Question No - 1

Reduce the following matrices to row echleon form and find their rank.

(1) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Sol^n

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A minor of order 2, $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

Hence, rank of the given matrix is 2.

(2) $\begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$

Sol^n

$$A = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{3}{5} R_1$$

$$R_3 \rightarrow R_3 - \frac{7}{5} R_1$$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & -11/5 & 1/5 & -3/5 \end{bmatrix}$$

$$R_3 \rightarrow R_2 + 11 R_3$$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Minor of order 2, is not equal to zero
Hence, the rank of the matrix is 2.

Question No - 2

Test for consistency and solve

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

Sol^n

$AX = B$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_2 \to R_2 - \frac{3}{5} R_1$

$R_3 \to R_3 - \frac{7}{5} R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & -11/5 & 1/5 & -3/5 \end{array}\right]$$

$R_3 \to 11R_3 + R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Notes

$\Rightarrow$ As the root of $\left[\frac{A}{B}\right] = [A] = 2$

The system is consistent and it has infinite sol^n.

On expansion,

$$5x + 3y + 7z = 4 \quad \text{———} \ (1)$$
$$\frac{121}{5}y - \frac{11}{5}z = \frac{33}{5} \quad \text{———} \ (2)$$

Put $x = \alpha$ in eq^n (2)

$$\frac{121}{5}y - \frac{11\alpha}{5} = \frac{33}{5}$$

$$11y - \alpha = 3 \Rightarrow y = \frac{3+\alpha}{11} \quad \text{———} \ (2)$$

Putting the value of y in eq^n (1)

$$5x + \frac{3(3+\alpha)}{11} + 7\alpha = 4$$

$$55x + 9 + 9\alpha + 7\alpha = 44$$
$$55x = 35 - 80\alpha$$
$$x = \frac{7-16\alpha}{11}$$

Hence the required sol^n of the eq^n system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (7-16\alpha)/11 \\ (3+\alpha)/11 \\ \alpha \end{bmatrix}$$

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Notes

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Question 3

Verify the caley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$ and find its inverse if it exists.

Sol^n

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$AX = \lambda X$

$(A - \lambda I) X = 0$ (characteristic polynomial)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & -1-\lambda & 1 \\ 1 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)(2+\lambda+2\lambda+\lambda^2) - 1(-1) + 2(1+\lambda) = 0$

$\Rightarrow (3-\lambda)(\lambda^2+3\lambda+2) + 1 + 2 + 2\lambda = 0$

$\Rightarrow 3\lambda^2 + 9\lambda + 6 - \lambda^3 - 3\lambda^2 - 2\lambda + 3 + 2\lambda = 0$

$-\lambda^3 + 9\lambda + 9 = 0$

$\lambda^3 - 9\lambda - 9 = 0$

According to caley Hamilton theorem every matrix satisfies its own characteristic eq^n.

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Notes

To show that, $A^3 - 9A - 9I = 0$ ——— (1)

LHS

$$A^2 = \begin{bmatrix} 11 & 2 & 3 \\ 1 & 1 & -3 \\ 1 & 1 & 6 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 36 & 9 & 18 \\ 0 & 0 & 9 \\ 9 & 0 & -9 \end{bmatrix}$$

Putting value in the expression.

$$\Rightarrow \begin{bmatrix} 36 & 9 & 18 \\ 0 & 0 & 9 \\ 9 & 0 & -9 \end{bmatrix} - 9\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 36-27-9 & 9-9 & 18-18 \\ 0 & 9-9 & 9-9 \\ 0 & 0 & -9+18-9 \end{bmatrix}$$

$\Rightarrow$ RHS

Hence Proved.

Finding A^{-1}, multiplying A^{-1} on both sides of (1)

$$A^2 - 9A^{-1} - 9I = 0$$

$$A^{-1} = \frac{1}{9}[A^2 - 9I]$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 11-9 & 2 & 3 \\ 1 & 1-9 & -3 \\ 1 & 1 & 6-9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2/9 & 2/9 & 3/9 \\ 1/9 & -8/9 & -1/3 \\ 1/9 & 1/9 & -1/3 \end{bmatrix}$$

Question No - 4

Show that the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is diagonilizable

Find the matrix P such that $P^{-1}AP$ is a diagonal matrix.

Sol^n

$AX = \lambda X$

$(A - \lambda I)X = 0$ (characteristic polynomial)

$|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} = 0$$

$(1-\lambda)(\lambda^2 - 4) - 2(-2-\lambda) - 2(\lambda+2) = 0$

$-\lambda^3 + \lambda^2 + 4\lambda - 4 = 0$

$(\lambda - 1)(4 - \lambda^2) = 0$

$(\lambda - 1)(\lambda + 2)(\lambda - 2) = 0$

$\lambda = 1, -2, 2$

for $\lambda = 1$, $X_1' = [x, y, z]$

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Notes

$(A-I)X_1 = 0$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2y - 2z = 0 \Rightarrow y = z$

$x + z = 0 \Rightarrow x = -z$

Putting $x = k$

$X_1' = [-k, k, k]$

put $k = 1$

$X_1' = [-1, 1, 1]$

For $\lambda_2 = +2$ $\quad X_2' = [x_2, y_2, z_2]$

$(A - 2I)X = 0$

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + 2y - 2z = 0$ —— (1)

$x - y + z = 0$ —— (2)

$x + 3y - 3z = 0$

Adding (1) and (2)

$y - z = 0 \Rightarrow y = z$

$x = 0$

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Notes

Put $x = m$

$X_2' = [0 \quad m \quad m]$

Put $m = 1$

$X_2' = [0 \quad 1 \quad 1]$

For $\lambda = -2$ $\quad X_3' = [x_3 \quad y_3 \quad z_3]$

$(A + 2I)X = 0$

$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_3 + 2y_3 - 2z_3 = 0$ —— 1

$x_3 + 3y_3 + z_3 = 0$ —— 2

$x_3 + 3y_3 + z_3 = 0$

1 + 2 * 2

$5x_3 + 8y_3 = 0$

$x_3 = -\frac{8}{5} y_3$

$-\frac{8}{5} y_3 + 3y_3 + z_3 = 0$

$-\frac{7}{5} y_3 = z_3.$

Put $y_3 = 5p.$

$X_3' = [-8p \quad 5p \quad -7p]$

$P = [X_1 \quad X_2 \quad X_3]$

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Notes

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Notes

to

$$\Rightarrow \begin{bmatrix} -1 & 0 & -8 \\ 1 & 1 & 5 \\ 1 & 1 & -7 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj } P}{|P|} = \frac{1}{12}\begin{bmatrix} -12 & -8 & 8 \\ 12 & 15 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2/3 & 2/3 \\ 1 & 5/4 & -1/4 \\ 0 & 1/12 & -1/12 \end{bmatrix}$$

Now $P^{-1}AP$ should be a diagonal matrix.

$$P^{-1}AP = \begin{bmatrix} -1 & -2/3 & 2/3 \\ 1 & 5/4 & -1/4 \\ 0 & 1/12 & -1/12 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & -3 \\ 1 & 1 & 5 \\ 1 & 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2/3 & 2/3 \\ 2 & 5/2 & -1/2 \\ 0 & -1/6 & 1/6 \end{bmatrix}\begin{bmatrix} -1 & 0 & -8 \\ 1 & 1 & 5 \\ 1 & 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Question No - 5

Find all the asymtotes of the curve

$$(x+y)^2(x+2y+2) = x+9y-2$$

Sol^n

$$(x^2+y^2+2xy)(x+2y+2) = x+9y-2$$

$$(x^3+2yx^2+2x^2+y^2x+2y^3+2y^2+2x^2y+4xy^2+4xy-x+9y-2$$

$$x^3+2y^3+5y^2x+4x^2y+2x^2+2y^2+4xy-x-9y+2=0$$

$$\phi_3(m) = 1+2m^3+5m^2+4m.$$

$$\phi_2(m) = 2+2m^2+4m.$$

$$\phi_1(m) = -1-9m.$$

$$\phi_3'(m) = 6m^2+10m+4.$$

$$\phi_3''(m) = 12m+10.$$

$$\phi_3(m) = 0$$

$$2m^3+5m^2+4m+1 = 0$$

$$(m+1)^2(m+1/2) = 0.$$

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$

For $m = -1/2$

$$c = -\frac{(2+2m^2+4m)}{6m^2+10m+4} = \frac{(\cancel{2}+1/2-\cancel{2})(-1)}{3/2-5+4} = \frac{-1}{2}\times 2 = -1$$

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Notes

For $m = -1$

$$\frac{c^2}{2}\phi_3''(m) + c(\phi_2(m)) + \phi_1(m) = 0$$

$$\frac{c^2}{2}(12m+10) + c(4m+4) + 1(-1-9m) = 0.$$

$$\frac{c^2}{2}(-2) + 1(+8) = 0$$

$$c^2 - 8 = 0 \Rightarrow c = \pm 2\sqrt{2}$$

Hence the asymtotes are -

$$y = -x + \sqrt{2}$$
$$y = -x - \sqrt{2}$$
$$y = -\frac{x}{2} - 1 \Rightarrow 2y + x + 2 = 0.$$

Question No - 6

Trace the curve $y^2(a-x) = x^2(a+x)$

$$y^2 = x^2\frac{(a+x)}{(a-x)}$$

- **Symmetry** - y appears in even powers of y, so curve is symmetric about x axis.

- **Nature at origin** - when $x = 0$, $y^2 = 0$

The curve passes through origin and origin is a double point

- Tangent at origin -

$$y^2 a = x^2 a$$
$$y = \pm x$$

Tangent at origin are distinct, hence curve is a node

- Asymtote -

$$y^2(a-x) = 0$$
$$x = a$$

- Behaviour -

$$y^2 = \frac{x^2(a+x)}{(a-x)}$$

$$2yy' = \frac{(a-x)(2xa+3a^2) - x^2(a+x)}{(a-x)^2}$$

$$y' = \frac{a^2 - ax - x^2}{(a-x)^{3/2}(a+x)^{1/2}}$$

when $y'=0$, $\quad x = -0.6a$ and $1.6a$

$a^2 - ax - x^2 = 0 \quad y = 0.3a$

- Special Point - Shifting origin to $(-a, 0)$ i.e

$$x = X - a$$

Eq^n becomes,

$$y^2 = (X-a)^2 \frac{(\cancel{a} + X - \cancel{a})}{(a - X + a)} = \frac{(X-a)^2 X}{(2a - X)}$$

$$y^2(2a - X) = X^3 + a^2 X - 2aX^2$$

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Notes

Equating the lowest degree term to 0 i.e $X=0$

$x+a=0$

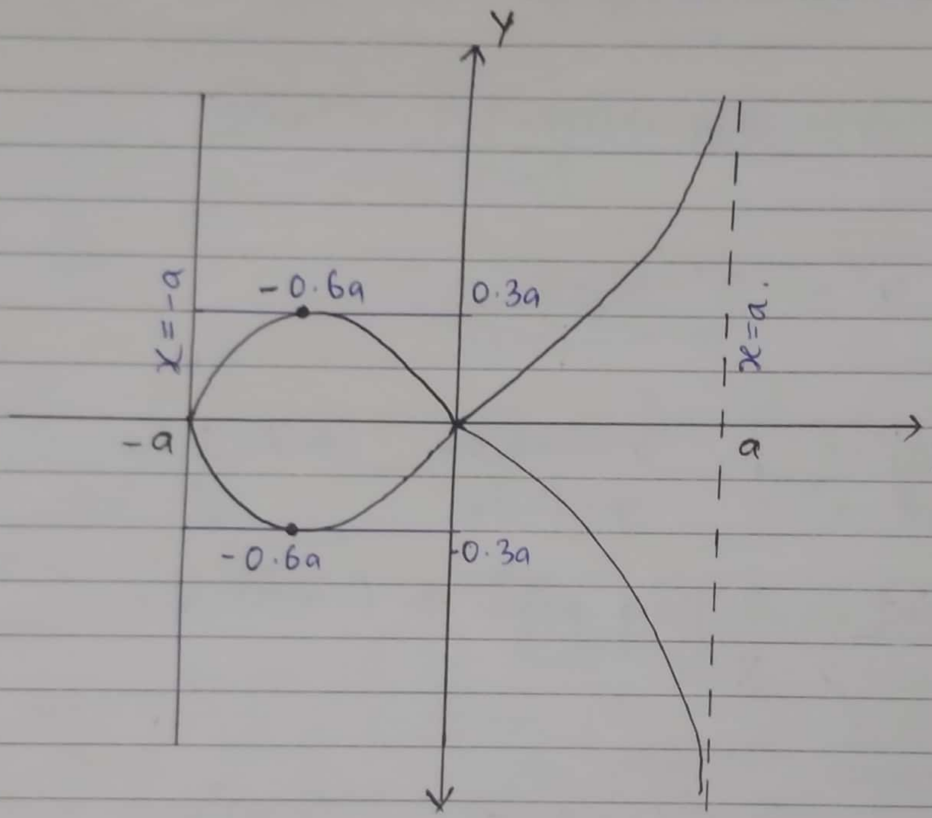
$x=-a$. is a tangent.

$y^2<0$, curve does not exist for $x<-a$.

$y^2>0$, curve is rising for $-a<x<-0.6a$.

$y^2<0$, curve is falling for $-0.6a<x<0$

$y^2>0$, curve is rising for $0<x<a$



Question No - 7

Show that the length of the arc measured from $\theta = 0$ to any point of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is $\frac{1}{2}\theta^2 a$.

Sol^n

$$\frac{dx}{d\theta} = -a\sin\theta + \theta\cos\theta + a\sin\theta = a\theta\cos\theta.$$

$$\frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\sin\theta\cdot\theta = a\theta\sin\theta.$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta}$$

$$(\because \sin^2\theta + \cos^2\theta = 1)$$

$$\frac{ds}{d\theta} = a\theta$$

$$S = \int_0^\theta a\theta$$

$$S = \frac{a\theta^2}{2},$$ Hence Proved.

Question No - 8

A segment is cut off from the sphere of radius 'a' by a plane at a distance $a/2$ from the centre. Show that the volume of the segment is $5/32$ of the volume of the sphere

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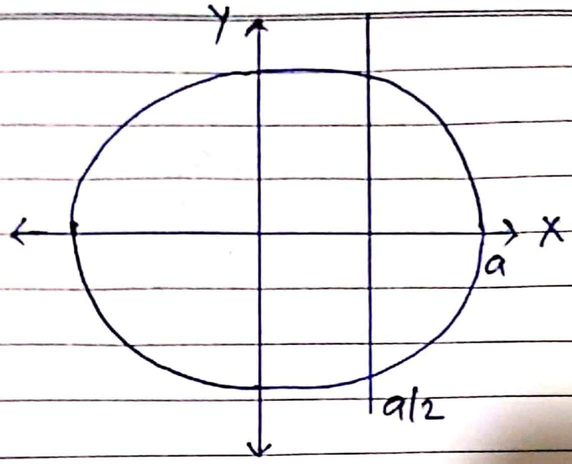
Notes

Sol^n



Volume required =

$$\pi \int_{a/2}^{a} y^2 \, dx$$

$x^2 + y^2 = a^2$ — (1)

$$\Rightarrow y^2 = a^2 - x^2$$

$$V = \pi \int_{a/2}^{a} (a^2 - x^2) \, dx$$

$$= \pi \left[a^2 x - \frac{x^3}{3} \right]_{a/2}^{a}$$

$$= \pi \left[a^3 - \frac{a^3}{3} - \left(\frac{a^3}{2} - \frac{a^3}{24} \right) \right] = \frac{\pi 5a^3}{24}$$

Total Volume $= \frac{4\pi a^3}{3}$.

$$\frac{\text{Volume of segment}}{\text{Volume of sphere}} = \frac{5\pi a^3}{24\ 8} \times \frac{\cancel{3}}{4\pi a^3} = \frac{5}{32}$$

Volume of segment $= \frac{5}{32}$ volume of sphere.

Hence Proved

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Notes

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Question No-9.

Discuss the convergence of the series.

$$\sum \frac{2n^2}{3n^4+5n^3+3}$$

Sol^n

$$U_n = \frac{2n^2}{3n^4+5n^3+3}$$

Let, $V_n = \frac{1}{n^2}$ be convergent series

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{2n^4}{3n^4+5n^3+3}$$

$$\Rightarrow \frac{2\cancel{n^4}}{\cancel{n^4}(3+5/n+3/n^4)}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{2}{3} \neq 0 \qquad (\because \text{as } \tfrac{1}{n} \to 0 \text{ for } n\to\infty)$$

since $\frac{2}{3}$ is finite, U_n is also convergent like V_n.

Question No-10

Find the surface of the solid generated by the revolution of the curve $r = 2a\cos\theta$ about the initial line.

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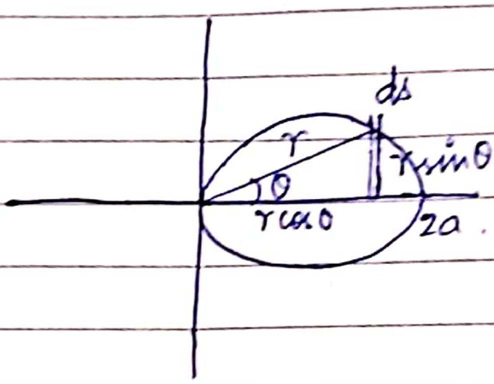
Notes

Solⁿ

$$r = 2a\cos\theta$$

$$\frac{dr}{d\theta} = -2a\sin\theta$$



$$S = 2\pi\int_0^{\pi/2} y\,ds.$$

$$= 2\pi\int_0^{\pi/2} r\sin\theta\,ds.$$

$$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}$$

$$ds = 2a\,d\theta$$

$$S = 2\pi\int_0^{\pi/2} 2a\,d\theta \times \sin\theta.$$

$$= 4\pi a\int_0^{\pi/2} 2a\cos\theta\sin\theta\,d\theta.$$

$$= 4\pi a^2\int_0^{\pi/2} \sin 2\theta\,d\theta$$

$$= 4\pi a^2\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2\pi a^2\left[-\cos\pi + \cos 0\right] = 4\pi a^2.$$

Teacher's Remarks :

Question No - 11

Find the surface area of the solid generated by revolving the curve $x = e^t \cos t$ and $y = e^t \sin t$
$0 \le t \le \pi/2$

Solⁿ

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t$$

$$\frac{dy}{dt} = e^t \sin t + e^t \cos t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}$$

$$\frac{ds}{dt} = \sqrt{2e^{2t}}$$

$$\frac{ds}{dt} = \sqrt{2}e^t \Rightarrow ds = \sqrt{2}e^t\,dt$$

$$S = 2\pi \int_0^{\pi/2} y\,ds$$

$$= 2\pi\sqrt{2} \int_0^{\pi/2} e^t \sin t\, e^t$$

$$= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \sin t\,dt$$

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Notes

Let $I = \int e^{2t} \sin t \, dt$

$$I = \frac{e^{2t}}{2}\sin t - \int \frac{e^{2t}}{2} \cos t \, dt$$

$$I = \frac{e^{2t}\sin t}{2} - \frac{e^{2t}\cos t}{4} + \int \frac{e^{2t}}{4}(-\sin t)$$

$$I = \frac{e^{2t}\sin t}{2} - \frac{e^{2t}\cos t}{4} - \frac{I}{4}$$

$$\frac{5I}{4} = \frac{e^{2t}\sin t}{2} - \frac{e^{2t}\cos t}{4}$$

$$I = \frac{4}{5}\left[\frac{e^{2t}\sin t}{2} - \frac{e^{2t}\cos t}{4}\right]$$

$$S = \frac{4\cancel{2}\sqrt{2}\pi}{5}\left[\frac{e^{2t}\sin t}{\cancel{2}} - \frac{e^{2t}\cos t}{\cancel{4}2}\right]_0^{\pi/2}$$

$$= \frac{4\sqrt{2}\pi}{5}\left[e^{\pi} - 0 + \frac{1}{2}\right]$$

$$S = \frac{2\sqrt{2}\pi(2e^{\pi}+1)}{5}$$

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