Assessment-1
Higher Mathematics (AMS-2610)
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1. Show that the function $f(z) = |xy|^{1/2}$, the Cauchy-Reimann equation are satisfied at origin. Does $f'(0)$ exist?

Sol.

For the given $f(x,y) = |xy|^{1/2}$

Let $u = |xy|^{1/2}$
and $v = 0$.

we have $\frac{\partial u}{\partial x}$ at $(0,0)$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{|(x+\Delta x).y|^{1/2} - |xy|^{1/2}}{\Delta x + i\Delta y}$$

at $(0,0)$

$$\lim_{\Delta x \to 0} = \left| \frac{0}{\Delta x + i\Delta y} = 0 \right|$$

Similarly,

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} = 0$$

$$\& \left| \frac{\partial v}{\partial x} = 0 \quad \text{as } v = 0 = \frac{\partial v}{\partial y} \right|$$

$\therefore$
$u_x = v_y = 0$
$\& \; u_y = -v_x = 0$

$\therefore$ Cauchy Reimann Eq holds at $(0,0)$.

$f'(z)$ at $z_0 = 0$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{|(x+\Delta y)(y+\Delta y)|^{1/2} - |xy|}{\Delta x + i\Delta y}$$

at $x = 0$ & $y = 0$ $\Rightarrow \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta y . \Delta x)^{1/2} - 0}{\Delta x + i\Delta y}$

let path chosen be $y = mx$

$\therefore$ $\Delta y = m\Delta x$

$$\therefore f'(0) = \frac{\cancel{\Delta y . m}(\Delta x . \Delta x . m)^{1/2}}{\Delta x(1+im)}$$

$$= \frac{m^{1/2} . \cancel{\Delta x}}{\cancel{\Delta x}(1+mi)}$$

$\Delta x \to$

$$\boxed{f'(0) = \frac{m^{1/2}}{1+im}}$$

Since

$f'(0)$ is a function of m $\therefore$ does not exist

2. If $f(z) = u(x,y) + iv(x,y)$, $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2}$, is continuous as a function of two variables z & $\bar{z}$, Show that

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4\partial^2 u}{\partial z . \partial \bar{z}}$.

Solⁿ (1).

$$x = \frac{\bar{z}+z}{2}$$

$$\boxed{\frac{\partial x}{\partial z} = \frac{1}{2} \ \& \ \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}} \quad \text{— (i)}$$

Similaly

$$\boxed{\frac{\partial y}{\partial z} = \frac{1}{2i} \ \& \ \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}} \quad \text{— (ii)}$$

Now, we know

$$\frac{\partial y}{\partial u} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial z} \quad \text{from (i) \& (ii)}$$

$$\frac{\partial y}{\partial a} = \frac{\partial u}{\partial x} \cdot \frac{1}{2} + \frac{\partial u}{\partial y} \cdot \frac{1}{2}$$

$$\frac{\partial^2 u}{\partial z . \partial \bar{z}} = \frac{1}{2} \cdot \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{1}{2i} \cdot \frac{\partial^2 u}{\partial y^2} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial^2 u}{\partial z . \partial \bar{z}} = \frac{1}{4} \frac{\partial^2 u}{\partial x^2} + \frac{1}{4} \cdot \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial^2 u}{\partial z . \partial \bar{z}}}$$

2. $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the Cauchy-Riemann Eq.

Soln.

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} \cdot \frac{\partial \cancel{x}}{\partial x} + \frac{\partial u}{\partial \bar{z}} \cdot \frac{\partial y}{\partial y}$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \frac{1}{2} + \frac{\partial y}{\partial y} \cdot \left(\frac{-1}{2i}\right)$$

$$\frac{\partial v}{\partial \bar{z}} = \frac{\partial v}{\partial x} \cdot \frac{1}{2} + \frac{\partial v}{\partial y} \cdot \left[\frac{-1}{2y}\right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \frac{1}{2} + \frac{\partial y}{\partial y}\left[\frac{-1}{2y}\right] + i\left[\frac{\partial v}{\partial x} \cdot \frac{1}{2} + \frac{\partial v}{\partial y}\left[\frac{1}{x}\right]\right]$$

$$\boxed{\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} + i\left[\frac{1}{2}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = 0}$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ are real part are equal

& $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ imaginary parts.

3. Find the value of a, b, & c so that following function are entire.

i) $f(z) = x + ay - i(bx + cy)$.

Sol.

$u = x + ay$

$v = -(bx + cy)$

$\frac{\partial u}{\partial x} = 1$ & $\frac{\partial v}{\partial x} = -b$

$\frac{\partial u}{\partial y} = a$ & $\frac{\partial v}{\partial y} = -c$

for the function to be entire,

$u_x = v_y$

$-c = 1$

$\boxed{c = -1}$

& $u_y = v_x$

$\boxed{a = b}$

ii) $f(z) = ax^2 - by^2 + icxy$

Sol. $u = ax^2 - by^2$

$v = cxy$

$\frac{\partial u}{\partial x} = 2ax$ & $\frac{\partial v}{\partial x} = cy$

$\frac{\partial u}{\partial y} = -2by$ & $\frac{\partial v}{\partial x} = cx.$

for the function to be entire,

$u_x = v_y$ & $u_y = -v_x$

$2a\cancel{x} = c\cancel{x}$ & $-2b\cancel{y} = -c\cancel{y}$

$\boxed{c = 2a = 2b}$

iii) $f(z) = e^x \cos ay + i e^x \sin(y+b) + c$

$u = e^x \cos ay + c$ & $v = e^x \sin(y+b)$

$\frac{\partial u}{\partial x} = e^x \cos ay$, $\frac{\partial v}{\partial x} = e^x \sin(y+b)$

$\frac{\partial u}{\partial y} = -a e^x \sin ay$ $\frac{\partial v}{\partial y} = e^x \cos(y+b)$

$u_x = v_y$ & $u_y = -v_x$

$\cos ay = \cos(y+b)$ — (i)

$a \sin ay = \sin(y+b)$. — (ii)

Sq (i) & (ii) & adding.

$\cos^2 ay + a^2 \sin^2 ay = 1$ for $a^2 = 1$

$\therefore \boxed{a = \pm 1}$

& $a^2 \sin^2 ay + \cos^2 ay = 1$

for $a = 0$

as ~~cos~~

$a^2 \sin^2 ay = \sin^2 ay$

$a(a^2-1) \cdot \sin^2 ay = 0$ for $a = 0$

($b = 0$ and $c =$ any No)

4. If f is analytic in Domain 'D', prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2|f(z)|^{n-2}\cdot|f'(z)|^2$$

Let

$$f(z) = u+iv.$$

$$\therefore |f(z)| = \sqrt{u^2+v^2}$$

$$\therefore |f(z)|^n = (u^2+v^2)^{n/2}$$

$$\frac{\partial}{\partial x}(u^2+v^2)^{n/2} = \frac{n}{2}(u^2+v^2)^{\frac{n-1}{2}}\left[2\frac{\partial u}{\partial x}.u + 2\frac{\partial v}{\partial x}.v\right]$$

$$\frac{\partial^2}{\partial x^2}|f(z)|^n = \frac{n}{2}\left[\frac{n}{2}-1\right](u^2+v^2)^{\frac{n}{2}-2}\left[2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right]^2 +$$

$$\frac{n}{2}(u^2+v^2)^{\frac{n}{2}-1}\left[2.\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial x^2}+\right.$$

$$\left.2\left[\frac{\partial v}{\partial x}\right]^2+2v\frac{\partial^2 v}{\partial x^2}\right] \quad -(1)$$

Similaly for,

$$\frac{\partial^2|f(z)|^n}{\partial y^2} = \frac{n}{2}\left[\frac{n}{2}-1\right](u^2+v^2)^{n/2-2}\left[2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}\right]^2+$$

$$\frac{n}{2}(u^2+v^2)^{n/2-1}\left[2.\left(\frac{\partial u}{\partial y}\right)^2+2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial v}{\partial y}\right)^2+2v.\frac{\partial^2 v}{\partial y^2}\right]$$

Adding (i) & (ii)

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^n = n\left(\frac{n}{2}-1\right)\left[u^2+v^2\right]^{\frac{n}{2}-2}\left[4u^2\left[\frac{\partial u}{\partial x}\right]^2+4v^2\left[\frac{\partial v}{\partial x}\right]^2\right.$$

$$\left.+4uv(u_x v_x)\right]$$

$$+ 4u^2\left[\frac{\partial u}{\partial y}\right]^2 + 4v^2\left[\frac{\partial v}{\partial y}\right]^2 + \cancel{4u.v(4y} \; 4(u\cdot v)(u_y\cdot v_y)\Big] +$$

$$\frac{n}{2}(u^2+v^2)^{n/2-1}\left[2\cdot\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] +$$

$$2u(u_{yy} + u_{xx}) + 2v(v_{xx} + v_{yy})\Big]$$

we know that u & v are hamonic,

$\because$ $f(z) = u + iv$

$\therefore$ $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. $|f'(z)| = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]^{1/2}$

$\therefore$

RHS

$$\frac{n}{2}\left[\frac{n}{2}-1\right]|f(z)|^{n-4}\left\{4u^2|f'(z)|^2 + 4v^2|f'(z)|^2 + 4vu(u_y v_x - v_y v_x)\right\}$$

$$+\frac{n}{2}|f(z)|^{n-2}\left\{4|f'(z)|^2\right\}.$$

$$= \frac{n}{2}\left[\frac{n}{2}-1\right]|f(z)|^{n-2}|f'(z)|^2 \times 4 + u^2 \cdot \frac{n}{2}|f(z)|^{n-2}\cdot|f'(z)|^2.$$

$$n[n-2+2]|f(z)|^{n-2}|f'(z)|^2$$

$$\left[n^2|f(z)|^{n-2}\,|f'(z)|^2\right]$$ Hence Proued.

Q5. Find the analytical function $f(z) = u+iv$, if $u+v = \frac{x}{x^2+y^2}$ & $f(1) = 1$.

Soln

we know,

$$u+v = \frac{x}{x^2+y^2}$$

$$\& \; f(z) = u+iv$$

$$\& \; if(z) = \underline{iu-v}$$

$$\underbrace{(u-v)}_{U} + i\underbrace{(u+v)}_{V} = \underline{f(z)+if(z)}.$$

$U = u-v$

$V = (u-v) . F(z) = (i+1) f(z)$

$V = \frac{x}{x^2+y^2} = u+v$ (given).

$\therefore$

$$\phi_1(x,y) = \frac{\partial V}{\partial y} = \frac{(x^2+y^2).0 - x.(2y)}{(x^2+y^2)^2} = \frac{-x.2y}{(x^2+y^2)^2}$$

$$\phi_1(z,0) = \left[\frac{\partial v}{\partial y}\right]_{(z,0)} = 0$$

$$\phi_2(x,y) = \left[\frac{\partial v}{\partial x}\right] = \frac{(x^2+y^2)\times 1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\phi = (z,0) = \left[\frac{\partial v}{\partial x}\right]_{z,0} \boxed{\frac{-1}{z^2}}$$

$$F'(z) = \phi_1(z,0) + i\phi_2(z,0)$$

$$\int F'(z)\,dz = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

$$F(z) = 0 + i\int \frac{-1}{z^2}\,dz$$

$$F(z) = \frac{i}{z} + c$$

$$(i+1)f(z) = \frac{i}{z} + c$$

$$f(z) = \frac{i+1}{z(i+1)} + c$$

at $f(1) = 1$

$$f(1) = \frac{i+1}{2} + c = 1$$

$$c = \frac{2-1-c}{2}$$

$$\boxed{c = \frac{1-i}{2}}$$

$$\therefore \boxed{f(z) = \frac{1+i}{2z} + \frac{1-i}{2}} \text{ Ans}$$

~~6. $u(x,y) = e$~~

6. Show that the given function $u(x,y) = e^{x^2-y^2}\cos 2xy$ is harmonic. Find the corresponding conjugate harmonic function $v(x,y)$ and construct the analytical function $f(z) = u+iv$.

Soln

For given $f(x)$ to be harmonic, $u_{xx}+u_{yy}=0$

So,

$$u_x = 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy$$

$$u_{xx} = 4x^2e^{x^2-y^2}\cos 2xy + 2xe^{x^2-y^2}\cos 2xy - 4xye^{x^2-y^2}\sin 2xy$$
$$-4xye^{x^2-y^2}\sin 2xy - 4y^2e^{x^2-y^2}\cos 2xy$$

& $$u_y = -2ye^{x^2-y^2}\cos 2xy - 2xe^{x^2-y^2}\sin 2xy$$

$$u_{yy} = -2e^{x^2-y^2}\cos 2xy + 4y^2e^{x^2-y^2}\cos 2xy + 4xye^{x^2-y^2}\sin 2xy$$
$$+ 4xye^{x^2-y^2}\sin 2xy - 4x^2e^{x^2-y^2}\sin 2xy$$

Here we have,

$$\boxed{u_{xx}+u_{yy}=0}$$

$$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy)$$

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy)$$

$$\phi_1(z,0) = 2e^{z^2}(z\times\cos 0 - 0)$$
$$\phi_2(z,0) = 2e^{z^2}(0\times 1 + z\times 0)$$

$$\boxed{\phi_1 = 2ze^{z^2} \;\&\; \phi_2 = 0}$$

$$f(z) = \int_c (\phi_1 - i\phi_2)\, dz + c$$

$$= \int_c 2z e^{z^2} dz + c$$

$$= \int_c e^t dt + c$$

$$= e^t + c$$

$$= \boxed{e^{z^2} + c} \quad \text{Ans.}$$

or $\boxed{f(z) = [e^{x^2-y^2}(\cos 2xy + i \sin 2xy)]}$

7. Find the Laurent's series of $f(z) = \frac{4-3z}{z(1-z)(2-z)}$

valid in the region,

i) $|z|<1$

$$f(z) = \frac{4-3z}{z(1-z)(2-z)}.$$

By Partial fraction

$$\frac{4-3z}{z(1-z)(2-z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{2-z}$$

$\boxed{A = \frac{4}{2} = 2}$

$\boxed{B = 1}$ & $C = \boxed{\frac{-2}{-2}} = 1$

$\therefore$

$$= \boxed{\frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}}$$

$\therefore$ for (i) $\frac{2}{z} + (1-z)^{-1} + (2-z)^{-1}$ | since $(1-x)^{-1} = 1+x+x^2+x^3\ldots$ for $|x|<1$.

$$= \frac{2}{z} + (1+z+z^2+z^3) + \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$= \frac{2}{z} + \frac{3}{2} + \frac{5z}{4} + \frac{9z^2}{8} + \frac{17z^3}{16} + \cdots$$

2. for $|z|>1$ & $|z|<2$.

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{2}{z} - \frac{1}{z(\frac{1}{z}-1)} + \frac{1}{2(1-z/2)}$$

as $\frac{1}{z}<1$ for $|z|>1$

& $|z/2|<1$ as $|z|<2$

$$\therefore \frac{2}{z} - \left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] + \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right]$$

$$= \left[\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}\cdots\right] + \frac{1}{2} + \left[\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right]$$

3. for $|z|>2$

$$\frac{2}{|z|}<1 \quad \& \quad \frac{1}{|z|}<1$$

$$\therefore f(z) = \frac{2}{z} - \frac{1}{z(1-1/z)} - \frac{1}{z(1-2/z)}$$

$$= \frac{2}{z} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{z[1-2/z]}$$

$$= \frac{2}{z} - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right]$$

$$= \frac{2}{z} - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} \cdots - \frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \frac{8}{z^4}$$

$$= \left[-\frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4}\cdots\right]$$

8

i)

ii)

8 Classify the singular points of the following functions in the finite complex plane.

i) $\dfrac{\sin z^2}{z(e^z-1)}$

for $z=0$ the function is not defined

but at $\lim_{z\to 0} \dfrac{\sin z^2}{z\left(\dfrac{e^z-1}{z}\right)z} = \dfrac{\sin z^2}{z^2\left(\dfrac{e^z-1}{z}\right)} = 1$

as $\lim_{a\to 0}\dfrac{\sin a}{a} = 1$

& $\lim_{x\to 0}\dfrac{e^x-1}{x} = 1$

$\therefore$

at $z=0$, we have $(z=0)$ as a removable singularity by defin.

$\boxed{f(x) \text{ at } 0 := 1}$

ii) $f(z) = \dfrac{1-\cos z}{\sin^3 z}$

for $z=0$ $f(z)$ is not defined,

as $1-\cos x = 2\sin^2 x/2$

$\therefore \dfrac{\cancel{2}\sin^2 z/2 \times z^2/4\, 2}{\sin^3 z \times \left(\dfrac{z^2}{4}\right)}$

$=$

$\to \lim_{z\to 0} \dfrac{\dfrac{\sin^2 z/2}{(z/2)^2}}{\dfrac{\sin^3 z}{z^2}}$ as $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$

$= \dfrac{\cancel{2}}{\sin \lim_{z\to 0}} \left|\dfrac{z^2}{\sin 3z}\right|$ is not defined as require z^3 but have z^2

but $\lim_{z\to 0} \dfrac{f(z)}{0} = 1$ $\therefore$ $[z=0$ is singularity of $f(z)$ of pole $=0$ & order $=-1]$

iii) $\dfrac{z^2}{e^{1/z}-1}$

let $z = 1/\lambda$

$$\therefore \; f(1/\lambda) = \frac{1/\lambda^2}{e^{\lambda}-1}$$

or $f(\lambda)$

$$= \frac{1}{\lambda^2(e^{\lambda}-1)}$$

at $\lambda = 0$, $f\left(\frac{1}{\lambda}\right)$ is not defined.

$$\Downarrow$$

$$f(\lambda)$$

but for $\lim\limits_{\lambda\to 0} \dfrac{1}{\lambda^2\left(\frac{e^{\lambda}-1}{\lambda}\right)\times\lambda} = \dfrac{1}{\lambda^3}$ is not defined

$\therefore$ $\lambda = 0$ or $z = \infty$ is an essential singularity for $f(z)$.

Q. Evaluate, $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$ using contour integration

we know, from euler's theo,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$\therefore \quad e^{i\sin\theta} = \underline{\cos(\sin\theta)} + i\sin(\sin\theta)$$

$$= R_e \left[\int_0^{2\pi} e^{\cos\theta} . \cos\theta(\sin\theta) \right] d\theta$$

$$= R_e \left[\int_0^{2\pi} e^{\cos\theta} . \left[e^{i\sin\theta} \right] d\theta \right]$$

$$= R_e \left[\int_0^{2\pi} e^{\underline{\cos\theta + i\sin\theta}}\, d\theta \right]$$

$$= R_e \left[\int_0^{2\pi} e^{e^{i\theta}}\, d\theta \right]$$

let $z = \underline{e^{i\theta}}$. $\quad dz = \underline{ie^{i\theta} d\theta}$

$$\therefore \int_0^{2\pi} e^{z} . \cancel{ie^{i\theta}} \frac{dz}{ie^{i\theta}}$$

$$= \int_0^{2\pi} e^{z} . \frac{dz}{iz}$$

$$= \frac{1}{i} \int_C \frac{e^{z}}{z} dz \qquad \underline{z=0} \text{ pole.}$$

$$\therefore \int_C \frac{e^{z}}{z} = 2\pi i (zR).$$

$$R = \lim_{z\to 0} \left[(z-0)\frac{e^{z}}{z} = 1 \right] \quad \therefore \int_0 \frac{e^{z}}{z} dz = \boxed{2\pi i}$$

$\frac{1}{i}\int_C \frac{e^z}{z}dz = 2\pi$ Hence, $Re\left[\frac{1}{i}\int_C \frac{e^z}{z}dz\right] = \boxed{2\pi}$.

10. Evaluate the integral using Contour integration.

$$f(z) = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4x+5}\,dx$$

$e^{i\theta} = \sin\theta + i\cos\theta$

$\therefore Re(e^{ix}) = \sin x$

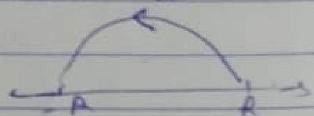
$\therefore$ let $\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+4x+5}\,dx$

Hence, $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4x+5}\,dx = Im\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+4x+5}\,dx$

Also, consider

$$f(z) = \frac{ze^{iz}}{z^2+4z+5}$$

$z = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2+i$ CR



$\boxed{z = -2+i}$ & $-2-i$

Pole lies in Contour C.

$$\int \frac{ze^{iz}}{z^2+4z+5}dz = \int_{-R}^{R} \frac{xe^{ix}}{x^2+4x+5}dx + \int \frac{ze^{iz}}{z^2+4z+5}dz$$

let $z = Re^{i\theta}$ $[dz = Rie^{i\theta}d\theta]$

$$\Rightarrow \int \frac{ze^{iz}}{z^2+4z+5}dz = \int \frac{Re^{i\theta}\cdot e^{iRe^{i\theta}}\cdot Re^{i\theta}d\theta}{R^2e^{i\theta}+4Re^{i\theta}+5}.$$

$$\left| \int_{C_R} \frac{z e^{iz}}{z^2+4z+5} dz \right| \le \frac{R^2}{R^2-4R+5} \int_0^{\pi} |e^{iR\cos\theta}| \times |e^{-R\sin\theta}| d\theta \le$$

$$\le \frac{2R^2}{R^2-4R+5} \int_0^{\pi/2} |e^{iR\cos\theta}| \times |e^{-R\sin\theta}| d\theta$$

$\therefore$

$$\le \frac{2R^2}{R^2-4R+5} \int_0^{\pi/2} e^{-R\sin\theta} d\theta$$

Applying Jordan's Lemma.

$$\int_0^{\pi/2} e^{-R\sin\theta} d\theta \le \frac{\pi}{2R}$$

$\therefore$

$$\le \frac{\cancel{2}R^{\cancel{2}}}{R^2-4R+5} \times \frac{\pi}{\cancel{2R}}$$

$$\le \frac{\pi}{R\left[1-\frac{4}{R}+\frac{5}{R^2}\right]}$$

for $R \to \infty$ $\;1/R \to 0 \;\therefore$

$$\le \frac{\pi}{R[1-0+0]} = 0$$

Hence,

$$\int \frac{z e^{iz}}{z^2+4z+5} dz = \int_{-R}^{R} \frac{x e^{ix}}{x^2+4x+5} dx$$

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+4x+5} dx = \boxed{\overset{R\to\infty}{2\pi i \sum R}}$$

Residue Theorem

$$= \lim_{z \to i-2} \frac{(z-i+2)\ ze^{iz}}{(z-i+2)(z\sqrt{i+2})}$$

$$= \frac{(i-2)\, e^{i(i-2)}}{(i-2)(i+2)} = \frac{(i-2)\, e^{i(i-2)}}{2i}$$

$$\int_c f(z)\,dz = 2\pi i \times \frac{i-2}{2i}\, e^{i(i-2)}$$

$$= \pi\left[(i-2)\, e^{-1-2i}\right]$$

$$= \frac{\pi}{e}(i-2)\left[\cos(-2) + i\sin(-2)\right]$$

$$= \frac{\pi}{e}\left[i\cos 2 + \sin 2 - 2\cos 2 + i\sin 2\right]$$

$$= \frac{\pi}{e}\left[(\sin 2 - 2\cos 2) + i(\cos 2 + 2\sin 2)\right]$$

$$\operatorname{Im}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+4x+5}\,dx = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4x+5}\,dx = \operatorname{Im}\int_c f(z)\,dz$$

$$= \operatorname{Im}\left[\frac{\pi}{e}(\sin 2 - 2(\cos 2) + i(\cos 2 + 2\sin 2)\right]$$

$$= \boxed{\frac{\pi}{e}\left[\cos 2 + 2\sin 2\right]}$$