Steady-State Low-Order Explicit (LOE) Runge-Kutta Schemes with Improved Convergence

AIAA Aviation

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June 12 - 16, 2023

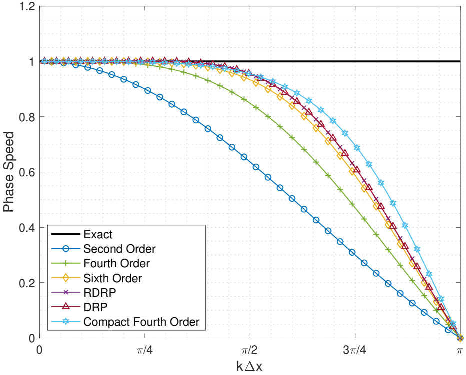
Background

- Computational Fluid Dynamics (CFD) is the study of fluid mechanics that predicts fluid flows using digital computers.
- To accomplish this goal, the analytic governing equations of fluid motion are discretized in space (using structured grid meshing), resulting in a system of coupled ordinary differential equations (ODE).
- ODE are integrated in the temporal direction using numerical integration schemes to advance the flow in time.
- The effectiveness of numerical integration depends on convergence, accuracy, and computational efficiency

Spatial Derivative

- Derivative of flow fluxes can be expressed with values of the function at the neighboring point by using the Taylor series expansion.
- The spatial derivative is calculated using a **finite difference** approach.

$$\left.\frac{\partial u}{\partial x}\right|_{Numerical} = \underbrace{\frac{\partial u}{\partial x}}_{\text{Exact}} \underbrace{\left[\frac{(k^{*}\Delta x)}{(k\Delta x)}\right]}_{\text{Error}}$$



Time Marching

- Starting with an initial state at time zero, given a boundary condition, we can "march" one time-step to a time later and use a numerical scheme to evaluate the solution at that time.
- This process, known as time marching, can be used to obtain steady-state solutions for steady flow problems and time-accurate solutions for unsteady flow problems.
- Time marching is subdivided into two schemes, **explicit** and **implicit**.
- Explicit schemes can be relativley simple to implement, but may require excessive time steps due a bounded stability limit.
- When solving viscous flow problems where the computational domain must be highly clustered near the body, the size of the time step an explicit scheme can take is restricted by the CFL condition.
- The stability restriction can be eliminated by using an implicit time marching scheme.

Implicit Time Marching

- To derive the implicit time marching algorithim, the one-dimensional Euler equation can be written as:

$$\frac{\partial Q}{\partial t} + \left.\frac{\partial E}{\partial x}\right|^{n+1} = 0 \qquad (1)$$

- Linearize the Taylor Series expansion of the fluxes, to rewrite the delta form of the equation as:

$$\underbrace{\left[I + \frac{\partial}{\partial x}\left(\Delta t \cdot \left.\frac{\partial E}{\partial Q}\right|_i^n \right)\right]}_{\text{Numerics}} \{\Delta Q_i\} = -\Delta t \cdot \underbrace{\left\{ \frac{\partial}{\partial x}\left(E_i^n\right)\right\}}_{\text{Physics}} \qquad (2)$$

- Much research has focused on approximating the Numerics due to complexities involved with inverting an unfactored implicit scheme in multi-dimensions

Implicit Time Marching - Prior Work

- **Alternating Direction Implicit** (ADI) is a practical implicit scheme that approximately factorizes the implicit operator in delta form
- The **Lower-Upper Symmetric-Gauss-Seidel** method (LU-SGS) is another commonn approximation where the numerics are preconditioned to a product of three matricie which provides computational efficiency
- A preconditioner, psuedo-time derivative, can be added to the equations to accelerate the time evolution scheme in the unsteady equation, an algorithim first proposed by Anthony Jameson known as **Dual-Time Stepping**.
- Implicit time marching has been introduced with higher-order differencing stencils by Miguel Visbal and Datta Gaitonde from the Air Force Research Lab and Ohio State University, **FDL3DI**

Implicit Time Marching - Preconditioning

- Instead of using:

$$\left[I + \Delta t \left(\frac{A_{i+3} - 8A_{i+2} + 37A_{i+1} - 37A_{i-1} + 8A_{i-2} - A_{i-3}}{48\Delta x} \right)^{n+1,l} \right] \Delta Q_i = -\Delta t \{RHS\}_i^{n+1,l} \quad (3)$$

is preconditioned as:

$$\left[I + \Delta t \left(\frac{A_{i+1} - A_{i-1}}{2\Delta x} \right)^{n+1,l} \right] \Delta Q_i = -\Delta t \{RHS\}_i^{n+1,l} \quad (4)$$

- This will result in a more efficient solver since the number of diagonals have been reduced.

Stability

- The equation is changed to add a scaling factor σ multiplying the spatial derivative on the LHS:

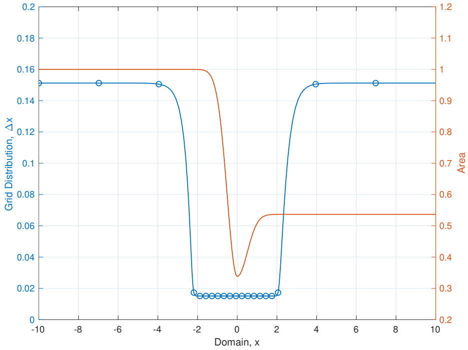
$$\left[I + \Delta t \sigma \left(\frac{A_{i+1} - A_{i-1}}{2\Delta x} \right)^{n+1,l} \right] \Delta Q_i = -\Delta t \{RHS\}_i^{n+1,l} \tag{5}$$

	σ_{E2}	σ_{E4}	σ_{E6}	σ_{DRP}	σ_{RDRP}	σ_{C4}
Periodic	*1.0*	*1.0*	$^{11}/_{10}$	*1.16682*	$^{7}/_{6}$	*1.49982*
Bounded	*1.0*	*2.121796*	*6.352796*	*4.9748714*	*6.3620687*	*3.80018*

Numerical Results for Benchmark Problems

- One Problem from the Third Computational Aeroacoustics (CAA) workshop is analyzed to validate the work done

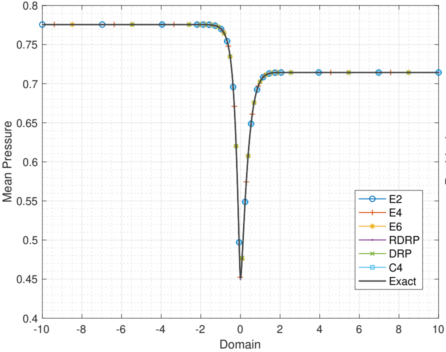
$$\left(\frac{\partial}{\partial t}\begin{Bmatrix} \rho \\ \rho u \\ E_{tot} \end{Bmatrix} + \frac{\partial}{\partial x}\begin{Bmatrix} \rho u \\ \rho u^2 + p \\ u(E_{tot}+p) \end{Bmatrix} + \frac{1}{A}\frac{\partial A}{\partial x}\begin{Bmatrix} \rho u \\ \rho u^2 \\ u(E_{tot}+p) \end{Bmatrix} \right) = 0$$

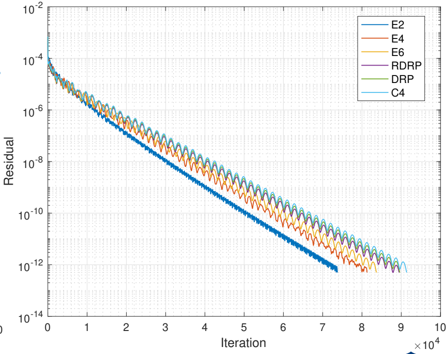


Category 1 Problem 1, Steady State Results

- The mean flow is set as:

$$\begin{Bmatrix} \bar{\rho} \\ \bar{u} \\ \bar{p} \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.4 \\ \frac{1}{\gamma} \end{Bmatrix}$$

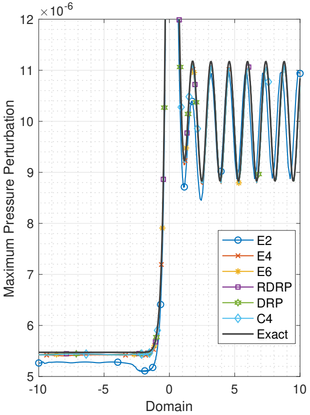


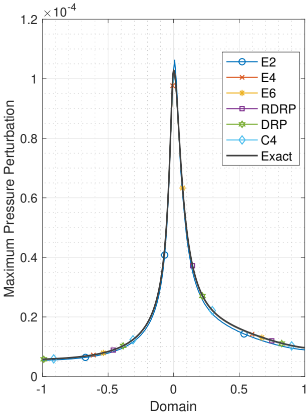


Category 1 Problem 1, Unsteady Results

Table: CFL Values for Category 1 Problem 1

	E2	E4	E6	DRP	RDRP	C4
$\nu_{physical}$	*12.521*	*17.518*	*19.858*	*20.834*	*20.834*	*22.237*





Conclusion

- The present research aimed to establish a highly efficient implicit solver while using higher-order spatial differencing schemes.
- Through scaling factors, the LHS matrix was replaced with an easier to solve matrix, cutting down on the computational work needed for the iterative steady-state solver.
- Although the current study is based on one-dimensional problems, the findings suggest stability is possible if scaling factors are used on the LHS spatial difference.
- The preconditioned matrix was tested against steady and unsteady CAA workshop data; the numerical solution agrees well with the exact.

Acknowledgments

- This work is supported by the NASA Advanced Air Transportation Technologies (AATT) Project.
- The authors would like to thank Dr. Edmane Envia of the NASA Glenn Research Center, who is the technical monitor of this work.

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