بنام خدا

تمرین چهارم

Subject: ....

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a) $x(t) = e^{-2|t-1|}$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} e^{-2|t-1|}\, e^{-j\omega t}\, dt =$$

$$\int_{-\infty}^{1} e^{-2+t(2-j\omega)}\, dt + \int_{1}^{\infty} e^{2+t(-2-j\omega)}\, dt =$$

$$\frac{e^{-2}}{2-j\omega} \cdot e^{t(2-j\omega)} \Big|_{-\infty}^{1} + \frac{e^{2}}{(-2-j\omega)} e^{t(-2-j\omega)} \Big|_{1}^{\infty} =$$

$$\frac{e^{-2}}{2-j\omega} \left[e^{2-j\omega} - 0 \right] + \frac{e^{2}}{-2-j\omega} \left[0 - e^{-2-j\omega} \right] =$$

$$\frac{e^{-2+2-j\omega}}{2-j\omega} + \frac{e^{2-2-j\omega}}{+2+j\omega} = e^{-j\omega}\left[\frac{1}{2-j\omega} + \frac{1}{2+j\omega}\right] = \frac{4e^{-j\omega}}{4+\omega^2}$$

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b) $x(t) = 1 + 3\cos(4\pi t + \frac{\pi}{3}) = 1 + \frac{3}{2} e^{(4\pi t + \frac{\pi}{3})j} + \frac{3}{2} e^{-(4\pi t + \frac{\pi}{3})j}$

periodic with $T = \frac{2\pi}{4\pi} = \frac{1}{2}$ and $w_0 = 4\pi$

$a_0 = 1$

$a_1 = \frac{3}{2} e^{\frac{\pi}{3}j}$, $a_{-1} = \frac{3}{2} e^{-\frac{\pi}{3}j}$

$X(jw) = 2\pi a_0 \delta(w_0) + 2\pi a_1 (w - w_0) + 2\pi a_{-1} \delta(w + w_0)$

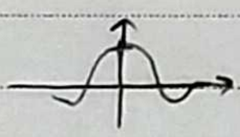
$= 2\pi \delta(w) + 3\pi e^{\frac{\pi}{3}j} \delta(w - 4\pi) + 3\pi e^{-\frac{\pi}{3}j} \delta(w + 4\pi)$

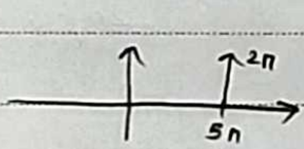
Subject: $T = \frac{2}{5}$

Year: Month: Day: ( ) $T = 1$

c) $x(t) = e^{5\pi j t} - \frac{\sin(2\pi t)}{\pi t} \Rightarrow T_0 = 2 \Rightarrow \omega_0 = \pi$

$e^{5\pi j t} \Rightarrow a_5 = 1 \Rightarrow X(j\omega) = 2\pi \times 1 \times \delta(\omega - 5\omega_0) = 2\pi\,\delta(\omega - 5\pi)$

$\frac{\sin(2\pi t)}{\pi t}$  $\Rightarrow X(j\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases}$



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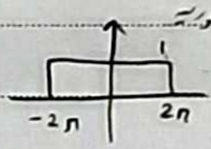
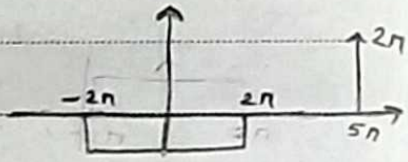
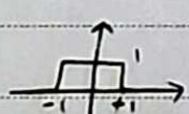
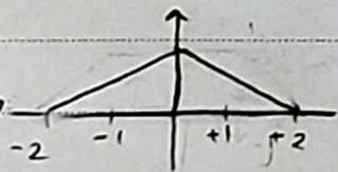


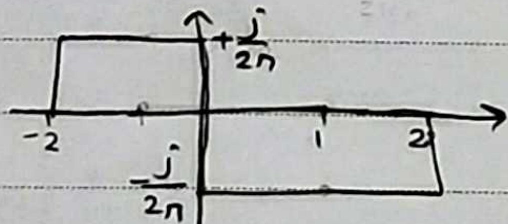
خطی بودن سری فوریه
$\Rightarrow$


d) $x(t) = \frac{1}{t}\left(\frac{t\sin(t)}{\pi t}\right)^2 = t\left(\frac{\sin t}{\pi t}\right)^2 \quad \frac{\sin t}{\pi t} = y(t)$

$y(t) \xrightarrow{F.t}$  $\Rightarrow y(t)\cdot y(t) \xrightarrow{F.t} \frac{1}{2\pi} \underbrace{(Y(j\omega) * Y(j\omega))}_{Z(j\omega)}$

$\Rightarrow$ 

$t.(y(t).y(t)) \xrightarrow{F.t} j\frac{d\,Z(j\omega)}{d\omega}$



$X(j\omega) = \begin{cases} j/2\pi & -2 \leq \omega < 0 \\ -j/2\pi & 0 < \omega < 2 \\ 0 & o/w \end{cases}$

e) $x(t) = \dfrac{4t}{(1+t^2)^2}$

$$e^{-|t|} \xrightarrow{F.t} \frac{2}{1+w^2}$$

$$t \cdot e^{-|t|} \xrightarrow{F.t} -\frac{4wj}{(1+w^2)^2} \xRightarrow{\text{duality}}$$

$$\frac{-4tj}{(1+t^2)^2} \xrightarrow{F.t} -2\pi w \cdot e^{-|w|} \xrightarrow[\text{ضربدر j}]{\text{دوطرف}} \frac{4t}{(1+t^2)^2} \xrightarrow{F.t} -2\pi j w \cdot e^{-|w|}$$

f) $x_3(t) = x(t) + x(t+1) \xrightarrow{F.t} X_3(j\omega) = X(j\omega) + e^{j\omega_0} X(j\omega)$

$T = 1 \rightarrow \omega_0 = 2\pi$

$$e^{-t} \xrightarrow{F.t} \frac{1}{1+j\omega} \Longrightarrow X_3(j\omega) = \frac{1}{1+j\omega} + e^{j2\pi} \frac{1}{1+j\omega} = \frac{2}{1+j\omega}$$

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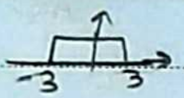
a) $X(j\omega) = 2\delta(\omega+6)$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\,\delta(\omega+6)\, e^{j\omega t}\, d\omega = \frac{1}{\pi}\, e^{-6jt}$

b) $X(j\omega) = 2\,\frac{\sin(3(\omega-2\pi))}{\omega-2\pi} = 2\,\frac{\sin(3z)}{z}$



دانیم که عکس تبدیل فوریه $2\,\frac{\sin 3z}{z}$ برابر $X(j\omega)$

$e^{j2\pi t} x_1(t) \longrightarrow X_1(j(\omega-2\pi)) \Rightarrow x(t) = e^{2\pi tj}\,(u(t-3)-u(t+3)$

$= \begin{cases} e^{2\pi tj} & |t|<3 \\ 0 & o/w \end{cases}$

c) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2[\delta(\omega-1)-\delta(\omega+1)] + 3[\delta(\omega-2\pi)+\delta(\omega+2\pi)]\, e^{j\omega t}\, d\omega$

$= \frac{1}{\pi}\left[e^{jt} - e^{-tj}\right] + \frac{3}{2\pi}\left(e^{2\pi jt} + e^{-2\pi jt}\right) = \frac{2j}{\pi}\sin(t) + \frac{3}{\pi}\cos(2\pi t)$

d) $X(j\omega) = \frac{7j\omega+46}{\underbrace{-\omega^2}_{(j\omega)^2}+13j\omega+42} = \frac{7j\omega+46}{(j\omega+6)(j\omega+7)}$

$= \frac{A}{j\omega+6} + \frac{B}{j\omega+7}$ $\qquad A = (j\omega+6)\times X(j\omega)\Big|_{j\omega=-6} = 4$

$B = (j\omega+7)\,X(j\omega)\Big|_{j\omega=-7} = 3$

$X(j\omega) = \frac{4}{j\omega+6} + \frac{3}{j\omega+7} \Rightarrow x(t) = 4e^{-6t}u(t) + 3e^{-7t}u(t)$

3/ الف) نادرست $f(x) = re(x) + i\, im(x)$

در این چون $f(x)$ موهومی است پس $f(x) = i\, im(x)$

$$F(q) = i\int_{-\infty}^{+\infty} im(x)\cos(qx)\,dx - \int_{-\infty}^{+\infty} im(x)\sin(qx)\,dx$$

از آنجا که $f(x)$ فرد است و $\cos(x)$ زوج است و حاصل زیر انتگرال فرد می‌شود $\Rightarrow$

خود انتگرال صفر شده در نتیجه $F(q) = -\int_{-\infty}^{+\infty} im(x)\sin(qx)\,dx$ می‌شود که فرد و حقیقی است

ب) درست $E(j\omega) * O(j\omega) = \int_{-\infty}^{+\infty} E(\tau)\, O(j\omega-\tau)\, d\tau$

$j\omega \to -j\omega \Rightarrow E(-j\omega) = E(j\omega) \Rightarrow$

$$E(j\omega) * O(-j\omega) = \int_{-\infty}^{+\infty} E(\tau)\, O(j\omega+\tau)\, d\tau \xrightarrow[d\mu = -d\tau]{\mu = -\tau}$$

$$= -\int_{+\infty}^{-\infty} E(-\mu)\, O(j\omega-\mu)\, d\mu = E(j\omega) * O(j\omega)$$ $\Rightarrow$ فرد است

1/2

4)

$y(t) = x(t) * h(t) \longrightarrow Y(j\omega) = X(j\omega) \cdot H(j\omega)$

$g(t) = x(3t) * h(3t) \longrightarrow G(j\omega) = X(3j\omega) \cdot H(3j\omega)$

به جای $j\omega$ در معادله اول $3j\omega$ می‌گذاریم

$\Rightarrow Y(3j\omega) = X(3j\omega) \cdot H(3j\omega) \Rightarrow Y(3j\omega) = G(j\omega)$

$\xrightarrow[\text{مقیاس پذیری}]{IFT} \frac{1}{3} y\left(\frac{t}{3}\right) = g(t)$ $A = \frac{1}{3}$ $B = \frac{1}{3}$

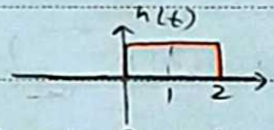
5)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[e^{0jt} + e^{\pi jt} + e^{5jt}\right]$$

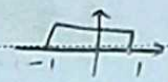
$$= \frac{1}{2\pi} + \underbrace{\frac{(\cos \pi t + j\sin \pi t)}{2\pi}}^{T_1} + \underbrace{\frac{(\cos 5t + j\sin 5t)}{2\pi}}^{T_2}$$

الف) نمی‌توان دوره تناوب پایه برای این دو پیدا کرد ← غیر متناوب : $T_2 = \frac{2\pi}{5}$ $\quad T_1 = 2$

ب) $h(t)$



همچنین می‌دانیم تبدیل فوریه



برابر $\frac{2\sin\omega}{\omega}$ است

پس از خاصیت time shifting استفاده می‌کنیم $\quad x(t) \rightarrow \frac{2\sin\omega}{\omega}$

$h(t) = x(t-1) \rightarrow 2e^{-j\omega}\frac{\sin\omega}{\omega}$

$$H(j\omega) = 2e^{-j\omega}\frac{\sin\omega}{\omega}$$

$$Y(j\omega) = \left[\delta(\omega) + \delta(\omega-\pi) + \delta(\omega-5)\right] \times 2 \times e^{-j\omega} \times \frac{\sin\omega}{\omega} = 2\delta(\omega) + \delta(\omega-5) \times 2\,\frac{\sin 5}{5}e^{-5j}$$

ضرب

$H(j\omega)$ به ازای $\omega = k\pi$ و $k \neq 0$ صفر است

$$\Rightarrow y(t) = \frac{1}{\pi} + \frac{\sin(5)e^{-5j}}{5}e^{5jt}$$

با $T_0 = \frac{2\pi}{5}$ متناوب است

ج) با توجه به الف و ب که نوشتیم دو سیگنال غیر متناوب می‌توانند متناوب باشند

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6)

$$Y(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega+4} = \frac{1}{(j\omega+3)(j\omega+4)}$$

از طرفی

$$X(j\omega) \cdot H(j\omega) = Y(j\omega)$$

$$X(j\omega) \cdot \frac{1}{j\omega+3} = \frac{1}{(j\omega+3)(j\omega+4)} \rightarrow X(j\omega) = \frac{1}{j\omega+4}$$

$$\xrightarrow{IFT} x(t) = e^{-4t} u(t)$$

7)

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

با استفاده از نتایج سوال 1.d :

$$x(t) = t \left(\frac{\sin t}{\pi t}\right)^2$$

$$X(j\omega) = \begin{cases} j/2\pi & -2<\omega<0 \\ -j/2\pi & 0<\omega<2 \\ 0 & o/w \end{cases}$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi} \left(-\frac{1}{4\pi^2} - \frac{1}{4\pi^2} \right) = -\frac{1}{4\pi^3}$$

8) الف

$$-\omega^2 Y(j\omega) + 6j\omega Y(j\omega) + 8 Y(j\omega) = 2 X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{\underbrace{-\omega^2}_{(j\omega)^2} + 6j\omega + 8} = \frac{2}{(j\omega+2)(j\omega+4)}$$

$$= \frac{A}{j\omega+2} + \frac{B}{j\omega+4} \rightarrow A = 1 \quad B = -1 \Rightarrow h(t) = e^{-2t} u(t) - e^{-4t} u(t)$$

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ب)

$$H(j\omega) = \frac{2}{(j\omega+2)(j\omega+4)}, \quad X(j\omega) = \frac{1}{(j\omega+2)^2} \quad \Bigg\} \quad X(j\omega) = H(j\omega) \cdot X(j\omega)$$

$$X(j\omega) = \frac{2}{(j\omega+2)^3(j\omega+4)} = \frac{A_{11}}{(j\omega+2)^1} + \frac{A_{12}}{(j\omega+2)^2} + \frac{A_{13}}{(j\omega+2)^3} + \frac{A_{21}}{(j\omega+4)}$$

$$A_{11} = \frac{1}{2}\left(\frac{d^2}{d(j\omega)^2}\,\frac{2}{j\omega+4}\right)\Bigg|_{j\omega=-2} = \frac{1}{2}\left(\frac{d}{d(j\omega)}\,\frac{-2}{(j\omega+4)^2}\right) = \frac{1}{2}\left(\frac{4(j\omega+4)}{(j\omega+4)^4}\right)$$

$$= \frac{1}{2} \times \frac{4\times 2}{2^4} = \frac{1}{4}$$

$$A_{12} = \frac{1}{1}\left[\frac{d}{d(j\omega)}\left(\frac{2}{j\omega+4}\right)\Bigg|_{j\omega=-2}\right] = \frac{-2}{(j\omega+4)^2}\Bigg|_{j\omega=-2} = -\frac{1}{2}$$

$$A_{13} = \frac{2}{j\omega+4}\Bigg|_{j\omega=-2} = 1$$

$$A_{21} = \frac{2}{(j\omega+2)^3}\Bigg|_{j\omega=-4} = -\frac{2}{8} = -\frac{1}{4}$$

$$y(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + t^2e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t)$$

9) الف)

$$H_1(j\omega)H_2(j\omega) = 1 \rightarrow H_1(j\omega) = \underset{\overset{\downarrow\,F.T}{\delta(t)}}{1} - \underset{\overset{\downarrow F.T}{3e^{-t}u(t)}}{\frac{3}{1+j\omega}}$$

$$H_2(j\omega) = \frac{1}{H_1(j\omega)} = \frac{1}{1-\frac{3}{j\omega+1}} = \frac{j\omega+1}{j\omega-2} = \frac{j\omega}{j\omega-2} + \frac{1}{j\omega-2}$$

$$= 1 + \frac{2}{j\omega-2} + \frac{1}{j\omega-2} = 1 + \frac{3}{j\omega-2}$$

$$\Rightarrow h_2(t) = \delta(t) + 3e^{2t}u(t)$$

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ب)

$$H(j\omega) = 1 - \frac{3}{j\omega+1} = \frac{j\omega-2}{j\omega+1}$$

$$X(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega+4} = \frac{1}{(j\omega+4)(j\omega+3)}$$

$$\Rightarrow Y(j\omega) = \frac{j\omega-2}{(j\omega+1)(j\omega+3)(j\omega+4)} = \frac{A}{j\omega+1} + \frac{B}{j\omega+3} + \frac{C}{j\omega+4}$$

$$A = \frac{-3}{2\times 3} = -\frac{1}{2} \qquad B = \frac{-5}{-2\times 1} = \frac{5}{2} \qquad C = \frac{-6}{-3\times -1} = 2$$

$$y(t) = -\frac{1}{2}e^{-t}u(t) + \frac{5}{2}e^{-3t}u(t) + 2e^{-4t}u(t)$$