Geometry of Multiple Views

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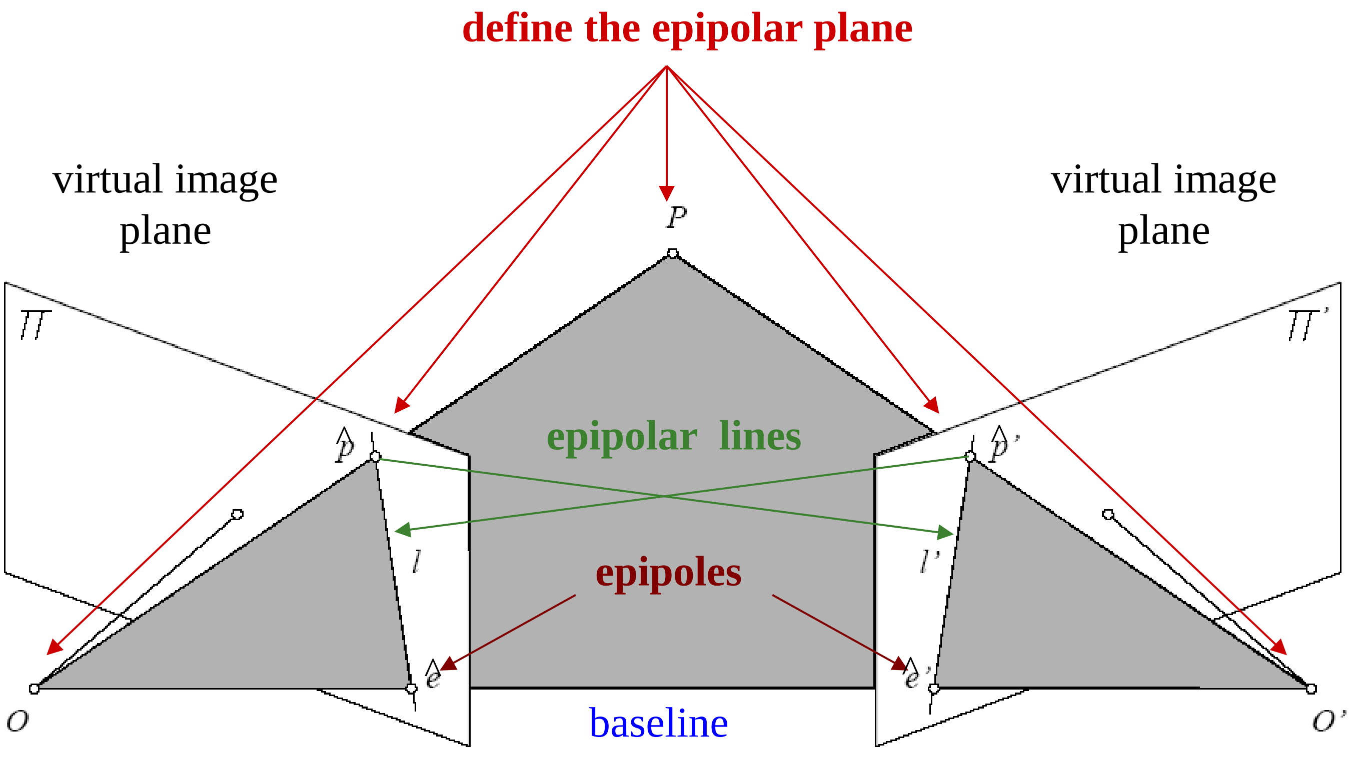
Objetive

This chapter elucidates the geometric and algebraic constraints that hold among two views of the same scene.

Geometry of Multiple Views

Contents:

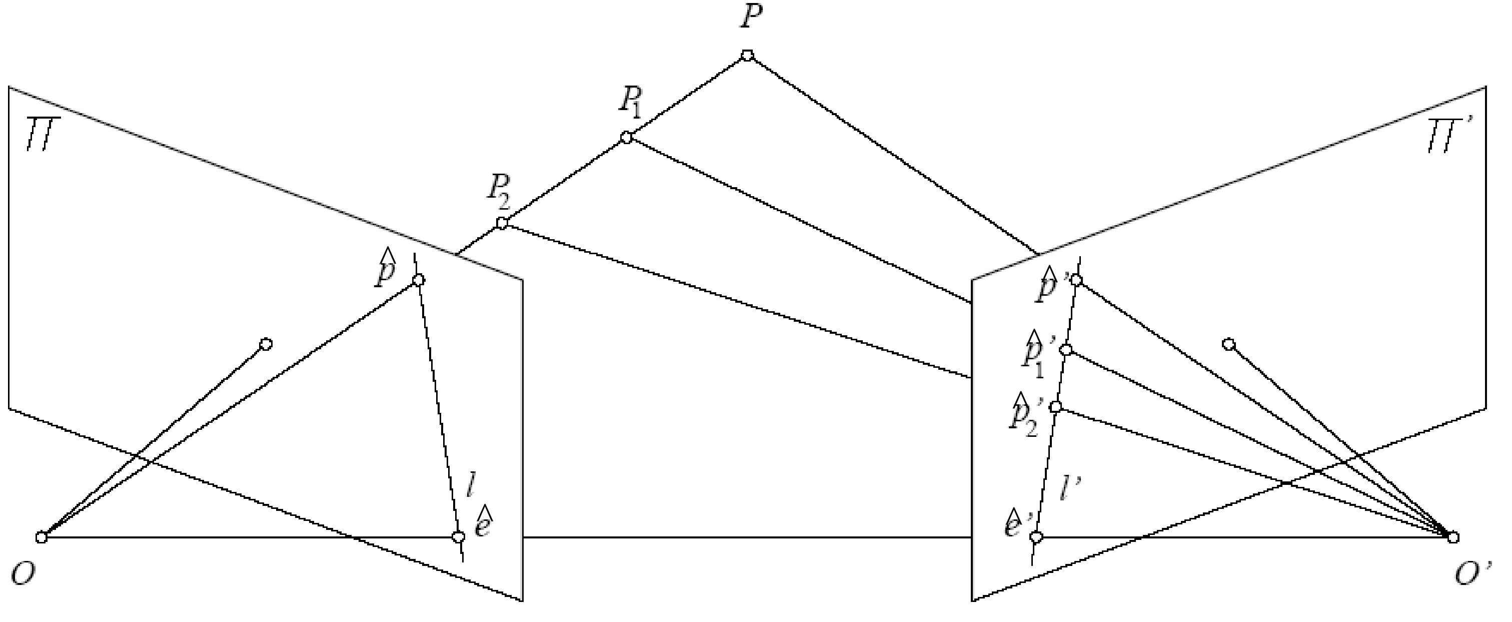
Epipolar Geometry



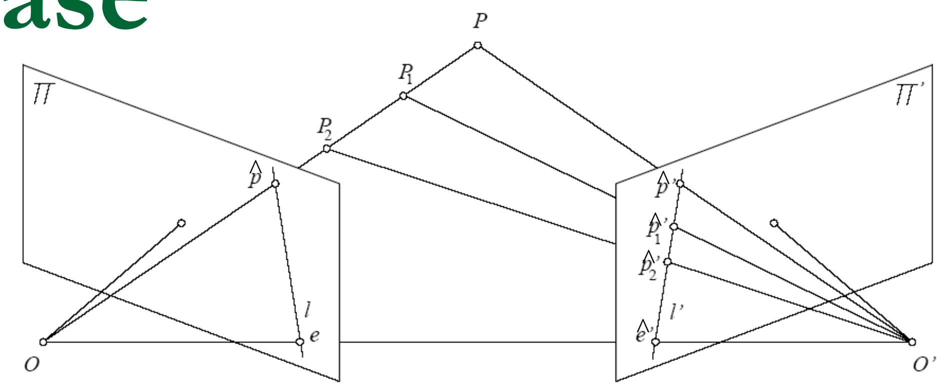
Epipolar Geometry

The Epipolar Constraint

The point corresponding to $\hat{p}$ on the right camera lies necessarily over the epipolar line l' of $\hat{p}$.



The Calibrated Case



The epipolar constraint implies that

$$\overrightarrow{O\hat{p}} \cdot \left[\overrightarrow{OO'} \times \overrightarrow{O'\hat{p}'}\right] = 0 \Rightarrow \hat{\boldsymbol{p}} \cdot \left[\boldsymbol{t} \times \mathcal{R}\hat{\boldsymbol{p}}'\right] = 0 \Rightarrow \hat{\boldsymbol{p}} \cdot \left[\boldsymbol{t}_\times\right] \mathcal{R}\, \hat{\boldsymbol{p}}' = \hat{\boldsymbol{p}}^T \mathcal{E}\, \hat{\boldsymbol{p}}' = 0$$

where:

$\hat{\boldsymbol{p}}$ is the homogeneous coordinate vector on the left normalized image

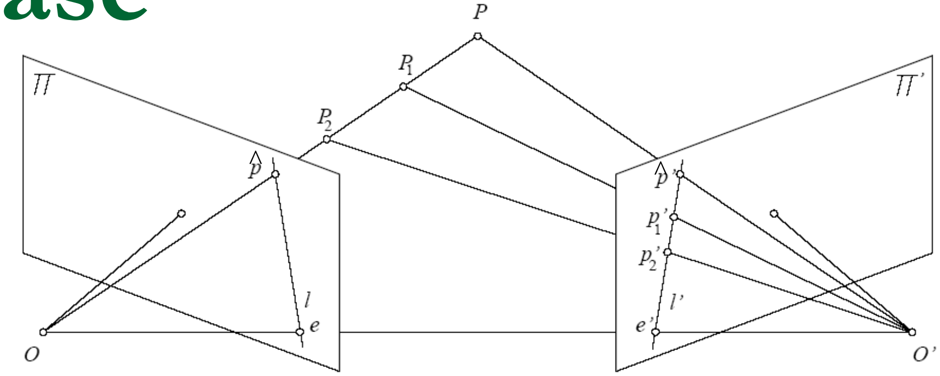
$\hat{\boldsymbol{p}}'$ is the homogeneous coordinate vector on the right normalized image

$\boldsymbol{t}$ is the coordinate vector of the translation from O to O'

$\mathcal{R}$ is the rotation matrix relating both camera frames

$[\boldsymbol{t}_\times]$ is the skew symmetric matrix such that $[\boldsymbol{t}_\times]\, \boldsymbol{a} = \boldsymbol{t} \times \boldsymbol{a}$

The Calibrated Case



The epipolar constraint implies that

$$\overrightarrow{O\hat{p}} \cdot \left[\overrightarrow{OO'} \times \overrightarrow{O'\hat{p}'}\right] = 0 \Rightarrow \hat{\boldsymbol{p}} \cdot \left[\boldsymbol{t} \times \mathcal{R}\hat{\boldsymbol{p}}'\right] = 0 \Rightarrow \hat{\boldsymbol{p}} \cdot [\boldsymbol{t}_\times]\mathcal{R}\,\hat{\boldsymbol{p}}' = \hat{\boldsymbol{p}}^T \mathcal{E}\,\hat{\boldsymbol{p}}' = 0$$

recall that

$$\text{if} \quad \boldsymbol{t} = [\, t_x \; t_y \; t_z]^T \;\; \text{then} \quad [\boldsymbol{t}_\times] = \begin{pmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{pmatrix}$$

The Calibrated Case

The Essential Matrix

- Definition:

$$\boxed{\mathcal{E} = [\boldsymbol{t}_\times]\,\mathcal{R}}$$

Properties:

- $\hat{\boldsymbol{p}}^T \mathcal{E}\, \hat{\boldsymbol{p}}' = 0$
- $\mathcal{E}\,\hat{\boldsymbol{p}}'$ ($\mathcal{E}^T \hat{\boldsymbol{p}}$) is the vector that represents the epipolar line associated to the point $\hat{\boldsymbol{p}}'$ ($\hat{\boldsymbol{p}}$) on the left (right) image.
- $\mathcal{E}$ is singular with two equal nonzero singular values.
- the eigenvector of $\mathcal{E}$ ($\mathcal{E}^T$) associated to eigenvalue equal to zero is the coordinate vector of the epipole $\hat{\boldsymbol{e}}'$ ($\hat{\boldsymbol{e}}$) of the left (right) camera.

The Uncalibrated Case

From the previous chapter

$$\boldsymbol{p} = \mathcal{K}\,\hat{\boldsymbol{p}} \quad \text{and} \quad \boldsymbol{p}' = \mathcal{K}'\,\hat{\boldsymbol{p}}'$$

This implies that

$$\boxed{\boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0}$$

where $\mathcal{F} = \mathcal{K}^{-T}\,\mathcal{E}\,\mathcal{K}'^{-1}$ is the fundamental matrix.

The Uncalibrated Case

The Fundamental Matrix

- Definition:

$$\boxed{\mathcal{F} = \mathcal{K}^{-T}\,\mathcal{E}\,\mathcal{K}'^{-1}}$$

Properties:

- $\boldsymbol{p}^T \mathcal{F}\, \boldsymbol{p}' = 0$
- $\mathcal{F}\,\boldsymbol{p}'$ ($\mathcal{F}^T\,\boldsymbol{p}$) is the vector that represents the epipolar line associated to the point $\boldsymbol{p}'$ ($\boldsymbol{p}$) on the left (right) image.
- $\mathcal{F}$ has rank equal to 2.
- the eigenvector of $\mathcal{F}$ ($\mathcal{F}^T$) associated to eigenvalue equal to zero is the coordinate vector of the epipole $\boldsymbol{e}'$ ($\boldsymbol{e}$) of the left (right) camera (why?).

The Uncalibrated Case

- $\mathcal{F}$ is defined up to a scale factor ($\boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0$)
- $\mathcal{F}$ has rank 2 $\longrightarrow$ det ($\mathcal{F}$)=0
- Thus $\mathcal{F}$ admits **seven** independent parameters.
- $\mathcal{F}$ can be parameterized this way:

$$\boxed{\mathcal{F} = \begin{pmatrix} b & a & -a\beta - b\alpha \\ -d & -c & c\beta + d\alpha \\ d\beta' - b\alpha' & c\beta' - a\alpha' & -c\beta\beta' - d\beta'\alpha + a\beta\alpha' + b\alpha\alpha' \end{pmatrix}}$$

where $\boldsymbol{e}=(\alpha,\beta)$ and $\boldsymbol{e}'=(\alpha',\beta')$ are the epipoles.

Computing $\mathcal{E}$ and $\mathcal{F}$

The Eight-point Algorithm

From the previous slide

$$\boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0 \quad \Rightarrow \quad (u, v, 1) \begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & \mathcal{F}_{13} \\ \mathcal{F}_{21} & \mathcal{F}_{22} & \mathcal{F}_{23} \\ \mathcal{F}_{31} & \mathcal{F}_{32} & \mathcal{F}_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

since this equation is homogeneous in the coefficients of $\mathcal{F}$ we can set $\mathcal{F}_{33}$=1 (scale ambiguity eliminated), and use eight point correspondences

Computing $\mathcal{E}$ and $\mathcal{F}$
The Eight-point Algorithm

For exactly 8 pairs of corresponding points:

$$\begin{pmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_8u_8' & u_8v_8' & u_8 & v_8u_8' & v_8v_8' & v_8 & u_8' & v_8' \end{pmatrix} \begin{pmatrix} \mathcal{F}_{11} \\ \mathcal{F}_{12} \\ \cdots \\ \mathcal{F}_{32} \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix}$$

Note that the rank 2 property of $\mathcal{F}$ is ignored!

Computing $\mathcal{E}$ and $\mathcal{F}$
The Eight-point Algorithm

For more than 8 pairs of corresponding points, apply least square to minimize

$$\sum_{i=1}^{p}\left(\boldsymbol{p}_i^T \mathcal{F} \boldsymbol{p}_i'\right)^2$$

which leads to the overconstrained homogeneous system

$$\begin{pmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' & 1 \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_pu_p' & u_pv_p' & u_p & v_pu_p' & v_pv_p' & v_p & u_p' & v_p' & 1 \end{pmatrix} \begin{pmatrix} \mathcal{F}_{11} \\ \mathcal{F}_{12} \\ \cdots \\ 1 \end{pmatrix} = \mathbf{0}$$

Note that the rank 2 property of $\mathcal{F}$ is ignored!

Computing $\mathcal{E}$ and $\mathcal{F}$

The Luong linear method

Step 1:

Starting with $\mathcal{F}$ produced by the eight-point algorithm, use least square to find the epipoles $\boldsymbol{e}=(\alpha,\beta)$ and $\boldsymbol{e}'=(\alpha',\beta')$ that minimize $|\mathcal{F}^T\boldsymbol{e}|^2$ and $|\mathcal{F}\boldsymbol{e}'|^2$.

Recall that $\boldsymbol{e}$ $(\boldsymbol{e}')|^2$ is the eigenvector of $\mathcal{F}(\mathcal{F}^T)$ corresponding to the zero eingenvalue.

Computing $\mathcal{E}$ and $\mathcal{F}$
The Luong linear method

Step 2:

Substitute α, β, α' and β' in equation 10.6 below

$$\mathcal{F} = \begin{pmatrix} b & a & -a\beta - b\alpha \\ -d & -c & c\beta + d\alpha \\ d\beta' - b\alpha' & c\beta' - a\alpha' & -c\beta\beta' - d\beta'\alpha + a\beta\alpha' + b\alpha\alpha' \end{pmatrix}$$

This leads to a linear parameterization of the matrix $\mathcal{F}$ in a, b, c and d.

Computing $\mathcal{E}$ and $\mathcal{F}$
The Luong linear method

Step 3 :

Estimate *a*, *b*, *c* and *d* using least square by minimizing

$$\sum_{i=1}^{p} \left(\boldsymbol{p}_i^T \mathcal{F} \boldsymbol{p}_i' \right)^2$$

Recall that the parameterization of eq. 10.6 guarantees the rank-2 property!

Computing $\mathcal{E}$ and $\mathcal{F}$

The Luong non linear method

Minimize

$$\boxed{\sum_{i=1}^{p}\left[\mathrm{d}^2\left(\boldsymbol{p}_i, \mathcal{F}\boldsymbol{p}'_i\right)+\mathrm{d}^2\left(\boldsymbol{p}', \mathcal{F}^T \boldsymbol{p}_i\right)\right]}$$

where

- $\mathrm{d}\left(\boldsymbol{p},\boldsymbol{l}\right)$ denotes the (signed geometric distance between the point $\boldsymbol{p}$ and the line $\boldsymbol{l}$, and
- $\mathcal{F}\boldsymbol{p}$ and $\mathcal{F}^T\boldsymbol{p}'$ are the epipolar lines associated to $\boldsymbol{p}$ and $\boldsymbol{p}'$.

The result of the linear algorithm is a good starting solution.

Computing $\mathcal{E}$ and $\mathcal{F}$

The Luong non linear method

The geometric distance in this case will be given by:

$$\mathrm{d}^2\left(\boldsymbol{p}_i, \mathcal{F}\boldsymbol{p}_i'\right) = \frac{\left(\boldsymbol{p}_i \mathcal{F}\boldsymbol{p}_i'\right)^2}{\left(\mathcal{F}\boldsymbol{p}_i'\right)_1^2 + \left(\mathcal{F}\boldsymbol{p}_i'\right)_2^2}$$

$$\mathrm{d}^2\left(\boldsymbol{p}_i', \mathcal{F}^T \boldsymbol{p}_i\right) = \frac{\left(\boldsymbol{p}_i' \mathcal{F}^T \boldsymbol{p}_i\right)^2}{\left(\mathcal{F}^T \boldsymbol{p}_i\right)_1^2 + \left(\mathcal{F}^T \boldsymbol{p}_i\right)_2^2} = \frac{\left(\boldsymbol{p}_i \mathcal{F}\boldsymbol{p}_i'\right)^2}{\left(\mathcal{F}^T \boldsymbol{p}_i\right)_1^2 + \left(\mathcal{F}^T \boldsymbol{p}_i\right)_2^2}$$

where the subscript indicates the component of a vector.

Computing $\mathcal{E}$ and $\mathcal{F}$

The Luong non linear method

Thus the function to be minimized is

$$\boxed{\sum_{i}^{p}\left[\frac{1}{(\mathcal{F}\boldsymbol{p}_i')_1^2+(\mathcal{F}\boldsymbol{p}_i')_2^2}+\frac{1}{(\mathcal{F}^T\boldsymbol{p}_i)_1^2+(\mathcal{F}^T\boldsymbol{p}_i)_2^2}\right](\boldsymbol{p}_i\mathcal{F}\boldsymbol{p}_i')^2}$$

Computing $\mathcal{E}$ and $\mathcal{F}$

The Hartley method

Step 1:

Transform the image coordinates using appropriate translation and scaling operator:

$$\tilde{\boldsymbol{p}}_i = \mathcal{T}\boldsymbol{p}_i \qquad \tilde{\boldsymbol{p}}'_i = \mathcal{T}'\boldsymbol{p}'_i$$

so that they are centered at the origin and have average distance to the origin equal to $\sqrt{2}$.

What are these transforms?

Computing $\mathcal{E}$ and $\mathcal{F}$

The Hartley method

Step 2:

Use linear least square to compute $\tilde{\mathcal{F}}$ minimizing

$$\sum_{\boldsymbol{i}=1}^{p}\left(\tilde{\boldsymbol{p}}_i^T\ \tilde{\mathcal{F}}\ \tilde{\boldsymbol{p}}_i'\right)^2$$

Step 3:

Construct the *singular value decomposition* of $\tilde{\mathcal{F}} = \mathcal{U}\mathcal{S}\mathcal{V}^T$; assuming that $\mathcal{S}$=diag (r,s,t) , where $t \leq s \leq r,$ compute

$$\overline{\mathcal{F}} = \mathcal{U}\mathrm{diag}\left(r,s,0\right)\mathcal{V}^T$$

Step 4:

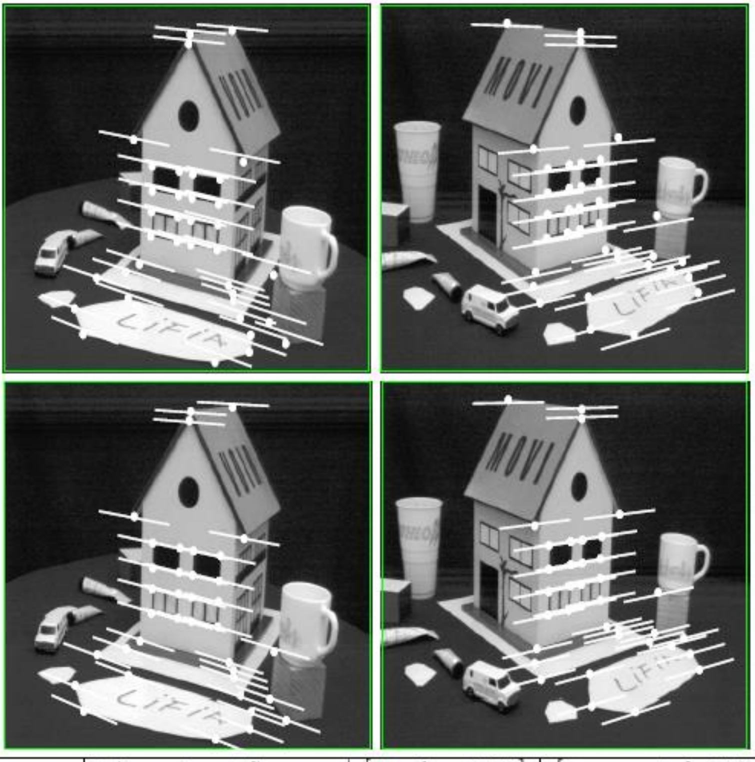
Compute the final estimate

$$\mathcal{F} = \mathcal{T}^T\,\overline{\mathcal{F}}\,\mathcal{T}'$$

Computing $\mathcal{E}$ and $\mathcal{F}$

(a)

(b)



	Linear Least Squares	[Hartley, 1995]	[Luong *et al.*, 1993]
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel

Figure 12.5. Weak calibration experiments using 37 point correspondences between two images of a toy house. The figure shows the epipolar lines found by (a) the least-squares version of the 8-point algorithm, and (b) the "normalized" variant of this method proposed by Hartley [1995]. Note for example the much larger error in (a) for the feature point close to the bottom of the mug. Quantitative comparisons are given in the table, where the average distances between the data points and corresponding epipolar lines are shown for both techniques as well as the non-linear distance minimization algorithm of Luong *et al.* [1993].

Computing $\mathcal{E}$ and $\mathcal{F}$

Using RANSAC to deal with outliers:

Determine:

n=9 – the smallest number of points required

k – the number of iterations required

t – the Sampson distance threshold (pixels) to identify a point that fits well

d – the number of nearby points required

Until *k* iterations have occurred

Draw a sample of *n* points from the data uniformly and at random

Apply **some algorithm** to compute the *Fundamental Matrix* that best fits that set of *n* points

For each data point outside the sample

Compute the *geometric distance* (slide 19) to the epipolar lines and test it against *t*

end

If there are *d* or more points *close* to the function, then there is a good fit

Refit the *geometric Matrix* using these points

end

Use the best fit from this collection, using the fitting error as criterion.

Exercise 1

Download the package of MATLAB functions that implement approaches for fundamental matrix estimation. *Test_RANSAC_Feature_Matching* is the key function. Read its help and follow the recommendations below:

- In the first run collect a set of at least 8 true correspondences and a couple of false correspondences;
- Test different approaches for fundamental matrix estimation;
- Disable the RANSAC by setting the input parameter *Dmax* to a very large number;
- Save the figures produced at the end of each run in a separate .fig file and compare the results.

Exercise 2

Compare the F matrix estimation obtained in previous exercise with the ones obtained with methods implemented in MATLAB (*estimateFundamentalMatrix*).

Next Topic

Stereopsis