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Q6. Evaluate the double integral $\iint_R \log\left(\frac{x-y}{x+y}\right) dy\, dx$; where R is the region bounded by the lines. $x-3y=1$; $x+y=1$ and $x=4$.

Soln.

$$I = \iint_R \log\left(\frac{x-y}{x+y}\right) dy\, dx$$

Let $u = x-y$ —ⓐ

& $v = x+y$ --ⓑ

from ⓐ and ⓑ

$$x = \frac{1}{2}(u+v)$$

$$y = \frac{1}{2}(v-u)$$

Now,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\Rightarrow J = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{4} - \left(\frac{-1}{4}\right) = \boxed{1/2}.$$

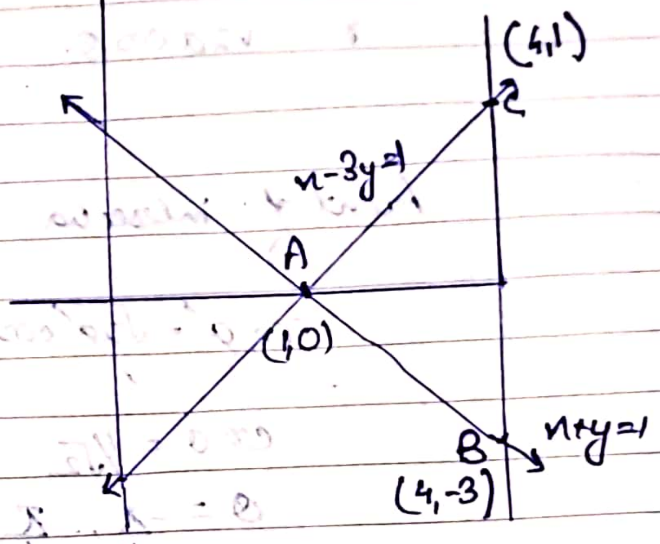
$(x-y)_{min} = 1 \Rightarrow u_{min} = 1$

$(x+y)_{max} = 7 \Rightarrow u_{max} = 7.$

$(x+y)_{min} = 1 \Rightarrow V_{min} = 1$

$(x+y)_{max} = 5 \Rightarrow V_{max} = 5.$

$$\therefore I = \iint_R f(x,y)\, dy\, dx = $$



$$\iint f(u,v)\, J\, du\, dv$$

$$\Rightarrow I = \int_1^7 \int_1^5 \log\left(\frac{u}{v}\right) \frac{1}{2}\, dv.\, du$$

$$I = \frac{1}{2} \int_1^7 du \int_1^5 \log(u) - \log v \; dv$$

$$I = \frac{1}{2} \int_1^7 du \left[v \log u - v \log v + v \right]_1^5 \qquad \boxed{\int \log x = x(\log x - 1)}$$

$$= \frac{1}{2} \int_1^7 du \left\{ (5 \log u - 5 \log 5 + 5) - (\log u - \log 1 + 1) \right\}$$

$$= \frac{1}{2} \int_1^7 4 \log u - 5 \log 5 + 4 \; du$$

$$= \frac{1}{2} \Big[4[\log u(u) - u] - 5 \log 5 \times u + 4u \Big]_1^7$$

$$= \frac{1}{2} \Big[\{4(7 \log 7 - 7) - (5 \log 5 - 4) \times 7\} - \{4(\log 1 - 1) - (5 \log 5 - 4) \times 1\} \Big]$$

$$= \frac{1}{2} \left[28 \log 7 - \cancel{28} - 35 \log 5 + \cancel{28} + \cancel{4} + 5 \log 5 - \cancel{4} \right]$$

$$= \boxed{14 \log 7 - 15 \log 5} \text{ Ans.}$$

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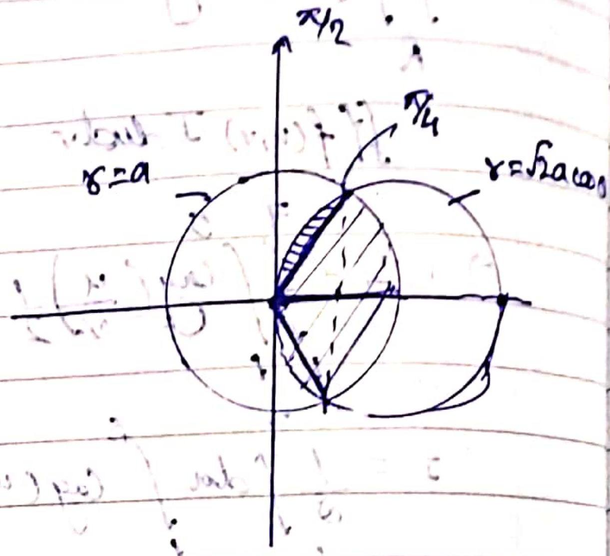
Q7. Find the area common to the circel $r=a$ & $r=\sqrt{2}a\cos\theta$



Point of intersecti

(1)

$r = a = \sqrt{2}a\cos\theta$

$\cos\theta = 1/\sqrt{2}$

$\theta = -\frac{\pi}{4},\ \frac{\pi}{4}$

$\therefore$

$$\text{Area} = \iint r\,dr\,d\theta$$

$$2\left[\int_0^{\pi/4}\int_0^{a} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}a\cos\theta} r\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{\pi/4} d\theta\ \frac{r^2}{2}\Big|_0^{a} + \int_{\pi/4}^{\pi/2} \frac{r^2}{2}\Big|_0^{\sqrt{2}a\cos\theta} d\theta\right]$$

$$= 2\left[\int_0^{\pi/4}\frac{a^2}{2}d\theta + \int_{\pi/4}^{\pi/2} a^2\cos^2\theta\, d\theta\right]$$

$$= \int_0^{\pi/4} a^2 d\theta + \int_{\pi/4}^{\pi/2} 2a^2\cos^2\theta\, d\theta$$

$$\Rightarrow \frac{\pi}{4}a^2 + 2a^2\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{\pi}{4}a^2 + 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

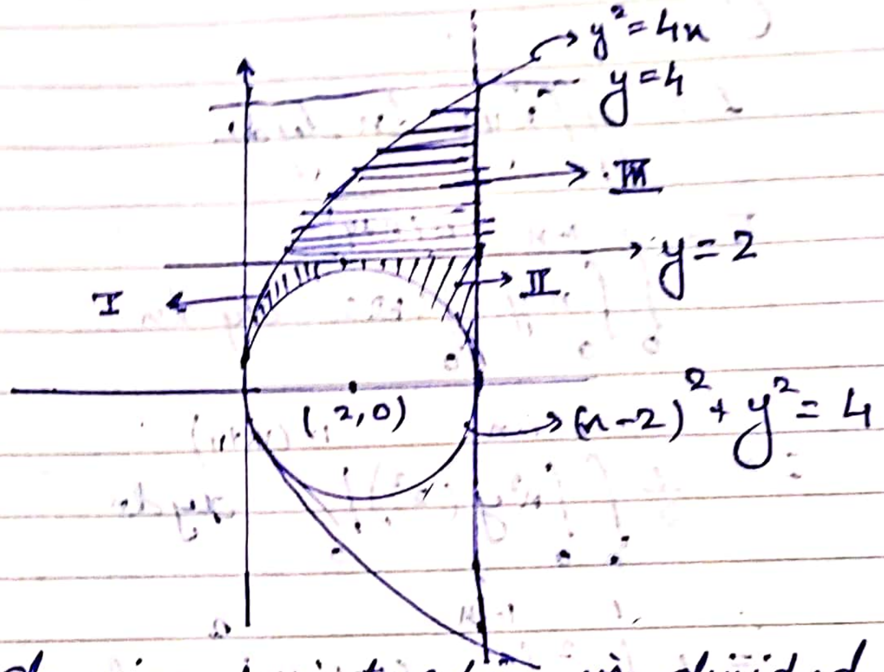
$$\Rightarrow \frac{\pi}{4}a^2 + a^2\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$$

$$= \frac{\pi a^2}{4} + a^2\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi a^2 \times \cancel{2}}{\cancel{4}\,2} - \frac{a^2}{2}$$

$$= \boxed{a^2\left(\frac{\pi}{2} - 1\right)}$$ Ans.

Q8. Change the order of integration $\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x,y)\, dy\, dx$



In this way domain of integration is divided into three sub-regions, I, II, III to each of which corresponds a double integral.

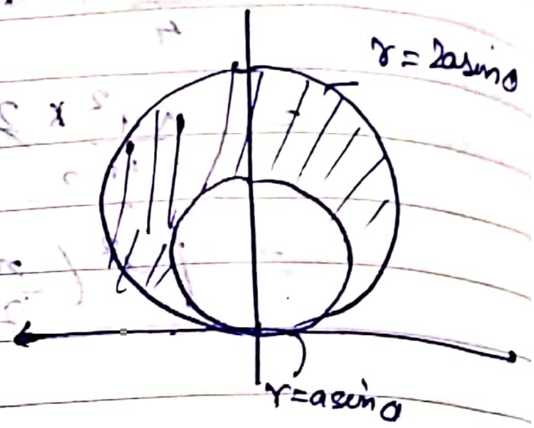
Thus, we get

$$\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x,y)\, dy\, dx = \underbrace{\int_0^2 \int_{\frac{y^2}{4}}^{2-\sqrt{4-y^2}} f(x,y)\, dy\, dx}_{\text{Part I}} +$$

$$\underbrace{\int_0^2 \int_{2+\sqrt{4-y^2}}^{4} f(x,y)\, dy\, dx}_{\text{Part II}} + \underbrace{\int_2^4 \int_{y^2/4}^{4} f(x,y)\, dy\, dx}_{\text{Part III}}$$

Q3. Evaluate $\iint r^2\,dr\,d\theta$, over the area between the circles $r = a\sin\theta$ and $r = 2a\sin\theta$

Soln.



$$\text{Area} = 2\int_0^{\pi/2}\int_{a\sin\theta}^{2a\sin\theta} r^2\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left.\frac{r^3}{3}\right|_{a\sin\theta}^{2a\sin\theta} d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} (2a\sin\theta)^3 - (a\sin\theta)^3\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} 8a^3\sin^3\theta - a^3\sin^3\theta\,d\theta$$

$$= \frac{2\times 7a^3}{3}\int_0^{\pi/2} \sin^3\theta\,d\theta$$

using Gamma function.

$$= \frac{14a^3}{3} \times \frac{\Gamma\frac{3+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{3+0+2}{2}}$$

$$= \frac{14a^3}{3} \times \frac{\Gamma 2 \cdot \Gamma \frac{1}{2}}{2\,\Gamma\frac{5}{2}}$$

$$= \frac{\overset{7}{\cancel{14}}a^3}{\cancel{3}} \times \frac{1! \times \cancel{\Gamma\frac{1}{2}}}{\cancel{2}\times\frac{\cancel{3}}{2}\times\frac{1}{2}\cancel{\Gamma\frac{1}{2}}}$$

$$\Rightarrow \boxed{\frac{28a^3}{9}\ \text{Ans}}$$

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Q10. Evaluate $\iiint x^2 y z \, dx\,dy\,dz$ over the region bounded by planes $x=0, y=0, z=0$ and $x+y+z=1$

<u>Solution:</u>

$$I = \iiint x^2 y z \, dx\,dy\,dz$$

$$I = \int_0^1 \int_0^{1-x} x^2 y \int_0^{1-(x+y)} z\,dz \; dy\,dx$$

$$I = \frac{1}{2} \int_0 \int_0 x^2 y (z^2) \Big|_0^{1-(x+y)} dy\,dx$$

$$= \frac{1}{2} \int_0^1 x^2 \int_0^{1-x} y (1-(x+y))^2 \, dy\,dx$$

$$= \frac{1}{2} \int_0^1 x^2 \int_0^{1-x} (1-x)^2 y + y^3 - 2(1-x) y^2 \, dy\,dx$$

$$= \frac{1}{2} \int_0^1 x^2 (1-x)^2 \frac{y^2}{2} + \frac{y^4}{4} - \frac{2}{3}(1-x) y^3 \Big|_0^{1-x}$$

$$= \frac{1}{2} \int_0^1 x^2 \Big[\frac{(1-x)^2 (1-x)^2}{2} + \frac{(1-x)^4}{4} - \frac{2}{3}(1-x).(1-x)^3 \Big] dx$$

$$= \frac{1}{2} \int_0^1 x^2 (1-x)^4 \Big\{ \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \Big\} dx$$

$$= \frac{1}{2} \times \frac{1}{12} \int_0^1 x^2 (1-x)^4 dx$$

using substitution method

$$1-x = t.$$

$$-dx = dt$$

$$I = \frac{1}{2} \times \frac{1}{12} \int_0^1 t^4 . (1+t^2 - 2t) \} dt$$

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$$= \frac{1}{24}\int_0^1 (t^4 + t^6 - 2t^5)\, dt$$

$$= \frac{1}{24}\left[\frac{t^5}{5} + \frac{t^7}{7} - \frac{2t^6}{6}\right]_0^1$$

$$= \frac{1}{24}\left[\frac{21+15-35}{7\times 5\times 3}\right]$$

$$= \frac{1}{24}\times\frac{1}{105}$$

$$= \frac{1}{2520} \text{ Ans.}$$

Q11. If $L(y) = \log\frac{(s+a)}{(s+b)}$ find y?

Let

$$F(s) = -\frac{d}{ds}(P(s)) \text{ where } P(s) = \ln\frac{(s+a)}{(s+b)}$$

$$F(s) = -\frac{d}{ds}[\ln(s+a) - \ln(s+b)]$$

$$F(s) = -\frac{1}{s+a} + \frac{1}{s+b} \quad \text{--- (i)}$$

we know,

$$L(y) = \int_s^\infty F(s^*)\,ds^* = -\int_s \frac{d}{ds}P(s^*)\,ds^* = \int_s^\infty \left[\frac{1}{s^*+b} - \frac{1}{s^*+a}\right]ds$$

$$= \left[\ln\frac{(s^*+b)}{(s^*+a)}\right]_s^\infty = \ln\frac{s+a}{s+b}$$

using theorem,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s^*)\,ds^* \qquad s > \alpha$$

$$y = \frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s^*)\,ds^*\right]$$

Now,

$$F(t) = L^{-1}\{f(s)\} = e^{-bt} - e^{-at} \text{ from (i)}$$

$$y(t) = \frac{f(t)}{t} = \frac{e^{-bt} - e^{-at}}{t}$$

$$\therefore \quad y(t) = \frac{1}{t}\left(e^{-bt} - e^{-at}\right)$$

Ans

Q12. Solve the integral equation using Laplace Transform.

$$y + \int_0^t y\,dt = 1 - e^{-t}$$

Soln. using Laplace transformation on both sides

$$\mathcal{L}\left\{y + \int_0^t y\,dt\right\} = \mathcal{L}\{1 - e^{-t}\}$$

$$\mathcal{L}\{y(t)\} + \mathcal{L}\left\{\int_0^t y\,dt\right\} = \mathcal{L}[1] - \mathcal{L}\{e^{-t}\} \quad \text{-(i)}$$

$$\mathcal{L}\left\{\int_0^t y\,dt\right\} = \frac{1}{s}\, y(s)$$

$$\mathcal{L}\{y(t)\} = y(s)$$

Now, $\mathcal{L}\{y(t)\} + \mathcal{L}\left\{\int_0^t y\,dt\right\} = \mathcal{L}\{1\} - \mathcal{L}\{e^{-t}\}$

$$y(s) + \frac{1}{s}y(s) = \frac{1}{s} - \frac{1}{s+1} \quad \left[\because \mathcal{L}(1) = \frac{1}{s}, \ \mathcal{L}\{e^{-t}\} = \frac{1}{s+1} \right.$$

$$y(s)\left[1 + \frac{1}{s}\right] = \frac{(s+1) - s}{s(s+1)} = \frac{1}{s(s+1)}$$

$$y(s)\left[\frac{s+1}{s}\right] = \frac{1}{s(s+1)}$$

$$y(s) = \frac{1}{(s+1)^2}$$

$$\mathcal{L}\{y(t)\} = y(s) = \frac{1}{(s+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right]$$ using theorem, $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

we know $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$ $\therefore$

$$y(t) = e^{-t} \cdot t$$

$$\therefore \boxed{y(t) = te^{-t}}$$

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Q13. Find Laplace Transform of:

i) $\frac{\cos 2t \cdot \sin 2t}{e^t}$

Solⁿ a) $f(t) = \frac{\cos 2t \cdot \sin 2t}{e^t} \times \frac{2}{2}$

$\Rightarrow$

$f(t) \quad \frac{\sin 4t}{2e^t} \Rightarrow \frac{1}{2} e^{-t} \cdot \sin 4t.$

$L\{f(t)\} = L\{\frac{1}{2} e^{-t} \sin 4t\}.$

using shifting theorem,

$$= \frac{1}{2}\int_0^{\infty} e^{-st} \cdot e^{-t} \sin 4t \, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(s+1)t} \cdot \sin 4t \cdot dt \quad \text{--- (ii)}$$

and $L\{e^{at} f(t)\} = F(s-a)$ $\quad a = -1$ from i)

$$= \frac{1}{2} \times \frac{4^2}{[s-(-1)]^2 + 16}$$

$$= \boxed{\frac{2}{(s+1)^2 + 16}} \text{ Ans.}$$

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b) $t\int_0^t e^{-4t}\sin 3t\, dt$

Using Convolution theorem, $L\{(f*g)t\} = L\{f(t)\}L\{g(t)\}$
$= F(s).G(s)$

we get $f(t) = t$ and $g(t) = \int_0^t e^{-4t}\sin 3t\, dt$

i) $L\{f(t)\} = L\{t\} = 1/s^2$

(ii) $L\{G(t)\} = L\{\int_0^t e^{-4t}\sin 3t\}/dt$

To solve this, using identity

$$L\{\int_0^t P(t)\,dt\} = \frac{1}{s}P(s)$$

$P(t) = e^{-4t}\sin 3t.$

& $L\{e^{at}\} = F(s-a).$

$\therefore$ $f(t) = \sin 3t.$ $e^{at} = e^{-4t}.$

$$L\{f(t)\} = \frac{3}{s^2+9}$$

&

replacing $s \to [s-a]$ and ~~F(s) =~~ $L\{y(t)\} = \frac{3}{s[(s+4)^2+9]}$

$$= \frac{3}{(s+4)^2+9}$$

$$\boxed{L\{(f*g)t\} = \frac{1}{s^2} \times \frac{3}{[(s+4)^2+9]s}}$$

Ans

14. Find Laplace Transform of

$$f(t) = \begin{cases} 1 & 0<t<1 \\ e^t & 1<t<4 \\ 0 & t>4. \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt.$$

$$= \int_0^1 f(t)dt\, e^{-st} + \int_1^4 f(t)dt\, e^{-st} + 0$$

$\hookrightarrow$ as $L\{0\} = 0$.

$$L\{f(t)\} = \int_0^1 1\, dt\, e^{-st} + \int_1^4 e^t . e^{-st}\, dt$$

$$= \cancel{\left. \frac{e^{-st}}{-s} \right|_0^1}$$

$$= \int_0^1 e^{-st}\, dt + \int_1^4 e^{(1-s)t}\, dt$$

$$= \frac{1}{s} + \left. \frac{e^{(1-s)t}}{1-s} \right|_1^4$$

$$= \frac{1}{s} + \frac{e^{(1-s)4}}{1-s} - \frac{e^{(1-s)1}}{1-s}$$

$$\therefore = \boxed{\frac{1}{s} + \frac{e^{4(1-s)} - e^{(1-s)}}{1-s}}$$ Ans.

15. Solve the differential equation using Laplace Transform.

$$\frac{dy}{dt} + y = \cos 2t \quad , \quad y(0) = 1$$

15.

Diff Eq is

$$y' + y = \cos 2t$$

Taking Laplace Trasform on both sides

$$L\{y' + y\} = L\{\cos 2t\} \quad \text{— (i)}$$

using Theorm,

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{n-1}(0)$$

$$L\{y'(t)\} = s L\{y(t)\} - s^0 y(0)$$

$$= s y(s) - 1 \quad \text{as} \quad L\{y(t)\} = y(s) / Y(s)$$

$$L\{y'(t)\} = s Y(s) - 1$$

from (i)

$$s Y(s) - 1 + Y(s) = \frac{s}{s^2+4} \quad \text{as} \quad L\{\cos 2t\} = \frac{s}{s^2+4}.$$

$$Y(s) = \frac{1}{s+1} + \frac{s}{s^2+4}$$

$$Y(s) = \frac{1}{s+1} + \frac{(s+1) - 1}{(s^2+4)} = \frac{1}{s+1} + \frac{1}{s^2+4} - \frac{1}{(s+1)(s^2+4)}$$

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using Partial fraction for $\frac{1}{(s+1)(s^2+4)}$

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$\frac{1}{(s+1)(s^2+4)} = \frac{A(s^2+4) + (Bs+C)(s+1)}{(s+1)(s^2+4)}$$

on comparing,

$A+B=0 \Rightarrow B=-A$

$B+C=0 \Rightarrow C=-B$

$4A+C=0 \Rightarrow 5A=1$ $\boxed{A=1/5}$

$$\therefore Y(s) = \frac{4}{5}\left(\frac{1}{s+1}\right) + \frac{4}{5}\left(\frac{1}{s^2+4}\right) + \frac{1}{5}\left(\frac{s}{s^2+4}\right)$$

using Inverse Laplace, as $L\{e^{at}\} = \frac{1}{s-(a)}$

$L\{\cos at\} = \frac{s}{s^2+a^2}$ & $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$y(t) = L^{-1}\{Y(s)\}$$

$$= \frac{4}{5} L^{-1}\left(\frac{1}{\underbrace{s+1}_{a=-1}}\right) + \frac{2}{5} L^{-1}\left(\frac{2}{s^2+4}\right) + \frac{1}{5} L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$y(t) = \frac{4}{5}e^{-t} + \frac{2}{5}\sin 2t + \frac{1}{5}\cos 2t.$$

$$\boxed{y(t) = \frac{1}{5}\left(4e^{-t} + 2\sin 2t + \cos 2t\right)}$$ Solution of diff Eq

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