TTK4255: Robotic Vision

Homework 3: Rigid motion

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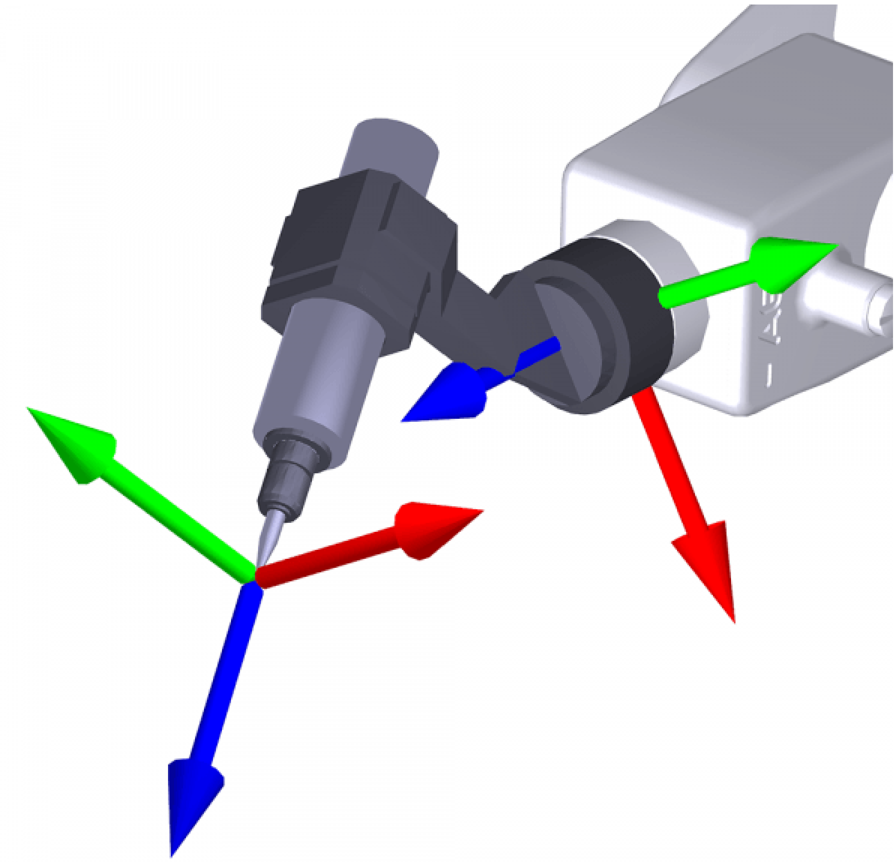


Figure 1: Souce of image - How is Orientation in Space Represented with Euler Angles?

Instructions

First make sure to read `about.pdf` in the "Course work" page on Blackboard. To get your assignment approved, you need to complete any 60%. Upload the requested answers and figures as a single PDF. You may collaborate with other students and submit the same report, but you still need to upload individually on Blackboard. Please write your collaborators' names on your report's front page. If you want detailed feedback, please indicate so on the front page.

About the assignment

The concept of rotation can be represented in various forms, such as matrices, quaternions, rotation vectors, and Euler angles. In this assignment, we will delve deeper into matrices and Euler angles, as well as their interrelation. This is particularly relevant to our course, where we predominantly represent rotation using matrices in conjunction with Euler angles. This assignment focuses solely on mathematics, with no coding required.

Notation

Vector:

In this exercise, a vector is defined as a directional entity with magnitude, akin to an arrow pointing in a certain direction as Figure 2. It does not have a specific starting location. In our notation, a vector is represented by placing an arrow over a variable, such as $\vec{a}$.

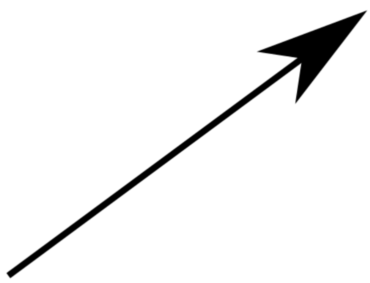


Figure 2: An arrow in space.

Vector's Components:

A vector can be decomposed into a set of other vectors. For example, the vector $\vec{x}$ can be expressed as $x_1^A\vec{a}_1 + x_2^A\vec{a}_2 + x_3^A\vec{a}_3$. Here, x_1^A, x_2^A, and x_3^A are the components of the vector $\vec{x}$, represented in the basis set $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

The components of a vector, such as $\vec{x}$, are often represented using a column matrix. For instance:

$$\mathbf{x}^A := \begin{bmatrix} x_1^A \\ x_2^A \\ x_3^A \end{bmatrix}.$$

The superscript ‘A’ indicates the specific set of basis vectors being referred to. It is important to note that $\vec{x}$ and $\mathbf{x}^A$ represent two distinct mathematical entities. $\vec{x}$ is the vector itself, characterized by its direction and magnitude. On the other hand, $\mathbf{x}^A$ denotes the components of the vector $\vec{x}$ when represented in the basis ‘A’. This distinction is crucial in understanding the relationship between a vector and its representation in different basis sets.

Definition of a new operation:

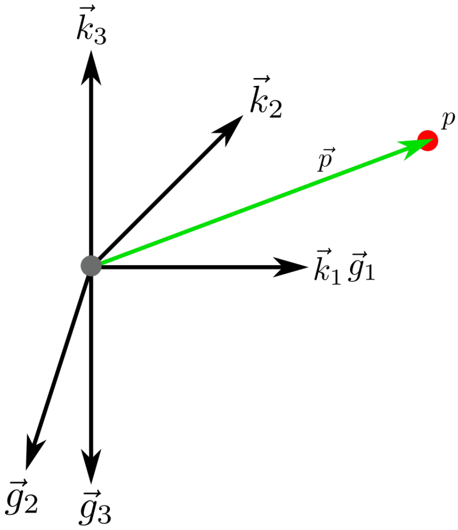
To facilitate a shorter and more concise notation, we extend matrix operations to incorporate vectors as elements within a matrix. Consider the following expression:

$$\begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} := \begin{bmatrix} \vec{a}\cdot\vec{x} & \vec{a}\cdot\vec{y} & \vec{a}\cdot\vec{z} \\ \vec{b}\cdot\vec{x} & \vec{b}\cdot\vec{y} & \vec{b}\cdot\vec{z} \\ \vec{c}\cdot\vec{x} & \vec{c}\cdot\vec{y} & \vec{c}\cdot\vec{z} \end{bmatrix},$$

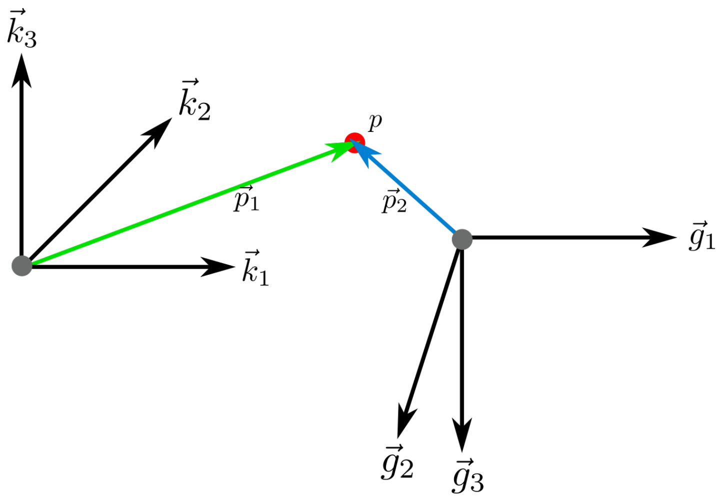
where $\vec{a}\cdot\vec{x} = ||\vec{a}||\,||\vec{x}||\cos\theta$. This formulation does not require coordinate information, relying solely on the magnitudes of the vectors and the relative angle between them.

Part 1 Representation of a Point as Vector(s) (10%)

Task 1.1: (5%) Figure 3a illustrates two coordinate frames with a common origin, K and G. Conversely, Figure 3b depicts the coordinate frames K and G with distinct origins. In both figures, a point p is presented. Explain why K and G can use the same vector to represent p in Figure 3a, but not in Figure 3b.



(a) Coordinate frame K and G share the same origin.



(b) Coordinate frame K and G does not share the same origin.

Figure 3: Both sets of basis vectors, $\{\vec{k}_1, \vec{k}_2, \vec{k}_3\}$ and $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$, consist of unit vectors, with $\vec{k}_1 \times \vec{k}_2 = \vec{k}_3$, and $\vec{g}_1 \times \vec{g}_2 = \vec{g}_3$

Task 1.2: (5%) In Figure 3a, does $\mathbf{p}^K$ equal $\mathbf{p}^G$, even though the same vector can be used to represent point p in this context? Here, $\mathbf{p}^K$ represents the vector components of $\vec{p}$ expressed in the coordinate frame K, while $\mathbf{p}^G$ represents the vector components of $\vec{p}$ in the coordinate frame G.

Part 2 Pure Rotation (80%)

In this part, we assume that two coordinate frames, K and G, share the same origin as Figure 3a. Given a vector $\vec{r}$ and we expressed it into the frame K and G in the following:

$$\begin{aligned} \vec{r} = \begin{bmatrix} \vec{k}_1 & \vec{k}_2 & \vec{k}_3 \end{bmatrix} \begin{bmatrix} r_1^K \\ r_2^K \\ r_3^K \end{bmatrix} && = \begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \end{bmatrix} \begin{bmatrix} r_1^G \\ r_2^G \\ r_3^G \end{bmatrix} \\ = r_1^K \vec{k}_1 + r_2^K \vec{k}_2 + r_3^K \vec{k}_3 && = r_1^G \vec{g}_1 + r_2^G \vec{g}_2 + r_3^G \vec{g}_3. \end{aligned}$$

We want to find the relation between $[r_1^K, r_2^K, r_3^K]^T$ and $[r_1^G, r_2^G, r_3^G]^T$. We proceed as follows:

$$\begin{bmatrix} \vec{k}_1 & \vec{k}_2 & \vec{k}_3 \end{bmatrix} \begin{bmatrix} r_1^K \\ r_2^K \\ r_3^K \end{bmatrix} = \begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \end{bmatrix} \begin{bmatrix} r_1^G \\ r_2^G \\ r_3^G \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \vec{k}_1 \\ \vec{k}_2 \\ \vec{k}_3 \end{bmatrix} \begin{bmatrix} \vec{k}_1 & \vec{k}_2 & \vec{k}_3 \end{bmatrix} \begin{bmatrix} r_1^K \\ r_2^K \\ r_3^K \end{bmatrix} = \begin{bmatrix} \vec{k}_1 \\ \vec{k}_2 \\ \vec{k}_3 \end{bmatrix} \begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \end{bmatrix} \begin{bmatrix} r_1^G \\ r_2^G \\ r_3^G \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} r_1^K \\ r_2^K \\ r_3^K \end{bmatrix} = \mathbf{R}_{K,G} \begin{bmatrix} r_1^G \\ r_2^G \\ r_3^G \end{bmatrix}, \tag{3}$$

and the definition of $\mathbf{R}_{K,G}$

$$\mathbf{R}_{K,G} := \begin{bmatrix} \vec{k}_1 \\ \vec{k}_2 \\ \vec{k}_3 \end{bmatrix} [\vec{g}_1, \vec{g}_2, \vec{g}_3] \tag{4}$$

$$= \begin{bmatrix} \vec{k}_1 \cdot \vec{g}_1 & \vec{k}_1 \cdot \vec{g}_2 & \vec{k}_1 \cdot \vec{g}_3 \\ \vec{k}_2 \cdot \vec{g}_1 & \vec{k}_2 \cdot \vec{g}_2 & \vec{k}_2 \cdot \vec{g}_3 \\ \vec{k}_3 \cdot \vec{g}_1 & \vec{k}_3 \cdot \vec{g}_2 & \vec{k}_3 \cdot \vec{g}_3 \end{bmatrix}. \tag{5}$$

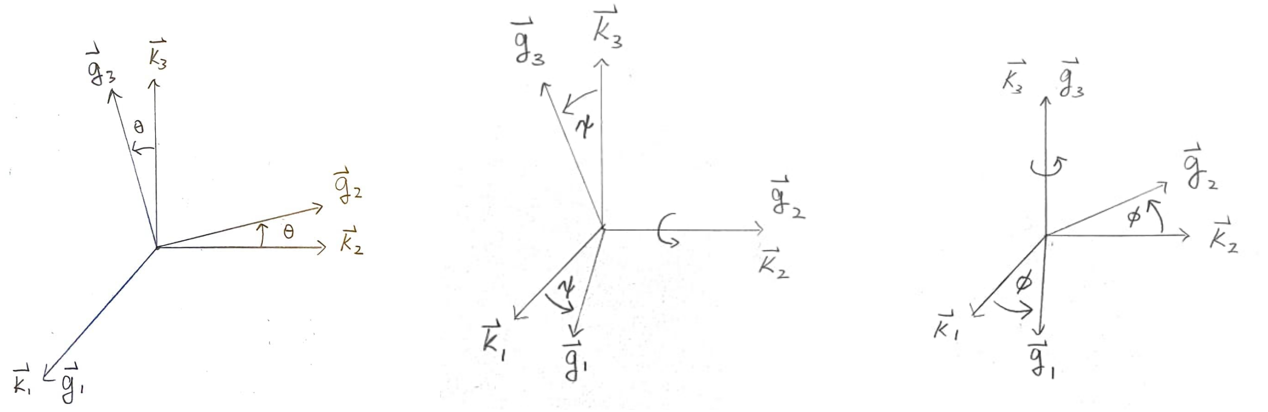
From the above equation, we can see the rotation matrix is actually specified by the inner product between two basis vectors.

Task 2.1: (5%) Utilize the rotation matrix presented in Equation 5 to demonstrate that the rotation matrix representing the orientation in Figure 4a is given by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \tag{6}$$

Task 2.2: (5%) Utilize the rotation matrix presented in Equation 5 to demonstrate that the rotation matrix representing the orientation in Figure 4b is given by:

$$\begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}. \tag{7}$$



(a) Rotation with respect to $\vec{k}_1$. (b) Rotation with respect to $\vec{k}_2$. (c) Rotation with respect to $\vec{k}_3$.

Figure 4: Rotation with Euler angles

Task 2.3: (5%) Utilize the rotation matrix presented in Equation 5 to demonstrate that the rotation matrix representing the orientation in Figure 4c is given by:

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

Task 2.4: (5%) We define

$$\mathbf{R}_{\vec{k}_1}(\theta) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \mathbf{R}_{\vec{k}_2}(\psi) := \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}, \mathbf{R}_{\vec{k}_3}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The sequence (θ, ψ, ϕ) are generally referred as Euler angles. A complete consecutive rotation can be described by the following equation:

$$\mathbf{R}_{\vec{k}_3}(\phi)\mathbf{R}_{\vec{k}_2}(\psi)\mathbf{R}_{\vec{k}_1}(\theta). \tag{9}$$

Give an example to show that $\mathbf{R}_{\vec{k}_3}(\phi)\mathbf{R}_{\vec{k}_2}(\psi)\mathbf{R}_{\vec{k}_1}(\theta)$ do not commute. For example, given a sequence (θ, ψ, ϕ), show that $\mathbf{R}_{\vec{k}_3}(\phi)\mathbf{R}_{\vec{k}_2}(\psi)\mathbf{R}_{\vec{k}_1}(\theta) \neq \mathbf{R}_{\vec{k}_1}(\theta)\mathbf{R}_{\vec{k}_2}(\psi)\mathbf{R}_{\vec{k}_3}(\phi)$

Task 2.5: Rotation: intrinsic vs extrinsic (10%) This task is designed to explore the conceptual differences between intrinsic (mobile frame) rotation and extrinsic (fixed frame) rotation. Please visit the provided How is Orientation in Space Represented with Euler Angles?, and scroll down to the section displaying Figure 5.

- First, slide α from 0 to 45 degrees using the blue sliding bar. Then, slide β from 0 to 45 degrees. Observe the arrows indicating the rotation direction: is it the light green or the dark green arrow? This rotation is known as intrinsic.
- Set α and β to zero. Double-click the area in the red box of Figure 5 such that the symbol apostrophe (') is gone.

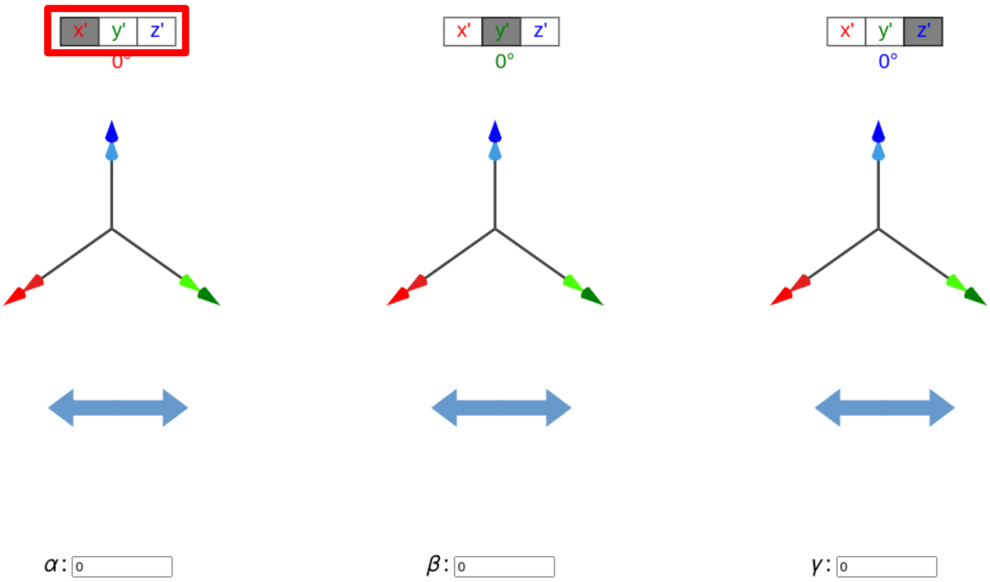
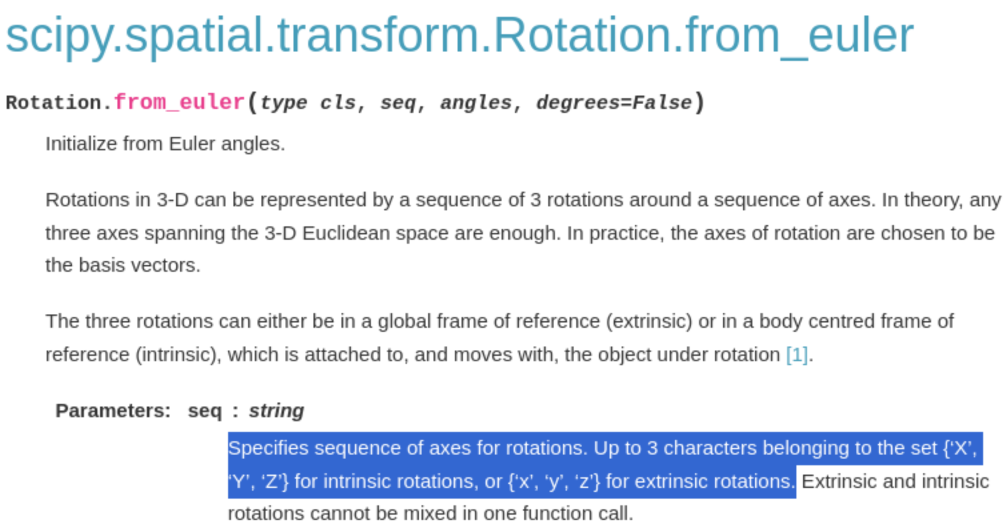


Figure 5

- First, slide α from 0 to 45 degrees using the blue sliding bar. Then, slide β from 0 to 45 degrees. Observe the arrows indicating the rotation direction: is it the light green or the dark green arrow? This rotation is known as extrinsic.

Explain the difference between intrinsic and extrinsic rotation in just a few sentences.

In Scipy, it is necessary to specify whether the Euler angles represent intrinsic or extrinsic rotations, as illustrated in Figure 6.



scipy.spatial.transform.Rotation.from_euler

`Rotation.from_euler(type cls, seq, angles, degrees=False)`

Initialize from Euler angles.

Rotations in 3-D can be represented by a sequence of 3 rotations around a sequence of axes. In theory, any three axes spanning the 3-D Euclidean space are enough. In practice, the axes of rotation are chosen to be the basis vectors.

The three rotations can either be in a global frame of reference (extrinsic) or in a body centred frame of reference (intrinsic), which is attached to, and moves with, the object under rotation [1].

Parameters: **seq** : ***string***

Specifies sequence of axes for rotations. Up to 3 characters belonging to the set {'X', 'Y', 'Z'} for intrinsic rotations, or {'x', 'y', 'z'} for extrinsic rotations. Extrinsic and intrinsic rotations cannot be mixed in one function call.

Figure 6: scipy.spatial.transform.Rotation.fromEuler

Task 2.6: Rotation: passive vs active (10%) Use Equation 1 and 3, show that:

$$\begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \end{bmatrix} = \begin{bmatrix} \vec{k}_1 & \vec{k}_2 & \vec{k}_3 \end{bmatrix} \mathbf{R}_{K,G}, \tag{10}$$

which depicts the direct relation between the basis vectors of frame K and the basis vectors of frame G.

Summary:

1. $\mathbf{R}_{K,G}$ transforms the coordinate of a point **from G to K**.(**coordinate transformation or passive rotation**). See Equation 3.

2. $\mathbf{R}_{K,G}$ transforms the set of basis vectors **from K to G**. (**basis vectors transformation or active rotation**). See Equation 10.

The same rotation matrix affects a vector and its components in opposite manners. Confusion arises when using this matrix to transform coordinates from system G to K, while simultaneously deriving the matrix as moving basis vectors from G to K. Consequently, when the rotation matrix does not produce the correct result, employing its inverse often yields the correct outcome. However, we would avoid the confusion in the first place.

Understanding the distinction between coordinate transformation and basis vector transformation is crucial. In coordinate transformation, no actual rotation of the object occurs, whereas, in basis vector transformation, the basis vectors themselves are rotated. In this course, we often focus on coordinate transformation since we focus on expressing a 3D point in multiple coordinate frames.

Recover Euler angles from rotation matrix

Sometimes, one might need to retrieve Euler angles from a matrix. Let's explore whether this is always possible. Let us consider an example:

$$\begin{aligned}
\mathbf{R}_{K,G} &= \mathbf{R}_{\vec{k}_3}(\phi)\mathbf{R}_{\vec{k}_2}(\psi)\mathbf{R}_{\vec{k}_1}(\theta) \\
&= \begin{bmatrix} \cos\phi\cos\psi & \cos\phi\sin\psi\sin\theta - \sin\phi\cos\theta & \cos\phi\sin\psi\cos\theta + \sin\phi\sin\theta \\ \sin\phi\cos\psi & \sin\phi\sin\psi\sin\theta + \cos\phi\cos\theta & \sin\phi\sin\psi\cos\theta - \cos\phi\sin\theta \\ -\sin\psi & \cos\psi\sin\theta & \cos\psi\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}.
\end{aligned} \tag{11}$$

Task 2.7: Recover θ (10%) Could we recover θ from the last matrix in equation 11? Study it in the case $|\psi| = \frac{\pi}{2}$, case $|\psi| < \frac{\pi}{2}$ and case $|\psi| > \frac{\pi}{2}$.

Tips: Use two arguments arctan:

$$\theta_r = \text{arctan2}(r_{3,2}, r_{3,3}) = \text{arctan2}(\cos\psi\sin\theta, \cos\psi\cos\theta) \tag{12}$$

Will $\theta_r = \theta$?

Task 2.8: Recover ψ (10%) Could we recover ψ from the last matrix in equation 11? Study it in the case $|\psi| = \frac{\pi}{2}$, case $|\psi| < \frac{\pi}{2}$ and case $|\psi| > \frac{\pi}{2}$.

Tips: Use two arguments arctan:

$$\psi_r = \text{arctan2}(-r_{3,1}, \sqrt{r_{3,2}^2 + r_{3,3}^2}) = \text{arctan2}(\sin\psi, |\cos\psi|). \tag{13}$$

Will $\psi_r = \psi$?

Task 2.9: Recover ϕ (10%) Could we recover ϕ from the last matrix in equation 11? Study it in the case $|\psi| = \frac{\pi}{2}$, case $|\psi| < \frac{\pi}{2}$ and case $|\psi| > \frac{\pi}{2}$.

Tips: Use two arguments arctan:

$$\phi_r = \arctan2(r_{2,1}, r_{1,1}) = \arctan2(\sin\phi\cos\psi, \cos\phi\cos\psi) \tag{14}$$

Will $\phi_r = \phi$?

Part 3 Rotation and Translation (20%)

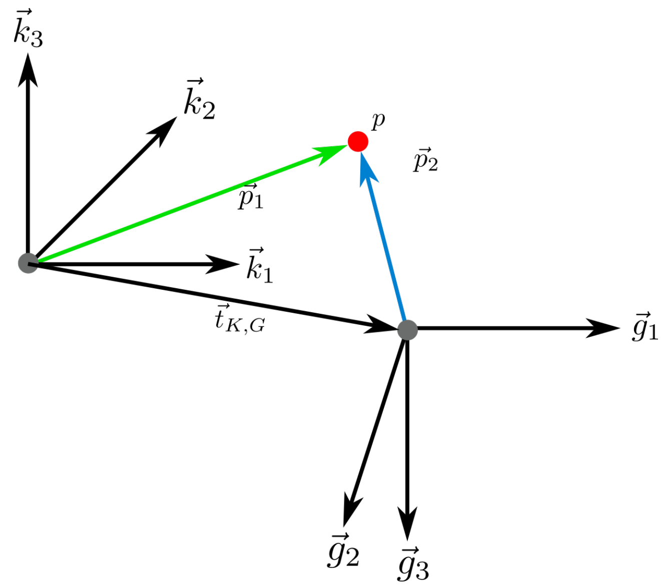


Figure 7: Transformation from K to G. Both rotation and translation are involved.

Besides rotation, there can be a translation between two coordinate frames. In Figure 7, there is a translation from frame K to frame G; The translation is expressed by a vector $\vec{t}_{K,G}$ pointing from the origin K to origin G. We can specify the vector's components of into coordinate frame K with upper script K in this way:

$$\mathbf{t}^K_{K,G} = \begin{bmatrix} t^K_1 \\ t^K_2 \\ t^K_3 \end{bmatrix}.$$

Given a point p and is expressed as $\mathbf{p}^G$ (in coordinate frame G), and we can find $\mathbf{p}^K$ in the following equation:

$$\mathbf{p}^K = \mathbf{R}_{K,G}\mathbf{p}^G + \mathbf{t}^K_{K,G}. \tag{15}$$

Task 3.1: (5%) Express the above equation into matrix multiplication with homogenous coordinates.

Task 3.2: (5%) In equation 15, explain why it will be wrong if we replace $\mathbf{t}^K_{K,G}$ with $\mathbf{t}^G_{K,G}$.

Task 3.3: (10%) Given

$$\mathbf{p}^G = \mathbf{R}_{G,K}\mathbf{p}^K + \mathbf{t}^G_{G,K}, \tag{16}$$

show that $\mathbf{t}^G_{G,K} = -(\mathbf{R}_{K,G})^T\mathbf{t}^K_{K,G}$.

Tips: $(\mathbf{R}_{K,G})^{-1} = (\mathbf{R}_{K,G})^T$