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1. For the function, $f(z) = |xy|^{1/2}$
$u = |xy|^{1/2}$ and $v = 0$
we have $\frac{\partial u}{\partial x}$ at $(0,0)$ to be

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{|(x+\Delta x)y|^{1/2} - |xy|^{1/2}}{\Delta x + i\Delta y}$$

at $(0,0)$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x + i\Delta y} = 0$$

Similarly

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{|(x)(y+\Delta y)|^{1/2} - |xy|^{1/2}}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0}{\Delta x + i\Delta y} = 0$$

and $\frac{\partial v}{\partial x} = 0 \quad (\because v = 0)$

$\frac{\partial v}{\partial y} = 0 \quad (\because v = 0)$

So, $u_x = v_y = 0$
and $u_y = -v_x = 0$

Hence, Cauchy-Reimon equations are satisfied at $(0,0)$

Now, $f'(z)$ at $z_0 = 0$

$$f'(z) = \lim_{\Delta x \to 0, \Delta y \to 0} \frac{|(x+\Delta x)(y+\Delta y)|^{1/2} - |xy|}{\Delta x + i\Delta y}$$

at $x=0$ and $y=0$, $\Rightarrow \lim_{\Delta x \to 0, \Delta y \to 0} \frac{|\Delta x \Delta y|^{1/2}}{\Delta x + i\Delta y}$

Let the path chosen be $y = mx$
$\Rightarrow \Delta y = m\Delta x$

$$f'(0) = \frac{|m|^{1/2}\Delta x}{\Delta x(1+im)}$$

$$f'(0) = \frac{|m|^{1/2}}{1+im}$$

As it depends on $m$ so, $f'(0)$ doesnot exist.

2 (i) We have been given $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

So, we have

$$\frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$$

$$\frac{\partial y}{\partial z} = \frac{1}{2i}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$$

Now,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} \times \frac{1}{2} + \frac{\partial u}{\partial y} \times \frac{1}{2}$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2} \times \frac{\partial^2 u}{\partial x^2} \times \frac{\partial x}{\partial \bar{z}} + \frac{1}{2i} \times \frac{\partial^2 u}{\partial y^2} \times \frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4} \times \frac{\partial^2 u}{\partial x^2} + \frac{1}{4} \times \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}$$ , Hence proved.

(ii) $\frac{\partial f}{\partial \bar{z}} = 0$ , $\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}}$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \times \frac{1}{2} + \frac{\partial u}{\partial y} \times \left[\frac{-1}{2i}\right]$$

$$\frac{\partial v}{\partial \bar{z}} = \frac{\partial v}{\partial x} \times \frac{1}{2} + \frac{\partial v}{\partial y} \times \left[\frac{-1}{2i}\right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial x} \times \frac{1}{2} + \frac{\partial u}{\partial y} \left[\frac{-1}{2i}\right] + i\left[\frac{\partial v}{\partial x} \times \frac{1}{2} + \frac{\partial v}{\partial y} \left[\frac{-1}{2i}\right]\right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\times\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} + i\left[\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\times\frac{\partial u}{\partial y}\right] = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{(real part)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{(Imaginary part)}$$

3. Find the values of a, b & c so that functions are entire:

(i) $f(z) = x + ay - i(bx + cy)$

$u = x + ay$

$v = -(bx + cy)$

$\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = -b$

$\frac{\partial u}{\partial y} = a$ and $\frac{\partial v}{\partial y} = -c$

for the functions to be entire

$u_x = v_y$

$-c = 1$

$\boxed{c = -1}$

and

$u_y = -v_x$

$\boxed{a = b}$

(ii) $f(z) = ax^2 - by^2 + icxy$

$u = ax^2 - by^2$

$v = cxy$

$\frac{\partial u}{\partial x} = 2ax$ $\quad$ $\frac{\partial v}{\partial x} = cy$

$\frac{\partial u}{\partial y} = -2by$ $\quad$ $\frac{\partial v}{\partial y} = cx$

for the function to be entire,

$u_x = v_y$ and $u_y = -v_x$

$2ax = cx$ $\quad$ $-2by = -cy$

$c = 2a$ $\quad$ $c = 2b$

$\boxed{c = 2a = 2b}$

(iii) $f(z) = e^x \cos ay + ie^x \sin(y+b) + c$

$u = e^x \cos ay + c$ and $v = e^x \sin(y+b)$

$\frac{\partial u}{\partial x} = e^x \cos ay$ ; $\frac{\partial u}{\partial y} = -ae^x \sin ay$

$\frac{\partial v}{\partial x} = e^x \sin(y+b)$ ; $\frac{\partial v}{\partial y} = e^x \cos(y+b)$

$u_x = v_y$ and $u_y = -v_x$

$\cos ay = \cos(y+b)$ -(I)

$a \sin ay = \sin(y+b)$ (II)

Squaring (I) & (II) and add

$a^2 \sin^2 ay + \cos^2 ay = 1$

$a^2 \sin^2 ay = \sin^2 ay$

$\sin^2 ay(a^2-1) = 0$

$\sin^2 ay = 0$

$a = 0$ & $a = \pm 1$

$b = 0$
$c = d$ (any no.)

4. $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |f(z)|^n = n^2 |f(z)|^{n-2} |f'(z)|^2$

We have, $f(z) = u + iv$

$|f(z)| = \sqrt{u^2+v^2} = (u^2+v^2)^{1/2}$

$|f(z)|^n = (u^2+v^2)^{n/2}$

$$\frac{\partial (u^2+v^2)^{n/2}}{\partial x} = \frac{n}{2}(u^2+v^2)^{\frac{n}{2}-1}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]$$

$$\frac{\partial^2 |f(z)|^n}{\partial x^2} = \frac{n}{2}\left[\frac{n}{2}-1\right](u^2+v^2)^{\frac{n}{2}-2}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]^2 + \frac{n}{2}(u^2+v^2)^{\frac{n}{2}-1}\left[2\left[\frac{\partial u}{\partial x}\right]^2 + 2u\frac{\partial^2 u}{\partial x^2} + 2\left[\frac{\partial v}{\partial x}\right]^2 + 2v\frac{\partial^2 v}{\partial x^2}\right] - (1)$$

Similarly,
for
$$\frac{\partial^2 |f(z)|^n}{\partial y^2} = \frac{n}{2}\left[\frac{n}{2}-1\right][u^2+v^2]^{\frac{n}{2}-2}\left[2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right] + \frac{n}{2}(u^2+v^2)^{\frac{n}{2}-1}\left[2\left[\frac{\partial u}{\partial y}\right]^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2\left[\frac{\partial v}{\partial y}\right]^2 + 2v\frac{\partial^2 v}{\partial y^2}\right] - (11)$$

Adding (1) & (11)

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |f(z)|^n = \frac{n}{2}\left[\frac{n}{2}-1\right][u^2+v^2]^{\frac{n}{2}-2}\left[4u^2\left[\frac{\partial u}{\partial x}\right]^2 + 4v^2\left[\frac{\partial v}{\partial x}\right]^2 + 4uv(u_x v_x)\right]$$

$$+4u^2\left[\frac{\partial u}{\partial y}\right]^2 + 4v^2\left[\frac{\partial v}{\partial y}\right]^2 + 4uv(u_y \cdot v_y)\Big]$$

$$+\frac{n}{2}(u^2+v^2)^{\frac{n}{2}-1}\left[2\left[\left[\frac{\partial u}{\partial x}\right]^2+\left[\frac{\partial u}{\partial y}\right]^2+\left[\frac{\partial v}{\partial x}\right]^2+\left[\frac{\partial v}{\partial y}\right]^2\right]+\right.$$

$$\left. 2u(\underbrace{u_{yy}+u_{xx}}_{0}) + 2v(\underbrace{v_{xx}+v_{yy}}_{0})\right]$$

we know that u and v are harmonic

$$f(z) = u+iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \circ \ |f'(z)| = \left[\left[\frac{\partial u}{\partial x}\right]^2+\left[\frac{\partial v}{\partial x}\right]^2\right]^{1/2}$$

So, our R.H.S will be

$$\frac{n}{2}\left[\frac{n}{2}-1\right]|f(z)|^{n-4}\left\{4u^2|f'(z)|^2+4v^2|f'(z)|^2+ \atop 4uv(u_y v_x - v_y v_x)\right\}$$

$$+\frac{n}{2}|f(z)|^{n-2}\left\{4|f'(z)|^2\right\}$$

$$=\frac{n}{2}\left[\frac{n}{2}-1\right]|f(z)|^{n-2}|f'(z)|^2\times 4+\frac{2}{4}\times\frac{n}{2}|f(z)|^{n-2}|f'(z)|^2$$

$$n\left[n-2+2\right]|f(z)|^{n-2}|f'(z)|^2$$

$$=n^2|f(z)|^{n-2}|f'(z)|^2. \text{ hence proved.}$$

5. we have

$f(z) = u + iv$

$i f(z) = iu - v$

$(u-v) + i(u+v) = (i+1) f(z)$

$U = (u-v), \quad V = (u-v) \; F(z) = (i+1) f(z)$

Given $V = \frac{x}{x^2+y^2}$

$\phi_1(x,y) = \left[\frac{\partial v}{\partial y}\right] = \frac{(x^2+y^2)0 - x(2y)}{(x^2+y^2)^2} = \frac{-x(2y)}{(x^2+y^2)^2}$

$\phi_1(z,0) = \left[\frac{\partial v}{\partial y}\right]_{(z,0)} = 0$

$\phi_2(x,y) = \left[\frac{\partial v}{\partial x}\right] = \frac{(x^2+y^2)\times 1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\phi_2(z,0) = \left[\frac{\partial v}{\partial x}\right]_{(z,0)} = \frac{-1}{z^2}$

$F'(z) = \phi_1(z,0) + i\phi_2(z,0)$

$\int F'(z)dz = \int \phi_1(z,0)dz + i\int \phi_2(z,0)dz$

$F(z) = 0 + i\int \frac{-1}{z^2} dz$

$F(z) = i\frac{1}{z} + c$

$(1+i) f(z) = i\frac{1}{z} + c$

$f(z) = \frac{1+i}{2z} + c$

at $f(1) = 1$

$f(1) = \frac{1+i}{2} + c = 1$

$c = \frac{2-1-i}{2}$

$c = \frac{1-i}{2}$

$f(z) = \frac{1+i}{2z} + \frac{1-i}{2}$ ,, Ans

6. $u(x,y) = e^{x^2-y^2}\cos 2xy$

for given fn to be harmonic, $u_{xx}+u_{yy}=0$

So,

$u_x = 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy$

$u_{xx} = 4x^2e^{x^2-y^2}\cos 2xy + 2e^{x^2-y^2}\cos 2xy - 4xye^{x^2-y^2}\sin 2xy - 4xye^{x^2-y^2}\sin 2xy - 4y^2e^{x^2-y^2}\cos 2xy$

$u_y = -2ye^{x^2-y^2}\cos 2xy - 2xe^{x^2-y^2}\sin 2xy$

$u_{yy} = -2e^{x^2-y^2}\cos 2xy + 4y^2e^{x^2-y^2}\cos 2xy + 4xye^{x^2-y^2}\sin 2xy + 4xye^{x^2-y^2}\sin 2xy - 4x^2e^{x^2-y^2}\sin 2xy$

Here we have,

$u_{xx}+u_{yy} = 0$

$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy)$

$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy)$

$\phi_1(z,0) = 2e^{z^2}(z \times \cos 0 - 0 \times 0)$

$\phi_2(z,0) = 2e^{z^2}(0 \times 1 + \cancel{z} \times 0)$

$\phi_1 = 2ze^{z^2}$

$\phi_2 = 0$

$f(z) = \int_c (\phi_1 - i\phi_2)\,dz + c$

$\int_c 2ze^{z^2}dz + c$

$\int e^t dt + c$

$= e^t + c$

$\Rightarrow e^{z^2} + c$

OR

$f(z) = e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$ . Ans

7. $f(z) = \frac{4 \cdot 3z}{z(1-z)(2-z)}$

(i) $|z| < 1$

$f(z) = \frac{4 \cdot 3z}{z(1-z)(2-z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{2-z}$

$A = \frac{4}{2} = 2.$

$B = 1$ & $C = \frac{-2}{-2} = 1$

$$= \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

① $|z| < 1$

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{2}{z} + (1-z)^{-1} + (2-z)^{-1}$$

$$\Rightarrow \frac{2}{z} + (1 + z + z^2 + z^3 \dots) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right)$$

$$\Rightarrow \frac{2}{z} + \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \frac{17}{16}z^3 + \dots$$

② $|z| > 1$ & $|z| < 2$

$$\frac{1}{|z|} < 1 \qquad \frac{|z|}{2} < 1$$

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{2}{z} - \frac{1}{z(1-1/z)} + \frac{1}{2(1-z/2)}$$

$$= \frac{2}{z} - \left[1 - \frac{1}{z}\right]^{-1} + \frac{1}{2}\left[1 - \frac{z}{2}\right]^{-1}$$

$$= \frac{2}{z} - \left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right] + \frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} \dots\right]$$

$$= \left[\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} \dots\right] + \frac{1}{2} + \left[\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right]$$

(3) $|z|>2$

$\frac{2}{|z|}<1$ . Also $\frac{1}{|z|}<1$

$f(z)=\frac{2}{z}+\frac{1}{1-z}+\frac{1}{2-z}$

$=\frac{2}{z}-\frac{1}{z[1-1/z]}-\frac{1}{z[1-2/z]}$

$=\frac{2}{z}-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}-\frac{1}{z[1-2/z]}$

$=\frac{2}{z}-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]-\frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right]$

$=\frac{2}{z}-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}\cdots-\frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\frac{8}{z^4}-\cdots$

$=-\frac{3}{z^2}-\frac{5}{z^3}-\frac{9}{z^4}-\cdots$

8. (i) $\dfrac{\sin z^2}{z(e^z-1)}$

At $z=0$, the fn $f(z)$ is not defined

but $\lim\limits_{z\to 0} f(z) = \dfrac{\sin z^2}{z(e^z-1)}$

$$= \frac{\sin z^2}{z^2 \frac{(e^z-1)}{z}}$$

$$\lim_{z\to 0} \frac{\sin z^2}{z^2} = \frac{1}{z^2}\left[z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} \ldots\right] = 1$$

&

$$\lim_{z\to 0} \frac{e^z-1}{z} = \frac{\left[1+z+\frac{z^2}{2!}+\frac{z^3}{3!} \ldots\right]-1}{z}$$

$$= \frac{\left[z+\frac{z^2}{2!}+\frac{z^3}{3!} \ldots\right]}{z} = 1$$

So, $\lim\limits_{z\to 0} f(z) = 1$

We find that at $z=0$ we have $(z=0)$ as a removable singularity.

(ii) $\dfrac{1-\cos z}{\sin^3 z}$

at $z=0$, $f(z)$ is not defined

but $\lim\limits_{z\to 0} f(z) = \dfrac{1-\cos z}{\sin^3 z} = \dfrac{2\sin^2 z/2}{\sin^3 z}$

$$= \frac{\sin^2(z/2)}{2(z/2)^2 \frac{(\sin^3 z)}{z^2}}$$

We know that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ 9.

So $\lim_{x \to 0} \frac{\sin^2(z/2)}{(z/2)^2} = 1$

but for $\lim_{z \to 0} \frac{\sin^3(z)}{z^2}$ to exist we have to have power of 3 over z so as to become one.

So, $\lim_{z \to 0} f(z)$ is not defined but $\lim_{z \to 0} \frac{f(z)}{0} = 1$

So, $z=0$ is singularity of $f(z)$ of pole $= 0$ and order $-1$.

(iii) $\frac{z^2}{e^{1/z} - 1}$

Here f(z)
Let us have $z = \frac{1}{w}$,

$f(w) = \frac{1}{w^2(e^w - 1)}$

At $w = 0$, $f(w)$ is not defined but let us see for $\lim_{w \to 0} f(w)$.

$f(w) = \lim_{w \to 0} \frac{1}{w^3 \frac{(e^w - 1)}{w}}$ = Not defined

So, we find that $w=0$ or $z=\infty$ is an essential singularity for $f(z)$.

9. $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$

$Re \left[ \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta)} d\theta \right]$

$\left( \because e^{i\sin\theta} = \underbrace{\cos(\sin\theta)}_{Re} + i\sin(\sin\theta) \right)$

$= Re \left[ \int_0^{2\pi} e^{\cos\theta + i\sin\theta} d\theta \right.$

$= Re \left[ \int_0^{2\pi} e^{e^{i\theta}} d\theta \right]$ $\left[ \because e^{i\theta} = \cos\theta + i\sin\theta \right]$

Now, $z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$

$d\theta = \frac{dz}{iz}$

$= Re \left[ \int_c \frac{e^z dz}{iz} \right]$

$= Re \left[ \frac{1}{i} \int_c \frac{e^z}{z} dz \right]$

$z = 0$, pole

Hence $\int_c \frac{e^z}{z} = 2\pi i (\Sigma R)$

$R = \lim_{z \to 0} (z-0) \frac{e^z}{z} = e^0 = 1$

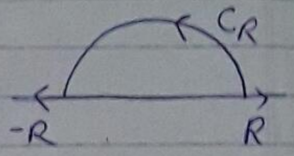
$\int_c \frac{e^z}{z} dz = 2\pi i$ ~~ans~~

$$\frac{1}{i}\int_C \frac{e^z}{z}\,dz = 2\pi$$

Hence, $\mathrm{Re}\left[\frac{1}{i}\int_C \frac{e^z}{z}\,dz\right] = 2\pi$, Ans

10. $$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4x+5}\,dx$$

Consider, $$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+4x+5}\,dx$$




$$\int_{\infty}^{\infty} \frac{x\sin x\,dx}{x^2+4x+5} = \mathrm{Im}\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+4x+5}\,dx$$

Also, Consider

$$f(z) = \frac{z e^{iz}}{z^2+4z+5}$$

$$z = -4 \pm \sqrt{16-20} = \frac{-4\pm 2i}{2} = -2\pm i$$

$z = -2+i$, $-2-i$ × (rejected because it lies in lower half)

Pole lies in Contour C

$$\int \frac{z e^{iz}}{z^2+4z+5}\,dz = \int_{-R}^{R} \frac{x e^{ix}}{x^2+4x+5}\,dx + \int \frac{z e^{iz}}{z^2+4z+5}\,dz$$

$$\Rightarrow \int \frac{z e^{iz}}{z^2+4z+5}\,dz = \int_{\pi} \frac{Re^{i\theta}\, e^{iRe^{i\theta}}\, Re^{i\theta}\,d\theta}{R^2 e^{i\theta} + 4Re^{i\theta}+5}$$

$\because$ $z = Re^{i\theta}$
$dz = Rie^{i\theta} d\theta$

$$\left| \int_{C_R} \frac{ze^{iz}}{z^2+4z+5} dz \right| \leq \frac{R^2}{R^2-4R+5} \int_0^{\pi} |e^{iR\cos\theta}| \times |e^{-R\sin\theta}| d\theta$$

$$\leq \frac{2R^2}{R^2-4R+5} \int_0^{\pi/2} |e^{iR\cos\theta}| \times |e^{-R\sin\theta}| d\theta$$

$$\leq \frac{2R^2}{R^2-4R+5} \int_0^{\pi/2} e^{-R\sin\theta} d\theta$$

Using Jordan's Lemma

$$\int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq \pi/2R$$

$$\leq \frac{2R^2}{R^2-4R+5} \times \frac{\pi}{2R}$$

$$\leq \frac{\pi}{R\left[1 - \frac{4}{R} + \frac{5}{R^2}\right]}$$

As, $R \to \infty$ $\int_{\wedge} f(z) dz = 0$

Hence, $\displaystyle\int \frac{ze^{iz}}{z^2+4z+5} dz = \int_{-R}^{R} \frac{xe^{ix}}{x^2+4x+5} dx$

$R \to \infty$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+4x+5} dx = \underbrace{2\pi i \sum R}_{\text{Residue Theorem}}$$

$$\int_R \frac{ze^{iz}}{z^2+4z+5}\,dz = \lim_{z\to i-2} (z-i+2)\frac{ze^{iz}}{(z-(i+2))(z+i+2)}$$

$$= \frac{(i-2)e^{i(i-2)}}{(i-2+i+2)} = \frac{(i-2)e^{i(i-2)}}{2i}$$

$$\int_C f(z)dz = 2\pi i \times \frac{(i-2)}{2i} e^{i(i-2)}$$

$$= \pi\left[(i-2)e^{-1-2i}\right]$$

$$= \frac{\pi}{e}(i-2)(\cos(-2) + i\sin(-2)]$$

$$= \frac{\pi}{e}\left[i\cos 2 + \sin 2 - 2\cos 2 + 2i\sin 2\right]$$

$$= \frac{\pi}{e}\left[(\sin 2 - 2\cos 2) + i(\cos 2 + 2\sin 2)\right]$$

$$\mathrm{Im}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+4x+5}dx = \int_{-\infty}^{\infty}\frac{x\sin x}{x^2+4x+5}dx = \mathrm{Im}\int_C f(z)dz$$

$$= \mathrm{Im}\left[\frac{\pi}{e}(\sin 2 - 2\cos 2) + i(\cos 2 + 2\sin 2)\right]$$

$$= \frac{\pi}{e}(\cos 2 + 2\sin 2) \quad \text{Ans}$$