Unit - IV

Laplace Transformation

Definition : Let $f(t)$ be a function of t defined for all positive values of t. The Laplace transform $\bar{f}(s)$ of the function is defined by

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

provided that the integral exists.

NOTE 1 $\bar{f}(s) = L\{f(t)\}$, we also write

$f(t) = L^{-1}\{\bar{f}(s)\}$ is known as the inverse Laplace transform of $\bar{f}(s)$.

2. The parameter s is a real or complex number. we shall generally take it to be a real number
3. The Laplace transform replaces a function $f(t)$ by another function $\bar{f}(s)$ which is simpler in nature than $f(t)$.
4. It converts an ordinary differential equation into an algebraic equation. So the Laplace transform can be conveniently used in solving the differential equations, specially useful for initial value problem.

Laplace Transform of Some elementary functions.

(1) Let $f(t) = 1$ Constant function

$$L\{1\} = \int_0^\infty e^{-st}.1\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, \text{ if } s>0.$$

(ii) let $f(t) = t^n$, power function

$$L\{t^n\} = \int_0^\infty e^{-st}.t^n\,dt$$

Put $st = u \Rightarrow s\,dt = du \Rightarrow dt = \frac{1}{s}du$

$$= \int_0^\infty e^{-u}.\frac{u^n}{s^n}.\frac{1}{s}du = \int_0^\infty \frac{e^{-u}u^n}{s^{n+1}}du$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \text{ if } s>0 \text{ and } n>-1.$$

If n is a positive integer, we have

$$L\{t^n\} = \frac{\lfloor n}{s^{n+1}}.$$

(iii) let $f(t) = e^{at}$, exponential function

$$L\{e^{at}\} = \int_0^\infty e^{-st}.e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{s-a}, \text{ if } s>a.$$

(iv) $L\{e^{iat}\} = \frac{1}{s-ia}$

$$L\{\cos at + i\sin at\} = \frac{1}{(s-ia)}.\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}$$

Separating the real and imaginary parts we get,

$$L\{\cos at\} = \frac{s}{s^2+a^2} \text{ and } L\{\sin at\} = \frac{a}{s^2+a^2}$$

(v) $L\{f(t)+g(t)\} = \int_0^\infty e^{-st}(f(t)+g(t))\,dt$

$$= \int_0^\infty e^{-st}f(t)\,dt + \int_0^\infty e^{-st}g(t)\,dt$$

$$= L\{f(t)\} + L\{g(t)\}.$$

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Similarly, $L\{c f(t)\} = c L\{f(t)\}$

where c is constant.

Ex1. Find the Laplace transform of $f(t) = 5 - 3t - 2e^{-t}$

Soln $L\{5 - 3t - 2e^{-t}\} = 5L\{1\} - 3L\{t\} - 2L\{e^{-t}\}$

$$= 5 \cdot \frac{1}{s} - 3\frac{1}{s^2} - 2\frac{1}{s+1}$$

$$= \frac{5}{s} - \frac{3}{s^2} - \frac{2}{s+1} = \frac{5s(s+1) - 3(s+1) - 2s^2}{s^2(s+1)}$$

$$= \frac{3s^2 + 2s - 3}{s^2(s+1)}.$$

Ex2 Find $L\{\sin 2t \cos 3t\} = \frac{1}{2} L\{\sin 5t - \sin t\}$

$$= \frac{1}{2}\left(\frac{5}{s^2+5^2} - \frac{1}{s^2+1}\right) = \frac{2s^2 - 10}{(s^2+25)(s^2+1)}.$$

NOTE (i) $L\{\sinh at\} = \frac{a}{s^2 - a^2}$ if $s > |a|$

(ii) $L\{\cosh at\} = \frac{s}{s^2 - a^2}$ if $s > |a|$

Definition A function $f(t)$ is said to be of exponential order 'a' as $t \to \infty$, if there exists a number 'a' such that $\lim_{t\to\infty} e^{-at} f(t)$ is finite.

Existance Condition of Laplace transform of a function f(t)

Theorem: If $f(t)$ is continuous and of exponential order 'a' as $t \to \infty$, then $\bar{f}(s)$ exists for $s > a$.

Remark The conditions stated in the above theorem are sufficient for the existance of $\bar{f}(s)$ but not the necessary conditions.

e.g. $L\{t^n\}$ exists for $-1 < t < 0$, even though the function

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is not continuous at $t=0$. Similarly a function (of exponential order a) having a finite discontinuities will have a L.T. if $s>a$.

Theorem (First Shifting Theorem) If $\bar{f}(s)$ is the Laplace transform of a function $f(t)$, then $\bar{f}(s-a)$ is the Laplace transform of $e^{at}.f(t)$.

Proof $L\{e^{at} f(t)\} = \int_0^\infty e^{-st}.e^{at}.f(t)dt.$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt.$$

Put $s-a=p$ we have

$$= \int_0^\infty e^{-pt} f(t)dt = \bar{f}(p) = \bar{f}(s-a).$$

Remark Application of F.S.T gives a number of useful results.

For examples

(i) $L\{e^{at}.t^n\} = \dfrac{\lfloor n}{(s-a)^{n+1}}$, $s>a$ Since $L\{t^n\} = \dfrac{\lfloor n}{s^{n+1}}$

(ii) $L\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$, $s>a$ Since $L\{\cos bt\} = \dfrac{s}{s^2+b^2}$

(iii) $L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$, $s>a$ Since $L\{\sin bt\} = \dfrac{b}{s^2+b^2}$

Ex1. Find $L\{e^{2t}(\cos 4t+3\sin 4t)\} = L\{e^{2t}\cos 4t\} + 3L\{e^{2t}\sin 4t\}$

$$= \frac{s-2}{(s-2)^2+4^2} + 3\,\frac{4}{(s-2)^2+4^2} = \frac{s+10}{s^2-4s+20}.$$

Ex2. Find $L\{t\sin at\}$ and $L\{t\cos at\}$

Soln. (i) we know that $L\{t\} = \dfrac{1}{s^2}$,

$\therefore$ by F.S. Theorem $L\{t\,e^{iat}\} = \dfrac{1}{(s-ia)^2}$

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$$L\{t(\cos at + i\sin at)\} = \frac{1}{(s-ia)^2}\cdot\frac{(s+ia)^2}{(s+ia)^2}$$

$$= \frac{(s+ia)^2}{(s^2+a^2)^2} = \frac{(s^2-a^2)+2asi}{(s^2+a^2)^2}$$

Separating the real and imaginary parts we get -

$$L\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2} \quad \text{and} \quad L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

Example 8

Q2 $\quad L\{A + Bt + Ct^{-1/2}\} = L(A) + B\,L\{t\} + C\,L\{t^{-1/2}\}.$

$$= \frac{A}{s} + \frac{B}{s^2} + C\,\frac{\Gamma(-\frac{1}{2}+1)}{s^{-1/2+1}} = \frac{A}{s} + \frac{B}{s^2} + C\,\frac{\Gamma\frac{1}{2}}{s^{1/2}}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C\sqrt{\pi}}{\sqrt{s}}$$

Q4. Find $L\{t^2 e^{-at}\}$

we know that $\quad L\{t^2\} = \frac{2}{s^3}$, by first shifting theorem

$$L\{e^{-at}t^2\} = \frac{2}{(s+a)^3}$$

Q9. Find $L\{\cosh at \cos at\}$

Soln $\quad L\{\cosh at \cdot \cos at\} = L\{\frac{1}{2}(e^{at}+e^{-at})\cos at\}$

$$= \frac{1}{2}L\{e^{at}\cos at\} + \frac{1}{2}L\{e^{-at}\cos at\}.$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right] = \,? \quad \text{(Simplify)}$$

Q11. Find $L\{t\,e^{at}\sin at\}$

Soln $\quad L\{t\,e^{iat}\} = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{(s^2+a^2)^2} = \frac{(s^2-a^2)+2asi}{(s^2+a^2)^2}$

Equating the imaginary parts on both sides we get -

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$$L\{t \sin at\} = \frac{2as}{(s^2+a^2)^2}$$

By first shifting theorem,

$$L\{e^{at}(t\sin at)\} = \frac{2a(s-a)}{[(s-a)^2+a^2]^2} = ? \quad (\text{simplify})$$

Q13. Find $L\left\{e^{-at} - 2e^{at/2}\cos\frac{1}{2}\sqrt{3}\,at\right\}$

Soln. $L\left\{e^{-at} - 2e^{at/2}\cos\frac{1}{2}\sqrt{3}\,at\right\}$

$$= L\{e^{-at}\} - 2L\left\{e^{at/2}\cos\frac{\sqrt{3}}{2}at\right\}$$

$$= \frac{1}{s+a} - 2\frac{\left(s-\frac{a}{2}\right)}{\left(s-a/2\right)^2+\left(\frac{\sqrt{3}}{2}a\right)^2}$$

$$= \frac{1}{s+a} - \frac{2s-a}{(s-a/2)^2+\frac{3}{4}a^2} = ? \quad (\text{Simplify.})$$

Q14 Find the L.T. of $f(t) = \begin{cases} t/T, & 0<t<T \\ 1, & t>T \end{cases}$

Soln $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st}\cdot\frac{t}{T}\,dt + \int_T^\infty e^{-st}\cdot 1\,dt$

$$= \left[\frac{t\,e^{-st}}{(-s)T} - \int_0^T \frac{e^{-st}}{-s}dt\right]_0^T + \left[\frac{e^{-st}}{-s}\right]_T^\infty$$

$$\cancel{= \frac{T\,e^{-sT}}{T(-s)} + \frac{t}{Ts} + \frac{e^{-sT}}{-s} = -\frac{e^{-sT}}{s}}$$

$$= \left[\frac{t\,e^{-st}}{T(-s)} - \frac{e^{-st}}{Ts^2}\right]_0^T + \left[\frac{e^{-st}}{-s}\right]_T^\infty$$

$$= \frac{\cancel{T}\,e^{-sT}}{\cancel{T}(-s)} - \frac{e^{-sT}}{Ts^2} + \frac{1}{Ts^2} + \cancel{\frac{e^{-sT}}{s}} = \frac{1-e^{-sT}}{Ts^2}$$

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Q Find the L.T. of the following functions:

(i) $L\{\sin^3 2t\}$ (ii) $L\{\cos^3 5t\}$ (iii) $L\{\sin t \sin 2t \sin 3t\}$

Soln (i) $L\{\sin^3 2t\} = L\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\} = \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$

(ii) $L\{\cos^3 5t\} = L\{\frac{3}{4}\cos 5t + \frac{1}{4}\cos 15t\}$

$= \frac{3}{4}\left(\frac{s}{s^2+25}\right) + \frac{1}{4}\left(\frac{s}{s^2+225}\right)$

✓(iii) $\sin t \sin 2t \sin 3t = \frac{1}{2}\sin t\,(2\sin 3t \sin 2t)$

$= \frac{1}{2}\sin t\,[\cos t - \cos 5t] = \frac{1}{2}[\sin t \cos t - \sin t \cos 5t]$

$= \frac{1}{4}[2\sin t \cos t - 2\sin t \cos 5t]$

$= \frac{1}{4}[\sin 2t - \sin 6t + \sin 4t]$

$= \frac{1}{4}\left[\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right]$

Theorem 1° If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Ex Since $L\{\sin t\} = \frac{1}{s^2+1}$, then

$L\{\sin 3t\} = \frac{1}{3}\,\frac{1}{(s/3)^2+1} = \frac{3}{s^2+9}$

Theorem 2° If $L\{f(t)\} = \bar{f}(s)$, then $L\{t\,f(t)\} = -\frac{d}{ds}(\bar{f}(s))$

Ex Find $L\{t\cos at\}$

Soln we know that $L\{\cos at\} = \frac{s}{s^2+a^2}$, Then by Theorem 2°

$L\{t\cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\frac{(s^2+a^2)\cdot 1 - s\times 2s}{(s^2+a^2)^2}$

$= \frac{s^2-a^2}{(s^2+a^2)^2}$.

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Theorem 3: If $L\{f(t)\} = \bar{f}(s)$ and $\lim_{t\to 0} \frac{f(t)}{t}$ exists, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(u)\,du$$

Ex 1 Find $L\left\{\frac{1}{t}(1-e^{t})\right\}$

Soln we know that $L\{1-e^{t}\} = \left(\frac{1}{s} - \frac{1}{s-1}\right)$

By Theorem 3:

$$L\left\{\frac{1}{t}(1-e^{t})\right\} = \int_s^{\infty}\left(\frac{1}{u} - \frac{1}{u-1}\right)du = \Big[\log u - \log(u-1)\Big]_s^{\infty}$$

$$= \left[\log\left(\frac{u}{u-1}\right)\right]_s^{\infty} = 0 - \log\left(\frac{s}{s-1}\right) = \log\left(\frac{s-1}{s}\right)$$

$$\lim_{u\to\infty} \log\frac{u}{u-1} = \lim_{u\to\infty} \log\frac{1}{1-\frac{1}{u}} = \log 1 = 0$$

Q Find (i) $L\left\{\frac{1-\cos t}{t}\right\}$ (ii) $L\left\{\frac{1-\cos t}{t^2}\right\}$

Inverse Laplace Transform

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