Ex. 12

Q7. A body falls from rest in a liquid whose density is one-fifth that of the body. If the liquid offers resistance proportional to the ~~body~~ velocity and the velocity approaches a limiting value of 10 meters per second, find the distance fallen in 10 seconds.

Soln The equation of the motion of the body is

$$m\frac{dv}{dt} = mg - \frac{1}{4}mg - mkv$$

or
$$\frac{dv}{dt} + kv = \frac{3}{4}g \quad \text{---} (1)$$

The initial conditions are $v=0$ when $t=0$.

where mg, weight acting downwards.

$\frac{1}{4}mg$, upthrust acting upwards.

mkv, resistance acting upwards.

$\therefore$ The Laplace transform of equation (1) is

$$s\bar{v} + k\bar{v} = \frac{3}{4}g/s$$

$$\bar{v} = \frac{3g}{4}\Big/ s(s+k) = \frac{3g}{4k}\left(\frac{1}{s} - \frac{1}{s+k}\right) \quad \text{---} (2)$$

Taking inverse L.T. of (2) we get.

Hence
$$v = \frac{3g}{4k}\left(1 - e^{-kt}\right) \quad \text{---} (3)$$

when $s \to \infty$, $v \to \frac{3g}{4k}$, given that the limiting velocity is 10 meters per second. so

$$\frac{3g}{4k} = 10 \quad \text{or} \quad k = \frac{3g}{40}$$

Equation (3) becomes

$$\frac{dx}{dt} = \frac{3g}{4\times\frac{3g}{40_{10}}}\left(1 - e^{-kt}\right) = 10\left(1 - e^{-kt}\right) \quad \text{---} (4)$$

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and the initial conditions is $x=0$ when $t=0$.

Laplace transform of equation (4) is

$$s\bar{x} = 10\left(\frac{1}{s} - \frac{1}{s+k}\right)$$

i.e. $$\bar{x} = 10\left(\frac{1}{s^2} - \frac{1}{s(s+k)}\right) = 10\left[\frac{1}{s^2} - \frac{1}{k}\left(\frac{1}{s} - \frac{1}{s+k}\right)\right]$$

Taking inverse L.T. we get

$$x = 10\left[t - \frac{1}{k}\left(1 - e^{-kt}\right)\right]$$

$$x = 10\left[t - \frac{40}{3g}\left(1 - e^{-\frac{3g}{40}t}\right)\right]$$

Put. $t=10$ and $g = 9.8\ m/sec^2$ we get

$$x = 10\left[10 - \frac{40}{3\times 9.8}\left(1 - e^{-\frac{3\times 9.8}{40}\times 10}\right)\right]$$

$x = ?$ (Simplify)

Q12. A voltage Ee^{-at} is applied at $t=0$ to a circuit of inductance L and resistance R. Show that the current at time t is

$$\frac{E}{R-aL}\left(e^{-at} - e^{-\frac{R}{L}t}\right).$$

Soln. Let I be the current in the circuit. Then the potential difference across the inductance L is $L\frac{dI}{dt}$ and across the resistance R is RI

$\therefore$ The equation of the electric circuit is

$$L\frac{dI}{dt} + RI = Ee^{-at} \quad \text{--- (1)}$$

Laplace transform of equation (1) is

$$(Ls+R)\bar{I} = \frac{E}{s+a}$$

$$\bar{I} = \frac{E}{(Ls+R)(s+a)} = \frac{E/L}{(s+R/L)(s+a)} \quad \text{--- (2)}$$

$$\bar{I} = \frac{E/L}{\frac{R}{L} - a}\left(\frac{1}{s+a} - \frac{1}{s+R/L}\right)$$

$$\bar{I} = \frac{E}{R-aL}\left(\frac{1}{s+a} - \frac{1}{s+R/L}\right) \quad \text{--- (3)}$$

Taking inverse L.T. of (3) we get.

$$I = \frac{E}{R-aL}\left(e^{-at} - e^{-\frac{R}{L}t}\right).$$

Q14. An alternating voltage $E \sin \omega t$ is applied at $t=0$ to a circuit of inductance L and resistance R. If the initial current be zero, show that the current at time t is

$$\frac{E}{\sqrt{R^2+L^2\omega^2}}\left[e^{-\frac{R}{L}t}\sin\gamma + \sin(\omega t - \gamma)\right],$$

where $\tan\gamma = \frac{L\omega}{R}$.

Soln. The equation of the electric circuit is

$$L\frac{dI}{dt} + RI = E\sin\omega t \quad \text{--- (1)}$$

where $L\frac{dI}{dl}$ is the potential difference across the inductance L

RI " " " the resistance R

I be the current in the circuit.

Taking L.T. of equation (1) we get -

$$(Ls+R)\bar{I} = \frac{E\omega}{s^2+\omega^2}$$

$$\bar{I} = \frac{E\omega}{(Ls+R)(s^2+\omega^2)} = \frac{\omega E/L}{(s+R/L)(s^2+\omega^2)}$$

$$= \frac{E\omega/L}{\omega^2+R^2/L^2}\left(\frac{1}{s+R/L} - \frac{s-R/L}{s^2+\omega^2}\right) \quad \text{by inspection}$$

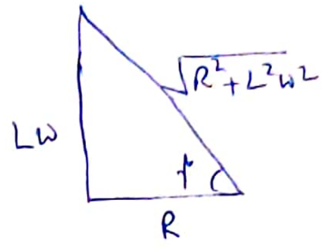
$$\bar{I} = \frac{LEw}{L^2w^2+R^2}\left(\frac{1}{s+R/L} - \left(\frac{s-R/L}{s^2+w^2}\right)\right) \quad — ②$$

$$\left(\frac{s}{s^2+w^2} - \frac{R/L}{s^2+w^2}\right)$$

Taking inverse L.T. of ② we get

$$I = \frac{LEw}{L^2w^2+R^2}\left(e^{-\frac{R}{L}t} - \cos wt + \frac{R}{Lw}\sin wt\right)$$

$\tan\phi = \frac{Lw}{R}$ (given)



$\therefore$ $\sin\phi = \frac{Lw}{\sqrt{R^2+L^2w^2}} \Rightarrow Lw = \sqrt{R^2+L^2w^2}\,\sin\phi$

$\cos\phi = \frac{R}{\sqrt{R^2+L^2w^2}} \Rightarrow R = \sqrt{R^2+L^2w^2}\,\cos\phi$

$$I = \frac{E\cancel{Lw}\sqrt{R^2+L^2w^2}\,\sin\phi}{R^2+L^2w^2}\left(e^{-\frac{R}{L}t} - \cos wt + \frac{\cos\phi}{\sin\phi}\cdot\sin wt\right)$$

$$= \frac{E}{\sqrt{R^2+L^2w^2}}\left[e^{-\frac{R}{L}t}\sin\phi - \cos wt\sin\phi + \cos\phi\sin wt\right]$$

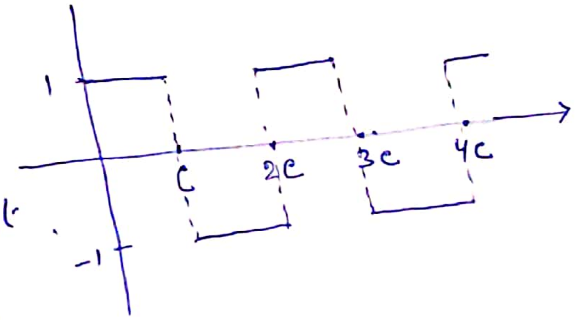
$$= \frac{E}{\sqrt{R^2+L^2w^2}}\left[e^{-\frac{R}{L}t}\sin\phi + \sin(wt-\phi)\right].$$

Laplace Transform of a periodic functions

Definition A function $f(x)$ is said to be periodic, if and only if $f(x+p) = f(x)$ —① is true for some value of p and every value of x. The smallest positive value of p for which equation ① is true for every value of x will be called the periodic function of period p.

Laplace transform of the square wave function

Consider the square wave function with period $2c$. Then its Laplace transform is equal to



$$\int_0^c e^{-st}\cdot 1\,dt + \int_c^{2c} e^{-st}(-1)\,dt + \int_{2c}^{3c} e^{-st}\cdot 1\,dt + \int_{3c}^{4c} e^{-st}(-1)\,dt + \cdots$$

$$= \int_0^c e^{-st}\,dt - \int_c^{2c} e^{-st}\,dt + \int_{2c}^{3c} e^{-st}\,dt - \int_{3c}^{4c} e^{-st}\,dt + \cdots$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^c - \left[\frac{e^{-st}}{-s}\right]_c^{2c} + \left[\frac{e^{-st}}{-s}\right]_{2c}^{3c} - \left[\frac{e^{-st}}{-s}\right]_{3c}^{4c} + \cdots$$

$$= \frac{1}{s}\left[(1 - e^{-cs}) - (e^{-cs} - e^{-2cs}) + (e^{-2cs} - e^{-3cs}) - (e^{-3cs} - e^{-4cs}) + \cdots\right]$$

$$= \frac{1}{s}\left[1 - 2e^{-cs} + 2e^{-2cs} - 2e^{-3cs} + 2e^{-4cs} - \cdots\right]$$

$$= \frac{1}{s}\left[\frac{2}{1+e^{-cs}} - 1\right] = \frac{1}{s}\left[\frac{1-e^{-cs}}{1+e^{-cs}}\right] = \frac{1}{s}\left[\frac{e^{cs/2} - e^{-cs/2}}{e^{cs/2} + e^{-cs/2}}\right] = \frac{1}{s}\tanh\frac{cs}{2}.$$

Ex 1: Show that the Laplace transform of a periodic function $f(t)$ of period c is

$$L\{f(t)\} = \frac{1}{1-e^{-cs}} \int_0^c e^{-st} \cdot f(t)\,dt$$

Soln

By definition of L.T.

$$\cancel{\int_0^\infty e^{-st} f(t)\,dt = \int_0^c e^{-st} f(t)\,dt + \int_c^{2c} e^{-st} f(t)\,dt + \int_{2c}^{3c} e^{-st} f(t)\,dt + \cdots} \quad \cancel{—(1)}$$

~~Now put $t = u + nc$ we get.~~

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^c e^{-st} f(t)\,dt + \int_c^\infty e^{-st} f(t)\,dt \quad —(1)$$

In the second integral of equation (1), put $t = u+c$ $\therefore dt = du$

$$\therefore L\{f(t)\} = \int_0^c e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+c)} f(u+c)\,du$$

$$= \int_0^c e^{-st} f(t)\,dt + e^{-sc} \int_0^\infty e^{-su} f(u)\,du$$ (Since $f(u+c) = f(u)$) periodic

$$\therefore L\{f(t)\} = \cancel{\int_0^c e^{-su} f(u)\,du + e^{-sc} \int_0^\infty e^{-su} f(u)\,du}$$

$$\cancel{= (1-e^{-sc}) \int_0^c e^{-su} f(u)\,du}$$

$$= \int_0^c e^{-st} f(t)\,dt + e^{-cs} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^c e^{-st} f(t)\,dt + e^{-cs} L\{f(t)\}$$

$$\Rightarrow (1-e^{-cs}) L\{f(t)\} = \int_0^c e^{-st} f(t)\,dt$$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-cs}} \int_0^c e^{-st} f(t)\,dt.$$

Q. Find the L.T. of the square wave function of period $2c$, using the above example 1°.

Ex2° Find the Laplace transform of a square wave function

$$f(t) = \begin{cases} E, & \text{for } 0 \le t \le a/2 \\ -E, & \text{for } a/2 \le t < a \end{cases} \quad \text{and } f(t+a) = f(t).$$

Soln. Since $f(t)$ is a periodic function of period a, by Ex1° above

$$L\{f(t)\} = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt \quad —①$$

Now $$\int_0^a e^{-st} f(t)\, dt = \int_0^{a/2} e^{-st} f(t)\, dt + \int_{a/2}^{a} e^{-st} f(t)\, dt.$$

$$= \int_0^{a/2} e^{-st} E\, dt + \int_{a/2}^{a} e^{-st} (-E)\, dt.$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - E\left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}$$

$$= E\left[\frac{1-e^{-sa/2}}{s}\right] - E\left[\frac{e^{-sa/2} - e^{-sa}}{s}\right]$$

$$= \frac{E}{s}\left[1 - 2e^{-sa/2} + e^{-sa}\right] = \frac{E}{s}\left(1-e^{-sa/2}\right)^2$$

From ①
$$L\{f(t)\} = \frac{E(1-e^{-sa/2})^2}{s(1-e^{-sa})} = \frac{E(1-e^{-sa/2})^{\cancel{2}}}{s\,\cancel{(1-e^{-sa/2})}(1-e^{-sa/2})}$$

$$= \frac{E}{s}\left[\frac{1-e^{-sa/2}}{1+e^{-sa/2}}\right] = \frac{E}{s}\left[\frac{e^{sa/4} - e^{-sa/4}}{e^{sa/4} + e^{-sa/4}}\right] = \frac{E}{s}\tanh\frac{sa}{4}$$

Q. Find the Laplace transform of the function $f(t)$ with period $2\pi/\omega$ where

$$f(t) = \begin{cases} \sin\omega t, & \text{for } 0 < t < \pi/\omega \\ 0, & \text{for } \omega < t < 2\pi/\omega \end{cases}.$$

Q. Find the Laplace transform of the function

$$f(t) = \begin{cases} t, & \text{for } 0 < t < a \\ 2a - t, & \text{for } a < t < 2a \end{cases} \quad \text{and } f(t+2a) = f(t).$$

Soln. Since $f(t)$ is a periodic function of period $2a$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt \quad \text{— (1)}$$

Now $$\int_0^{2a} e^{-st} f(t)\,dt = \int_0^a e^{-st} \cdot t\, dt + \int_a^{2a} e^{-st}(2a-t)\,dt$$

$$= \int_0^a \underset{II}{e^{-st}} \cdot \underset{I}{t}\, dt + \int_a^{2a} e^{-st} \cdot 2a\, dt - \int_a^{2a} \underset{II}{e^{-st}} \cdot \underset{I}{t}\, dt$$

$$= \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]_0^a + 2a\left[\frac{e^{-st}}{-s}\right]_a^{2a} - \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]_a^{2a}$$

$$= \left\{-\frac{a}{s} e^{-as} - \frac{1}{s^2}(1 - e^{-as})\right\} + \frac{2a\, e^{-as}}{s}(1 - e^{-as})$$

$$+ \left\{\frac{2a}{s} e^{-2as} + \frac{1}{s^2} e^{-2as} - \frac{a}{s} e^{-as} - \frac{1}{s^2} e^{-as}\right\}$$

$$= \frac{1}{s^2}\left(1 - 2e^{-as} + e^{-2as}\right) = \frac{(1-e^{-as})^2}{s^2}.$$

Q. Find the Laplace transform of $f(t) = \sin\left(\frac{\pi t}{a}\right)$ for $0 < t < a$, the rectified wave function of period a.

Soln. $$L\{f(t)\} = \int_0^a e^{-st} \sin\left(\frac{\pi t}{a}\right) dt = \text{Imaginary part of } \int_0^a e^{-st} e^{i\frac{\pi t}{a}}\, dt$$

$$= \text{Im. part}\left[\frac{e^{(-s+i\frac{\pi}{a})t}}{-s+i\pi/a}\right]_0^a = \text{Im part of}\left[\frac{e^{(-s+\frac{i\pi}{a})a} - 1}{-s+i\pi/a}\right]$$

$$= \text{Im. part of}\left[\frac{e^{-as}(\cos\pi + i\sin\pi) - 1}{-s + i\pi/a}\right]$$

$$= \text{Im. part of} \left[\frac{1 - e^{-as}(\cos \pi + i \sin \pi)}{(s - i\pi/a)} \right]$$

$$= \text{Im part of} \left[\frac{(1 + e^{-as})(s + i\pi/a)}{(s - i\pi/a)(s + i\pi/a)} \right]$$

$$= \cancel{\text{Im part of}} \left[\frac{(1 + e^{-as})\frac{\pi}{a}}{(a^2 s^2 + \pi^2)/a^2} \right] = \frac{(1 + e^{-as})\, a\pi}{a^2 s^2 + \pi^2}$$

$$\therefore \quad L\{f(t)\} = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-as}} \cdot \frac{(1 + e^{-as})\, a\pi}{a^2 s^2 + \pi^2}$$

$$= \frac{e^{as/2} + e^{-as/2}}{e^{as/2} - e^{-as/2}} \cdot \frac{a\pi}{a^2 s^2 + \pi^2}$$

$$= \frac{a\pi \coth \frac{as}{2}}{a^2 s^2 + \pi^2}.$$