Steady-State Low-Order Explicit (LOE) Runge-Kutta Schemes with Improved Convergence

AIAA Aviation

Zaid Sabri
Ray Hixon

University of Toledo
Toledo Ohio, 43606 USA

June 12 - 16, 2023

Table of Contents

Background

- Computational Fluid Dynamics (CFD) is the study of fluid mechanics that predicts fluid flows using digital computers.
- To accomplish this goal, the analytic governing equations of fluid motion are discretized in space (using structured grid meshing), resulting in a system of coupled ordinary differential equations (ODE).

Implicit Time Marching

- The one-dimensional Euler equation can be written as:

$$\frac{\partial Q}{\partial t} + \left.\frac{\partial E}{\partial x}\right|^{n+1} = 0 \tag{1}$$

- In these equations, the flux E is a nonlinear function of the flow vector Q.
- We linearize the Taylor Series expansion of the fluxes, to rewrite the delta form of the equation as:

$$\underbrace{\left[I + \frac{\partial}{\partial x}\left(\Delta t \cdot \left.\frac{\partial E}{\partial Q}\right|_i^n \right) \right]}_{\text{Numerics}} \{\Delta Q_i\} = -\Delta t \cdot \underbrace{\left\{ \frac{\partial}{\partial x}\left(E_i^n\right) \right\}}_{\text{Physics}} \tag{2}$$

Implicit Time Marching - Preconditioning

- Instead of using:

$$\left[I + \Delta t \left(\frac{A_{i+3} - 8A_{i+2} + 37A_{i+1} - 37A_{i-1} + 8A_{i-2} - A_{i-3}}{48\Delta x}\right)^{n+1,l}\right] \Delta Q_i \;=\; -\Delta t \left\{RHS\right\}_i^{n+1,l} \quad (3)$$

is preconditioned as:

$$\left[I + \Delta t \left(\frac{A_{i+1} - A_{i-1}}{2\Delta x}\right)^{n+1,l}\right] \Delta Q_i \;=\; -\Delta t \left\{RHS\right\}_i^{n+1,l} \quad (4)$$

- This will result in a more efficient solver since the number of diagonals have been reduced.

Stability

- The equation is changed to add a scaling factor σ multiplying the spatial derivative on the LHS:

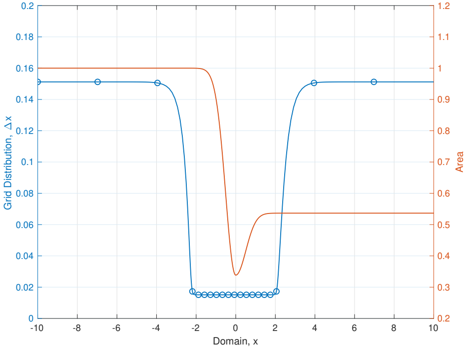
$$\left[I + \Delta t \sigma \left(\frac{A_{i+1} - A_{i-1}}{2\Delta x} \right)^{n+1,l} \right] \Delta Q_i \;=\; -\Delta t \left\{ RHS \right\}_i^{n+1,l} \tag{5}$$

	σ_{E2}	σ_{E4}	σ_{E6}	σ_{DRP}	σ_{RDRP}	σ_{C4}
Periodic	*1.0*	*1.0*	$^{11}/_{10}$	*1.16682*	$^{7}/_{6}$	*1.49982*
Bounded	*1.0*	*2.121796*	*6.352796*	*4.9748714*	*6.3620687*	*3.80018*

Numerical Results for Benchmark Problems

- One Problem from the Third Computational Aeroacoustics (CAA) workshop is analyzed to validate the work done
- The Quasi-1D nonlinear Euler equations are used to solve this problem. The equations are given in the conservative variables as:

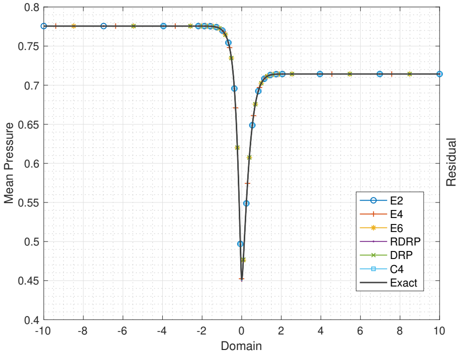
$$\left(\frac{\partial}{\partial t}\begin{Bmatrix} \rho \\ \rho u \\ E_{tot} \end{Bmatrix} + \frac{\partial}{\partial x}\begin{Bmatrix} \rho u \\ \rho u^2 + p \\ u(E_{tot} + p) \end{Bmatrix} + \frac{1}{A}\frac{\partial A}{\partial x}\begin{Bmatrix} \rho u \\ \rho u^2 \\ u(E_{tot} + p) \end{Bmatrix} \right) = 0$$

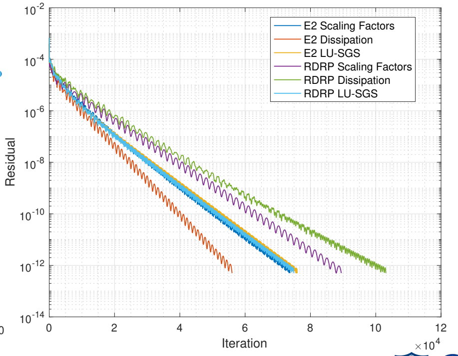


Category 1 Problem 1, Steady State Results

- The mean flow is set as:

$$\begin{Bmatrix} \bar{\rho} \\ \bar{u} \\ \bar{p} \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.4 \\ \frac{1}{\gamma} \end{Bmatrix}$$

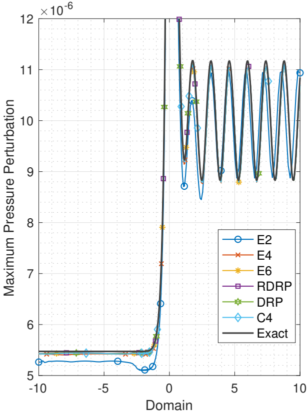


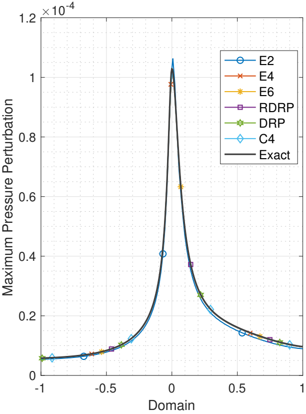


Category 1 Problem 1, Unsteady Results

Table: CFL Values for Category 1 Problem 1

	E2	E4	E6	DRP	RDRP	C4
$\nu_{physical}$	*12.521*	*17.518*	*19.858*	*20.834*	*20.834*	*22.237*





Conclusion

- The present research aimed to establish a highly efficient implicit solver while using higher-order spatial differencing schemes.
- Through scaling factors, the LHS matrix was replaced with an easier to solve matrix, cutting down on the computational work needed for the iterative steady-state solver.
- Although the current study is based on one-dimensional problems, the findings suggest stability is possible if scaling factors are used on the LHS spatial difference.
- The preconditioned matrix was tested against steady and unsteady CAA workshop data; the numerical solution agrees well with the exact.

Acknowledgments

- This work is supported by the NASA Advanced Air Transportation Technologies (AATT) Project.
- The authors would like to thank Dr. Edmane Envia of the NASA Glenn Research Center, who is the technical monitor of this work.

Zaid Sabri
University of Toledo
Toledo Ohio, 43606 USA
+1 (567)377-9009
zaid.sabri@rockets.utoledo.edu