This shows that if we draw a circle with its centre at origin and radius $|z_1|$, the series is absolutely convergent at every point inside this circle. The largest such circle inside which the series (3) converges, is called the *circle of convergence*.

Let $|z|=R$ be the circle of convergence and $|z|=R_1$ a circle within it. Then from the above proof

$$|a_n z^n|<M(R_1/R)^n$$

for every point inside the circle $|z|=R_1$. Therefore, by Wierstrass's M test, the series is also uniformly convergent inside this circle. As in the case of real series, it can be shown that a uniformly convergent series of continuous complex functions is itself continuous, and is term by term integrable. It can be further shown that this integration can be extended to *all* points within the circle of convergence.

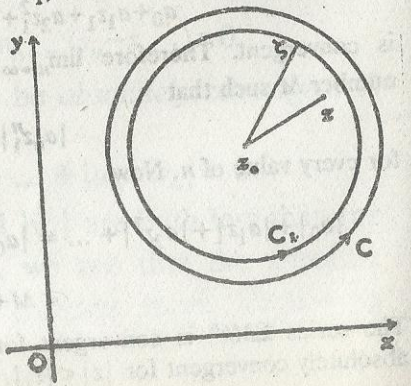
5.61. Taylor's Series. *A function $f(z)$, analytic inside a circle C with centre z_0, can be expanded in the series*

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\ldots$$
$$+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\ldots, \quad (1)$$

which is convergent at every point inside C.

Take any point z inside C. Draw a circle C_1 concentric with C and inside it, but with a radius larger than $|z-z_0|$. Let ζ be a point on C_1; then

$$|z-z_0|<|\zeta-z_0|. \quad (2)$$



Binomial expansion gives

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0-(z-z_0)}$$

$$=\frac{1}{\zeta-z_0}\left(1-\frac{z-z_0}{\zeta-z_0}\right)^{-1}$$

$$=\frac{1}{\zeta-z_0}\left\{1+\frac{z-z_0}{\zeta-z_0}+\left(\frac{z-z_0}{\zeta-z_0}\right)^2+\ldots+\left(\frac{z-z_0}{\zeta-z_0}\right)^n+\ldots\right\}. \quad (3)$$

This series is uniformly convergent as

$$\left|\frac{z-z_0}{\zeta-z_0}\right|<1, \text{ by (2).}$$

So we can multiply both sides of (3) by $f(\zeta)$ and integrate over C_1; we obtain

$$\int_{C_1}\frac{f(\zeta)}{\zeta-z}d\zeta=\int_{C_1}\frac{f(\zeta)}{\zeta-z_0}d\zeta+(z-z_0)\int_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^2}d\zeta+\ldots$$
$$+(z-z_0)^n\int_{C_1}\frac{f(z)d\zeta}{(\zeta-z_0)^{n+1}}+\ldots \quad (4)$$

Since $f(\zeta)$ is analytic on and inside C_1, we can apply the results of §§ 5.54 and 5.55 to these integrals. This gives us the expansion (1).

5.62. Laurent's Series. *Let C_1 and C_2 be two concentric circles of radii R_1 and R_2 ($R_1>R_2$) with centre z_0; and let $f(z)$ be analytic on C_1 and C_2, and in the annular region R between them. Then for any point z in R*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}, \quad (1)$$

where

$$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}},\ b_n=\frac{1}{2\pi i}\int_{C_2}(\zeta-z_0)^{n-1}f(\zeta)d\zeta,$$

each integral being taken anticlockwise.

The series (1) is known as *Laurent's series*.

By giving it a cut along AB, the region R can be converted to a simply connected region bounded by curve C' (fig. 2). Hence, if z be any point in R', we have

$$f(z)=\frac{1}{2\pi i}\int_{C'}\frac{f(\zeta)}{\zeta-z}d\zeta.$$

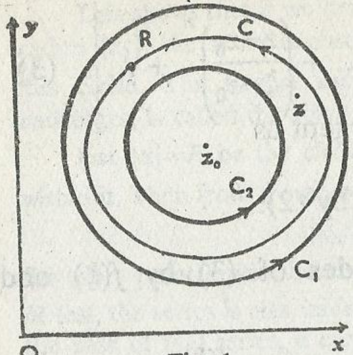


Fig. 1

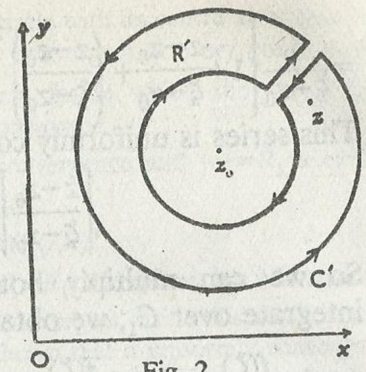


Fig. 2

Now C' comprises of the curve C_1 traversed anticlockwise, the curve C_2 traversed clockwise and the cut AB traversed once in direction AB and once in direction BA. The integrals over the last two portions cancel each other. Therefore

$$f(z)=\frac{1}{2\pi i}\left\{\int_{C_1}\frac{f(\zeta)}{\zeta-z}d\zeta-\int_{C_2}\frac{f(\zeta)}{\zeta-z}d\zeta\right\}, \tag{2}$$

where both C_1 and C_2 are traversed anticlockwise in (2).

For the first integral $1/(\zeta-z)$ can be expanded exactly as in the previous article. This gives

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{\zeta-z}d\zeta=\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}. \tag{3}$$

In the second integral, ζ lies on C_2. Therefore

$$|\zeta-z_0|<|z-z_0|, \quad \text{or} \quad \left|\frac{\zeta-z_0}{z-z_0}\right|<1.$$

So in this integral we put

$$\frac{1}{\zeta-z}=\frac{1}{(\zeta-z_0)-(z-z_0)}=-\frac{1}{(z-z_0)}\left(1-\frac{\zeta-z_0}{z-z_0}\right)^{-1}$$

$$=-\frac{1}{z-z_0}\left\{1+\frac{\zeta-z_0}{z-z_0}+\left(\frac{\zeta-z_0}{z-z_0}\right)^2+\ldots+\left(\frac{\zeta-z_0}{z-z_0}\right)^{n-1}+\ldots\right\}.$$

This gives* $-\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta$

$$=\sum_{n=1}^{\infty}\frac{1}{(z-z_0)^n}\frac{1}{2\pi i}\int_{C_2}(\zeta-z_0)^{n-1}f(\zeta)\,d\zeta. \tag{4}$$

Putting the results (3) and (4) in (2), we get the required expansion (1).

NOTES. 1. Let C be a curve lying between C_1 and C_2 such as shown in fig. 1. Then by § 5.53

$$\int_{C_1}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}=\int_C\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}},$$

as the integrand is analytic in the region between C_1 and C. For the same reason

$$\int_{C_2}(\zeta-z_0)^{n-1}f(\zeta)\,d\zeta=\int_C(\zeta-z_0)^{n-1}f(\zeta)\,d\zeta.$$

Hence the coefficients in the expansion may he modified, and we may write

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n,$$

where
$$a_n=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta.$$

2. The evaluation of the coefficients by contour integration often becomes cumbersome. As far as possible other methods, such as binomial expansion, should be used for finding the coefficients.

Ex. Expand $f(z)=1/(z+1)(z+3)$ in a Laurent's series valid for

(*a*) $1<|z|<3$, (*b*) $|z|>3$,
(*c*) $0<|z+1|<2$, (*d*) $|z|<1$.

(*a*) By partial fractions

$$\frac{1}{(z+1)(z+3)}=\frac{1}{2}\left\{\frac{1}{z+1}-\frac{1}{z+3}\right\} \tag{1}$$

$$=\frac{1}{2}\left\{\frac{1}{z(1+z^{-1})}-\frac{1}{3(1+\frac{1}{3}z)}\right\} \tag{2}$$

For $1<|z|<3$, both $|z^{-1}|$ and $|\frac{1}{3}z|$ are less than 1. Hence (2) gives on expansion

*Term by term integration is permissible as the series is uniformly convergent.

$$f(z)=\tfrac{1}{2}z^{-1}(1-z^{-1}+z^{-2}-z^{-3}+\ldots)-\tfrac{1}{6}(1-\tfrac{1}{3}z+\tfrac{1}{9}z^{2}-\tfrac{1}{27}z^{3}+\ldots).$$

(*b*) For $|z|>3$, we can write (1) as

$$f(z)=\frac{1}{2}\left\{\frac{1}{z(1+z^{-1})}-\frac{1}{z(1+3z^{-1})}\right\}$$
$$=\tfrac{1}{2}z^{-1}(1-z^{-1}+z^{-2}-z^{-3}+\ldots)-\tfrac{1}{2}z^{-1}(1-3z^{-1}+9z^{-2}-27z^{-3}+\ldots)$$
$$=z^{-2}-4z^{-3}+13z^{-4}-\ldots+(-1)^{n+1}\tfrac{1}{2}(3^{n}-1)z^{-n-1}+\ldots.$$

(*c*) For $0<|z+1|<2$, we write (1) as

$$f(z)=\frac{1}{2}\left\{\frac{1}{z+1}-\frac{1}{(z+1)+2}\right\}=\frac{1}{2}\left[\frac{1}{z+1}-\frac{1}{2\{1+\frac{1}{2}(z+1)\}}\right]$$
$$=\tfrac{1}{2}(z+1)^{-1}-\tfrac{1}{4}\{1-\tfrac{1}{2}(z+1)+\tfrac{1}{4}(z+1)^{2}-\tfrac{1}{8}(z+1)^{3}+\ldots\}.$$

The expansion is valid since $|\tfrac{1}{2}(z+1)|<1$.

(*d*) For $|z|<1$, we write (1) as

$$f(z)=\tfrac{1}{2}(1+z)^{-1}-\tfrac{1}{6}(1+\tfrac{1}{3}z)^{-1}$$
$$=\tfrac{1}{2}(1-z+z^{2}-z^{3}+\ldots)-\tfrac{1}{6}(1-\tfrac{1}{3}z+\tfrac{1}{9}z^{2}-\tfrac{1}{27}z^{3}+\ldots)$$
$$=\tfrac{1}{3}-\tfrac{4}{9}z+\tfrac{13}{27}z^{2}-\ldots+(-1)^{n}\tfrac{1}{2}(1-3^{-n-1})z^{n}+\ldots.$$

For $|z|<1$, the series becomes a Taylor's series.

5.63. Alternatiave Proof. In obtaining Laurent's series we have used the result that a uniformly convergent complex series can be integrated term by term, without actually proving it. This can be avoided by proceeding as follows.

We have, by actual division,

$$\frac{1}{1-\alpha}=1+\alpha+\alpha^{2}+\ldots+\alpha^{n-1}+\frac{\alpha^{n}}{1-\alpha}.$$

Using this for expanding $1/(\zeta-z)$, we get

$$\frac{1}{\zeta-z}=\frac{1}{(\zeta-z_0)-(z-z_0)}=\frac{1}{\zeta-z_0}\cdot\frac{1}{1-(z-z_0)/(\zeta-z_0)}$$
$$=\frac{1}{\zeta-z_0}+\frac{z-z_0}{(\zeta-z_0)^{2}}+\frac{(z-z_0)^{2}}{(\zeta-z_0)^{3}}+\ldots+\frac{(z-z_0)^{n-1}}{(\zeta-z_0)^{n}}+\frac{(z-z_0)^{n}}{(\zeta-z_0)^{n}(\zeta-z)},$$

for ζ on C_1; and similarly

$$-\frac{1}{\zeta-z}=\frac{1}{z-z_0}+\frac{\zeta-z_0}{(z-z_0)^{2}}+\frac{(\zeta-z_0)^{2}}{(z-z_0)^{3}}+\ldots+\frac{(\zeta-z_0)^{n-1}}{(z-z_0)^{n}}+\frac{(\zeta-z_0)^{n}}{(z-z_0)^{n}(z-\zeta)},$$

for ζ on C_2. Putting these values of $1/(\zeta-z)$ in the first and second integrals of equation (2), § 5.62, respectively, we get

$$f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^{2}+\ldots+a_{n-1}(z-z_0)^{n-1}+R_n$$
$$+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^{2}}+\ldots+\frac{b_n}{(z-z_0)^{n}}+Q_n,$$

where $a_0, a_1, \ldots, b_1, b_2, \ldots$ are as given in the theorem and

$$R_n=\frac{(z-z_0)^{n}}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n}(\zeta-z)},$$

$$Q_n=\frac{1}{2\pi i(z-z_0)^{n}}\int_{C_2}\frac{(\zeta-z_0)^{n}f(\zeta)}{z-\zeta}d\zeta.$$

Let $|z-z_0|=r$; then $R_2<r<R_1$. Let M_1 be the maximum value of $|f(\zeta)|$ on C_1. The maximum value of $1/(\zeta-z)$ on C_1 is $1/(R_1-r)$. Also $|\zeta-z_0|=R_1$ on C_1. Therefore by § 5.51 (v)

$$|R_n|\le\left(\frac{r}{R_1}\right)^{n}\frac{M_1R_1}{R_1-r}.$$

Similarly $$|Q_n|\le\left(\frac{R_2}{r}\right)^{n}\frac{M_2R_2}{r-R_2},$$

where M_2 is the maximum value of $|f(\zeta)|$ on C_2. By taking n sufficiently large we can make R_n and Q_n as small as we please. Therefore the infinite series (1) of § 5.62 convergs to $f(z)$.

A similar proof can be given for Taylor's theorem also.

5.64. Zeros of an Analytic Function. If a function $f(z)$, analytic in a region R, is zero at a point z_0 in R, then z_0 is called a *zero* of $f(z)$.

If $f(z_0)=0$ but $f'(z_0)\neq0$, z_0 is called a *simple zero* or a zero of the first order. If

$$f(z_0)=f'(z_0)=\ldots=f^{(n-1)}(z_0)=0,\quad f^{(n)}(z_0)\neq0,$$

z_0 is called a *zero of order n*.

Thus the function $z^2 \sin z$ has a zero of third order at $z=0$ and simple zeros at $z=\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, ... The function $1-\cos z$ has second order zeros at $z=0$, $\pm 2\pi$, $\pm 4\pi$, etc.

Suppose an analytic function $f(z)$ has a zero of order n at $z=z_0$. Then its Taylor's series is of the form

$$\begin{aligned} f(z) &= b_n(z-z_0)^n+b_{n+1}(z-z_0)^{n+1}+b_{n+2}(z-z_0)^{n+2}+\ldots \\ &= (z-z_0)^n[b_n+(z-z_0)\{b_{n+1}+b_{n+2}(z-z_0)+\ldots\}] \\ &= (z-z_0)^n[b_n+(z-z_0)S], \text{ say.} \end{aligned}$$

By taking $|z-z_0|$ sufficiently small we can make

$$|b_n|>|z-z_0| \,.\, |S|.$$

Thus a neighbourhood of z_0 can be found where

$$b_n+(z-z_0)S$$

is not zero. Consequently, $f(z)$ cannot have another zero in this neighbourhood.

We express this by saying that *the zeros of an analytic function are isolated.*

5.7. Singular Points. A point z_0 at which a function $f(z)$ is not analytic is known as a *singular point* or singularity of $f(z)$. If the singular point is such that there is no other singular point in its neighbourhood, we call it an *isolated* singular point.

For example, the function $1/(z-a)$ is analytic everywhere except at $z=a$. Thus it has an isolated singularity at $z=a$. The function

$$\frac{z^2+1}{(z-1)^2(z^2+4)}$$

has three isolated singular points, namely $z=1$, $z=2i$ and $z=-2i$. The function

$$\frac{1}{\sin(\pi/z)}$$

has an infinite number of isolated singularities at $z=\pm1$, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{4}$, ... The origin $z=0$ is also a singular point, but it is not isolated

since there are other singular points in its neighbourhood, however small we choose the neighbourhood.

The function $\sqrt{z}$ has a singularity at $z=0$ which is not isolated. The other singularities near origin arise due to the discontinuity suffered by a single value branch of $\sqrt{z}$ on crossing the cut in the z-plane (cf § 5.41, III). Singularities of this nature always arise at 'branch points' where two branches of a multivalued function have the same value. Other examples of such a singularity are $\log z$ at $z=0$ and $\sin^{-1} z$ at $z=\pm1$.

Near an isolated singular point z_0, we can always find a region $0<|z-z_0|<R_1$ in which the function $f(z)$ is analytic. So it can be represented in this region by a Laurent's series

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}. \tag{1}$$

In some cases it may happen that the second series terminates due to coefficients b_{m+1}, b_{m+2}, ... all being zero after some particular term b_m. In such a case Laurent's series reduces to

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\ldots+\frac{b_m}{(z-z_0)^m}. \tag{2}$$

Such a singularity is called a *pole of order m*. A pole of order one is called a *simple pole*.

If the series of negative powers in expansion (1) does not terminate, z_0 is called an *essential singular point* of $f(z)$.

The coefficient b_1 in Laurent's expansion (1) of $f(z)$ about an isolated singular point z_0, is called the *residue* of $f(z)$ at $z=z_0$. From the definition of the coefficients, we see that the residue

$$b_1=\frac{1}{2\pi i}\int_{C_2} f(z)\,dz.$$

If z_0 is a simple pole,

$$f(z)=\sum_{n=0}^{\infty} a(z-z_0)^n+\frac{b_1}{z-z_0}.$$

Hence $\quad (z-z_0)\,f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^{n+1}+b_1.$

The terms $\Sigma a_n(z-z_0)^{n+1}$ vanish when $z\to z_0$; therefore

$$b_1=\lim_{z\to z_0}\{(z-z_0)\,f(z)\}.$$

This gives a simple method of calculating the residue at z_0.

If z_0 is a pole of order m, Laurent's expansion of $f(z)$ is given by (2). This gives

$$(z-z_0)^m f(z)$$
$$=\Sigma a_n(z-z_0)^{n+m}+b_1(z-z_0)^{m-1}+b_2(z-z_0)^{m-2}+\dots+b_m. \quad (3)$$

Differentiating this $m-1$ times and taking the limit as $z\to z_0$, we get

$$b_1=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\}.$$

We note from (3) that

$$\lim_{z\to z_0}(z-z_0)^m f(z) \text{ is finite and non-zero.}$$

This can be used for determining the order of a pole.

Ex. 1. Determine the poles of the function

$$f(z)=\frac{z+2}{(z+1)^2(z-2)},$$

and the residue at each pole.

The function has a simple pole at $z=2$, since

$$\lim_{z\to 2}\{(z-2)f(z)\}=\lim_{z\to 2}\frac{z+2}{(z+1)^2}=\frac{4}{9},$$

which is finite and non-zero. This also shows that the residue at $z=2$ is $4/9$.

$f(z)$ has a pole of order two at $z=-1$, since $(z+1)^2f(z)$ tends to a finite non-zero limit as $z\to -1$. The residue at $z=-1$ is

$$\lim_{z\to -1}\frac{d}{dz}\left(\frac{z+2}{z-2}\right)=\lim_{z\to -1}\frac{-4}{(z-2)^2}=-\frac{4}{9}.$$

NOTE. Alternatively we could expand $f(z)$ by partial fractions.

Ex. 2. Determine the poles of the function $z/\cos z$, and the residue at each pole. *[Delhi, 1967]*

The denominator of $f(z)=z/\cos z$ vanishes at

$$z=(n+\tfrac{1}{2})\pi,\ n=0,\ \pm 1,\ \pm 2,\ \dots .$$

Denoting this value by z_0, we have

$$\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}\frac{z(z-n\pi-\frac{1}{2}\pi)}{\cos z}$$
$$=\lim_{z\to z_0}\frac{2z-n\pi-\frac{1}{2}\pi}{-\sin z},$$

by L'Hospital's rule,

$$=\frac{n\pi+\frac{1}{2}\pi}{-\sin(n\pi+\frac{1}{2}\pi)}=(-1)^{n+1}(n+\tfrac{1}{2})\pi.$$

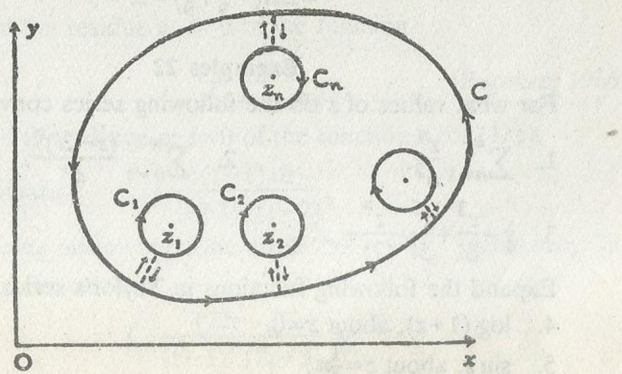
So $z=(n+\frac{1}{2})\pi$ gives the poles of $z/\cos z$. They are simple poles with residues $(-1)^{n+1}(n+\frac{1}{2})\pi$.

5.71. Residue Theorem.

If $f(z)$ be analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singular points $z_1, z_2, \dots, z_n$ inside C, then

$$\int_C f(z)\,dz=2\pi i \text{ (sum of the residues at } z_1, z_2, \dots, z_n).$$

Enclose the points $z_1, z_2, \dots, z_n$ by small non-intersecting circles $C_1, C_2, \dots, C_n$ lying wholly inside C (see figure). Then $f(z)$ is analytic in the multiply connected region lying between the curves C and $C_1, C_2, \dots, C_n$.



This region can be converted into a singly connected region by giving it suitable cuts (along the dotted lines). The boundary Γ of the singly connected region consists of the curve C (traversed anticlockwise), the circles C_1, ..., C_n (all traversed clockwise), and the cuts from C to these circles (traversed both ways).

By Cauchy's integral theorem

$$\int_\Gamma f(z)\,dz=0,$$

or, on omitting the integral over the cuts since it is zero,

$$\int_C f(z)\,dz-\int_{C_1} f(z)\,dz-\ldots-\int_{C_n} f(z)\,dz=0,$$

where now all the integrations are taken in anticlockwise direction. But

$$\int_{C_k} f(z)\,dz$$

is $2\pi i$ times the residue at z_k. Hence $\int_C f(z)\,dz$ is equal to $2\pi i$ times the sum of the residues at $z_1, z_2, \ldots, z_n$.

Ex. If C be the circle $|z|=3$, show that

$$\int_C \frac{z+2}{(z+1)^2(z-2)}\,dz=0.$$

The integrand is analytic on $|z|=3$ and at all points inside except the poles $z=-1$ and $z=2$. The residues at the poles are respectively $-4/9$ and $4/9$ (Ex. 1 p. 182). Hence by residue theorem, the required integral

$$=2\pi i(-\tfrac{4}{9}+\tfrac{4}{9})=0.$$

Examples 22

For what values of z do the following series converge ?

1. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

2. $\sum_{n=1}^{\infty} \frac{(z-2i)^n}{4^n}$.

3. $z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots$.

Expand the following functions in Taylor's series :

4. $\log(1+z)$, about $z=0$.

5. $\sin z$, about $z=\frac{1}{4}\pi$.

6. $1/(z+1)$, about $z=1$.

7. Show that when $|z+1|<1$,

$$\frac{1}{z^2}=1+\sum_{n=1}^{\infty}(n+1)(z+1)^n. \qquad \text{[Gorakhpur, 1972]}$$

Expand the following functions in Laurent's series :

8. $1/(z-1)(z-2)$, for $1<|z|<2$. [*Bhopal*, 1978]

9. $1/z(z-1)(z-2)$, for $|z|>2$. [*Jodhpur*, 1969]

10. $1/z(z-1)(z-2)$, for $|z-1|<1$. [*Roorkee*, 1981]

11. $(z-1)/z^2$, for $|z-1|>1$. [*Allahabad, M. E.*, 1977]

12. $e^z/(z-1)^2$, about $z=1$. [*Punjab, M.Sc. (Ch. Eng.)*, '78]

13. $\sin z/(z-\pi)$, about $z=\pi$. [*Roorkee, M.E.*, 1979]

14. Find five terms in the Laurent's expansion of $1/z(e^z-1)$ for the region $0<|z|<2\pi$. [*Roorkee*, 1980]

Locate the zeros of the following functions and determine their order.

15. $z\tan^2 z$.

16. $(z^2-3z+2)(z^2+4z+5)$.

17. $(z^3-1)/(z^3+1)$.

Find the poles of the following functions. Find the order of each pole and the residue at it.

18. $(z+1)/(z^2-2z)$. [*Kurukshetra*, 1975]

19. $z^2/(z-1)^2(z+2)$. [*Punjab, M.Sc. (Ch. Eng.)*, 1980]

20. $1/(z^4+1)$. [*Gorakhpur*, 1972]

21. $ze^{iz}/(z^2+a^2)$. [*Punjab, M.Sc. (Ch. Eng.)*, '79]

22. $e^z/(1+z)^2$. [*Rajasthan, M. E.*, 1980]

23. $(1-e^{2z})/z^4$. [*Alld.*, 66]

24. $e^{2z}/(1+e^z)$.

25. $z/\sin z$. [*Roorkee*, 1978]

26. $\cot z$.

27. Find the residue at $z=0$ of the function

$$\frac{1+e^z}{\sin z+z\cos z}. \qquad \text{[Roorkee, 1966]}$$

28. Find the residue at $z=0$ of the function $z\cos(1/z)$.

29. Evaluate $\int \frac{(z-1)\,dz}{(z+1)^2\,(z-2)}$, the integral being taken round the circle $|z-i|=2$. [*Banaras*, '78]

30. Find the value of

$$\int_C \frac{(12z-7)}{(z-1)^2\,(2z+3)}\,dz$$

where C is circle (a) $|z|=2$; (b) $|z+i|=\sqrt{3}$. [*Roorkee*, 1979]

31. Evaluate the integral of $z \exp(1/z)$ in the positive sense around the unit circle about the origin. [*Vikram*, 1964]

32. Find the value of $\int_C \frac{dz}{\sinh z}$, where C is the circle $|z|=4$ described in the positive sense. [*Roorkee*, 1967]

5.8. Evaluation of real integrals. The residue theorem gives a simple and elegant method for evaluating certain types of real integrals. Some of these are illustrated below.

I. *Integrals of type* $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$, *where* $F(\cos\theta, \sin\theta)$ *is a rational function of* $\cos\theta$ *and* $\sin\theta$.

On the unit circle $z=e^{i\theta}$. Therefore

$$\cos\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}) = \tfrac{1}{2}\left(z+\frac{1}{z}\right),$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right),$$

and $d\theta = dz/iz$. Also as θ varies from 0 to 2π, the point z moves once round the unit circle C. Hence

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_C F\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}.$$

The residue theorem can be used to evaluate the integral on the right.

Ex. 1. Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $a^2<1$. [*Bhopal*, 1978]

On using the above substitutions, the integral becomes

$$\int_C \frac{dz/iz}{1-2a\{(z^2+1)/2z\}+a^2} = \int_C \frac{dz}{i\{(1+a^2)z-az^2-a\}}$$
$$= \int_C \frac{dz}{i(1-az)(z-a)}.$$

The integrand has poles at $z=a$ and $1/a$, of which only the first lies inside the unit circle C. The residue at this pole is

$$\lim_{z\to a} \frac{1}{i(1-az)} \text{ i.e. } \frac{1}{i(1-a^2)}.$$

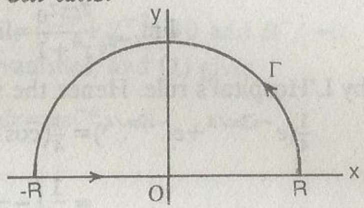
Hence by the residue theorem, the value of the integral is

$$\frac{2\pi}{1-a^2}.$$

II. *Integrals of type* $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx$, *where* $f(x)$ *and* $F(x)$ *are polynomials in* x *such that*

$$xf(x)/F(x) \to 0 \text{ as } x\to\infty,$$

and $F(x)$ *has no zeros on the real axis.*



Consider the integral

$$\int_C \frac{f(z)}{F(z)}\, dz \qquad (1)$$

over the closed contour C consisting of the real axis from $-R$ to R and the semicircle Γ of radius R in the upper half plane. For sufficiently large R, the value of the integral is equal to

$2\pi i$ [sum of the residues of $f(z)/F(z)$ in upper half plane]. (2)

$$\text{Now } \int_C \frac{f(z)}{F(z)}\, dz = \int_{-R}^{R} \frac{f(z)}{F(z)}\, dz + \int_\Gamma \frac{f(z)}{F(z)}\, dz$$
$$= \int_{-R}^{R} \frac{f(x)}{F(x)}\, dx + \int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta}\, i\, d\theta, \quad (3)$$

since $z=Re^{i\theta}$ on Γ. For large R,

$$\left|\int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta}\, i d\theta\right| \text{ is of the order of } \frac{Rf(R)}{F(R)}.$$

Therefore the second integral in (3) vanishes when $R\to\infty$, and we are left with

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx.$$

It is thus equal to (2).

This method can be applied to some cases where $f(z)$ contains trigonometric functions also, as in Ex. 3 below.

Ex. 2. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$. [*Bhopal*, 1978]

The function $1/(z^4+1)$ has 4 poles at points given by

$$z^4=-1=e^{i(2r+1)\pi},$$

or

$$z=e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}.$$

Of these only the first two lie in the upper half plane.

The residue at a pole z_0 is

$$\lim_{z\to z_0}\frac{z-z_0}{z^4+1}=\lim_{z\to z_0}\frac{1}{4z^3}=\frac{1}{4}(z_0)^{-3},$$

by L'Hospital's rule. Hence the sum of residues at $e^{i\pi/4}$ and $e^{i3\pi/4}$ is

$$\tfrac{1}{4}(e^{-i3\pi/4}+e^{-i9\pi/4})=\tfrac{1}{4}(\cos\tfrac{3}{4}\pi-i\sin\tfrac{3}{4}\pi+\cos\tfrac{9}{4}\pi-i\sin\tfrac{9}{4}\pi)$$

$$=\frac{1}{4}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=-\frac{i}{2\sqrt{2}}.$$

Integrating $1/(z^4+1)$ over the semi-circular contour C of fig. on p. 187 we get

$$\int_C\frac{dz}{z^4+1}=2\pi i\left(-\frac{i}{2\sqrt{2}}\right)=\frac{\pi}{\sqrt{2}}.$$

Now

$$\int_C\frac{dz}{z^4+1}=\int_{-R}^{R}\frac{dx}{x^4+1}+\int_0^{\pi}\frac{Re^{i\theta}i\,d\theta}{R^4e^{4i\theta}+1}.$$

Taking the limit as $R\to\infty$, we see that

$$\frac{Re^{i\theta}}{R^4e^{4i\theta}+1}=\frac{R^{-3}e^{i\theta}}{e^{4i\theta}+R^{-4}}\to 0;$$

so the second integral vanishes.

Therefore

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+1}=\frac{\pi}{\sqrt{2}}.$$

Ex. 3. Evaluate $\int_0^{\infty}\frac{\cos ax}{x^2+1}\,dx,\ a\geq 0.$ [*Roorkee*, 1972]

Since $\cos ax$ is the real part of e^{iax}, we consider the function

$$\frac{e^{iaz}}{z^2+1}.$$

This has poles at $z=i, -i$; only the first being in the upper half plane.

The residue at $z=i$ is

$$\lim_{z\to i}\frac{(z-i)e^{iaz}}{z^2+1}=\lim_{z\to i}\frac{e^{iaz}}{z+i}=\frac{e^{-a}}{2i}.$$

Hence

$$\int_C\frac{e^{iaz}}{z^2+1}\,dz=2\pi i\,\frac{e^{-a}}{2i}=\pi e^{-a}, \quad \ldots \quad (1)$$

C being the contour of fig. on p. 187. On the semicircle Γ,

$$\frac{e^{iaz}}{z^2+1}z=\frac{e^{iaR(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta}+1}Re^{i\theta}$$

$$=\frac{(e^{iaR\cos\theta}\,.\,e^{i\theta})e^{-aR\sin\theta}\,.\,R^{-1}}{e^{2i\theta}+R^{-2}}.$$

This tends to zero as $r\to\infty$, since

$$|e^{iaR\cos\theta}\,.\,e^{i\theta}|\leq 1,\ |e^{-aR\sin\theta}|\leq 1,\ |e^{2i\theta}+R^{-2}|\neq 0 \text{ and } R^{-1}\to 0.$$

Therefore the integral over Γ vanishes, and (1) gives

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}\,dx=\pi e^{-a},$$

or, on equating the real parts,

$$\int_{-\infty}^{\infty}\frac{\cos ax}{x^2+1}\,dx=\pi e^{-a}.$$

Hence

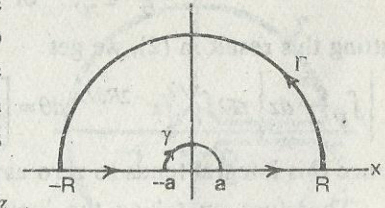
$$\int_0^{\infty}\frac{\cos ax}{x^2+1}\,dx=\frac{\pi e^{-a}}{2}.$$

III. *Integrals of type* $\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\,dx$, *when* $F(x)$ *has zeros on the real axis.*

In such cases we proceed in a manner similar to above, except that the contour C is indented at the singularities on the real axis by small semicircles to avoid including these singularities inside C (see fig. below).

Ex. 4. Evaluate $\int_0^{\infty}\frac{\sin x}{x}\,dx.$ [*Garakhpur*, 1971]

Evaluate the function e^{iz}/z over the contour C shown in the figure consisting of parts of the real axis from $-R$ to $-a$ and a to R, the small semicircle γ and the large semicicle Γ. The function has no singularity within this contour. Therefore



$$\int_C\frac{e^{iz}}{z}\,dz=0,$$

i.e.

$$\int_{-R}^{-a}\frac{e^{ix}}{x}\,dx+\int_{\gamma}\frac{e^{iz}}{z}\,dz+\int_a^{R}\frac{e^{ix}}{x}\,dx+\int_{\Gamma}\frac{e^{iz}}{z}\,dz=0. \quad (1)$$

Substitutiug $-x$ for x in the first integral and combining it with the third, we get

$$\int_a^R \frac{e^{ix}-e^{-ix}}{x}\,dx, \text{ i.e. } 2i\int_a^R \frac{\sin x}{x}\,dx.$$

The second integral

$$\int_\gamma \frac{e^{iz}}{z}\,dz=\int_\gamma \frac{1}{z}\,dz+\int_\gamma \frac{e^{iz}-1}{z}\,dz.$$

On γ, $z=ae^{i\theta}$; therefore

$$\int_\gamma \frac{dz}{z}=\int_\pi^0 \frac{ae^{i\theta}\,i\,d\theta}{ae^{i\theta}}=-\int_0^\pi i\,d\theta=-i\pi.$$

Also $$\left|\int_\gamma \frac{e^{iz}-1}{z}\,dz\right| \le M\int_\gamma \frac{|dz|}{|z|}=\pi M,$$

where M is the maximum value on γ of

$$|e^{iz}-1|, \text{ i.e. } |e^{ia(\cos\theta+i\sin\theta)}-1|.$$

Evidently $M\to 0$ as $a\to 0$.

Putting $z=Re^{i\theta}$ in the integral over Γ, we get

$$\int_\Gamma \frac{e^{iz}}{z}\,dz=\int_0^\pi \frac{e^{iR(\cos\theta+i\sin\theta)}}{Re^{i\theta}}\,Re^{i\theta}\,i\,d\theta$$
$$=i\int_0^\pi e^{iR\cos\theta}\,.\,e^{-R\sin\theta}\,d\theta.$$

Since $|e^{iR\cos\theta}|\le 1$, therefore

$$\left|\int_\Gamma \frac{e^{iz}}{z}\,dz\right| \le \int_0^\pi e^{-R\sin\theta}\,d\theta=2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta. \qquad (2)$$

Now the function $(\sin\theta)/\theta$ continually decreases from 1 to $2/\pi$ as θ increases from 0 to $\frac{1}{2}\pi$. Hence, for $0\le\theta\le\frac{1}{2}\pi$,

$$\frac{\sin\theta}{\theta}\ge\frac{2}{\pi}, \quad\text{or}\quad \sin\theta\ge\frac{2\theta}{\pi}.$$

Putting this result in (2), we get

$$\left|\int_\Gamma \frac{e^{iz}}{z}\,dz\right| \le 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta=\left[-\frac{\pi}{R}e^{-2R\theta/\pi}\right]_0^{\pi/2}=\frac{\pi(1-e^{-R})}{R}.$$

Hence this integral tends to zero as $R\to\infty$.

Therefore on taking the limit of (1) as $a\to 0$ and $R\to\infty$, we obtain

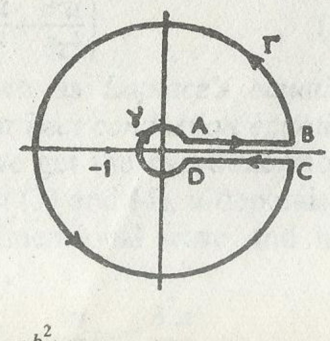
$$2i\int_0^\infty \frac{\sin x}{x}\,dx-i\pi=0, \quad\text{or}\quad \int_0^\infty \frac{\sin x}{x}\,dx=\frac{1}{2}\pi.$$

Examples 23

1. $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. [*Allahabad, M. E.*, 1970]
2. $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$, $a>|b|$. [*Roorkee, M. E.*, 1980]
3. $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$. [*Banaras*, 1978]
4. $\int_0^{\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$. [*Roorkee*, 1981]
5. $\int_0^{\infty} \frac{dx}{(x^2+1)^2}$. [*Roorkee*, 1980]
6. $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx$. [*Ranchi*, 1976]
7. $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$. [*Bhopal*, 1976]
8. $\int_0^{\infty} \frac{x^2\,dx}{x^6+1}$. [*Punjab, M.Sc. (Ch. Eng.)*, 1978]
9. $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$, $a>b>0$. [*Rajasthan, M.E.*, 1980]
10. $\int_0^{\infty} \frac{x\sin x}{x^2+a^2}\,dx$, $a>0$. [*Ranchi*, 1976]
11. $\int_0^{\infty} \frac{\sin mx}{x}\,dx$, $m>0$. [*Roorkee*, 1981]
12. $\int_0^{\infty} \frac{\sin mx\,dx}{x(x^2+a^2)}$, $m>0$, $a>0$. [*Kanpur, M.E.*, 1969]
13. $\int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx$, $0\le n\le 1$.

[*Aligarh, M.Sc. (Engg.)*, 1966]

[Hint. Integrate along the contour shown in the figure. The integrals round the circles γ and Γ vanish as $r\to 0$ and $R\to\infty$. On AB, $z=xe^{i0}$; and on CD, $z=xe^{i(\theta+2\pi)}$.]



14. Show that

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx=\tfrac{1}{2}\sqrt{\pi}e^{-b^2}.$$ [*Banaras*, 1978]