Lecture 7 Notes (Dr. Khaled Adarbeh) / Modern Algebra(I)

"Another Subgroup Test"

Theorem:

Let H , K be a subgroup of a group G

1. *$H \cap K$ is a subgroup.*
2. *$H \cup K$ is a subgroup $\iff H \subseteq K$ or $K \subseteq H$*
3. *$\{hk / h \in H, k \in K\}$*

Proof:

1

$1 \in H \cap K$:

$$\left.\begin{array}{l} 1 \in K \text{ (Subgroup)} \\ 1 \in H \text{ (Subgroup)} \end{array}\right\} \implies 1 \in H \cap K$$

$$\left.\begin{array}{ll} x, y \in H \cap K \implies \left.\begin{array}{l} x \in H \\ y \in H \end{array}\right\} \implies xy \in H \\ \boxed{\text{and}} \qquad\qquad\qquad \left.\begin{array}{l} x \in K \\ y \in K \end{array}\right\} \implies xy \in K \end{array}\right\} \implies xy \in H \cap K$$

$$x \in H \cap K \implies \left.\begin{array}{l} x \in H \implies x^{-1} \in H \\ x \in K \implies x^{-1} \in K \end{array}\right\} \implies x^{-1} \in H \cap K$$

2

"$\Longleftarrow$" Trivial

$H \subseteq K \implies H \cup K = K$, where K is subgroup

$\therefore$ $H \cup K$ is subgroup

"$\Longrightarrow$" Assume that:

- $H \cup K$ is subgroup
- $H \not\subseteq K$ (i.e. $\exists\, h_0 \in H \quad k$)

we want to show $K \subseteq H$:

$$\left.\begin{array}{l} \text{Let } k_0 \in K \text{ (arbitrary)} \implies k_0 \in H \cup K \\ \quad h_0 \in K \quad \implies h_0 \in H \cup K \end{array}\right\} \implies k_0 h_0 \in H \cup K \text{ "because } H \cup K \text{ subgroup"}$$

$k_0 h_0 \in H$	or	$k_0 h_0 \in K$
H is subgroup		K is subgroup
$\implies h_0^{-1} \in H$		$\implies k_0^{-1} \in H$
$\implies k h_0 h_0^{-1} \in H$		$\implies k_0^{-1} k h_0 \in k$
$\implies k_0 \in H$		$\implies h_0 \in k$

we have a contradiction.

"$H \not\subseteq K$ *i.e.* $\exists\, h_0 \in H \underbrace{\;/\;}_{\text{except}} K$

Thus $h \cup K$ is subgroup $\iff H \subseteq K$ or $K \subseteq H$

Note:

$H \cup K$ is subgroup in general?

No.

"Center of a group"

Definition:

$z(G) := \{a \in G /\ ax = xa \quad \forall\, x \in G\}$

Note:

this is for the whole group.

Theorem:

$z(G)$ is an abelian subgroup of G

Proof:

$1 \in z(G)$???

$1.x = x.1 = x$ it is true $\therefore 1 \in z(G)$

$\underline{a, b \in z(G) \overset{??}{\Longrightarrow} ab \in z(G)}$

$ax = xa$
$bx = xb$
$(ab)(x) = a(bx)$ "b in center" "Asspciative"
$= a(xb)$ $b \in z(G)$
$= (ax)b$ "Associative"
$= x(ab)$ $a \in z(G)$

$\underline{a \in z(G) \overset{??}{\Longrightarrow} a^{-1} \in z(G)}$

$ax = xa \quad \forall\, x \in G$
$(ax = xa) \times (a^{-1})$ "we multiply from both right and left"
$\Longrightarrow a^{-1}axa^{-1} = a^{-1}xaa^{-1}$
$\Longrightarrow exa^{-1} = a^{-1}xe$
$\Longrightarrow xa^{-1} = a^{-1}x \qquad \forall\, x \in G$

<u>Abelian ??</u>

$ab = ba \quad \forall\, a, b \in z(G) \Longrightarrow z(G)$ Abelian.

<u>Note:</u>

- $z(G)$ is maximum with this property (Abelian) meaning that, $z(G)$ is the largest subgroup we can obtain from G
- if G is abelian group $\Longrightarrow \quad z(G) = G$

"Centralizer of an element"

<u>Definition:</u>

G *group* , $a \in G$
$C(a) = \{x \in G \,/\, ax = xa\}$

<u>Note:</u>

- $C(a)$ this is for each element in the group.
- $z(G) \subseteq C(a) \quad \forall\, a \in G$
- Centralizer is greater than center of a group.

Note

$$z(G) = \bigcap_{a \in G} C(a)$$

Proof:

"$\Longrightarrow$" $\subseteq$

$$\begin{aligned} x \in z(G) &\Longrightarrow xa = ax \\ &\Longrightarrow x \in C(a) \\ &\Longrightarrow x \in \textstyle\bigcap_{a \in G} C(a) \end{aligned}$$

"$\Longleftarrow$" $\supseteq$

$$\begin{aligned} x \in \textstyle\bigcap_{a \in G} C(a) &\Longrightarrow x \in C(a)\,,\ \forall\, a \in G \\ &\Longrightarrow xa = ax\,,\ \forall\, a \in G \\ &\Longrightarrow x \in z(G) \end{aligned}$$

Note:

- $C(a)$ is a subgroup

Example:

Is $C(a)$ Abelian?

Solution:

let G a non abelian group

$e \in G$, $ea = ae = a$

$C(e) = a = G$ "But, G is non abelian from the assumption above"

$\therefore$ $C(e)$ non abelian

Note:

- G abelian $\Longleftrightarrow C(a) = G\,,\ \forall\, a \in G$
  $\Longleftrightarrow z(G) = G$

Example:

$\boxed{\text{Ex: 37}}$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	7	6	5	4	3
3	3	4	5	6	7	8	1	2
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	4	3	2	1	8	7
7	7	8	1	2	3	4	5	6
8	8	7	6	5	4	3	2	1

$$\begin{aligned}
C(1) &= G \\
C(2) &= \{1,2,5,6\} \\
C(3) &= \{1,3,5,7\} \\
C(4) &= \{1,4,5,8\} \\
C(5) &= \{1,5,2,3,4,6,7,8\} \\
C(6) &= \{1,6,5,2\} \\
C(7) &= \{1,7,3,5\} \\
C(8) &= \{1,8,4,5\}
\end{aligned}$$

$z(G) = \bigcap_{\forall a} C(a) = \{1,5\}$ "The intersection of all above sets"

Note: $5^2 = 1$

Example:

$\boxed{\textit{Ex: 79}}$

$G = GL(2,\mathbb{R})$

Find:

1) $C\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right)$ 2) $C\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right)$ 3) $2(G)$

Solution:

1. $$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}$$

$$\begin{bmatrix}a+b & a\\c+d & c\end{bmatrix} = \begin{bmatrix}a+c & b+d\\a & b\end{bmatrix}$$

$$\begin{aligned}
&\Longrightarrow a+b = a+c \Longrightarrow b = c \\
&\Longrightarrow a = b+d \Longrightarrow d = a-b
\end{aligned}$$

$\implies c + d = a$
$\implies b + d = c + d \implies b = c$

$$\therefore \begin{bmatrix} a & b \\ b & a-b \end{bmatrix} \qquad \text{s.t.} \quad a(a-b) - b^2 \neq 0$$

because we want to make sure it belongs to $GL(2, \mathbb{R})$ so $det\left(\begin{bmatrix} a & b \\ b & a-b \end{bmatrix}\right) \neq 0$

$\implies a^2 - ab - b^2 \neq 0$

So, $C\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} a & b \\ b & a-b \end{bmatrix} \Big/ a^2 - ab - b^2 \neq 0\right\}$

2. Answer is:

$$\left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} \Big/ a^2 - b^2 \neq 0\right\}$$

3. Since $z(G) = C_1 \cap C_2 \cap \ldots\ldots \cap C(a)$

we have, $\begin{bmatrix} a & b \\ b & a-b \end{bmatrix} \in C_1$

and since belongs to C_2
$\iff a = a - b$
$\iff b = 0$
$\therefore C_1 \cap C_2 = \left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \Big/ a \neq 0\right\} = aI_n$

Which is $z(G)$

Note

Scalar matrices are abelian
& each scalar multiple is Center