Inverse Laplace Transform

Example 9

Q.1 Find $L^{-1}\left\{\frac{1}{s(s-1)}\right\}$

Soln $\frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}$ By partial fraction

$L^{-1}\left\{\frac{1}{s(s-1)}\right\} = e^{t} - t$

Q2 Find $L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\}$

Soln $\frac{a^2}{s(s+a)^2} = \frac{A}{s} + \frac{B}{(s+a)^2} + \frac{C}{s+a}$

Multiplying both sides by $(s+a)^2$ and taking $s=0$ we get -

$A = 1$

Multiplying both sides by $(s+a)^2$ and put $s=-a$ we get -

$B = -a$

multiplying both sides both sides by $(s+a)$ and taking limit - $s \to \infty$ we get $C = -1$

$\therefore \frac{a^2}{s(s+a)^2} = \frac{1}{s} - \frac{a}{(s+a)^2} - \frac{1}{s+a}$

$L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\} = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{a}{(s+a)^2}\right\} - L^{-1}\left\{\frac{1}{s+a}\right\}$

$= 1 - ate^{-at} - e^{-at}$

Q7 Find $L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}$

Soln we know that $\frac{s}{(s+1)^2(s^2+1)} = \frac{s}{(s^2+2s+1)(s^2+1)}$

$= \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{1}{(s+1)^2}\right]$

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= \frac{1}{2}\sin t - \frac{1}{2}t\,e^{-t}.$$

Q10 Find $L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Soln

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} - \frac{s-2}{s^2+2s+5}$$ By partial fraction

$$= \frac{1}{s-1} - \frac{s+1}{s^2+2s+5} + \frac{3}{s^2+2s+5}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{(s+1)^2+2^2}$$

$$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + L^{-1}\left\{\frac{3}{(s+1)^2+2^2}\right\}$$

$$= e^{t} - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t.$$

Q15 Find $L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}$

Soln.

$$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+1)^2-s^2} = \frac{s}{(s^2-s+1)(s^2+s+1)}$$

$$= \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right]$$ By inspection

$$\therefore L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}\left((s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2\right)}\right\} - \frac{1}{2}L^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}\left((s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2\right)}\right\}$$

$$= \frac{1}{2}\times\frac{2}{\sqrt{3}}\,e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}\times\frac{2}{\sqrt{3}}\,e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{\sqrt{3}}\left[\sin\frac{\sqrt{3}}{2}t\left(2\times\frac{e^{t/2}-e^{-t/2}}{2}\right)\right] = \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\,\sinh\frac{t}{2}$$

Q18 Find $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}$

Soln $\frac{1}{(s^2+2s+5)^2} = \frac{1}{(s+1)^2+2^2}$

$$L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} = \frac{1}{16}(\sin 2t - 2t\cos 2t)$$

Q. Find $L^{-1}\left\{\log\left(1+\frac{a}{s}\right)\right\}$

Soln Let $F(s) = \log\left(1+\frac{a}{s}\right) = \log\left(\frac{s+a}{s}\right) = \log(s+a) - \log s$

Differentiating w.r.t. s we get.

$$F'(s) = \frac{1}{s+a} - \frac{1}{s}$$

we know that

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s}\right\} = -\frac{1}{t}\left(e^{-at} - 1\right)$$

Q. Find $L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}$

Soln Let $F(s) = \tan^{-1}\left(\frac{1}{s}\right)$

$$F'(s) = \frac{1}{1+\left(\frac{1}{s}\right)^2}\cdot\left(\frac{-1}{s^2}\right) = -\frac{1}{\left\{1+\left(\frac{1}{s}\right)^2\right\}s^2} = -\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{-1}{s^2+1}\right\} = \frac{1}{t}\sin t$$

Laplace Transforms of derivatives and Integrals

Let $f(t)$ be a function whose transform is $\bar{f}(s)$ and for which $\lim_{t\to\infty} e^{-st} f(t) = 0$, then Laplace transform of the differential Co-efficient $\frac{d}{dt}[f(t)]$ is $s\bar{f}(s) - f(0)$

In general $L\{D^n f(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0)$
$- \dots f^{(n-1)}(0)$

Theorem 1: If $\bar{f}(s)$ is the transform of $f(t)$, then $\frac{1}{s}\bar{f}(s)$ is the transform of $\int_0^t f(u)\,du$.

or $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s} L\{f(t)\}$.

Ex 1. Find $L\left\{\int_0^t \frac{\sin u}{u}\,du\right\}$

Soln we know that $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$

$\therefore L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s} L^{-1}\left\{\frac{\sin t}{t}\right\} = \frac{1}{s}\cot^{-1} s$.

Theorem 2: (Convolution Theorem) If $\bar{f}_1(s)$ and $\bar{f}_2(s)$ are the transforms of $f_1(t)$ and $f_2(t)$, then $\bar{f}_1(s)\cdot\bar{f}_2(s)$ is the transform of $\int_0^t f_1(u)\, f_2(t-u)\,du$.

Examples 10

Q2. Given that $L\left\{2\sqrt{t/\pi}\right\} = \frac{1}{s^{3/2}}$, then

$$L\left\{2\frac{d}{dt}\sqrt{t/\pi}\right\} = s\cdot\frac{1}{s^{3/2}} - 2\sqrt{0/\pi} = \frac{1}{s^{1/2}}$$

$$L\left\{\frac{1}{\pi}\frac{1}{\sqrt{t/\pi}}\right\} = \frac{1}{s^{1/2}}$$

$$\therefore\ L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{s^{1/2}}.$$

Find $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$

Q5 we know that

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = L^{-1}\left\{\frac{1}{a}\cdot\frac{a}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

$$\therefore\ L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin au\,du = \frac{1}{a}\left[-\frac{\cos au}{a}\right]_0^t$$

$$= \frac{1}{a^2}(1-\cos at)$$

Again

$$L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t \frac{1}{a^2}(1-\cos au)\,du$$

$$= \frac{1}{a^2}\left[u - \frac{\sin au}{a}\right]_0^t = \frac{1}{a^3}(at - \sin at).$$

Q6. Find $L\{t e^{at}\sin at\}$ by applying the above theorem, ?.

Soln we know that

$$L\{e^{at}\sin at\} = \frac{a}{(s-a)^2+a^2} = \frac{a}{s^2-2as+2a^2}$$

$$L\{t e^{at}\sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2-2as+2a^2}\right)$$

$$= \frac{2a(s-a)}{(s^2+2as+2a^2)^2}.$$

Q10. Find the Laplace transform of $\frac{1}{t}(e^{-at} - e^{-bt})$

Soln we know that

$$L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\therefore L\left\{\frac{1}{t}(e^{-at} - e^{-bt})\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds.$$

(by Theorem $L\{\frac{1}{t}f(t)\} = \int_s^\infty L\{f(t)\} ds$)

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

Q14. Apply convolution Theorem to evaluate-

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

Soln we know that $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ and

$$L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$$

$\therefore$ By convolution theorem

$$L^{-1}\left\{\left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+b^2}\right)\right\} = \int_0^t \cos au \cos b(t-u)\, du.$$

$$= \frac{1}{2}\int_0^t 2\cos au \cos b(t-u)\, du$$

$$= \frac{1}{2}\left[\int_0^t \{\cos(au - bu + bt) + \cos(au + bu - bt)\}\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(au - bu + bt)}{a-b} + \frac{\sin(au + bu - bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b} \right]$$

$$= \frac{a \sin at - b \sin bt}{a^2 - b^2}$$

Q15 Find $L^{-1}\left\{ \frac{1}{s(s^2-a^2)} \right\}$ by convolution theorem.

Soln we know that. $L^{-1}\left\{ \frac{1}{s+a} \right\} = e^{-at}$, then.

$$L^{-1}\left\{ \frac{1}{s(s+a)} \right\} = \int_0^t e^{-at}\, dt = \frac{1}{a}(1 - e^{-at})$$

Also. $L^{-1}\left\{ \frac{1}{s-a} \right\} = e^{at}$.

Then by convolution theorem.

$$L^{-1}\left\{ \frac{1}{s(s+a)} \times \frac{1}{s+a} \right\} = \int_0^t \frac{1}{a}(1 - e^{-au})\, e^{a(t-u)}\, du$$

$$= \frac{1}{a}\left[\int_0^t \left\{ e^{a(t-u)} - e^{a(t-2u)} \right\} du \right.$$

$$= \frac{1}{a^2}\left[- e^{a(t-u)} + \frac{1}{2} e^{a(t-2u)} \right]_0^t$$

$$= \frac{1}{a^2}\left[-1 + \frac{1}{2} e^{-at} + e^{at} - \frac{1}{2} e^{at} \right]$$

Q. Solve the integral equation $y + \int_0^t y\, dt = 1 - e^{-t}$ —(1)

Soln. Taking L.T. of equation (1) we have

$$L(y) + L\left\{ \int_0^t y\, dt \right\} = L\{1\} - L\{e^{-t}\} \quad —(2)$$

let $L(y) = \bar{y}$, and we know that $L\left\{ \int_0^t y\, dt \right\} = \frac{L(y)}{s} = \frac{\bar{y}}{s}$

Hence (2) becomes

$$\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s} - \frac{1}{s+1} \quad \text{i.e.} \quad \frac{1}{s}(s+1)\,\bar{y} = \frac{(s+1)-s}{s(s+1)}$$

$$\therefore \quad \bar{y} = \frac{1}{(s+1)^2} \quad \text{Hence } y = t\,e^{-t}$$