Hence $\tan^{-1}(x+iy)=\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}+\frac{1}{2}i\tanh^{-1}\frac{2y}{1+y^2+y^2}$.

(iii) $$i^i=(e^{\log i})^i=e^{i\log i}.$$

Now $$i\log i = i\log\{\cos(2n+\tfrac{1}{2})\pi+i\sin(2n+\tfrac{1}{2})\pi\}$$
$$= i\log e^{i(4n+1)\pi/2}=i^2(4n+1)\pi/2$$
$$=-(4n+1)\pi/2.$$

Therefore $$i^i=e^{-(4n+1)\pi/2}.$$

Ex. 2. Solve the equation $\cos z=2$. [*Aligarh*, 1980]

We are given that $\frac{1}{2}(e^{iz}+e^{-iz})=2$,

or $(e^{iz})^2-4e^{iz}+1=0$, or $e^{iz}=2\pm\sqrt{(4-1)}$.

Therefore $iz=\log(2\pm\sqrt{3})+2n\pi i$,

or $z=2n\pi-i\log(2\pm\sqrt{3})=2n\pi\mp i\log(2+\sqrt{3})$.

Examples 18

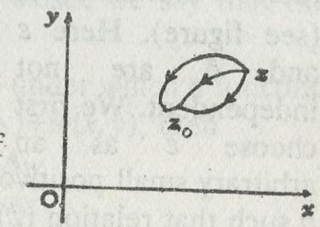
Express in the form $A+iB$:

1. $\cos(\frac{1}{4}\pi+\frac{1}{2}i)$. 2. $\log(1+i\sqrt{3})$.
3. $\tan(x+iy)$. [*Jodh.* '69] 4. $\text{sech}\,(x+iy)$.
5. $\cos^{-1}(x+iy)$. 6. $(1+i)^{1-i}$. [*Gor.* '71]
7. Evaluate $\log(-1+i)-\log(-1-i)$. [*Roorkee M.E.*, 1980]
8. Prove that $\sin(\log i^i)=-1$. [*Roorkee*, 1980]
9. If $u=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)$ prove that $\tanh\frac{1}{2}u=\tan\frac{1}{2}\theta$.
10. Prove that
$$|\sin z|^2=\sin^2 x+\sinh^2 y,$$
and
$$|\cosh z|^2=\sinh^2 x+\cos^2 y.$$ [*Roorkee*, 1966]
11. Find $|e^{2z+i}|$ and $|e^{z^2}|$, and show that
$$|e^{2z+i}+e^{z^2}|\le e^{2x}+e^{-2xy}.$$ [*Roorkee, M.E.*, 1979]
12. Show that $\tan^{-1}z=\frac{i}{2}\log\frac{i+z}{i-z}$. [*Roorkee, M.E.*, 1968]
13. Prove that $\tan^{-1}(2i)=\pm(n+\frac{1}{2}\pi)+i\log 3$ $(n=0, 1, 2, \ldots)$. [*Roorkee, M.E.*, 1979]
14. Prove that $\sinh^{-1}z=in\pi+(-1)^n\log\{z+\sqrt{(z^2+1)}\}$. [*Roorkee*, 1995]
15. If $\sin z=2i$, show that

$$z=n\pi+(-1)^{n+1}i\log(\sqrt{5}-2),$$
where $n=0, \pm1, \pm2, \ldots$. [*Roorkee*, 1989]

16. Solve the equation $\cosh z=-1$.
17. Find all the roots of the equation $\sin z=\cosh 4$.
18. If $u+iv=\sin^{-1}(x+iy)$, prove that
$$\frac{x^2}{\cosh^2 v}+\frac{y^2}{\sinh^2 v}=1 \quad\text{and}\quad \frac{x^2}{\sin^2 u}-\frac{y^2}{\cos^2 u}=1.$$ [*Allahabad, B. Tech.*, 1991]
19. If $\sin(x+iy)=re^{i\theta}$, show that
$$r=\sqrt{\{\tfrac{1}{2}(\cosh 2y-\cos 2x)\}} \text{ and } \theta=\tan^{-1}(\cot x\tanh y).$$
Deduce that $$y=\frac{1}{2}\log\frac{\cos(x-\theta)}{\cos(x+\theta)}.$$
20. If $x+iy=\tan(u+iv)$, prove that
$$\cot 2u=\frac{1-x^2-y^2}{2x};\ \coth 2v=\frac{1+x^2+y^2}{2y}.$$

5.3. Limits. We say that the complex number $z=x+iy$ approaches, or tends to, a fixed complex number $z_0=x_0+iy_0$ when $x\to x_0$ and $y\to y_0$. Graphically this means that the point z is in the neighbourhood of the point z_0 on the Argand diagram, and approaches it. It is evident that this approach can be made in an infinite number of ways from different directions.



Let $w=f(z)$ be a function of z, and let $u(x,y)$, $v(x,y)$ denote the real and imaginary parts of w. Let $w_0=u_0+iv_0$ be a fixed complex number. We say that *the limits of $f(z)$, as z tends to z_0, is w_0, and write*
$$\lim_{z\to z_0} f(z)=w_0, \tag{1}$$
when $u(x,y)\to u_0$ and $v(x,y)\to v_0$ as $x\to x_0$ and $y\to y_0$.

Graphically the relation $w=f(z)$ maps a point z in the z-plane to a corresponding point w in the w-plane. So the relation (1) means that the point w in the w-plane approaches the point w_0 when z approaches z_0.

The idea that the point z is in the neighbourhood of z_0 and approaches z_0 can be expressed by saying that

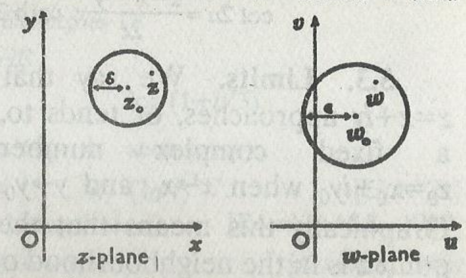
$$|z-z_0|<\delta,$$

where δ is an arbitrarily small positive number. This means that z lies within the circle of radius δ drawn with centre at z_0. By choosing δ smaller and smaller, z can be made to approach z_0 as nearly as we please.

The idea that $f(z)\to w_0$ as $z\to z_0$ can be expressed by saying that for every $\varepsilon>0$ there exists a δ such that

$$|f(z)-w_0|<\varepsilon \text{ when } |z-z_0|<\delta. \qquad (2)$$

This means that $f(z)$ lies within a circle of radius ε and centre w_0 when z lies within a circle of radius δ and centre z_0 (see figure). Here ε and δ are not independent. We first choose ε as an arbitrary small positive number; then we find a value of δ such that relation (2) is satisfied.



5.31. Continuity. The function $w=f(z)$ is said to be continuous at $z=z_0$ if

$$\lim_{z\to z_0} f(z)$$

exists and is equal to $f(z_0)$, the value of $f(z)$ at $z=z_0$.

In other words,

$$f(z)-f(z_0)\to 0 \text{ as } z\to z_0$$

for a function continuous at z_0. It is customary to write

$$z-z_0=\Delta z \text{ and } f(z)-f(z_0)=\Delta f(z).$$

So for a continuous function $\Delta f(z)\to 0$ as $\Delta z\to 0$.

We shall see from the definition of derivative given

in the next article, that a function should be continuous in order to possess a derivative.

5.32. Differential Coefficient. Let $w=f(z)$ be a function of the complex variable z, and let $z_0+\Delta z_0$ denote a point in the neighbourhood of a fixed point z_0. Then the *derivative* $f'(z)$ is defined by the relation

$$f'(z_0)=\lim_{\Delta z_0\to 0}\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0}.$$

The derivative (or differential coefficient) is also denoted by dw/dz; we can write

$$\frac{dw}{dz}=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z},$$

where Δw indicates the change in w corresponding to the change Δz in z. For this limit to exist, $\Delta w/\Delta z$ must approach the *same* limiting value when $\Delta z\to 0$ along different paths. When the limit exists, we say that the function is *differentiable* at $z=z_0$.

To obtain a set of conditions under which a function will be differentiable let $w=u(x,y)+iv(x,y)$; then

$$\frac{dw}{dz}=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y}.$$

Consider first this limit when $\Delta z\to 0$ along a path parallel to the real axis. In this case $\Delta y=0$ and $\Delta x\to 0$. Therefore

$$\frac{dw}{dz}=\lim_{\Delta x\to 0}\frac{\Delta u+i\Delta v}{\Delta x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

Next consider the limit when $\Delta z\to 0$ along a path parallel to the imaginary axis. Then $\Delta x=0$, so that

$$\frac{dw}{dz}=\lim_{\Delta y\to 0}\frac{\Delta u+i\Delta v}{i\Delta y}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}.$$

If the function is differentiable, these two limits must be equal, that is

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y},$$

or
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

These are known as *Cauchy-Riemann equations.* We have shown that they are *necessary* conditions for the existence of a derivative at a point. It can be shown that they are also *sufficient* conditions, provided that the first partial derivatives are continuous.

It is evident from the definition of a derivative that the various formulae obtained for the differential coefficients of real functions hold for functions of a complex variable also. One case is illustrated in the example below.

Ex. 1. Obtain the differential coefficient of z^3.

Let $w=z^3$; then $\Delta w=(z+\Delta z)^3-z^3$. Therefore

$$\frac{dw}{dz}=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{3z^2\Delta z+3z(\Delta z)^2+(\Delta z)^3}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\{3z^2+3z\Delta z+(\Delta z)^2\}=3z^2.$$

Ex. 2. Show that $\dfrac{d\bar{z}^2}{dz}$ does not exist at any point except $z=0$.

[*Banaras*, 1989]

Put $u+iv=\bar{z}^2=(x-iy)^2$;

then $u=x^2-y^2,\ v=-2xy$.

Therefore $\dfrac{\partial u}{\partial x}=2x,\ \dfrac{\partial v}{\partial y}=-2x;$

$$\frac{\partial u}{\partial y}=-2y,\ \frac{\partial v}{\partial x}=-2y.$$

So Cauchy-Riemann equations are not satisfied except when $x=0, y=0$. Hence $d\bar{z}^2/dz$ does not exist at any point, except possibly $z=0$. At the latter point the partial derivatives are continuous (and equal to 0). Hence $d\bar{z}^2/dz$ exists at $z=0$, and has the value 0.

5.33. Analytic Functions. *A function $f(z)$ of a complex variable z is said to be analytic at a point z_0 if its derivative $f'(z)$ exists at $z=z_0$ and at every point in some neighbourhood of z_0.* The function is said to be analytic in a region R if $f'(z)$ exists at every point in R. The terms *regular* or *holomorphic* are also used by some authors to denote analytic functions. From what we have seen in the preceding section, we get the following important theorem:

A necessary and sufficient condition for the function $f(z)=u(x,y)+iv(x,y)$ to be analyic in a region R in the z-plane, is that u, v and their first partial derivatives are continuous in R and the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\quad . . . \quad (1)$$

are satisfied at every point in R.

The real and imaginary parts of an analytic function are called *conjugate functions.* Thus if $f(z)=u(x,y)+iv(x,y)$ is an analytic function, then $u(x,y)$ and $v(x,y)$ are conjugate functions. The relation between two conjugate functions u and v is given by the Cauchy-Riemann equations (1).

We get from (1) on differentiation

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y},\quad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x}.$$

Addition gives

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.\quad . . . \quad (2)$$

We can obtain a similar result for v.

Thus *the real and imaginary parts of a complex function $f(z)=u(x,y)+iv(x,y)$, analyitc in a region R, are solutions in R of two dimensional Laplace's equation :*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

For this reason u and v are also known as *harmonic functions*.

Ex. 1. Determine the analytic function $f(z)=u+iv$, if

$$u=x^3-3xy^2+3x^2-3y^2+2x+1. \quad [\textit{Gorakhpur}, 1971]$$

By Cauchy-Riemann equations

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=-(-6xy-6y)=6xy+6y.$$

Integrating partially with respect to x, we get

$$v=3x^2y+6xy+\phi(y)+c, \qquad (1)$$

where $\phi(y)$ is some function of y alone and c is an arbitrary constant.

Again $$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=3x^2-3y^2+6x+2,$$

which gives on partial integration

$$v=3x^2y-y^3+6xy+2y+\psi(x)+c. \qquad (2)$$

Comparing (1) and (2), we see that $\phi(y)=-y^3+2y$ and $\psi(x)=0$.

Hence

$$v=3x^2y-y^3+6xy+2y+c,$$

and

$$\begin{aligned} f(z) &= u+iv=(x^3-3xy^2+3x^2-3y^2+2x+1) \\ &\quad +i(3x^2y-y^3+6xy+2y+c) \qquad (3) \\ &= (x+iy)^3+3(x+iy)^2+2(x+iy)+1+ic \\ &= z^3+3z^2+2z+1+ic. \qquad (4) \end{aligned}$$

[*Note that* (4) *can be obtained very easily from* (3) *by putting* $x=z$ *and* $y=0$ *in it.*]

ALTERNATIVE METHOD. Since

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y},$$

we have

$$\begin{aligned} f'(z) &= (3x^2-3y^2+6x+2)+i(6xy+6y) \\ &=3z^2+6z+2 \text{ (on putting } x=z \text{ and } y=0). \end{aligned}$$

Therefore

$$\begin{aligned} f(z) &=z^3+3z^2+2z+\text{constant}, \\ &=z^3+3z^2+2z+1+ic, \end{aligned}$$

on adjusting the real part of the constant, which is given to be 1.

Ex. 2. If $f(z)$ is an analytic function of z, then show that x and y can occur in $f(z)$ only in the combination $(x+iy)$. [*Roorkee*, 1977]

Let $f(z)=u+iv$, then u and v are functions of x and y. But $x=\frac{1}{2}(z+\bar{z})$ and $y=(z-\bar{z})/2i$. So we can consider u and v as functions of two independent variables z and $\bar{z}$. Therefore

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}+\frac{\partial u}{\partial \bar{z}}\cdot\frac{\partial \bar{z}}{\partial x}=\frac{\partial u}{\partial z}+\frac{\partial u}{\partial \bar{z}}$$

and

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}+\frac{\partial u}{\partial \bar{z}}\cdot\frac{\partial \bar{z}}{\partial y}=i\frac{\partial u}{\partial z}-i\frac{\partial u}{\partial \bar{z}}.$$

Similarly

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial z}+\frac{\partial v}{\partial \bar{z}} \text{ and } \frac{\partial v}{\partial y}=i\frac{\partial v}{\partial z}-i\frac{\partial v}{\partial \bar{z}}.$$

If $f(z)$ is analytic, then

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Therefore

$$\frac{\partial u}{\partial z}+\frac{\partial u}{\partial \bar{z}}=i\frac{\partial v}{\partial z}-i\frac{\partial v}{\partial \bar{z}}$$

and

$$i\frac{\partial u}{\partial z}-i\frac{\partial u}{\partial \bar{z}}=-\left(\frac{\partial v}{\partial z}+\frac{\partial v}{\partial \bar{z}}\right)=i^2\left(\frac{\partial v}{\partial z}+\frac{\partial v}{\partial \bar{z}}\right),$$

i.e.

$$\frac{\partial}{\partial z}(u-iv)+\frac{\partial}{\partial \bar{z}}(u+iv)=0$$

and

$$\frac{\partial}{\partial z}(u-iv)-\frac{\partial}{\partial \bar{z}}(u+iv)=0.$$

Subtraction gives

$$\frac{\partial}{\partial \bar{z}}(u+iv)=0, \quad \text{i.e.} \quad \frac{\partial f(z)}{\partial \bar{z}}=0.$$

Hence $f(z)$ is independent of $\bar{z}$; i.e. x and y occur in $f(z)$ in the combination $x+iy$ only.

5.34. Application to Potential Problems. We have seen that the real and imaginary parts of an analytic function are solutions of Laplace's equation in two dimensions. For this reason the conjugate functions provide solutions to a number of potential problems. In these problems the physical quantities are obtainable from a potential function which satisfies the Laplace's equation. We consider below some illustrative problems.

I. FLUID FLOW. Consider the two dimensional irrotational motion of an incompressible fluid. The motion is said to be *two-dimensional* when it is parallel to a given plane, and is the same in all planes parallel to this plane. Taking this plane as the xy-plane, the velocity $\mathbf{v}$ of a fluid particle can be written as

$$\mathbf{v}=\mathbf{i}v_x+\mathbf{j}v_y,$$

where v_x and v_y are functions of x and y only.

The motion is said to be *irrotational* when

$$\text{curl } \mathbf{v}=0.$$

This equation can be satisfied by taking $\mathbf{v}=\text{grad } \phi$, where ϕ is a scalar function of x and y. This gives

$$v_x=\frac{\partial \phi}{\partial x},\ v_y=\frac{\partial \phi}{\partial y}. \quad . \quad . \quad . \quad (1)$$

ϕ is called the *velocity potential.*

When the fluid is incompressible, its *divergence* is zero (§ 4.41). This gives

$$\text{div } \mathbf{v}=0, \quad \text{or} \quad \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0.$$

Substituting from (1), we therefore obtain

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0. \quad . \quad . \quad . \quad (2)$$

Since the real part ϕ of an analytic function

$$f(z)=\phi(x,y)+i\psi(x,y),$$

satisfies equation (2), we can take ϕ as the velocity potential of a fluid flow. To see the significance of the imaginary part ψ, consider the slope at any point of the curve $\psi(x,y)=c$. This is given by

$$\frac{dy}{dx}=-\frac{\partial \psi/\partial x}{\partial \psi/\partial y}=\frac{\partial \phi/\partial y}{\partial \phi/\partial x}, \text{ by } C\text{-}R \text{ equations,}$$

$$=\frac{v_y}{v_x}, \text{ by (1).}$$

Hence *the resultant velocity of a particle is along the tangent to the curve* $\psi(x,y)=c$. So the particle moves on the curve $\psi(x,y)=c$. These curves are called *stream lines*, and $\psi(x,y)$ is known as the *stream function*. The curves $\phi(x,y)=c'$ are called *equipotential lines*. They cut the stream lines orthogonally.

The choice of the function $f(z)$ describing a particular flow depends on the boundary conditions. Since the flow cannot cross a bounding wall, *such a boundary must be a stream line.*

We give below a few examples illustrating the use of conjugate functions in fluid flow.

(i) *Flow at a corner.* The analytic function

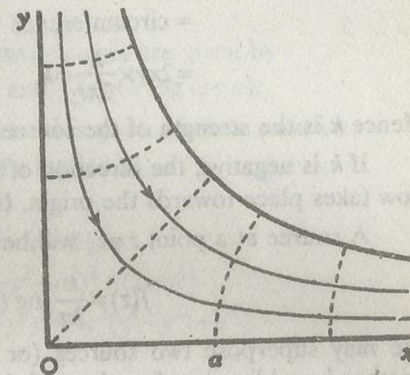
$$f(z)=z^2=x^2-y^2+2ixy$$

describes a flow whose equipotential lines are the hyperbolas

$$\phi=x^2-y^2=c,$$

and whose stream lines are the hyperbolas

$$\psi=2xy=c'.$$



Any two of the stream lines could be taken as the bounding walls of the flow. The figure shows the flow in a channel bounded by the axes (stream line $2xy=0$) and the hyperbola $xy=a^2$. The continuous lines are the stream lines and the dotted ones the eqipotential lines.

The velocity components at a point are

$$v_x=\frac{\partial \phi}{\partial x}=2x,\ v_y=\frac{\partial \phi}{\partial y}=-2y,$$

and the resultant velocity has the magnitude $2\sqrt{(x^2+y^2)}$.

Similarly, the function $f(z)=z^n$ will describe the flow at a corner enclosing an angle π/n.

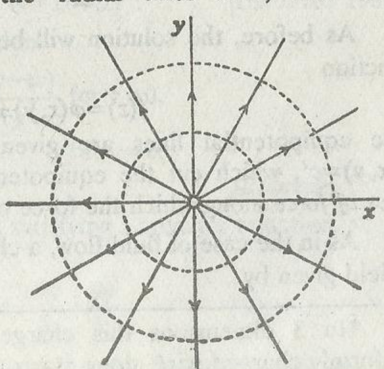
(ii) *Source or sink.* The function

$$f(z)=\frac{k}{2\pi}\log z=\frac{k}{2\pi}\log (re^{i\theta})$$

gives a flow for which

$$\phi=\frac{k}{2\pi}\log r \quad \text{and} \quad \psi=\frac{k}{2\pi}\theta.$$

Hence the stream lines are the radial lines $\theta=c$, and the equipotentials are the concentric circles $r=c'$ (see figure).



This describes a flow emanating from the origin, which is called a *source.* The velocity at any point has the radial and transverse components

$$v_r=\frac{\partial \phi}{\partial r}=\frac{k}{2\pi r}$$

and

$$v_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}=0.$$

The flow across any circle $r=c'$

$$=\text{circumference} \times \text{velocity}$$
$$=2\pi r\times\frac{k}{2\pi r}=k.$$

Hence k is the strength of the source.

If k is negative, the direction of the velocity is reversed and the flow takes place towards the origin. In this case it is called a *sink*.

A source at a point $z=z_1$ will be described by the function

$$f(z)=\frac{k}{2\pi}\log(z-z_1).$$

We may superpose two sources (or a source and a sink) on one another by adding two functions of the above type.

II. Electrostatic field. The force of attraction (or repulsion) between two electrically charged particles is governed by Coulomb's law:

force $\propto$ (product of charges)/(distance between them)^2.

This force can be expressed as the gradient of a scalar function ϕ, which is called the *electrostatic potential*. For charges distributed over a mass it is convenient to obtain the potential ϕ by summation and then obtain the force from grad ϕ. It will be shown in chapter VII that the electrostatic potential satisfies Laplace's equation. For the two dimensional case ϕ is a solution of

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0.$$

As before, the solution will be the real part of some analytic function

$$f(z)=\phi(x,y)+i\psi(x,y).$$

The equipotential lines are given by $\phi(x,y)=c$; and the lines $\psi(x,y)=c'$, which cut the equipotential lines orthogonally, are the *lines of force* along which the force of attraction acts.

As in the case of fluid flow, a charge at the origin* will produce a field given by

*In 3 dimensions this charge is due to an infinitely long uniformly charged wire along z-axis. In the system of units generally employed, the charge per unit length is $\frac{1}{2}k$.

$$f(z)=k\log z.$$

Ex. Find the potential due to two charged lines at $(a,0)$ and $(-a,0)$ having the same charge.

The potentials due to the two charges are given by

$$f_1(z)=k\log(z-a) \text{ and } f_2(z)=k\log(z+a),$$

respectively. The combined potential is given by their sum, i.e.,

$$\begin{aligned} f(z) &= k\log(z-a)+k\log(z+a)\\ &= k\log(z^2-a^2)\\ &= k\log(x^2-y^2-a^2+2ixy).\end{aligned}$$

Hence
$$\begin{aligned}\phi &= k\log\sqrt{\{(x^2-y^2-a^2)^2+(2xy)^2\}}\\ &= \tfrac{1}{2}k\log\{(x^2+y^2)^2-2a^2(x^2-y^2)+a^4\}.\end{aligned}$$

Examples 19

1. Show that if $f(z)=u(x,y)+iv(x,y)$, then
$$f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}.$$ [*Roorkee*, 1968]

2. If $f(z)=x^2+iy^2$, does $f'(z)$ exist at any point? [*Alld.*, '66]

3. Determine the points where the function $f(z)=x^4-i(y+1)^3$ is differentiable. Also find its derivative there. [*Roorkee*, 1979]

4. Show that continuity does not imply differentiability by considering the function $|z|^2$. [*Roorkee, M.E.*, 1979]

5. Determine where the Cauchy-Riemann equations are satisfied for the following functions—
(i) $w=1/z$, (ii) $w=2x+ixy^2$. [*Banaras*, 1964]

6. Verify whether the function
$$f(z)=\frac{x^3y(y-ix)}{x^6+y^2} \text{ for } z\neq 0,$$
$$=0 \qquad \text{for } z=0,$$
is analytic at $z=0$. [*Roorkee*, 1977]

7. If ϕ and ψ are functions satisfying Laplace's equation, show that $s+it$ is analytic, where
$$s=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x} \text{ and } t=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}.$$ [*Bhopal*, 1976]

8. Show that the polar form of Cauchy-Riemann equation is
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

Use the result to show that $\log z$ is analytic. [*Alld.*, 1977]

9. If $f(z)=u+iv$ is an analytic function of z, show that the curves $u=$ const. and $v=$ const. cut orthogonally. [*Roorkee*, 1980]

10. If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2. \quad \text{[Punjab, 1979]}$$

Show that the following functions are harmonic and determine their conjugate functions—

11. $u=2x(1-y)$. [*Banaras*, 1979]

12. $u=2x-x^3+3xy^2$. [*Bhopal*, 1978]

13. $u=\frac{1}{2}\log(x^2+y^2)$. [*Banaras*, 1975]

14. $v=-y/(x^2+y^2)$. [*Punjab, M.Sc. (Ch. Eng.)*, 1980]

15. $v=\sinh x\cos y$.

Determine the analytic function $w=u+iv$, if

16. $v=\dfrac{x-y}{x^2+y^2}$. [*Allahabad*, 1977]

17. $u=e^{2x}(x\cos 2y-y\sin 2y)$. [*Roorkee, M.E.*, 1980]

18. $u=\dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$. [*Rajasthan, M.E.*, 1980]

19. If $w=f(z)=u+iv$ and $u-v=(x-y)(x^2+4xy+y^2)$, find w in terms of z. [*Roorkee*, 1996]

20. If $f(z)=\phi+i\psi$ represents the complex potential for an electric field and

$$\psi=x^2-y^2+\frac{x}{x^2+y^2},$$

determine the conjugate function ϕ and $f(z)$. [*Roorkee*, 1994]

21. Show that the function

$$w=A\left(z+\frac{a^2}{z}\right)$$

may be used to represent the flow of a perfect incompressible fluid past a circular cylinder. [*Bhopal*, 1978]

22. If
$$w=A\log\frac{z-a}{z+a},$$
where a is real, show that the lines $u=$ const., $v=$ const., form two orthogonal families of coaxial circles. Hence show that this function may be used to represent a source and sink combination.

[*Roorkee*, 1979]

23. Two concentric circular cylinders of radii a, b ($a<b$) are kept at potentials ϕ_1 and ϕ_2 respectively. Use the complex function $w=A\log z+c$ to prove that the capacitances per unit length of the capacitor formed by them is

$$\frac{2\pi K}{\log(a/b)},$$

where K is the dielectric constant of the medium. [*Bhopal*, 1978]

[Capacitance is the charge required to maintain a unit potential difference. Thus capacitance = (charge)/(potential difference).

A medium of dielectric constant K increases the potential difference K times to that in vaccum (for the same charge).]

24. Two semi-infinite conducting planes intersect at angle α ($<\pi$) in a straight line which is the common insulated edge. The planes are kept at potentials ϕ_1, ϕ_2 ($\phi_1>\phi_2$). Use the complex function $w=A\log z+iB$ to prove that the charge density at a point distant r from the line of intersection on the plane at potential ϕ_1 is

$$\sigma=\frac{K(\phi_1-\phi_2)}{ar},$$

K being the dielectric constant of the medium. [*Roor., M.E.*, '59]

5.4. Conformal transformation. It has been said in Art. 5.2 that a relation $w=f(z)$ between two complex variables is represented graphically by taking two separate Argand diagrams, the z-plane and w-plane. For every point $x+iy$ in the z-plane the relation $w=f(z)$ defines a corresponding point $u+iv$ in the w-plane. We call this a *mapping* of the z-plane into the w-plane, and say that the point z_0 maps into the point $w_0=f(z_0)$. w_0 is also known as the *image* of z_0.

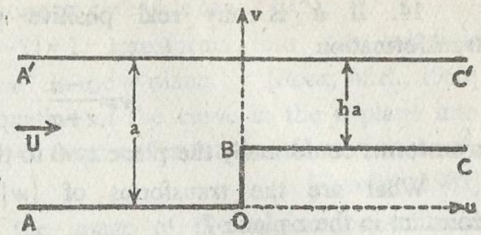
Let a point P move along a curve C in z- plane. Then the different positions of P will map into different images which will lie, in general, on some curve C'. Thus the relation $w=f(z)$ maps a curve C in z-plane into a curve C' in the w-plane. Similarly, different regions in z-plane will map into other regions in w-plane.

Consider for example the relation $w=z^2$, i.e.

$$u+iv=(x+iy)^2. \quad . \quad . \quad (1)$$

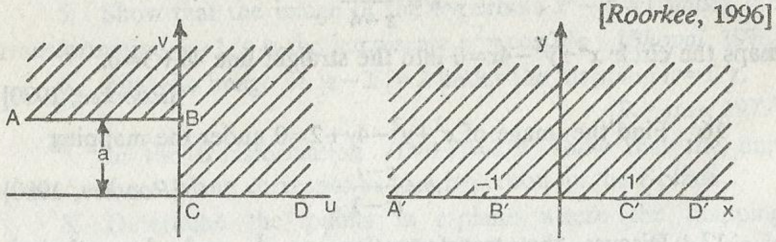
channel shown in the figure. The velocity at $u=-\infty$ is given to be U.

[Hint. Map $w=-\infty$ at $z=0$, $w=0$ at $z=c^2$ and B at $z=1$. Conditions of flow will give $c=h$.]



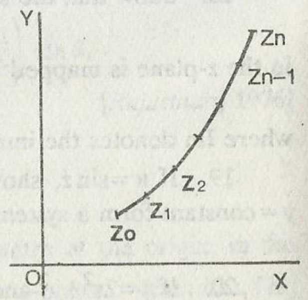
24. Find the Schwarz-Christoffel transformation which maps the shaded region of w-plane onto the upper half of the z-plane.

[*Roorkee*, 1996]



5.5. Integration.

5.5. Integration. The complex function $f(z)$ is a function of two independent variables. Hence its integral can be defined only as a line integral. Let C be any continuous curve in the z-plane. Divide C into n parts by the points $z_1, z_2, ..., z_{n-1}$ as in the figure, and put $\Delta z_r = z_r - z_{r-1}$. Then the limit of the sum



$$\sum_{r=1}^{n} f(z_r)\,\Delta z_r \qquad (1)$$

as $n \to \infty$ and each $\Delta z_r \to 0$, if it exists, is known as the *integral* of $f(z)$ over the *contour* C. It is denoted by

$$\int_C f(z)\,dz.$$

By putting $z_r = x_r + iy_r$ in (1), and then taking the limits, we easily see that

$$\int_C f(z)\,dz = \int_C (u+iv)\,d(x+iy) \qquad (2)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy).$$

Furthermore, if the parametric equation of C be

$$x=\phi(t),\ y=\psi(t),$$

then $u(x,y)+iv(x,y)$ and $x+iy$ can both be expressed in terms of t. Then (2) gives

$$\int_C f(z)\,dz = \int_a^b f\{z(t)\}\frac{dz}{dt}\,dt,$$

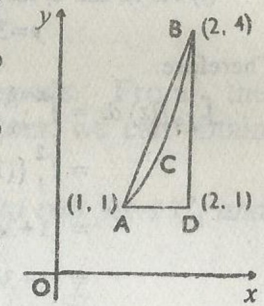
where $t=a$ and $t=b$ are the extremities of the curve C.

Ex. 1. Integrate $f(z)=z^2$ from $A(1,1)$ to $B(2,4)$ along

(i) the line-segment AD parallel to x-axis and DB parallel to y-axis;

(ii) the straight line AB joining the two points;

(iii) the curve $C: x=t,\ y=t^2$.



(i) Since $z=x+i\,.\,1$ for a point on AD, and $z=2+iy$ for a point on DB, we have

$$\int_{ADB} f(z)\,dz$$

$$= \int_{AD} z^2\,dz + \int_{DB} z^2\,dz = \int_1^2 (x+i)^2\,dx + \int_1^4 (2+iy)^2\,i\,dy$$

$$= \left[\tfrac{1}{3}(x+i)^3\right]_1^2 + \left[\tfrac{1}{3}(2+iy)^3\right]_1^4$$

$$= \tfrac{1}{3}[(2+i)^3-(1+i)^3+(2+4i)^3-(2+i)^3]$$

$$= \tfrac{1}{3}[(2+4i)^3-(1+i)^3] = -28\tfrac{2}{3}-6i.$$

(ii) The equation of AB is

$$y-1=3(x-1), \text{ or } y=3x-2.$$

Therefore for a point on AB

$$z=x+i(3x-2).$$

Hence $\displaystyle\int_{AB} z^2\,dz = \int_{x=1}^{x=2} \{x+i(3x-2)\}^2\,d\{x+i(3x-2)\}$

$$= \left[\tfrac{1}{3}\{x+i(3x-2)\}^3\right]_1^2$$

$$= \tfrac{1}{3}[(2+4i)^3-(1+i)^3] = -28\tfrac{2}{3}-6i.$$

(iii) $$\int_C z^2\,dz = \int_C (x+iy)^2\,d(x+iy) \quad . \quad . \quad . \quad (3)$$

$$= \int_{t=1}^{t=2} (t+it^2)^2\,d(t+it^2)$$

$$= \int_1^2 (t^2-t^4+2it^3)(1+2it)\,dt$$
$$= \int_1^2 \{t^2-5t^4+i(4t^3-2t^5)\}\,dt = \left[\tfrac{1}{3}t^3-t^5+i(t^4-\tfrac{1}{3}t^6)\right]_1^2$$
$$= -28\tfrac{2}{3}-6i.$$

Ex. 2. Integrate $f(z)=x^2+ixy$ from $A(1, 1)$ to $B(2, 4)$ along
(i) the straight line AB joining the two points; and
(ii) the curve $C: x=t,\ y=t^2$. [*Jodhpur*, 1969]

(i) As in Ex. 1, for a point on AB,
$$y=3x-2 \text{ and } z=x+i(3x-2).$$
Therefore
$$\int_{AB} f(z)\,dz = \int_{x=1}^{x=2} \{x^2+ix(3x-2)\}\,d\{x+i(3x-2)\}$$
$$= \int_1^2 \{(1+3i)x^2-2ix\}(1+3i)\,dx$$
$$= (1+3i)\left[(1+3i)\tfrac{1}{3}x^3-ix^2\right]_1^2$$
$$= (1+3i)[(1+3i)\tfrac{7}{3}-3i] = -9\tfrac{2}{3}+11i.$$

(ii) $$\int_C (x^2+ixy)\,d(x+iy) = \int_{t=1}^{t=2} (t^2+it^3)\,d(t+it^2)$$
$$= \int_1^2 (t^2+it^3)(1+2it)\,dt$$
$$= \int_1^2 (t^2-2t^4+3it^3)\,dt = \left[\tfrac{1}{3}t^3-\tfrac{2}{5}t^5+\tfrac{3}{4}it^4\right]_1^2$$
$$= -10\tfrac{1}{15}+11\tfrac{1}{4}i.$$

NOTE. We see that all the three integrals in Ex. 1 are equal. In fact relation (3) shows that the integral is equal to the difference of $\frac{1}{3}(x+iy)^3$ evaluated at the two ends, and does not depend on the paths in between. Such a result will always hold if $f(z)$ is analytic (see next article). In such cases the integral can be evaluated by the formulae for integration of functions of a single variable. In Ex. 2 the integral depends on the contour. Here $f(z)$ is not analytic.

Ex. 3. Integrate over a circle C of radius a and centre z_0 in anticlockwise direction the function

(i) $f(z)=1/(z-z_0)$; (ii) $f(z)=(z-z_0)^m$,

where m is an integer $\neq -1$.

A point on the circle C is
$$z=z_0+a(\cos\theta+i\sin\theta)=z_0+ae^{i\theta}.$$

Therefore
$$\int_C \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ae^{i\theta}\,.\,i\,d\theta}{ae^{i\theta}}$$
$$= \int_0^{2\pi} i\,d\theta = 2\pi i.$$
Again $$\int_C (z-z_0)^m\,dz = \int_0^{2\pi} (ae^{i\theta})^m\,ae^{i\theta}\,i\,d\theta$$
$$= ia^{m+1}\int_0^{2\pi} e^{i(m+1)\theta}\,d\theta$$
$$= ia^{m+1}\left[\frac{e^{i(m+1)\theta}}{i(m+1)}\right]_0^{2\pi} = 0.$$

5.51. Properties of contour integrals. From the definition of the integral over a contour, we can obtain a number of simple properties.

(i) *If the contour C be divided into two parts C_1 and C_2 then*
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

(ii) *If the sense of integtration is reversed, the sign of the integral changes*; thus
$$\int_{z_0}^{Z} f(z)\,dz = -\int_Z^{z_0} f(z)\,dz,$$
where z_0 and Z are the two ends of contour C.

(iii) $\int_C \{f_1(z)+f_2(z)\}\,dz = \int_C f_1(z)\,dz + \int_C f_2(z)\,dz.$

(iv) $\int_C kf(z)\,dz = k\int_C f(z)\,dz.$

(v) *If L be the length of the contour C, and if M be a positive number such that $|f(z)| \leq M$ everywhere on C, then*
$$\left|\int_C f(z)\,dz\right| \leq ML.$$

To prove this we notice that
$$|f(z_1)\Delta z_1+f(z_2)\Delta z_2+\ldots+f(z_n)\Delta z_n|$$
$$\leq |f(z_1)\Delta z_1|+|f(z_2)\Delta z_2|+\ldots+|f(z_n)\Delta z_n|$$
$$\leq |f(z_1)|\ |\Delta z_1|+|f(z_2)|\ |\Delta z_2|+\ldots+|f(z_n)|\ |\Delta z_n|$$

$$\leq M(|\Delta z_1|+|\Delta z_2|+\ldots+|\Delta z_n|)$$
$$\leq ML.$$

Taking the limit as $n \to \infty$ and $\Delta z_r \to 0$, we get the result.

5.52. Cauchy's Integral Theorem. *If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C, then*

$$\int_C f(z)\,dz = 0.$$

Denote the region enclosed by the curve C by R, and let $f(z)=u+iv$. Then

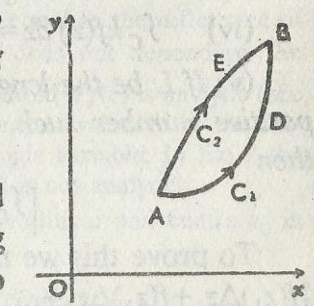
$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$
$$= \iint_R \left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) dx\,dy,$$

by Green's theorem (§ 4.61). Since $f(z)$ is analytic everywhere in R, therefore by Cauchy-Riemann equations both the integrands are zero. Hence the given integral is zero.

The above proof of the theorem is due to Cauchy*. A later proof given by Goursat does not require the continuity of $f'(z)$ but this proof is beyond our present scope.

Corollary. *If $f(z)$ is analytic in R, and if two points A and B in R are joined by two different curves C_1 and C_2 lying wholly in R, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$



Let C_1 be the curve ADB and C_2 the curve AEB. Applying Cauchy's integral theorem to

*Augustin-Louis Cauchy (1789-1857) was a versatile French mathematician, famous for his researches in complex functions, calculus, differential equations and convergence.

curve $ADBEA$, we have

$$\int_{ADBEA} f(z)\,dz = 0;$$

or $\qquad \int_{ADB} f(z)\,dz + \int_{BEA} f(z)\,dz = 0,$

or $\qquad \int_{ADB} f(z)\,dz - \int_{AEB} f(z)\,dz = 0,$

which proves the theorem.

We thus see that *the integral of an analytic function $f(z)$ from A to B does not depend upon the contour.*

5.53. Multiply connected regions. Cauchy's integral theorem has been proved above for a simply connected region enclosed by a simple curve. A simple curve is one which does not cross itself (fig. 1). A multiple curve crosses itself (fig. 2). A simple curve encloses a

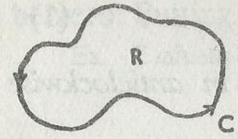


Fig. 1
A simple curve C enclosing a simply connected region R.

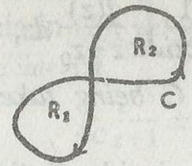


Fig. 2
A multiple curve. It encloses more than one separate region.

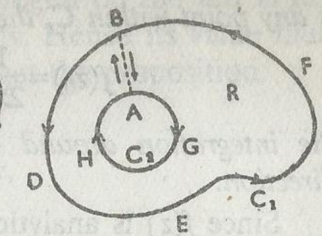


Fig. 3
A multiply connected region. It is bounded by more than one curve.

simply connected region. A simply connected region is a region such that *every* closed curve lying in it can be contracted indefinitely without passing out of it. A region in which this is not possible is called a multiply connected region. Fig. 3 gives an example of such a region R. Here R is enclosed between two separate curves C_1 and C_2. (There could be more than two separate curves.) We can convert a multiply connected region into a simply connected one, by giving it one or more cuts (e.g. along the dotted line AB in fig. 3).

Suppose a function $f(z)$ is analytic in a multiply connected region R and on its boundary C_1 and C_2 (fig. 3). Giving R a cut along AB we can convert it into a simply connected region bounded by the curve $BDEFBAGHAB$. Denoting this curve by C, we have by Cauchy's integral theorem

$$\int_C f(z)\,dz=0, \qquad \ldots \quad (1)$$

where the curve is traversed in the direction indicated by the arrows in the figure. On separating curve C into its component parts, equation (1) may be written as

$$\int_{C_1} f(z)\,dz+\int_{BA} f(z)\,dz-\int_{C_2} f(z)\,dz+\int_{AB} f(z)\,dz=0,$$

where the integrals over C_1 and C_2 are both taken in anticlockwise direction. The second and fourth integrals cancel each other. So we get

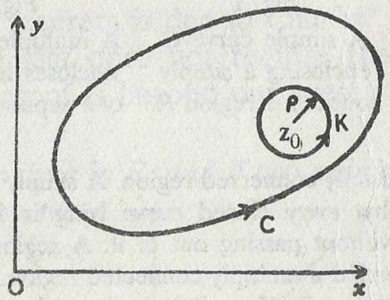
$$\int_{C_1} f(z)\,dz=\int_{C_2} f(z)\,dz.$$

5.54. Cauchy's Integral Formula. *If $f(z)$ is analytic inside and on a simple closed curve C, and if z_0 is any point within C, then*

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz, \qquad \ldots \quad (1)$$

the integration around C being taken in anticlockwise direction.

Since $f(z)$ is analytic inside C, therefore $f(z)/(z-z_0)$ also is analytic everywhere inside C except the point $z=z_0$. Take a small circle K of radius ρ and centre z_0 lying entirely within C. Then, since $f(z)/(z-z_0)$ is analytic in the region between C and K, we have



$$\int_C \frac{f(z)}{z-z_0}\,dz=\int_K \frac{f(z)}{z-z_0}\,dz$$

$$=\int_K \frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\,dz$$

$$=\int_K \frac{f(z)-f(z_0)}{z-z_0}\,dz+f(z_0)\int_K \frac{dz}{z-z_0}. \quad (2)$$

On puttinge $z=z_0+\rho e^{i\theta}$ in the second integral it can

be evaluated, as in Ex. 3 on page 164. Thus the second term is equal to $2\pi i\, f(z_0)$.

To evaluate the first integral, we note that $f(z)$ is continuous at z_0. Therefore if we take any small positive number ε, we can find another positive number δ such that

$$|f(z)-f(z_0)|<\varepsilon \text{ whenever } |z-z_0|\le\delta.$$

Taking $\rho=\delta$, and applying § 5.5 (v), we get

$$\left|\int_K \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \frac{\varepsilon}{\rho}(2\pi\rho)=2\pi\varepsilon.$$

So the absolute value of the integral on the left is less than an arbitrarily small quantity. Hence its value must be zero. Putting this in (2), we get the proposition.

Ex. Evaluate the integral

$$\int_C \frac{e^{-z}}{z+1}\,dz,$$

where C is the circle $|z|=2$.

Since e^{-z} is an analytic function and $z=-1$ is inside C, we have by Cauchy's integral formula

$$\int_C \frac{e^{-z}}{z+1}\,dz=2\pi i[e^{-z}]_{z=-1}=2\pi e i.$$

5.55. Derivatives of an analytic function. *If $f(z)$ is analytic inside and on a closed curve C, then*

$$f'(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz,$$

$$f''(z_0)=\frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}\,dz,$$

and in general

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

the integrations around C being taken in anticlockwise direction.

By Cauchy's integral formula

$$\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0}$$

$$=\frac{1}{\Delta z_0}\cdot\frac{1}{2\pi i}\left\{\int_C \frac{f(z)}{z-(z_0+\Delta z_0)}\,dz-\int_C \frac{f(z)}{z-z_0}\,dz\right\}$$

$$=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z_0)(z-z_0)}\,dz. \qquad (1)$$

Taking the limit as $\Delta z_0 \to 0$, we get (see note below)

$$f'(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz \qquad (2)$$

Replacing z_0 by $z_0+\Delta z_0$ in this,

$$f'(z_0+\Delta z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z_0)^2}\,dz. \qquad (3)$$

Subtracting (2) from (3) and dividing by Δz_0, we get

$$\frac{f'(z_0+\Delta z_0)-f'(z_0)}{\Delta z_0}=\frac{1}{2\pi i}\int_C \frac{2(z-z_0)-\Delta z_0}{(z-z_0-\Delta z_0)^2(z-z_0)^2}\,f(z)\,dz$$

Taking the limit as $\Delta z_0 \to 0$ we have

$$f''(z_0)=\frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}\,dz.$$

The general result can be proved by induction, proceeding in a manner similar to above.

It is clear from the above theorem that an analytic function $f(z)$ has derivatives of every order.

NOTE. The conclusion that integral (1) tends to (2) as $\Delta z_0 \to 0$ requires the uniform continuity of the former. To prove that this is so, we consider the difference of these two integrals. The modulus of this difference is

$$\left|\int_C \frac{\Delta z_0 f(z)}{(z-z_0-\Delta z_0)(z-z_0)^2}\,dz\right| \le \frac{|\Delta z_0|ML}{d^3},$$

where M is the maximum value of $f(z)$ on C, L is the length of contour C, and d is the shortest distance between C and a fixed small circle K (as in fig. on p. 168) within which $z_0+\Delta z_0$ always lies. Therefore the difference tends to zero as $\Delta z_0 \to 0$.

Examples 21

1. Evaluate $\int_{1-i}^{2+i}(2x+2iy+3)\,dz$

   (i) along the path $x=t+1, y=2t^2-1$;

   (ii) along the straight line joining $1-i$ to $2+i$.

2. Evaluate $\int_{1-i}^{2+i}(2x+iy+1)\,dz$, along the two paths of Ex. 1.

3. Evaluate $\int_C |z|\,dz$, where C is the contour

   (i) straight line AB from $z=-i$ to $z=i$;

   (ii) left half of the unit circle $|z|=1$ from $z=-i$ to $z=i$.

   [Roorkee, 1996]

4. Evaluate $\int_C \frac{dz}{z-z_0}$ where C is any simple closed curve and z_0 is (a) outside C, (b) inside C. *[Roorkee, M.E., 1980]*

5. Evaluate $\int_C \frac{dz}{(z-z_0)^n}$ over a simple closed curve C, where n is an integer ≥ 2 and z_0 is inside C.

6. Let $P(z)=a+bz+cz^2$, and

$$\int_C \frac{P(z)}{z}\,dz=\int_C \frac{P(z)}{z^2}\,dz=\int_C \frac{P(z)}{z^3}\,dz=2\pi i,$$

   where C is the circle $|z|=1$. Evaluate $P(z)$. *[Roorkee, 1981]*

7. Evaluate $\int_C \frac{e^{-z}}{z^2}\,dz$ round the contour C, where C is the circle $|z|=1$. *[Punjab, M.Sc. (Ch. Eng.), 1978]*

8. Use Cauchy's integral formula to evaluate

$$\int_C \frac{3z^2+z}{z^2-1}\,dz,$$

   where C is the circle $|z|=2$.

   [Hint. Break the integrand into partial fractions.]

9. Evaluate the integrals

$$\text{(i) } \int_C \frac{\cos \pi z}{z-1}\,dz; \qquad \text{(ii) } \int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$$

   where C is the circle $|z|=3$. *[Roorkee, 1978]*