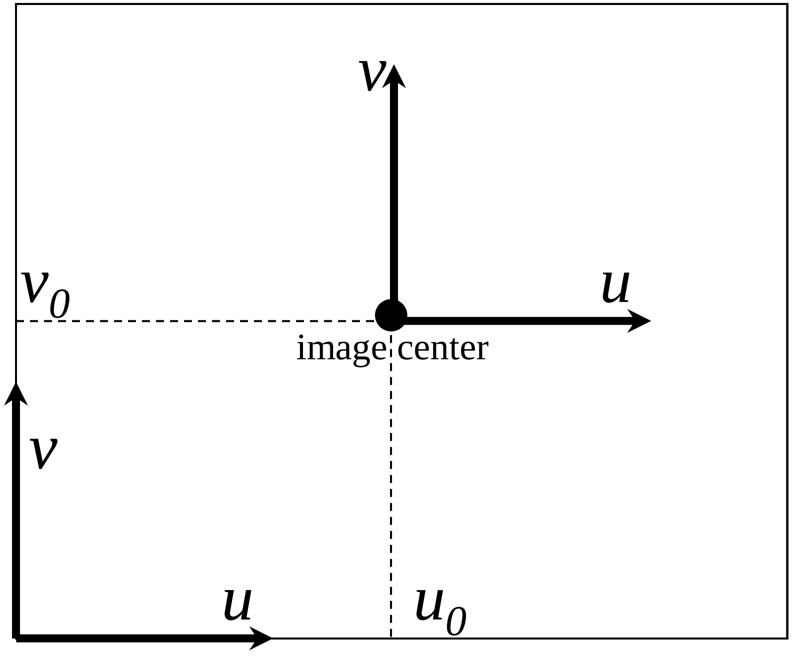
$$\begin{cases} u = kf\ \hat{u} \\ v = lf\ \hat{v} \end{cases}$$

$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & 1 \end{pmatrix} \hat{\boldsymbol{p}}$$

where $\alpha = kf$ and $\beta = lf$.

Intrinsic Camera Parameters

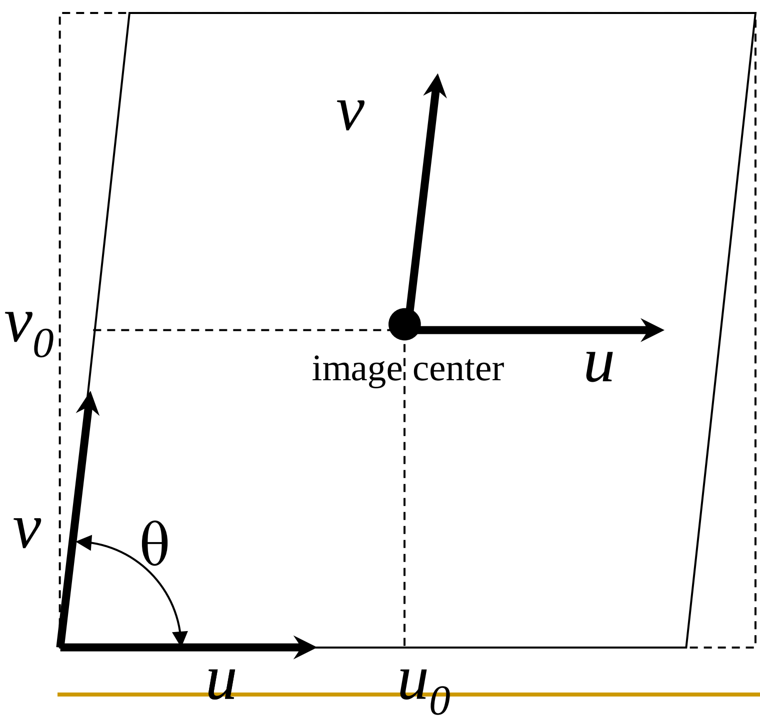
If the origin of the camera coordinate system is not at the center of the retina but at (u_0, v_0) .



$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & & u_0 \\ & \beta & v_0 \\ & & 1 \end{pmatrix} \hat{\boldsymbol{p}}$$

Intrinsic Camera Parameters

If the sensor matrix is skewed by an angle θ different from 90°



$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ & \beta/\sin\theta & v_0 \\ & & 1 \end{pmatrix}}}_{\mathcal{K}} \hat{\boldsymbol{p}}$$

camera calibration matrix

Intrinsic Camera Parameters

The calibration matrix or the matrix of intrinsic camera parameters

$$\boxed{\boldsymbol{p} = \mathcal{K}\hat{\mathbf{p}} = \frac{1}{z}\left(\mathcal{K}\ \mathbf{0}\right)\,{}^{C}\boldsymbol{P}}$$

where

$$\boxed{\mathcal{K} = \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \beta/\sin\theta & v_0 \\ 0 & 0 & 1 \end{pmatrix}}$$

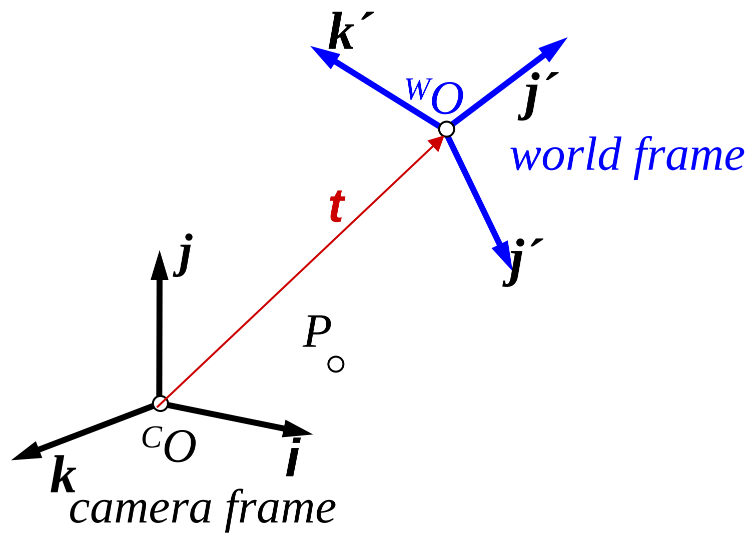
There are 5 intrinsic parameters:

- the magnifications α and β (in pixels/m) ,
  or alternatively focus length f (in pixels) and τ aspect ratio,
- the skew angle θ
  or alternatively α cot θ, and
- the position of the principal point u_0,v_0 (in pixels) .

Extrinsic Camera Parameters

From slide 12 :

$$ {}^{C}\boldsymbol{P} = \begin{pmatrix} {}^{C}_{W}\mathcal{R} & {}^{C}O_{W} \\ \mathbf{0}^{T} & 1 \end{pmatrix} {}^{W}\boldsymbol{P} = \begin{pmatrix} {}^{C}_{W}\mathcal{R} & \boldsymbol{t} \\ \mathbf{0}^{T} & 1 \end{pmatrix} {}^{W}\boldsymbol{P} $$



There are 6 extrinsic parameters :

- 3 independent parameters of the rotation matrix ${}^{C}_{W}\mathcal{R}$ and
- 3 components of the translation vector ***t***.

Perspective Projection

Combining the results of the two previous slides:

$$\boldsymbol{p} = \frac{1}{z}\begin{pmatrix}\mathcal{K} & \mathbf{0}\end{pmatrix}\begin{pmatrix} {}^{C}_{W}\mathcal{R} & \boldsymbol{t} \\ \mathbf{0}^{T} & 1 \end{pmatrix} {}^{W}\boldsymbol{P} = \frac{1}{z}\mathcal{K}\begin{pmatrix} {}^{C}_{W}\mathcal{R} & \boldsymbol{t}\end{pmatrix} {}^{W}\boldsymbol{P}$$

$$\boxed{\mathcal{M} = \mathcal{K}\begin{pmatrix} {}^{C}_{W}\mathcal{R} & \boldsymbol{t}\end{pmatrix}}$$

is the perspective projection matrix

Thus

$$\boxed{\boldsymbol{p} = \frac{1}{z}\mathcal{M}\, {}^{W}\boldsymbol{P}}$$

Perspective Projection Matrix

Let $\boldsymbol{m}_1^T, \boldsymbol{m}_2^T, \boldsymbol{m}_3^T$ denote the rows of $\mathcal{M}$

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\,{}^W\boldsymbol{P} \Rightarrow \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \frac{1}{z}\begin{pmatrix} \boldsymbol{m}_1 \cdot {}^W\boldsymbol{P} \\ \boldsymbol{m}_2 \cdot {}^W\boldsymbol{P} \\ \boldsymbol{m}_3 \cdot {}^W\boldsymbol{P} \end{pmatrix}$$

Therefore $z = \boldsymbol{m}_3 \cdot {}^W\boldsymbol{P}$ and

$$\boxed{\begin{cases} u = \dfrac{\boldsymbol{m}_1 \cdot {}^W\boldsymbol{P}}{\boldsymbol{m}_3 \cdot {}^W\boldsymbol{P}} \\[2ex] v = \dfrac{\boldsymbol{m}_2 \cdot {}^W\boldsymbol{P}}{\boldsymbol{m}_3 \cdot {}^W\boldsymbol{P}} \end{cases}} \qquad \text{(Eq.2.16)}$$

Perspective Projection Matrix

Writing the perspective projection matrix $\mathcal{M}$ as a function of the camera parameters

$$\boxed{\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_o \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_o t_z \\ \dfrac{\beta}{\operatorname{sen}\theta} \boldsymbol{r}_2^T + v_o \boldsymbol{r}_3^T & \dfrac{\beta}{\operatorname{sen}\theta} t_y + v_o t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}} \quad \text{(Eq.2.17)}$$

where $\boldsymbol{r}_1^T$, $\boldsymbol{r}_2^T$, $\boldsymbol{r}_3^T$ are the rows of ${}^C_W\mathcal{R}$.

Characterization of the Perspective Projection Matrix

Theorem: Let $\mathcal{M}=(\mathcal{A}\ \boldsymbol{b})$ be a 3×4, matrix and let $\boldsymbol{a}_i^T$ ($i=1,2,3$) denote the rows of the matrix $\mathcal{A}$ formed by the leftmost columns of $\mathcal{M}$.

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\det(\mathcal{A})\neq0$
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\det(\mathcal{A})\neq0$ and

$$(\boldsymbol{a}_1\times \boldsymbol{a}_3)\cdot(\boldsymbol{a}_2\times \boldsymbol{a}_3)=0$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero-skew and unit aspect ratio is that $\mathrm{Det}(\mathcal{A})\neq0$ and

$$(\boldsymbol{a}_1\times \boldsymbol{a}_3)\cdot(\boldsymbol{a}_2\times \boldsymbol{a}_3)=0 \text{ e}$$
$$(\boldsymbol{a}_1\times \boldsymbol{a}_3)\cdot(\boldsymbol{a}_1\times \boldsymbol{a}_3)= (\boldsymbol{a}_2\times \boldsymbol{a}_3)\cdot(\boldsymbol{a}_2\times \boldsymbol{a}_3)$$

Camera Anatomy

1. The vector $\boldsymbol{t}$ is the origin of the camera frame $\boldsymbol{O}$ expressed in the world frame.

   In non-homogeneous coordinates

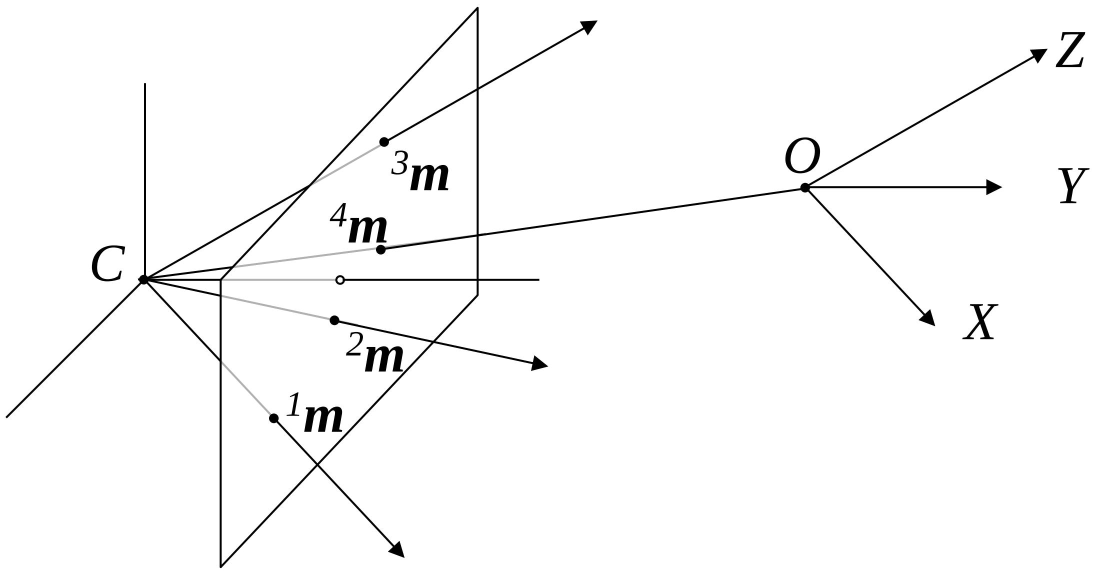
$$\mathbf{0} = \mathcal{R}\,{}^{\mathrm{W}}\boldsymbol{C} + \boldsymbol{t} \quad \rightarrow \; \boldsymbol{t} = -\mathcal{R}\,{}^{\mathrm{W}}\boldsymbol{C}$$

   Thus,

$$\mathcal{M} = \mathcal{K}\,\begin{pmatrix}\mathcal{R} & \boldsymbol{t}\end{pmatrix} = \mathcal{K}\,\begin{pmatrix}\mathcal{R} & -\mathcal{R}\,{}^{\mathrm{W}}\boldsymbol{C}\end{pmatrix}$$

Camera Anatomy

2. If ${}^{i}\boldsymbol{m}$ is the i-th column of $\mathcal{M}=({}^{1}\boldsymbol{m}\ {}^{2}\boldsymbol{m}\ {}^{3}\boldsymbol{m}\ {}^{4}\boldsymbol{m})$, then ${}^{1}\boldsymbol{m}$, ${}^{2}\boldsymbol{m}$ and ${}^{3}\boldsymbol{m}$ are the vanishing points of the world coordinate X, Y and Z axes respectively and ${}^{4}\boldsymbol{m}$ is the image of the world origin. Why?

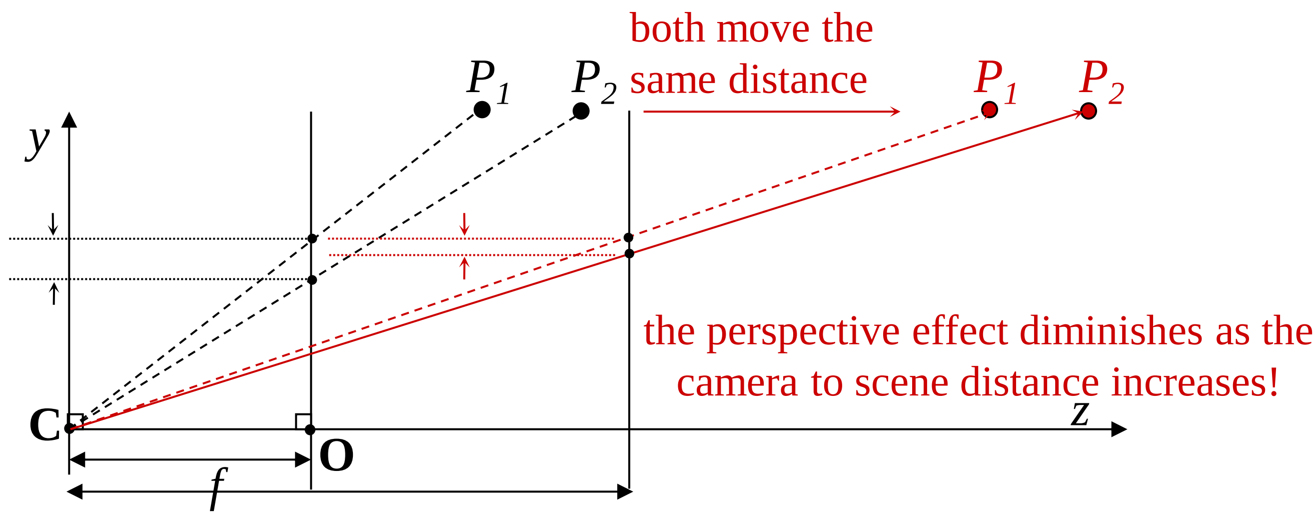


Content:

- Elements of Analytical Euclidean Geometry
- Camera Parameters and the Perspective Projection
- Affine Cameras and Affine Projection Equations

Affine Cameras

When a scene's relief is small compared with the average scene's depth relative to the camera, *affine projection models* can be used to approximate the imaging process.

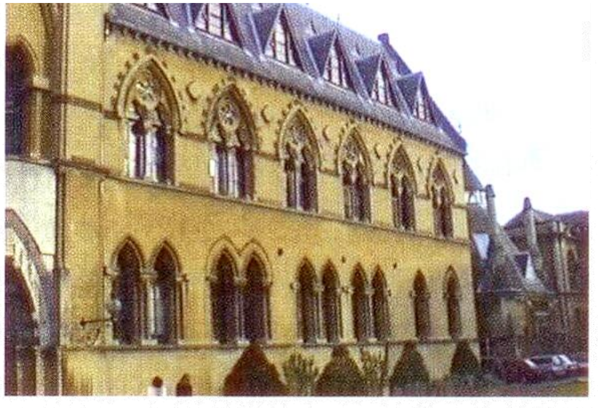


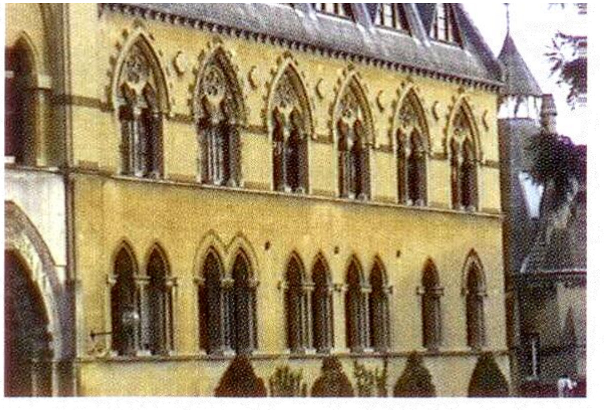
Affine Cameras

When a scene's relief is small compared with the average scene's depth relative to the camera, *affine projection models* can be used to approximate the imaging process.

increasing distance from camera →

increasing focal length →





Affine Cameras

Recall that for **projective** cameras

$$\begin{cases} \hat{u} = x \\ \hat{v} = y \end{cases} \rightarrow \hat{\boldsymbol{p}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix} = \cancel{\frac{1}{z}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cancelto{0}{1} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

In **affine** cameras the projection on the normalized image plane is assumed to be parallel to the *z* axis,

$$\boxed{\begin{cases} \hat{u} = x \\ \hat{v} = y \end{cases} \rightarrow \hat{\boldsymbol{p}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}}$$

Affine Cameras

The **orthographic** camera is an affine camera whose calibration matrix has $\alpha=\beta=1$, $\theta=90°$ and $u_0=v_0=0$, or equivalently, $\mathcal{K}=\mathrm{I}$. Thus

$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \hat{\boldsymbol{p}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Affine Cameras

For a **general orthographic** the orthographic projection is preceded by a 3D Euclidian coordinate change

$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} \boldsymbol{r}_1^T & t_x \\ \boldsymbol{r}_2^T & t_y \\ \boldsymbol{0}^T & 1 \end{pmatrix}}}_{\mathcal{M}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

where $\boldsymbol{r}_1^T$ and $\boldsymbol{r}_2^T$ are the first and second rows of $\mathcal{R}$

t_x and t_y are the first and second components of $\boldsymbol{t}$

→ five degrees of freedom.

Affine Cameras

A **scaled orthographic** camera is an orthographic camera followed by isotropic scaling. It is an affine camera whose calibration matrix has $\alpha=\beta\neq 1$, $\theta=90°$ and $u_0=v_0=0$. Thus

$$\boldsymbol{p}=\begin{pmatrix}u\\v\\1\end{pmatrix}=\begin{pmatrix}\alpha & & \\ & \alpha & \\ & & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{r}_1^T & t_x\\ \boldsymbol{r}_2^T & t_y\\ \boldsymbol{0}^T & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\\1\end{pmatrix}=\underbrace{\boxed{\begin{pmatrix}\alpha\boldsymbol{r}_1^T & \alpha t_x\\ \alpha\boldsymbol{r}_2^T & \alpha t_y\\ \boldsymbol{0}^T & 1\end{pmatrix}}}_{\mathcal{M}}\begin{pmatrix}x\\y\\z\\1\end{pmatrix}$$

→ six degrees of freedom.

Affine Cameras

A **weak perspective** camera is an orthographic camera followed by anisotropic scaling. It is an affine camera whose calibration matrix has $\alpha \neq \beta$, $\theta = 90°$ and $u_0 = v_0 = 0$. Thus

$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_1^T & t_x \\ \boldsymbol{r}_2^T & t_y \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} \alpha \boldsymbol{r}_1^T & \alpha t_x \\ \beta \boldsymbol{r}_2^T & \beta t_y \\ \boldsymbol{0}^T & 1 \end{pmatrix}}}_{\mathcal{M}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

→ seven degrees of freedom.

Affine Cameras

A **general affine** camera is an orthographic camera with a general matrix of intrinsic parameters ($u_0 = v_0 = 0$)

$$\boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} \alpha & -\alpha\cot\theta & 0 \\ & \beta/\sin\theta & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_1^T & t_x \\ \boldsymbol{r}_2^T & t_y \\ \boldsymbol{0} & 1 \end{pmatrix}}}_{\mathcal{M}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$\rightarrow$ eight degrees of freedom.

Affine Cameras

Summary of **affine mappings**

$$\boldsymbol{p} = \mathcal{M}\, {}^{W}\boldsymbol{P}, \quad \text{where}$$

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{r}_1^T & t_x \\ \boldsymbol{r}_2^T & t_y \\ \mathbf{0} & 1 \end{pmatrix}$$

orthographic - 5 dof

$$\mathcal{M} = \alpha \begin{pmatrix} \boldsymbol{r}_1^T & t_x \\ \boldsymbol{r}_2^T & t_y \\ \mathbf{0} & 1/\alpha \end{pmatrix}$$

scaled orthographic – 6 dof

$$\mathcal{M} = \begin{pmatrix} \alpha\boldsymbol{r}_1^T & \alpha t_x \\ \beta\boldsymbol{r}_2^T & \beta t_y \\ \mathbf{0} & 1 \end{pmatrix}$$

weak-perspective - 7 dof

$$\mathcal{M} = \begin{pmatrix} \alpha & -\alpha \cot\theta & 0 \\ & \beta/\sin\theta & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_1^T & t_x \\ \boldsymbol{r}_2^T & t_y \\ \mathbf{0} & 1 \end{pmatrix}$$

general affine - 8 dof

Note that in all cases the 3rd row is =(0 0 0 1)!

Next Topic

Geometric

Camera

Calibration