Faster Exact and Approximate Algorithms for k-Cut

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Abstract

In the k-CUT problem, we are given an edge-weighted graph G and an integer k, and have to remove a set of edges with minimum total weight so that G has at least k connected components. The current best algorithms are an $O(n^{(2-o(1))k})$ randomized algorithm due to Karger and Stein, and an $\tilde{O}(n^{2k})$ deterministic algorithm due to Thorup. Moreover, several 2-approximation algorithms are known for the problem (due to Saran and Vazirani, Naor and Rabani, and Ravi and Sinha).

It has remained an open problem to (a) improve the runtime of exact algorithms, and (b) to get better approximation algorithms. In this paper we show an $O(k^{O(k)} n^{(2\omega/3+o(1))k})$-time algorithm for k-CUT. Moreover, we show an $(1+\varepsilon)$-approximation algorithm that runs in time $O((k/\varepsilon)^{O(k)} n^{k+O(1)})$, and a 1.81-approximation in fixed-parameter time $O(2^{O(k^2)} \text{poly}(n))$.

1 Introduction

In this paper we consider the k-CUT problem: given an edge-weighted graph $G = (V, E, w)$ and an integer k, delete a minimum-weight set of edges so that G has at least k connected components. This problem is a natural generalization of the global min-cut problem, where the goal is to break the graph into $k = 2$ pieces. This problem has been actively studied in theory of both exact and approximation algorithms, where each result brought new insights and tools on graph cuts.

It is not *a priori* clear how to obtain poly-time algorithms for any constant k, since guessing one vertex from each part only reduces the problem to the NP-hard MULTIWAY CUT problem. Indeed, the first result along these lines was the work of Goldschmidt and Hochbaum [GH94] who gave an $O(n^{(1/2-o(1))k^2})$-time exact algorithm for k-CUT. Since then, the exact exponent in terms of k has been actively studied. The current best runtime is achieved by an $\widetilde{O}(n^{2(k-1)})$ randomized algorithm due to Karger and Stein [KS96], which performs random edge contractions until the remaining graph has k nodes, and shows that the resulting cut is optimal with probability at least $\Omega(n^{-2(k-1)})$. The asymptotic runtime of $\tilde{O}(n^{2(k-1)})$ was later matched by a deterministic algorithm of Thorup [Tho08]. His algorithm was based on tree-packing theorems; it showed how to efficiently find a tree for which the optimal k-cut crosses it $2k - 2$ times. Enumerating over all possible $2k - 2$ edges of this tree gives the algorithm.

These elegant $O(n^{2k})$-time algorithms are the state-of-the-art, and it has remained an open question to improve on them. An easy observation is that the problem is closely related to k-CLIQUE, so we may not expect the exponent of n to go below $(\omega/3)k$. Given the interest in fine-grained analysis of algorithms, where in the range $[(\omega/3)k, 2k - 2]$ does the correct answer lie?

Our main results give faster deterministic and randomized algorithms for the problem.

Theorem 1.1 (Faster Randomized Algorithm). *Let W be a positive integer. There is a randomized algorithm for k-CUT on graphs with edge weights in $[W]$ with runtime*

$$\widetilde{O}(k^{O(k)} n^{k + \lfloor (k-2)/3 \rfloor \omega + 1 + ((k-2) \bmod 3)} W) \approx O(k^{O(k)} n^{(1+\omega/3)k}),$$

that succeeds with probability $1 - 1/\mathrm{poly}(n)$.

Theorem 1.2 (Even Faster Deterministic Algorithm). *Let W be a positive integer. For any $\varepsilon > 0$, there is a deterministic algorithm for exact k-CUT on graphs with edge weights in $[W]$ with runtime*

$$k^{O(k)} n^{(2\omega/3+\varepsilon)k + O(1)} W \approx O(k^{O(k)} n^{(2\omega/3)k}).$$

In the above theorems, ω is the matrix multiplication constant, and $\widetilde{O}$ hides polylogarithmic terms. While the deterministic algorithm from Theorem 1.2 is asymptotically faster, the randomized algorithm is better for small values of k. Indeed, using the current best value of $\omega < 2.373$ [LG14], Theorem 1.1 gives a randomized algorithm for exact k-CUT on unweighted graphs which improves upon the previous best $\widetilde{O}(n^{2k-2})$-time algorithm of Karger and Stein for all $k \in [8, n^{o(1)}]$. For $k \leq 6$, faster algorithms were given by Levine [Lev00].

Approximation algorithms. The k-CUT problem has also received significant attention from the approximation algorithms perspective. There are several $2(1 - 1/k)$-approximation algorithms that run in time $\mathrm{poly}(n, k)$ [SV95, NR01, RS08], which cannot be improved assuming the Small Set Expansion Hypothesis [Man17]. Recently, we gave an 1.9997-approximation algorithm that runs in $2^{O(k^6)} n^{O(1)}$ [GLL18]. In this current paper, we give a $(1 + \varepsilon)$-approximation algorithm for this problem much faster than the current best exact algorithms; prior to our work, nothing better was known for $(1 + \varepsilon)$-approximation than for exact solutions.

Theorem 1.3 (Approximation). *For any $\varepsilon > 0$, there is a randomized (combinatorial) algorithm for k-CUT with runtime $(k/\varepsilon)^{O(k)} n^{k+O(1)}$ time on general graphs, that outputs a $(1+\varepsilon)$-approximate solution with probability $1 - 1/\mathrm{poly}(n)$.*

The techniques from the above theorem, combined with the previous ideas in [GLL18], immediately give an improved FPT approximation guarantees for the k-CUT problem:

Theorem 1.4 (FPT Approximation). *There is a deterministic 1.81-approximation algorithm for the k-CUT problem that runs in time $2^{O(k^2)} \cdot n^{O(1)}$.*

Limitations. Our exact algorithms raise the natural question: how fast can exact algorithms for k-CUT be? We give a simple reduction showing that while there is still room for improvement in the running time of exact algorithms, such improvements can only improve the constant in front of the k in the exponent, assuming a popular conjecture on algorithms for the CLIQUE problem.

Claim 1.5 (Relationship to Clique). *Any exact algorithm for the k-CUT problem for graphs with edge weights in $[n^2]$ can solve the k-CLIQUE problem in the same runtime. Hence, assuming k-CLIQUE cannot be solved in faster than $n^{\omega k/3}$ time, the same lower bound holds for the k-cut problem.*

1.1 Our Techniques

Our algorithms build on the approach pioneered by Thorup: using tree-packings, he showed how to find a tree T such that it crosses the optimal k-cut at most $2k - 2$ times. (We call such a tree a *Thorup tree*, or *T-tree*.) Now brute-force search over which edges to delete from the T-tree (and how to combine the resulting parts together) gave an $\widetilde{O}(n^{2k-2})$-time deterministic algorithm. This last step, however, raises the natural question—*having found such a T-tree, can we use the structure of the k-CUT problem to beat brute force?* Our algorithms answer the question in the affirmative, in several different ways. The main ideas behind our algorithm are dynamic programming and fast matrix-multiplication, carefully combined with the fixed-parameter tractable algorithm technique of color-coding, and random sampling in general.

Fast matrix multiplication. Our idea to apply fast matrix multiplication starts with the crucial observation that if (i) the T-tree T is "tight" and crosses the optimal k-cut only $k - 1$ times, and (ii) these edges are "incomparable" and do not lie on a root-leaf path, then the problem of finding these $k - 1$ edges can be modeled as a max-weight clique-like problem! (And hence we can use matrix-multiplication ideas to speed up their computation.) An important property of this special case is that choosing an edge e to cut fixes one component in the k-CUT solution — by incomparability, the subtree below e cannot be cut anymore. The cost of a k-cut can be determined by the weight of edges between each pair of components (just like being a clique is determined by pairwise connectivity), so this case can be solved via an algorithm similar to k-CLIQUE.

Randomized algorithm. Our randomized algorithm removes these two assumptions step by step. First, while the above intuition crucially relies on assumption (ii), we give a more sophisticated dynamic program using color-coding schemes for the case where the edges are not incomparable. Moreover, to remove assumption (i), we show a randomized reduction that given a tree that crosses the optimal cut as many as $2k - 2$ times, finds a "tight" tree with only $k - 1$ crossings (which is the least possible), at the expense of a runtime of $O(k^2 n)^{k-1}$. Note that guessing which edges to delete is easily done in n^{k-1} time, but adding edges to regain connectivity while not increasing the number of crossings can naively take a factor of m^{k-1} more time. We lose only a $k^{2(k-1)}$ factor using our random-sampling based algorithm, using that in an optimal k-CUT a split cluster should have more edges going to its own parts than to other clusters.

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Deterministic algorithm. The deterministic algorithm proceeds along a different direction and removes both assumptions (i) and (ii) at once. We show that by deleting some $O(\log k)$ carefully chosen edges from the T-tree T, we can break it into three forests such that we only need to delete about $2k/3$ edges from each of these forests. Such a deletion is not possible when T is a star, but appropriately extending T by introducing Steiner nodes admits this deletion. (And $\Theta(\log k)$ is tight in this extension.) For each forest, there are $n^{2k/3}$ ways to cut these edges, and once a choice of $2k/3$ edges is made, the forest will not be cut anymore. This property allows us to bypass (ii) and establish desired pairwise relationships between choices to delete $2k/3$ edges in two forests. Indeed, we set up a tripartite graph where one part corresponds to the choices of which $\leq 2k/3$ edges to cut in one forest and the cost of the min k-cut is the weight of the min-weight triangle, which we find efficiently using fast matrix multiplication. Some technical challenges arise because we need to some components for some forests may only have Steiner vertices, but we overcome these problems using color-coding.

Approximation schemes. The $(1+\varepsilon)$-approximation algorithm again uses the $O(k^2 n)^{k-1}$-time randomized reduction, so that we have to cut exactly $k - 1$ edges from a "tight" T-tree T. An exact dynamic program for this problem takes time $\approx n^k$ — as it should, since even this tight case captures clique, when T is a star and hence these $k - 1$ edges are incomparable. And again, we need to handle the case where these $k - 1$ edges are not incomparable. For the former problem, we replace the problem of finding cliques by approximately finding "partial vertex covers" instead. (In this new problem we find a set of $k - 1$ vertices that minimize the total number of edges incident to them.) Secondly, in the DP we cannot afford to maintain the "boundary" of up to k edges explicitly any more. We show how to maintain an "ε-net" of nodes so that carefully "rounding" the DP table to only track a small $f(k)$-sized set of these rounded subproblems incurs only a $(1+\varepsilon)$-factor loss in quality.

Our approximate DP technique turns out to be useful to get a 1.81-approximation for k-CUT in FPT time, improving on our previous approximation of ≈ 1.9997 [GLL18]. In particular, the *laminar cut problem* from [GLL18] also has a tight T-tree structure, and hence we can use (a special case of) our approximate DP algorithm to get a $(1+\varepsilon)$-approximation for laminar cut, instead of the $2 - \varepsilon$-factor previously known. Combining with other ideas in the previous paper, this gives us the 1.81-approximation.

1.2   Related Work

The first non-trivial exact algorithm for the k-CUT problem was by Goldschmidt and Hochbaum, who gave an $O(n^{(1/2-o(1))k^2})$-time algorithm [GH94]; this is somewhat surprising because the related MULTIWAY CUT problem is NP-hard even for $k = 3$. They also proved the problem to be NP-hard when k is part of the input. Karger and Stein improved this to an $O(n^{(2-o(1))k})$-time randomized Monte-Carlo algorithm using the idea of random edge-contractions [KS96]. Thorup improved the $O(n^{4k+o(1)})$-time deterministic algorithm of Kamidoi et al. [KYN07] to an $\tilde{O}(n^{2k})$-time deterministic algorithm based on tree packings [Tho08]. Better algorithms are known for small values of $k \in [2, 6]$ [NI92, HO94, BG97, Kar00, NI00, NKI00, Lev00].

Approximation algorithms. The first such result for k-CUT was a $2(1-1/k)$-approximation of Saran and Vazirani [SV95]. Later, Naor and Rabani [NR01], and also Ravi and Sinha [RS08] gave 2-approximation algorithms using tree packing and network strength respectively. Xiao et al. [XCY11] extended Kapoor [Kap96] and Zhao et al. [ZNI01] and generalized Saran and Vazirani to give an $(2 - h/k)$-approximation in time $n^{O(h)}$. On the hardness front, Manurangsi [Man17]

showed that for any $\varepsilon > 0$, it is NP-hard to achieve a $(2 - \varepsilon)$-approximation algorithm in time $\text{poly}(n, k)$ assuming the Small Set Expansion Hypothesis.

In recent work [GLL18], we gave a 1.9997-approximation for k-Cut in FPT time $f(k)\text{poly}(n)$; this does not contradict Manurangsi's work, since k is polynomial in n for his hard instances. We improve that guarantee to 1.81 by getting a better approximation ratio for the "laminar" k-cut subroutine, improving from $2 - \varepsilon$ to $1 + \varepsilon$. This follows as a special case of the techniques we develop in §4; the rest of the ideas in this current paper are orthogonal to those in [GLL18].

FPT algorithms. Kawarabayashi and Thorup give the first $f(\mathsf{Opt}) \cdot n^2$-time algorithm [KT11] for unweighted graphs. Chitnis et al. [CCH$^+$16] used a randomized color-coding idea to give a better runtime, and to extend the algorithm to weighted graphs. Here, the FPT algorithm is parameterized by the cardinality of edges in the optimal k-Cut, not by the number of parts k. For more details on FPT algorithms and approximations, see the book [CFK$^+$15], and the survey [Mar07].

1.3 Preliminaries

For a graph $G = (V, E, w)$, consider some collection of disjoint sets $\mathcal{S} = \{S_1, \ldots, S_r\}$. Let $E_G(\mathcal{S}) = E_G(S_1, \ldots, S_r)$ denote the set of edges in $E_G[\cup_{i=1}^{r} S_r]$ (i.e., among the edges both of whose endpoints lie in these sets) whose endpoints belong to different sets S_i. For any vertex set S, let ∂S denote the edges with exactly one endpoint in S; hence $E_G(\mathcal{S}) = \cup_{S_i \in \mathcal{P}} \partial S_i$. For a collection of edges $F \subseteq E$, let $w(F) := \sum_{e \in F} w(e)$ be the sum of weights of edges in F. In particular, for a k-Cut solution $\{S_1, \ldots, S_k\}$, the value of the solution is $w(E_G(S_1, \ldots, S_k))$.

For a rooted tree $T = (V_T, E_T)$, let $T_v \subseteq V_T$ denote the subtree of T rooted at $v \in V_T$. For an edge $e \in E_T$ with child vertex v, let $T_e := T_v$. Finally, for any set $S \subseteq V_T$, $T_S = \sum_{v \in S} T_v$.

For some sections, we make no assumptions on the edge weights of G, while in other sections, we will assume that all edge weights in G are integers in $[W]$, for a fixed positive integer W. We default to the former unrestricted case, and explicitly mention transitioning to the latter case when needed.

2 A Fast Randomized Algorithm

In this section, we present a randomized algorithm to solve k-Cut exactly in time $\widetilde{O}(k^{O(k)} n^{(1+\omega/3)k})$, proving Theorem 1.1. Section 2.1 introduces our high-level ideas based on Thorup's tree packing results. Section 2.2 shows how to refine Thorup's tree to a good tree that crosses the optimal k-cut exactly $k - 1$ times, and Section 2.3 presents an algorithm given a good tree.

2.1 Thorup's Tree Packing and Thorup's Algorithm

Our starting point is a transformation from the general k-Cut problem to a problem on trees, inspired by Thorup's algorithm [Tho08] based on greedy tree packings. We will be interested in trees that cross the optimal partition only a few times. We fix an optimal k-Cut solution, $\mathcal{S}^* := \{S_1^*, \ldots, S_k^*\}$. Let $OPT := E_G(S_1^*, \ldots, S_k^*)$ be edges in the solution, so that $w(OPT)$ is the solution value.

Definition 2.1 (T-trees). *A tree T of G is a ℓ-T-tree if it crosses the optimal cut at most ℓ times; i.e., $E_T(S_1^*, \ldots, S_k^*) \leq \ell$. If $\ell = 2k - 2$, we often drop the quantification and call it a T-tree. If $\ell = k - 1$, the minimum value possible, then we call it a tight T-tree.*

Our first step is the same as in [Tho08]: we compute a collection $\mathcal{T}$ of $n^{O(1)}$ trees such that there exists a T-tree, i.e., a tree $T \in \mathcal{T}$ that crosses OPT at most $2k - 2$ times.

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Theorem 2.2 ([Tho08], Theorem 1). *For $\alpha \in (0, \frac{9}{10})$, let $\mathcal{T}$ be a greedy tree packing with at least $3m(k/\alpha)^3 \ln(nmk/\alpha)$ trees. Then, on the average, the trees $T \in \mathcal{T}$ cross each minimum k-cut less than $2(k - 1 + 2\alpha)$ times. Furthermore, the greedy tree packing algorithm takes $\widetilde{O}(k^3 m^2)$ time.*

The running time comes from the execution of $\widetilde{O}(k^3 m)$ minimum spanning tree computations. Note that, since our results are only interesting when $k \geq 7$, resulting in algorithms of running time $\Omega(n^{7+2\omega})$, we can completely ignore the running time of the greedy tree packing algorithm, which is only run once. Letting $\alpha := 1/8$, we get the following corollary.

Corollary 2.3. *We can find a collection of $\widetilde{O}(k^3 m)$ trees such that for a random tree $T \in \mathcal{T}$, $|E_T(S_1^*, \ldots, S_k^*)| \leq 2k - 3/2$ in expectation. In particular, there exists a T-tree $T \in \mathcal{T}$.*

In other words, if we choose such a T-tree $T \in \mathcal{T}$, we get the following problem: find the best way to cut $\leq 2k - 2$ edges of T, and then merge the connected components into exactly k components $S_1, \ldots, S_k$ so that $E_G(S_1, \ldots, S_k)$ is minimized. Thorup's algorithm accomplishes this task using brute force: try all possible $O(n^{2k-2})$ ways to cut and merge, and output the best one. This gives a runtime of $\widetilde{O}(k^3 n^{2k-2} m)$, or even $\widetilde{O}(n^{2k-2} m)$ with a more careful analysis [Tho08]. The natural question is: can we do better than brute-force?

For the min-cut problem (when $k = 2$), Karger was able to speed up this step from $O(n^{2k-2}) = O(n^2)$ to $\widetilde{O}(n)$ using dynamic tree data structures [Kar00]. However, this case is special: since there are ≤ 3 components produced from cutting the $\leq 2k - 2 = 2$ tree edges, only one pair of components need to be merged. For larger values of k, it is not clear how to generalize the use of clever data structures to handle multiple merges.

Our randomized algorithm gets the improvement in three steps:

- First, instead of trying all possible trees $T \in \mathcal{T}$, we only look at a random subset of $\Omega(k \log n)$ trees. By Corollary 2.3 and Markov's inequality, the probability that a random tree satisfies $|E_T(S_1^*, \ldots, S_k^*)| \geq 2k - 1$ is $\leq (2k - 3/2)/(2k - 1) = 1 - \Omega(1/k)$. Therefore, by trying $\Omega(k \log n)$ random trees, we find a T-tree T w.h.p.

- Next, given a T-tree T from above, we show how to find a collection of $\approx n^{k-1}$ trees such that, with high probability, one of these trees T' is a tight T-tree, i.e., it intersects OPT in exactly $k - 1$ edges. We show this in §2.2.

- Finally, given a tight T-tree T' from the previous step, we show how to solve the optimal k-CUT in time $\approx O(n^{(\omega/3)k})$, much like the k-CLIQUE problem [NP85]. The runtime is not coincidental; the $W[1]$ hardness of k-CUT derives from k-CLIQUE, and hence techniques for the former must work also for the latter. We show this in §2.3.

2.2 A Small Collection of "Tight" Trees

In this section we show how to find a collection of $\approx n^{k-1}$ trees such that, with high probability, one of these trees T' is a tight T-tree. Formally,

Lemma 2.4. *There is an algorithm that takes as input a tree T such that $|E_T(S_1^*, \ldots, S_k^*)| \leq 2k - 2$, and produces a collection of $k^{O(k)} n^{k-1} \log n$ trees, such that one of the new trees T' satisfies $|E_{T'}(S_1^*, \ldots, S_k^*)| = k - 1$ w.p. $1 - 1/\text{poly}(n)$. The algorithm runs in time $k^{O(k)} n^{k-1} m \log n$.*

The algorithm proceeds by iterations. In each iteration, our goal is to remove one edge of T and then add another edge back in, so that the result is still a tree. In doing so, the value of $|E_T(S_1^*, \ldots, S_k^*)|$ can either decrease by 1, stay the same, or increase by 1. We call an iteration *successful* if $|E_T(S_1^*, \ldots, S_k^*)|$ decreases by 1. Throughout the iterations, we will always refer to T as the current tree, which may be different from the original tree. Finally, if $|E_T(S_1^*, \ldots, S_k^*)| = \ell$ initially, then after $\ell - (k-1)$ consecutive successful iterations, we have the desired tight T-tree T'.

Assume we know ℓ beforehand; we can easily discharge this assumption later. For an intermediate tree T in the algorithm, we say that component S_i^* is **unsplit** if S_i^* induces exactly one connected component in T, and **split** otherwise. Initially, there are at most $(k-1) - \ell$ split components, possibly fewer if some components induce many components in T. Moreover, if all $\ell - (k-1)$ iterations are successful, all components are unsplit at the end.

Figure 2.1: The red edges are deletion-worthy edges in this T-tree; the dashed lines mark the optimal components.

Lemma 2.5. *The probability of any iteration being successful, i.e., reducing the number of tree-edges belonging to the optimal cut, is at least $\Omega(1/nk^2)$.*

Proof. Each successful iteration has two parts: first we must delete a "deletion-worthy" edge (which happens with probability $1/(n-1)$), and then we add a "good" connecting edge (which happens with probability $\Omega(1/k^2)$). The former just uses that a tree has $n-1$ edges, but the latter must use that there are many good edges crossing the resulting cut—a naive analysis may only give $\Omega(1/m)$ for the second part.

We first describe the edges in T that we would like to delete. These are the edges such that if we delete one of them, then we are likely to make a successful iteration (after selectively adding an edge back in). We call these edges **deletion-worthy**. Let us first root the tree $T = (V, E_T)$ at an arbitrary, fixed root $v_r \in V$. For any edge e, let T_e denote the subtree below it obtained by deleting the edge e.

Definition 2.6. *A* deletion-worthy *edge $e \in E_T$ satisfies the following two properties:*

    (1) The edge crosses between two parts of the optimal partition, i.e., $e \in E_T(S_1^*, \ldots, S_k^*)$.

    (2) There is exactly one part $S_i^* \in \mathcal{S}^*$ satisfying $S_i^* \cap T_e \neq \emptyset$ and $S_i^* - T_e \neq \emptyset$. In other words, exactly one component of $\mathcal{S}^*$ intersects T_e but is not completely contained in T_e. Note that, by condition (1), S_i^* is necessarily split.

Claim 2.7. *If there is a split component S_i^*, there exists a deletion-worthy edge $e \in E_T$.*

Proof. For each S_i^*, contract every connected component of S_i^* induced in T, so that split components contract to multiple vertices. Root the resulting tree at v_r, and take a vertex $v \in V$ of maximum depth whose corresponding component S_i^* is split. It is easy to see that $v \neq v_r$ and the parent edge of v in the rooted tree is deletion-worthy. $\square$

Finally, we describe the deletion part of our algorithm. The procedure is simple: *choose a random edge in T to delete.* With probability $\geq 1/(n-1)$, we remove a deletion-worthy edge in T. This gives rise to the n^{-1} factor in the probability of a successful iteration.

Now we show that, conditioned on deleting a deletion-worthy edge, we can selectively add an edge to produce a successful iteration with probability $k^{-O(1)}$. In particular, we add a random edge in
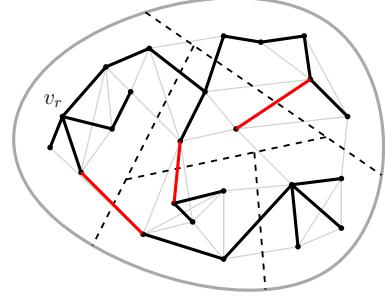
$E_G(T_e, V - T_e)$—i.e., an edge from subtree under e to the rest of the vertices—where the probability is weighted by the edge weights in $E_G(T_e, V - T_e)$. We show that this makes the iteration successful with probability $\Omega(1/k^2)$. (Recall that the iteration is successful if the number of tree edges lying in the optimal cut decreases by 1.)

First of all, it is clear that adding any edge in $E_G(T_e, V - T_e)$ will get back a tree. Next, to lower bound the probability of success, we begin with an auxiliary lemma.

Claim 2.8. *Given a set of $k+1$ components $S_1, \ldots, S_{k+1}$ that partition V, we have*

$$w(OPT) \leq \left(1 - \binom{k+1}{2}^{-1}\right) \cdot w(E_G(S_1, \ldots, S_{k+1})).$$

Proof. Consider merging two components S_i, S_j uniformly at random. Every edge in $E(S_1, \ldots, S_{k+1})$ has probability $\binom{k+1}{2}^{-1}$ of disappearing from the cut, so the expected new cut is

$$\left(1 - \binom{k+1}{2}^{-1}\right) \cdot E(S_1, \ldots, S_{k+1}),$$

and $w(OPT)$ can only be smaller. $\square$

For convenience, define $C := S_i^* \cap T_e$, where S_i^* is the split component corresponding to the deletion-worthy edge e we just deleted. Observe that the only edges in $E(T_e, V - T_e)$ that are not in OPT must be in $E(C, S_i^* - C)$; this is because, of the components S_j^* intersecting T_e, only S_i^* is split. Therefore,

$$w(E(T_e, V - T_e)) \leq w(OPT) + w(E(C, S_i^* - C)),$$

and the probability of selecting an edge in $E(C, S_i^* - C)$ is

$$\frac{w(E(C, S_i^* - C))}{w(E(T_e, V - T_e))} \geq \frac{w(E(C, S_i^* - C))}{w(OPT) + w(E(C, S_i^* - C))}. \tag{2.1}$$

Claim 2.9. $w(E(C, S_i^* - C)) \geq \left(\left(1 - \binom{k+1}{2}^{-1}\right)^{-1} - 1\right) \cdot w(OPT) = \Omega(1/k^2) \cdot w(OPT)$.

Proof. The set of edges $OPT \cup E(C, S_i^* - C)$ cuts the graph G into $k+1$ components. Claim 2.8 implies this set has total weight $\geq \left(1 - \binom{k+1}{2}^{-1}\right)^{-1} w(OPT)$. Observing that the edges of OPT and $E(C, S_i^* - C)$ are disjoint from each other completes the proof. $\square$

Using the above claim in (2.1) means the probability of selecting an edge in $E(C, S_i^* - C)$ is $\Omega(1/k^2)$. Hence the probability of an iteration being successful is $\Omega(1/(nk^2))$, completing the proof of Lemma 2.5. $\square$

Since we have ℓ iterations, the probability that each of them is successful is $\ell^{-O(\ell)} n^{-\ell}$. If we repeat this algorithm $\ell^{O(\ell)} n^{\ell} \log n$ times, then with probability $1 - 1/\text{poly}(n)$, one of the final trees T' will satisfy $|E_{T'}(S_1^*, \ldots, S_k^*)| = k - 1$. We can remove the assumption of knowing ℓ by trying all possible values of $\ell \in [k - 1, 2k - 2]$, giving a collection of $k^{O(k)} n^{k-1} \log n$ trees in running time $k^{O(k)} n^{k-1} m \log n$. This completes the proof of Lemma 2.4.

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2.3 Solving k-Cut on "Tight" Trees

In the previous section, we found a collection of $\approx n^k$ trees such that, with high probability, the intersection of one of these trees with the optimal k-cut OPT consists of only $k-1$ edges. In this section, we show that given this tree we can find the optimal k-cut in time $\approx n^{\omega k/3}$. This will follow from Lemma 2.10 below. In this section, we restrict the edge weights of our graph G to be positive integers in $[W]$.

Lemma 2.10. *There is an algorithm that takes a tree T and outputs, from among all partitions $\{S_1, \ldots, S_k\}$ that satisfy $|E_T(S_1, \ldots, S_k)| = k-1$, a partition $\mathcal{S}^\dagger := \{S_1^\dagger, \ldots, S_k^\dagger\}$ minimizing the number of inter-cluster edges $E_G(S_1^\dagger, \ldots, S_k^\dagger)$, in time $\widetilde{O}(k^{O(k)} n^{\lfloor (k-2)/3 \rfloor \omega + 2 + (k-2) \bmod 3} W)$.*

Given a tree $T = (V, E_T)$ and a set $F \subseteq E_T$ of tree edges, deleting these edges gives us a vertex partition $\mathcal{S}_F = \{S_1, \ldots, S_{|F|+1}\}$. Let $\mathsf{Cut}(F)$ be the set of edges in G that go between the clusters in $\mathcal{S}_F$; i.e.,

$$\mathsf{Cut}(F) := E(S_1, \ldots, S_{|F|+1}). \tag{2.2}$$

Put another way, these are the edges $(u, v) \in E$ such that the unique u-v path in T contains an edge in F. Note that Lemma 2.10 seeks a set $E^\dagger \subseteq E_T$ of size $k-1$ that minimizes $w(\mathsf{Cut}(F))$.

2.3.1 A Simple Case: Incomparable Edges

Our algorithm builds upon the algorithm of Nešetřil and Poljak [NP85] for k-CLIQUE, using Boolean matrix multiplication to obtain the speedup from the naive $O(n^k)$ brute force algorithm. It is instructive to first consider a restricted setting to highlight the similarity between the two algorithms. This setting is as follows: we are given a vertex $v_r \in V$ and the promise that if the input tree $T = (V, E_T)$ is rooted at v_r, then the optimal $k-1$ edges $E^\dagger := E_T(S_1^\dagger, \ldots, S_k^\dagger)$ to delete are *incomparable*. By incomparable, we mean any root-leaf path in T contains at most one edge in $E^\dagger$.

Like the algorithm of [NP85], our algorithm creates an auxiliary graph $H = (V_H, E_H)$ on $O(n^{\lceil k/3 \rceil})$ nodes. Our graph construction differs slightly in that it always produces a tripartite graph, and that this graph has edge weights. In this auxiliary graph, we will call the vertices *nodes* in order to differentiate them from the vertices of the tree.

- The nodes in graph H will form a tripartition $V_1 \cup V_2 \cup V_3 = V_H$. For each r, let $\mathcal{F}_r \subseteq 2^E$ be the family of all sets of exactly r edges in E_T that are pairwise incomparable in T. For each $i = 1, 2, 3$, define $r_i := \lfloor \frac{(k-1)+(i-1)}{3} \rfloor$ so that $r_1 + r_2 + r_3 = k-1$. For each $i = 1, 2, 3$ and each $F \in \mathcal{F}_{r_i}$, add a node v_i^F to V_i representing set F.

- Consider a pair (V_a, V_b) of parts in the tripartition with $(a, b) \in (1, 2), (2, 3), (3, 1)$. Consider a pair of sets $F^a := \{e_1^a, \ldots, e_{r_a}^a\} \in \mathcal{F}_{r_a}$, $F^b := \{e_1^b, \ldots, e_{r_b}^b\} \in \mathcal{F}_{r_b}$; recall these are sets of r_a and r_b incomparable edges in T. If the edges in F^a are also pairwise incomparable with the edges in F^b, then add an edge $(v_a^{F^a}, v_b^{F^b}) \in V_a \times V_b$ of weight

$$w_H(v_a^{F^a}, v_b^{F^b}) := \sum_{i=1}^{r_a} w(E(T_{e_i^a}, V - T_{e_i^a})) - \sum_{i=1}^{r_a} \sum_{j=i+1}^{r_a} w(E(T_{e_i^a}, T_{e_j^a})) - \sum_{i=1}^{r_a} \sum_{j=1}^{r_b} w(E(T_{e_i^a}, T_{e_j^b})).$$

Observe that every triple of nodes in graph H that form a triangle together represent $r_1 + r_2 + r_3 = k-1$ many incomparable edges. Moreover, the weights are set up so that for any triangle $(v_1^{F^1}, v_2^{F^2}, v_3^{F^3}) \in$

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$V_1 \times V_2 \times V_3$ such that $F := F^1 \cup F^2 \cup F^3 = \{e_1, \ldots, e_{k-1}\}$, the total weight of the edges is equal to

$$w_H(v_1^{F^1}, v_2^{F^2}) + w_H(v_2^{F^2}, v_3^{F^3}) + w_H(v_3^{F^3}, v_1^{F^1}) = \sum_{i=1}^{k-1} w(E(T_{e_i}, V - T_{e_i})) - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k-1} w(E(T_{e_i}, T_{e_j})).$$
(2.3)

A straightforward counting argument shows that this is exactly $w(E(T_{e_1}, \ldots, T_{e_{k-1}})) = \mathsf{Cut}(F)$, the solution value of cutting the edges in F.

Hence, the problem reduces to computing a minimum weight triangle in graph H. While the minimum weight triangle problem is unlikely to admit an $O(N^{3-\varepsilon})$ time algorithm on a graph with N vertices with arbitrary edge weights, the problem does admit an $\widetilde{O}(MN^\omega)$ time algorithm when the graph has integral edge weights in the range $[-M, M]$ [WW10]. Since the original graph G has integral edge weights in $[W]$, the edge weights in H must be in the range $[-O(Wm), O(Wm)]$. Therefore, we can set $N := O(n^{\lceil (k-1)/3 \rceil})$ and $M := O(Wm)$ to obtain an $\widetilde{O}(Wn^{\lceil (k-1)/3 \rceil \omega}m)$ time algorithm in this restricted setting.

2.3.2 The General Algorithm

Now we prove Lemma 2.10 in full generality, and show how to find $E^\dagger$. The ideas we use here will combine the matrix-multiplication idea from the restricted case of incomparable edges, together with dynamic programming.

> Given a tree edge $e \in E_T$, and an integer $s \in [k-2]$, let $\mathsf{State}(e, s)$ denote a set of edges F in subtree T_e such that $|F| = s - 1$ and $\mathsf{Cut}(\{e\} \cup F)$ is minimized.

In other words, $\mathsf{State}(e, s)$ represents the optimal way to cut edge e along with $s - 1$ edges in T_e. For ease of presentation, we assume that this value is unique. Observe that, once all of these states are computed, the remaining problem boils down to choosing an integer $\ell \in [k-1]$, integers $s_1, \ldots, s_\ell$ whose sum is $k - 1$, and incomparable edges $e_1, \ldots, e_\ell$ that minimizes

$$\mathsf{Cut}\left(\bigcup_{i=1}^{\ell} \mathsf{State}(e_i, s_i)\right) = \sum_{i=1}^{k-1} \mathsf{State}(e_i, s_i) - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k-1} w(E(T_{e_i}, T_{e_j})).$$

Comparing this expression to (2.3) suggests that this problem is similar to the incomparable case in §2.3.1, a connection to be made precise later.

We now compute states for all edges $e \in E_T$, which we do from bottom to top (leaf to root). When e is a leaf edge, the states are straightforward: $\mathsf{State}(e, 1) = \mathsf{Cut}(\{e\})$ and $\mathsf{State}(e, s) = \infty$ for $s > 1$. Also, for each edge $e \in E_T$, define $\mathsf{desc}(e)$ to be all "descendant edges" of e, formally defined as all edges $f \in E_T - e$ whose path to the root contains edge e.

Fix an edge $e \in E_T$ and an $s \in [k-2]$, for which we want to compute $\mathsf{State}(e, s)$. Suppose we order the edges in T_e in an arbitrary but fixed order. Let us now figure out some properties for this (unknown) value of $\mathsf{State}(e, s)$. As a thought experiment, let $F^\dagger$ be the list of all the "maximal" edges in $\mathsf{State}(e, s)$—in other words, $f \in F^\dagger$ iff $f \in \mathsf{State}(e, s)$ and $f \notin \mathsf{desc}(f')$ for all $f' \in \mathsf{State}(e, s)$. Let $\ell^\dagger := |F^\dagger|$ and $F^\dagger = (e_1^\dagger, \ldots, e_{\ell^\dagger}^\dagger)$ be the sequence in the defined order, and for each $e_i^\dagger$, let $s_i^\dagger := 1 + |\mathsf{desc}(e_i^\dagger) \cap \mathsf{State}(e, s)|$. Observe that $\sum_i s_i^\dagger = s - 1$, and that we must satisfy

$$\mathsf{State}(e, s) = \bigcup_{i=1}^{\ell^\dagger} \left(\{e_i^\dagger\} \cup \mathsf{State}(e_i^\dagger, s_i^\dagger)\right).$$
(2.4)

Also,

$$w(\mathsf{State}(e,s)) = E(T_e, V - T_e) + \sum_{i=1}^{\ell^\dagger} w\left(E(G[T_e]) \cap \mathsf{Cut}\left(\{e_i^\dagger\} \cup \mathsf{State}(e_i^\dagger, s_i^\dagger)\right)\right)$$

$$- \sum_{i=1}^{\ell^\dagger} \sum_{j=i+1}^{\ell^\dagger} w(E_{G[T_e]}[T_{e_i^\dagger}, T_{e_j^\dagger}]),$$

since the only edges double-counted in the first summation of $w(\mathsf{State}(e,s))$ are those connecting different $T_{e_i^\dagger}, T_{e_j^\dagger}$.

Given these "ideal" values $\ell^\dagger$ and $\{s_i^\dagger\}$, our algorithm repeats the following procedure multiple times:

- Pick a number ℓ uniformly at random in $[s-1]$. Then, let function $\sigma : [\ell] \to [s-1]$ be chosen uniformly at random among all $\le (s-1)^\ell$ possible functions satisfying $\sum_{i=1}^\ell \sigma(i) = s - 1$. With probability $\ge (s-1)^{-(\ell^\dagger+1)} = k^{-O(k)}$, we correctly guess $\ell = \ell^\dagger$ and $\sigma(i) = s_i^\dagger$ for each $i \in [\ell]$.[1]
- Construct an auxiliary graph H as follows. As in §2.3.1, H has a tripartition $V_1 \cup V_2 \cup V_3 = V_H$, and assume there is an arbitrary but fixed total ordering on the edges of the tree. For each r, let $\mathcal{F}_r \subseteq 2^E$ be the family of all sets of exactly r edges in E_T that are pairwise incomparable in T. For each $i = 1, 2, 3$, let $r_i := \lfloor \frac{\ell+(i-1)}{3} \rfloor$ so that $r_1 + r_2 + r_3 = \ell$, and for each $F \in \mathcal{F}_{r_i}$, add a node v_i^F to V_i representing the edges F as a sequence in the total order.
  Also, define $R_i := \sum_{j=1}^{i-1} r_i$ for $i = 1, 2, 3, 4$. Note that $R_1 = 0$ and $R_4 = r_1 + r_2 + r_3 = \ell$. Our intention is map the integer values $\{\sigma(R_i+1), \sigma(R_i+2), \dots, \sigma(R_{i+1})\}$ to the sequences represented by nodes in V_i, as we will see later. Consider each tripartition pair (V_a, V_b) with $(a, b) \in (1, 2), (2, 3), (3, 1)$. For each pair $F^a \in \mathcal{F}_{r_a}$, $F^b \in \mathcal{F}_{r_b}$ represented as ordered sequences $F^a = (e_1^a, \dots, e_{r_a}^a)$ and $F^b = (e_1^b, \dots, e_{r_b}^b)$, if the edges in F^a are pairwise incomparable with the edges in F^b, then add an edge $(v_a^{F^a}, v_b^{F^b}) \in V_a \times V_b$ in the auxiliary graph of weight

$$w_H(v_a^{F^a}, v_b^{F^b}) := \sum_{i=1}^{r_a} w\left(\mathsf{State}\left(e_i^a, \sigma(R_a + i)\right)\right)$$

$$- \sum_{i=1}^{r_a} \sum_{j=i+1}^{r_a} w(E_{G[T_e]}(T_{e_i^a}, T_{e_j^a})) - \sum_{i=1}^{r_a} \sum_{j=1}^{r_b} w(E_{G[T_e]}(T_{e_i^a}, T_{e_j^b})). \quad (2.5)$$

For any triangle $(v_1^{F^1}, v_2^{F^2}, v_3^{F^3}) \in V_1 \times V_2 \times V_3$ such that $F := F^1 \cup F^2 \cup F^3$ has ordered sequence $(e_1, \dots, e_\ell)$, the total weight of the edges is equal to

$$w_H(v_1^{F^1}, v_2^{F^2}) + w_H(v_2^{F^2}, v_3^{F^3}) + w_H(v_3^{F^3}, v_1^{F^1})$$

$$= \sum_{i=1}^\ell w(\mathsf{State}(e_i^a, \sigma(i))) - \sum_{i=1}^\ell \sum_{j=i+1}^\ell w(E_{G[T_e]}(T_{e_i}, T_{e_j})). \quad (2.6)$$

A straightforward counting argument shows that this is exactly

$$w\left(\mathsf{Cut}\left(\{e\} \cup \bigcup_{i=1}^\ell \mathsf{State}(e_i, \sigma(i))\right)\right) - w(E(T_e, V - T_e)).$$

---

[1] Of course, we could instead brute force over all $k^{O(k)}$ possible choices of ℓ and σ.

Thus, the weight of each triangle, with $w(E(T_e, V - T_e))$ added to it, corresponds to the cut value of one possible solution to $\mathsf{State}(e, s)$. Moreover, if we guess ℓ and $\sigma : [\ell] \to [s - 1]$ correctly, then this triangle will exist in auxiliary graph H, and we will compute the correct state if we compute the minimum weight triangle in $\widetilde{O}(W n^{\lceil \ell/3 \rceil \omega} m)$ time. Since the probability of guessing $\ell, \sigma(\cdot)$ correctly is $k^{-O(k)}$, we repeat the guessing $k^{O(k)} \log n$ times to succeed w.h.p. in time $\widetilde{O}(k^{O(k)} n^{\lceil (k-2)/3 \rceil \omega} mW)$. This concludes the computation of each $\mathsf{State}(e, s)$; since there are $O(kn)$ such states, the total running time becomes $\widetilde{O}(k^{O(k)} n^{\lceil (k-2)/3 \rceil \omega + 1} mW)$.

Lastly, to compute the final k-Cut value, we let $s := k - 1$ and construct the same auxiliary graph H, except that $k - 2$ is replaced by $k - 1$ and the relevant graph $G[T_e]$ becomes the entire G. By the same counting arguments, the weight of triangle $(v_1^{F^1}, v_2^{F^2}, v_3^{F^3}) \in V_1 \times V_2 \times V_3$ such that $F := F^1 \cup F^2 \cup F^3$ has ordered sequence $(e_1, \ldots, e_\ell)$ is exactly

$$w\Big(\mathsf{Cut}\big(\{e\} \cup \bigcup_{i=1}^{\ell} \mathsf{State}(e_i, \sigma(i))\big)\Big).$$

Again, by repeating the procedure $k^{O(k)} \log n$, we compute an optimal k-Cut w.h.p., in time $\widetilde{O}(k^{O(k)} n^{\lceil (k-1)/3 \rceil \omega} mW)$. Note that this time is dominated by the running time $\widetilde{O}(k^{O(k)} n^{\lceil (k-2)/3 \rceil \omega + 1} mW)$ of computing the states.

In order to get the runtime claimed in Theorem 1.1, we need a couple more ideas—however, they can be skipped on the first reading, and we defer them to the Appendix C.

3 A Faster Deterministic Algorithm

In this section, we show how to build on the randomized algorithm of the previous section and improve it in two ways: we give a deterministic algorithm, with a better asymptotic runtime. (The algorithm of the previous section has a better runtime for smaller values of k.) Formally, the main theorem of this section is the following:

Theorem 1.2 (Even Faster Deterministic Algorithm). *Let W be a positive integer. For any $\varepsilon > 0$, there is a deterministic algorithm for exact k-Cut on graphs with edge weights in $[W]$ with runtime*

$$k^{O(k)} n^{(2\omega/3 + \varepsilon)k + O(1)} W \approx O(k^{O(k)} n^{(2\omega/3)k}).$$

Our main idea is a more direct application of matrix multiplication, without paying the $O(n^k)$ overhead in the previous section. Instead of converting a given T-tree to a "tight" tree where matrix multiplication can be combined with dynamic programming, with only $n^{O(\log k)}$ overhead, we partition the given T-tree to subforests that are amenable to direct matrix multiplication approach.

As in §2 we build on the framework of Thorup [Tho08], where the k-Cut problem reduces to $n^{O(1)}$ instances of the following problem: given the graph G and a spanning tree T, find a way to cut $\leq 2k - 2$ edges from T, and then merging the connected components of T into k connected components, that minimizes the number of cut edges in G. Again, the optimal k-cut is denoted by $\mathcal{S}^* = \{S_1^*, \ldots, S_k^*\}$.

For the rest of this section, let T be some spanning tree in the instance that crosses the optimal k-cut in $(r - 1) \leq 2k - 2$ edges. If we delete these $r - 1$ edges from T, this gives us r components, which we denote by $C_1^*, \ldots, C_r^*$ — these are a refinement of $\mathcal{S}^*$, and hence can be then be merged together to give us $\mathcal{S}^*$. Let $E_T^* := E_T(C_1^*, \ldots, C_r^*) = E_T(S_1^*, \ldots, S_k^*)$ be these $r - 1$ cut edges in T.

3.1 Balanced Separators

We first show the existence of a small-size *balanced separator* in the following sense: there exist forests F_1, F_2, F_3 whose vertices partition $V(T)$, such that

   (i) we can delete $O(\log k)$ edges in T to get the forests, i.e., $|E(T) - \bigcup_{i=1}^{3} E(F_i)| = O(\log k)$, and

   (ii) we want to cut few edges from each forest, i.e., $|E(F_i) \cap E_T^*| \leq \lceil 2k/3 \rceil$ for each i.

Of course, small-size balanced edge separators typically do not exist in general trees, such as if the tree is a star. So we first apply a degree-reducing step. This operation reduces the maximum degree of the tree to 3, at a cost of introducing "Steiner" vertices, which are handled later.

Lemma 3.1 (Degree-Reduction). *Given a tree $T = (V_T, E_T)$, we can construct a tree $T' = (V_{T'}, E_{T'})$, where $V_{T'} = V_T \cup X$, where X are called the Steiner vertices, such that*

   1. T' has maximum degree 3.

   2. $|V(T')| \leq 2|V(T)|$

   3. For every way to cut r edges in T and obtain components $C_1, \ldots, C_{r+1}$, there is a way to cut r edges in T' and obtain components $C'_1, \ldots, C'_{r+1}$ such that each C_i is precisely $C'_i \cap V_T$.

Proof. Root the tree T at an arbitrary root, and select any non-Steiner vertex $v \in V_T$ with more than two children. Replace the star composed of v and its children with an arbitrary binary tree with v as the root and its children as the leaves. This process does not introduce any new vertex with more than two children, so we can repeat it until it terminates, giving us a tree T' of maximum degree 3. Every star of z edges adds exactly $z - 1$ Steiner nodes, and there are $\leq |V_T| - 1$ edges initially, so $\leq |V_T| - 2$ Steiner vertices are added throughout the process, and $|V_{T'}| \leq 2|V_T|$. Finally, if we cut some r edges $(u_i, v_i) \in E_T$ where v_i is the parent of u_i, then we can cut the parent edge of each u_i in T' to obtain the required components. $\square$

Having applied Lemma 3.1 to T to get T', Property (3) shows that we can still delete $\leq 2k - 2$ edges in T' to obtain the components of the optimal solution before merging. To avoid excess notation, we assume that T itself is a tree of degree ≤ 3, possibly with Steiner nodes. From now on, our task is to delete $\leq 2k - 2$ edges of T and merge them into k components, *each of which containing at least one non-Steiner vertex*, that minimizes the number of cut edges in G. To show that the aforementioned forests F_1, F_2, F_3 exist in the new tree T, we introduce the following easy lemma:

Lemma 3.2. *Let T be a tree of degree ≤ 3 and $F \subseteq E(T)$ be a subset of the edges. For any integer $r \in [1, |F| - 1]$, there exists a vertex partition A, B of $V(T)$ such that $|E_T(A, B)| = O(\log(r+1))$, and the induced subgraphs $T[A]$ and $T[B]$ have at most r and at most $|F| - r$ edges from F, respectively.*

Proof. We provide an algorithm that outputs a collection of $O(\log r)$ disjoint subtrees whose union comprises A. Root T at a degree-1 vertex, and find a vertex of maximal depth whose rooted subtree contains $> r$ edges from F. The degree condition ensures that v has ≤ 2 children, and by maximality, all of v's children have $\leq r$ edges in F in their subtrees. Moreover, the edges in T_v is precisely the union of the edge sets $E(T_u) \cup \{(u, v)\}$ for all children u of v. For convenience, define $E^+(T_u) := E(T_u) \cup \{(u, v)\}$ for a child u of v. So there must be a child u of v satisfying $|E^+(T_u) \cap F| \in (r/2, r]$.

If $|E^+(T_u) \cap F| = r$, then $(A, B) = (V(T_u), V(T) - V(T_u))$ is a satisfying partition with $|E_T(A, B)| = 1$, and we are done. Otherwise, recurse on the tree $T' := T[V(T) - V(T_u)]$ where we remove (u, v) and the subtree below it, with the parameters $r' := r - |E^+(T_u) \cap F|$ and $F' := F \setminus E^+(T_u)$ to get

partition (A', B'), and set $A := A' \cup V(T_u)$ and $B := B'$. By recursion, we guarantee that

$$
\begin{aligned}
|E(T[A]) \cap F| &\leq |E(T[A']) \cap F'| + |E^+(T_u) \cap F| \\
&\leq (r - |E^+(T_u) \cap F|) + |E^+(T_u) \cap F| = r
\end{aligned}
$$

and

$$
\begin{aligned}
|E(T[B]) \cap F'| = |E(T[B']) \cap F'| &\leq |F'| - (r - |E^+(T_u) \cap F|) \\
&= (|F| - |E^+(T_u) \cap F|) - (r - |E^+(T_u) \cap F|) = |F| - r.
\end{aligned}
$$

Since the value of r drops by at least half each time, there are $O(\log r)$ steps of the recursion. Each step can only add the additional edge (u, v) to $|E_T(A, B)|$, so $|E_T(A, B)| = O(\log r)$. $\qquad \square$

Corollary 3.3. *There exist forests F_1, F_2, F_3 whose vertices partition $V(T)$ such that*

(i) the number of crossing edges is $|E(T) - \bigcup_{i=1}^{3} E(F_i)| = O(\log |E_T^*|)$, and
(ii) $|E(F_i) \cap E_T^*| \leq \lceil |E_T^*|/3 \rceil$ for each i.

Proof. We apply Lemma 3.2 with $F := E_T^*$ and $r := \lceil |E_T^*|/3 \rceil$ to obtain the separation (A, B), and then set $F_1 := T[A]$. Before applying the lemma again on B, we first connect the connected components of B arbitrarily into a tree; let F^+ denote the added edges. Then, we apply with $F := E_T^* - E[F_1]$ and $r := \lceil |E_T^*|/3 \rceil$ to obtain separation (A', B'), and then set $F_2 := T[A'] - F^+$ and $F_3 := T[B'] - F^+$. $\qquad \square$

Given this result, our algorithm starts by trying all possible $n^{O(\log k)}$ ways to delete $O(\log k)$ edges of T and partition the connected components into three forests. By Corollary 3.3, one of these attempts produces the desired F_1, F_2, F_3 satisfying the two properties.

3.2 Matrix Multiplication

The balanced partitioning procedure from the previous section gives us three forests F_1, F_2, F_3, such that the optimal solution cuts at most $2k/3$ edges in each — and then combines the resulting pieces together. The algorithm now computes these solutions separately for each forest, and then uses matrix multiplication to combine these solutions together.

Indeed, for each $F_i \in \{F_1, F_2, F_3\}$, the algorithm computes all $O(n^{\lceil 2k/3 \rceil})$ ways to cut $\leq \lceil 2k/3 \rceil$ edges in F_i, followed by all $3^{O(k)}$ ways to label each of the $\leq \lceil 2k/3 \rceil + O(\log k)$ connected components with a label in $[k]$. For each one forest, note that some of these components might only contain Steiner vertices of the tree; we call these the *Steiner components*, and the other the *normal components*. For each subset $S \subseteq [k]$, let $\mathcal{F}_i^S$ denote all possible ways to cut and label F_i in the aforementioned manner such that the set of labels that are attributed to at least one normal component is precisely S.

The algorithm now enumerates over every possible triple of subsets $S_1, S_2, S_3 \subseteq [k]$ (not necessarily disjoint) whose union is exactly $[k]$. Note that there are at most 7^k of these triples. For each triple S_1, S_2, S_3, we construct the following tripartite auxiliary graph $H = (V_H, E_H)$ on $O(k^{O(k)} n^{\lceil 2k/3 \rceil})$ vertices,

with tripartition $V_H = V_1 \uplus V_2 \uplus V_3$. For each $i = 1, 2, 3$, each element in $\mathcal{F}_i^{S_i}$ is a tuple (X_i, σ_i) where $X_i \subseteq F_i$ is a set of edges that we cut from F_i, and σ_i is a labeling of the normal components in the resulting forest so that the label set is exactly S_i. Now for each $(X, \sigma) \in \mathcal{F}_i^{S_i}$, add a node $v_i^{(X,\sigma)}$ to V_i. Moreover, for each tripartition pair (V_a, V_b) with $(a, b) \in (1, 2), (2, 3), (3, 1)$, and for each way $(X_a, \sigma_a) \in \mathcal{F}_a^{S_a}$ to cut F_a into components $C_1^a, \ldots, C_{r_a}^a$ with labels $\sigma_a(1), \ldots, \sigma_a(r_a)$, and

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for each way $(X_b, \sigma_b) \in \mathcal{F}_b^{S_b}$ to cut F_b into components $C_1^b, \ldots, C_{r_b}^b$ with labels $\sigma_b(1), \ldots, \sigma_b(r_b)$, we add an edge $(v_a^{(X_a, \sigma_a)}, v_b^{(X_b, \sigma_b)}) \in V_a \times V_b$ of weight

$$
w_H(v_a^{(X_a, \sigma_a)}, v_b^{(X_b, \sigma_b)}) := \sum_{i=1}^{r_a} \sum_{j=i+1}^{r_a} \mathbb{1}[\sigma_a(i) \neq \sigma_a(j)] \cdot w(E_G[C_i^a, C_j^a])
$$

$$
+ \sum_{i=1}^{r_a} \sum_{j=1}^{r_b} \mathbb{1}[\sigma_a(i) \neq \sigma_b(j)] \cdot w(E_G[C_i^a, C_j^b]), \quad (3.7)
$$

where $\mathbb{1}$ is the indicator function, taking value 1 if the corresponding statement is true and 0 otherwise. Finally, the algorithm computes the minimum weight triangle in H.

A straightforward counting argument shows that the weight of each triangle $(v_1^{(X_1, \sigma_i)}, v_2^{(X_2, \sigma_2)}, v_3^{(X_3, \sigma_3)})$ in H is exactly the value of the cut in G obtained by merging all components in F^1, F^2, F^3 with the same label together. In particular, for the correct triple S_1, S_2, S_3 for E_T^*, there is a triangle in H whose weight is the cost of the optimal solution, and the algorithm will find it, proving the correctness of the algorithm.

As for running time, the algorithm has an $n^{O(\log k)} 7^k$ overhead for the guesswork of finding the forests (F_1, F_2, F_3) and the correct triple (S_1, S_2, S_3) of subsets of labels. This is followed by computing matrix multiplication on a graph with $k^{O(k)} n^{\lceil 2k/3 \rceil}$ nodes, with edge weights in $[-Wm, Wm]$. Altogether, this takes $k^{O(k)} n^{(2\omega/3 + \varepsilon)k + O(1)} W$ for any $\varepsilon > 0$, proving Theorem 1.2.

4 An $(1 + \varepsilon)$-Approximation Algorithm

We now give a $(1 + \varepsilon)$-approximation algorithm for the k-cut problem that achieves a running time better than both the previous algorithms. Moreover, the ideas we develop here allow us to get a better constant-factor approximation for k-CUT in FPT time. The main theorem we prove is the following:

Theorem 1.3 (Approximation). *For any $\varepsilon > 0$, there is a randomized (combinatorial) algorithm for k-CUT with runtime $(k/\varepsilon)^{O(k)} n^{k + O(1)}$ time on general graphs, that outputs a $(1 + \varepsilon)$-approximate solution with probability $1 - 1/\text{poly}(n)$.*

The runtime of n^k comes from the reduction given in Lemma 2.4 that, given a T-tree—i.e., a tree that crosses the optimal k-cut in $\leq 2k - 2$ edges—alters it to return a collection of $n^{k + O(1)}$ trees that contain at least one tight T-tree T, i.e., one that crosses the optimal k-cut in exactly $k - 1$ edges. How do we find the right $k - 1$ edges to cut, to minimize the total weight of edges in G that go between different components? It is this problem that we give an FPT-PTAS for: we show how to approximate k-CUT on tight T-trees to within a $(1 + \varepsilon)$-factor in time FPT in k:

Lemma 4.1. *Given a tree T satisfying $|E_T(S_1^*, \ldots, S_k^*)| = k - 1$, there is a deterministic $(1 + \varepsilon)$-approximation algorithm for the k-CUT problem with runtime $(k/\varepsilon)^{O(k)} \text{poly}(n)$.*

In this section, we only aim at a running time of $2^{\text{poly}(k/\varepsilon)} \text{poly}(n)$, in an effort to display our main ideas in a more streamlined fashion. To prove the running time required for Lemma 4.1, we defer the additional ideas to §E.

Firstly, we need an estimate for $w(OPT)$, for which a coarse approximation algorithm suffices. Indeed, let M be the value of a 2-approximation algorithm to k-CUT [SV95], so that our algorithm

knows M and $w(OPT) \leq M \leq 2w(OPT)$. Also, recall from (2.2) that given tree T and a set of edges $F \subseteq E(T)$, if $S_1, S_2, \ldots, S_{|F|+1}$ is the vertex partition obtained by deleting edges F from tree T, then $\mathsf{Cut}(F) = E_G(S_1, \ldots, S_{|F|+1})$ denotes the edges in the underlying graph G that cross this partition. We make the following simple observation.

Observation 4.2. *For each of the $k - 1$ edges $e \in E_T[S_1^*, \ldots, S_k^*]$, $\mathsf{Cut}(\{e\}) \leq M$.*

This allows us to contract all edges $e \in E(T)$ with $\mathsf{Cut}(\{e\}) > M$, since they cannot be cut in the optimal solution. Henceforth, assume that every edge $e \in E(T)$ has $\mathsf{Cut}(\{e\}) \leq M$.

4.1 The Game Plan

We want to apply dynamic programming on the tree T, which we root at an arbitrary vertex. The first question to ask is: for each subtree T_v, $v \in V(T)$, what dynamic programming states should we compute and store? As is typical in dynamic programming algorithms, we want our states to be as informative as possible, so that computing new states can be done efficiently. However, we also want a small number of states. Hence, we need to find a balance between a sparse representation of states and a fast way to compute them.

For each vertex $v \in V(T)$ and integer $s \in [k-1]$, we want to store a collection of states for v such that one of them provides information about $E_T[S_1^*, \ldots, S_k^*]$ when restricted to the scope of T_v. One way is the following: for $v, s, \{e_1, \ldots, e_\ell\}$, find the best way to cut s edges in the subtree below v, given that the cut edges closest to v are these ℓ incomparable edges. (We formalize this below in §4.2.) This dynamic program captures the problem exactly. But since ℓ could be close to k (for star-like graphs), there could be roughly n^{k-1} states, which would be no better than brute-force search. Indeed, the reduction from clique shows we do not expect to solve the problem exactly on stars faster than $n^{\omega k/3}$ time; see §A. Hence, we compress the number of states at a loss of a $(1 + \varepsilon)$-approximate factor. Indeed, we represent each "true" state $(u_1, \ldots, u_\ell)$ approximately with a "small" family of *representative* states—i.e., a family with size that is FPT in k.

4.2 The Ideal Dynamic Program

We extend the definition of Cut from a mapping for edge sets given in (2.2) to vertex sets: for a set of vertices $v_1, \ldots, v_\ell \in V(T) - \{v_r\}$ such that e_i is the parent edge of v_i in the rooted tree T, we define

$$\mathsf{Cut}(\{v_1, \ldots, v_\ell\}) := \mathsf{Cut}(\{e_1, \ldots, e_\ell\}). \tag{4.8}$$

For every subset $U \subseteq T_v$ of at most $k - 1$ incomparable vertices and integer $s \in [|U|, k-1]$, define $\mathsf{ExactDP}(v, s, U)$ to be the minimum value of $\mathsf{Cut}(U')$ over all subsets $U' \subseteq T_v$ of size exactly s whose "maximal" vertices are exactly U; in other words, $U' \supseteq U$ and every vertex in $U' - U$ is a descendant of (exactly) one vertex in U.

We now define a recursive statement for $\mathsf{ExactDP}(v, s, U)$. There are two cases, depending on whether $v \in U$ or not. If $v \notin U$, then the following recursive statement is true:

$$\mathsf{ExactDP}(v, s, U) := \min_{U', v_i, s_i, U_i'} \left(\sum_{i=1}^{\ell} \mathsf{ExactDP}(v_i, s_i, U_i') - \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} w(E(T_{U_i'}, T_{U_j'})) \right), \tag{4.9}$$

where the minimum is (i) over all $U' \supseteq U$ whose maximal vertices are exactly U, and $v_1, \ldots, v_\ell$ are the children of v whose subtrees T_{v_i} intersect U', and $U_i' := U' \cap T_{v_i}$; and (ii) over all positive

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integers $s_1, \ldots, s_\ell$ summing to s. Note that the weight in the double summation accounts for the double-counted edges, and is thus subtracted from the expression. If $v \in U$, then the recursion becomes

$$\mathsf{ExactDP}(v, s, U) := \mathsf{Cut}(\{v\}) + \min_{U', v_i, s_i, U_i'} \left(\sum_{i=1}^{\ell} \left(\mathsf{ExactDP}(v_i, s_i, U_i') - w(E(T_{U_i'}, V - T_v)) \right) \right.$$

$$\left. - \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} w(E(T_{U_i'}, T_{U_j'})) \right), \quad (4.10)$$

where the minimum is (ii) over all positive integers $s_1, \ldots, s_\ell$ summing to $s - 1$ this time, and with (i) the same. Again, all subtractions in the expression handle double-counted edges.

Observation 4.3. *Starting with the base states*

1. *$\mathsf{ExactDP}(v, 0, \emptyset) = 0$,*

2. *$\mathsf{ExactDP}(v, 1, \{v\}) = \mathsf{Cut}(\{v\})$, and*

3. *$\mathsf{ExactDP}(v, s, U) = \infty$ for $s \geq 2$, $U \subseteq \{v\}$*

for all leaves $v \in V(T)$, by applying DP with the recursions above, we can compute the correct values of $\mathsf{ExactDP}$.

In order to compress the number of states for this dynamic program we need the notion of *important nodes* and *representatives*, which we define in the next sections. Given a subtree T_v, the important nodes I_v should be thought of as a constant-sized family of consistent "samples" of nodes, such that we can "round" our guesses for which edges to delete to their nearest sample points. These rounded set of states are, loosely speaking, the representatives.

4.3   Important Nodes and Representatives

4.3.1   Important Nodes

Given tree T, some node $v \in V(T)$, we define a set of *important nodes* within $V(T_v)$, the nodes in the subtree below v. One can think of these essentially as an "ε-net" of the nodes in T, in a certain technical sense. For each node $u \in T_v$, assign a vertex weight $\phi_v(u)$ to u equal to the total weight of edges in G that connect u to vertices outside T_v; i.e.,

$$\phi_v(u) := w(E_G(\{u\}, V - T_v)). \quad (4.11)$$

Observe that the total $\phi_v(\cdot)$ weight of all vertices in T_v is exactly $E(T_v, V - T_v)$, which is at most M by Observation 4.2. We want a set $I_v \subseteq V(T_v)$ of *important nodes* for T_v such that

(P1) Every connected component in $T_v - I_v$ has total $\phi_v(\cdot)$ weight at most $W := \mathrm{poly}(\varepsilon/k)M$, whose exact value is determined later.

(P2) The size $|I_v|$ of any important set is at most $4M/W + 1 = \mathrm{poly}(k/\varepsilon)$.

(P3) For each pair of vertices $v, p \in V(T)$ where v is a descendant of p, $I_p \cap T_v \subseteq I_v$.
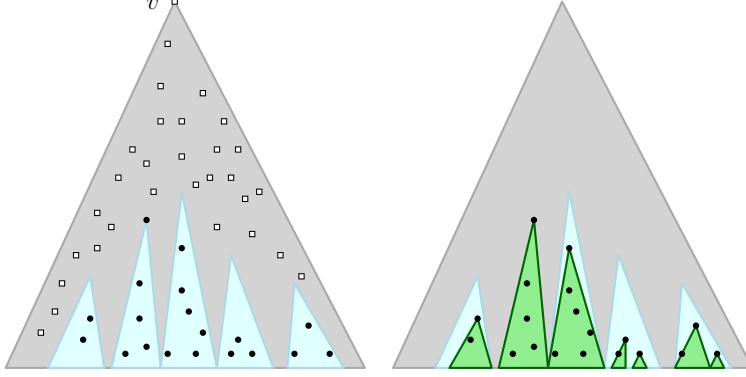
(P4) $v \in I_v$.

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Figure 4.2: The boxes and dots are the set of important nodes I_v. For the downwards-closed blue set S, its representative $R = \sigma_v(S)$ is the set of important nodes (dots) within it. The union of the green subtrees G then form the canonical set for R, and hence $\mathsf{round}_v(S) = G$.

Theorem 4.4 (Important Nodes). *Given T, there is a polynomial-time algorithm to find a set I_v of important nodes for each node $v \in V(T)$, satisfying properties (P1)-(P4) above.*

The approach is to start off with the empty set at the root, and proceed top-down, adding nodes to representative sets as the ϕ-weight increases. The details, along with the proof of why the size remains controlled, are deferred to §D, since they are orthogonal to the present story.

4.3.2 Representative States

The idea of representative states is simple: instead of keeping track of all possibly $\approx n^k$ states in ExactDP above, we "round" each state to a close-by representative, such that there are only a constant number of such representatives, but we incur only a small multiplicative error.

Fix a vertex $v \in V(T)$. We focus on computing representative states for v in the subtree T_v. Recall the notion of ϕ_v-weight from (4.11), and the properties (P1)-(P3) of important nodes $I_v \subseteq T_v$.

Definition 4.5 (Representatives). *The representative of a subset $S \subseteq T_v$ is $\sigma_v(S) := S \cap I_v$, i.e., the set of important nodes within S.*

The function $\sigma_v : 2^{T_v} \to 2^{I_v}$ maps sets to their representatives. Typically, we will deal with representatives R that are downward-closed, that is, if $u \in R$ then all of its descendants that are also in I_v also belong to R. Given a representative, the function σ_v^{-1} is an inverse of sorts, indicating a canonical set to consider for each representative. Figure 4.2 gives a pictorial depiction of this and following definitions.

Definition 4.6. *Given a representative R, define its canonical set $\sigma_v^{-1}(R) := T_R := \cup_{x \in R} T_x$.*

Observe that $\sigma_v(\sigma_v^{-1}(R)) = R$ for any downwards-closed representative R. We now show that any set of the form $S := \bigcup_{i=1}^{\ell} T_{u_i}$ is roughly equal to the canonical set of its representative $\sigma_v^{-1}(\sigma_v(S))$, in the following sense: $E(S, V - T_v) \approx E(\sigma_v^{-1}(\sigma_v(S)), V - T_v)$. For brevity, let us define

$$\boxed{\mathsf{round}_v(S) := \sigma_v^{-1}(\sigma_v(S)).} \tag{4.12}$$

Lemma 4.7. *Fix incomparable vertices $u_1, \ldots, u_\ell \in T_v$, and let $S := \bigcup_{i=1}^{\ell} T_{u_i}$. Then,*

(1) $\mathsf{round}_v(S) \subseteq S$

(2) $w(E(S - \mathsf{round}_v(S), V - T_v)) \leq \ell W$

Proof. Property (1) clearly holds by the definitions of σ_v and σ_v^{-1}. For property (2), suppose that $u \in S - \mathsf{round}_v(S)$. Recall that $T_v - I_v$ is a forest whose connected components each have weight $\leq W$. Then, u must be in a connected component of $T_v - I_v$ containing a vertex u_i, for some $i \in [\ell]$. Overall the vertices $u_1, \ldots, u_\ell$ are each responsible for one subtree of weight $\leq W$. Thus,

$$w(E(S - \mathsf{round}_v(S), V - T_v)) = \sum_{u \in S - \mathsf{round}_v(S)} \phi_v(u) \leq \ell W. \qquad \square$$

Lemma 4.7 suggests that "rounding" each state in ExactDP to its representative should changes the answer only by kW; we now proceed to the precise dynamic program.

4.4 A Smaller Dynamic Program

Recall the "ideal" DP ExactDP from §4.2: we want to approximate its states by a smaller set. The new DP SmallDP will still require $n^{\Omega(k)}$ time to compute, but will closely approximate ExactDP. In the next section we show the final link in the chain: how to compute a good approximation of SmallDP in FPT time.

We define $\mathsf{SmallDP}(v, s, R)$ for each $v \in V(T)$, $s \in [k - 1]$, and $R \subseteq I_v$, which we think of as the "rounded" DP states after approximating sets with their representatives. While we do not always enforce it, imagine that $R \subseteq I_v$ is always downward-closed inside I_v; that is, if $u \in R$, then $T_u \cap I_v \subseteq R$. In a perfect world, the state $\mathsf{SmallDP}(v, s, R)$ would equal the smallest value of $\mathsf{ExactDP}(v, s, U)$ over all U such that $\sigma_v(T_U) = R$. At a high level, we will argue that since representatives only cause a small additive error, $\mathsf{SmallDP}(v, s, R)$ will still be close to the smallest value of $\mathsf{ExactDP}(v, s, U)$.

The base states for SmallDP are the same: $\mathsf{SmallDP}(v, 0, \emptyset) = 0$ and $\mathsf{SmallDP}(v, 1, \{v\}) = \mathsf{Cut}(\{v\})$ for leaves $v \in V(T)$. Before we recursively define the DP states $\mathsf{SmallDP}(v, s, R)$, we first introduce the following lemma, which shows that the expressions for ExactDP from (4.9) and (4.10) do not change much if T_{U_i} is replaced by $\mathsf{round}_{v_i}(T_{U_i})$, and similarly for T_{U_j}. Note that we will be loose on the additive error bounds (such as the bound kW below), since it simplifies the argument and does not affect our running time asymptotically.

Lemma 4.8. *Consider a vertex $v \in V(T)$ and subset of incomparable vertices $U \subseteq T_v$ of size at most k. Let $v_1, \ldots, v_\ell$ be all the children of v whose subtrees T_{v_i} intersect U, and define $U_i := U \cap T_{v_i}$. Then,*

$$0 \leq w(E(T_{U_i}, V - T_v)) - w(E(\mathsf{round}_{v_i}(T_{U_i}), V - T_v)) \leq kW$$

and

$$0 \leq w(E(T_{U_i}, T_{U_j})) - w(E(\mathsf{round}_{v_i}(T_{U_i}), \mathsf{round}_{v_j}(T_{U_j}))) \leq 2kW.$$

Proof. For the first inequalities, it suffices to show that $T_{U_i} \supseteq \mathsf{round}_{v_i}(T_{U_i})$ and

$$w(E(T_{U_i} - \mathsf{round}_{v_i}(T_{U_i}), V - T_v)) \leq kW.$$

They follow immediately from Lemma 4.7 and the facts that $|U_i| \leq k$ and $V - T_v \subseteq V - T_{v_i}$. For the second, observe that

$$E(T_{U_i}, T_{U_j}) - E(\mathsf{round}_{v_i}(T_{U_i}), \mathsf{round}_{v_j}(T_{U_j}))$$

$$\subseteq \left(E(T_{U_i} - \mathsf{round}_{v_i}(T_{U_i}), V - V_{v_i})) \cup (E(T_{U_j} - \mathsf{round}_{v_j}(T_{U_j}), V - V_{v_j})) \right),$$

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which has total weight at most $2kW$, again by Lemma 4.7. □

We now define the SmallDP states. Recall that we have added vertex v to I_v for all v, so again, we have two cases. If $v \notin R$, then the recursion is

$$\mathsf{SmallDP}(v, s, R) := \min_{\ell, v_i, s_i, R_i} \left[\sum_{i=1}^{\ell} \mathsf{SmallDP}(v_i, s_i, R_i) - \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} w(E(\sigma_v^{-1}(R_i), \sigma_v^{-1}(R_j))) \right], \quad (4.13)$$

where the minimum is over all choices of $\ell \in [s]$, distinct children $v_1, \ldots, v_\ell$ of v, positive integers $s_1, \ldots, s_\ell$ whose sum is s, and representatives $R_1, \ldots, R_\ell$ of $v_i, \ldots, v_\ell$ such that $R_i \subseteq I_{v_i}$ and $R_i \cap I_v = R \cap T_{v_i}$ for each $i \in [\ell]$. For the last condition, $R_i \cap I_v = R \cap T_{v_i}$, observe that if $I_{v_i} = I_v \cap T_{v_i}$, then R_i must be $R \cap T_{v_i}$. However, $I_{v_i} \supseteq I_v \cap T_{v_i}$ in general, so we can view R_i as a "refinement" of R inside T_{v_i}.

If $v \in R$, then the s_i satisfy $\sum_i s_i = s - 1$ instead, and the recursion becomes

$$\mathsf{SmallDP}(v, s, R) := \mathsf{Cut}(\{v\}) + \min_{\ell, v_i, s_i, R_i} \left[\sum_{i=1}^{\ell} \left(\mathsf{SmallDP}(v_i, s_i, R_i) - w(E(\sigma_v^{-1}(R_i), V - T_v)) \right) \right.$$
$$\left. - \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} w(E(\sigma_v^{-1}(R_i), \sigma_v^{-1}(R_j))) \right];$$

Lemma 4.9. *For each vertex $v \in V(T)$, integer $s \in [k-1]$, and downward-closed subset $R \subseteq I_v$,*

$$\min_{U: \sigma_v(T_U) = R} \mathsf{ExactDP}(v, s, U) \leq \mathsf{SmallDP}(v, s, R) \leq \min_{U: \sigma_v(T_U) = R} \mathsf{ExactDP}(v, s, U) + (8s - 4)k^2 W.$$

Proof. We apply induction from the leaves of the tree to the root. Observe that if $U \subseteq T_v$ and $R \subseteq I_v$ satisfy $\sigma_v(T_U) = R$, then $v \in U \iff v \in R$, so we can separate the cases $v \in R$ and $v \notin R$.

Case 1: $v \notin R$. To show the first inequality, consider the values ℓ, v_i, s_i, R_i that achieve the minimum of $\mathsf{SmallDP}(v, s, R)$ in (4.13). By induction, for each $i \in [\ell]$, there exists U_i such that $\sigma_{v_i}(T_{U_i}) = R_i$ and $\mathsf{ExactDP}(v_i, s_i, U_i) \leq \mathsf{SmallDP}(v_i, s_i, R_i)$. Recalling the definition of round,

$$w(E(\sigma_v^{-1}(R_i), \sigma_v^{-1}(R_j))) = w(E(\mathsf{round}_{v_i}(T_{U_i}), \mathsf{round}_{v_j}(T_{U_j}))) \geq w(E(T_{U_i}, T_{U_j})),$$

using Lemma 4.8. Now matching the terms in the double summations of (4.9) and (4.13) gives $\mathsf{ExactDP}(v, s, U) \leq \mathsf{SmallDP}(v, s, R)$.

To show the second inequality, consider any U such that $\sigma_v(T_U) = R$ and $\mathsf{ExactDP}(v, s, U)$ is defined. We first consider the case $\ell = 1$ in (4.9): there is one child v_1 such that $U \subseteq T_{v_1}$, then we also have $R \subseteq T_{v_1}$, so by (4.9),

$$\mathsf{ExactDP}(v, s, U) = \mathsf{ExactDP}(v_1, s, U_1).$$

Setting $R_1 := \sigma_{v_1}(T_U)$, we have $R_1 \cap I_v = R = R \cap T_{v_1}$, where we used that R is downward-closed. By (4.13),

$$\mathsf{SmallDP}(v, s, R) \leq \mathsf{SmallDP}(v_1, s, R_1),$$

and by induction,

$$\mathsf{SmallDP}(v, s, R_1) \leq \mathsf{ExactDP}(v, s, U_1),$$

so putting the inequalities together gives $\mathsf{SmallDP}(v, s, R) \leq \mathsf{ExactDP}(v, s, U)$.

Now suppose that $\ell > 1$: let $v_1, \ldots, v_\ell$ be all the children of v whose subtrees T_{v_i} intersect U, and define $U_i := U \cap T_{v_i}$ and $s_i := |U_i|$. Again, we set $R_i := \sigma_{v_i}(T_{U_i})$, which satisfies $R_i \cap I_v = R \cap T_{v_i}$. By induction, for each $i \in [\ell]$,

$$\mathsf{SmallDP}(v, s, R_i) \leq \mathsf{ExactDP}(v, s, U_i) + (8s_i - 4)k^2 W.$$

By Lemma 4.8, the additive error of each of the terms in the double summations of (4.9) and (4.13) is at most $2kW$, and there are $\ell \leq k$ of them, incurring an additive error of at most $2k^2 W$. Altogether, we have

$$
\begin{aligned}
\mathsf{SmallDP}(v, s, R) &\leq \sum_{i=1}^{\ell} \mathsf{SmallDP}(v_i, s_i, R_i) - \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} w(E(\sigma_v^{-1}(R_i), \sigma_v^{-1}(R_j))) \\
&\leq \sum_{i=1}^{\ell} (\mathsf{ExactDP}(v_i, s_i, U_i) + (8s_i - 4)k^2 W) - \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} w(E(T_{U_i}, T_{U_j})) + 2k^2 W \\
&= \mathsf{ExactDP}(v, s, U) + \left(\sum_{i=1}^{\ell} (8s_i - 4) + 2 \right) k^2 W \\
&= \mathsf{ExactDP}(v, s, U) + (8s + 2 - 4\ell)k^2 W.
\end{aligned}
$$

Since $\ell \geq 2$, we have $8s + 2 - 4\ell \leq 8s - 4$, proving $\mathsf{SmallDP}(v, s, R) \leq \mathsf{ExactDP}(v, s, U) + (8s - 4)k^2 W$.

Case 2: $v \in R$. Most of the arguments are similar, so they are omitted. We only show the proof for the case $U = \{v\}$, which implies $R = I_v$ and $v \in R$. In this case, we have $\mathsf{ExactDP}(v, s, U) = \mathsf{SmallDP}(v, s, R) = \mathsf{Cut}(\{v\})$, so they are equal. $\qquad \square$

4.5   A Dynamic Program in FPT Time

In this section we compute a further approximation to $\mathsf{SmallDP}(v, s, R)$, called $\mathsf{PolyDP}(v, s, R)$, for each $v \in V(T)$, $s \in [k-1]$, and $R \subseteq I_v$. The advantage of this approximation is that we can compute it in time $(k/\varepsilon)^{O(k)} \mathrm{poly}(n)$. Our main goal is to show that $\mathsf{PolyDP}(v, s, R) \approx \mathsf{PolyDP}(v, s, R')$ up to a small additive factor.

4.5.1   Partial Vertex Cover

The base states for PolyDP are the same as those in $\mathsf{ExactDP}$ and $\mathsf{SmallDP}$. The computation of each recursive state, on the other hand, involves multiple calls to a problem well-studied in the FPT setting, known as *partial vertex cover*. We define a node-weighted version below.

Definition 4.10 ((Node-Weighted) Partial k-Vertex Cover)**.** *Given a graph G with node weights $\varphi : V \to [0, \infty)$ and edge weights $\psi : E \to [0, \infty)$, the* node-weighted partial k-vertex cover *problem is to find a set $S \subseteq V$ of exactly k nodes that minimizes $\varphi(S) + \psi(\bigcup_{v \in S} E(\{v\}, V - \{v\}))$.*

The following theorem essentially follows from [GLL18], with an improved running time from a more efficient coloring procedure and a separate, trivial case when $\ell = 1$. Its proof is deferred to §D.

Theorem 4.11. *There is an $(1 + \delta)$-approximation algorithm for node-weighted partial ℓ-vertex cover that runs in time $(k/\delta)^{O(k)} \mathrm{poly}(n)$. In addition, if $\ell = 1$, then the algorithm is optimal.*

Recall from (4.13) that, ideally, we want

$$\mathsf{SmallDP}(v, s, R) := \min_{\ell^\dagger, v_i^\dagger, s_i^\dagger, R_i^\dagger} \left[\sum_{i=1}^\ell \mathsf{SmallDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) - \sum_{i=1}^\ell \sum_{j=i+1}^\ell w(E(\sigma_v^{-1}(R_i^\dagger), \sigma_v^{-1}(R_j^\dagger))) \right],$$

over all choices of $\ell^\dagger \in [s]$, distinct children $v_1^\dagger, \ldots, v_{\ell^\dagger}^\dagger$ of v, positive integers $s_1^\dagger, \ldots, s_{\ell^\dagger}^\dagger$ whose sum is s, and representatives $R_1^\dagger, \ldots, R_{\ell^\dagger}^\dagger$ of $v_i^\dagger, \ldots, v_{\ell^\dagger}^\dagger$ such that $R_i^\dagger \subseteq T_{v_i^\dagger}$ and $T_{R_i^\dagger} \cap I_v = R \cap T_{v_i^\dagger}$ for each $i \in [\ell^\dagger]$. Of course, iterating over all possibilities may take $n^{\Omega(k)}$ time, which is where we obtain the speedup via partial vertex cover.

The intuition behind the partial vertex cover algorithm is as follows. For each child u_i of v, suppose we have guessed a value $s_i \in [s]$ and $R_i \in T_{u_i}$. Construct a node-weighted complete graph H' whose nodes are the children u_i of v. Let the weight of each node $u_i \in V(H')$ be

$$\varphi'(u_i) := \mathsf{SmallDP}(u_i, s_i, R_i),$$

and the weight of each edge $(u_i, u_j) \in E(H')$ be

$$\psi'(u_i, u_j) := -w(E(\sigma_v^{-1}(R_i), \sigma_v^{-1}(R_j))).$$

Observe how these weights relate to the expression in (4.13). Now suppose that we got lucky: for each child $v_i^\dagger$ that achieves the minimum in (4.13), we have guessed the correct corresponding values $s_i^\dagger, R_i^\dagger$ for that child, and moreover, we know the correct value $\ell^\dagger$. Then, if we run an algorithm that computes a node subset $S \subseteq V(H'')$ of size exactly $\ell^\dagger$ that minimizes

$$\sum_{u \in S} \varphi'(u) + \sum_{e \in E(H''[S])} \psi'(e),$$

then the optimal solution would return a set S containing exactly the values $v_i^\dagger$ minimizing (4.13).

Of course, the above problem is not an instance of partial vertex cover, and even if it is, computing a solution exactly is $W[1]$-hard. To solve these issues, we will first transform the instance to one of partial vertex cover, and then run a $(1+\delta)$-approximation algorithm instead of an exact one.

4.5.2   Defining PolyDP

The instance for computing $\mathsf{PolyDP}(v, s, R)$ is as follows. First, there is a guessing step, which is repeated multiple times. Let ℓ be a uniformly random value in $[s]$, and for each child u of v, let $s'(u)$ be a uniformly random value in $[s]$ and $R'(u) \subseteq I_u$ be chosen uniformly among those satisfying $R'(u) \subseteq T_u$ and $T_{R'(u)} \cap I_v = R \cap T_u$. We say that our guessing procedure succeeds if $\ell = \ell^\dagger$ and for each $v_i^\dagger$, we have $s'(v_i^\dagger) = s_i^\dagger$ and $R'(v_i^\dagger) = R_i^\dagger$; we make no assumption on the children not in $\{v_1^\dagger, \ldots, v_{\ell^\dagger}^\dagger\}$. Clearly, we succeed with probability at least

$$\frac{1}{s} \cdot \left(\frac{1}{s}\right)^\ell \cdot \left(\frac{1}{2^{|T_{v_i^\dagger}|}}\right)^\ell = 2^{-\mathrm{poly}(k/\varepsilon)}.$$

We will later repeat the procedure $2^{\mathrm{poly}(k/\varepsilon)} \log n$ times so that w.h.p., we succeed at least once. For a given procedure, since there are two cases depending on whether $v \in R$, we split into two cases.

Case 1: $v \notin R$. We construct the auxiliary graph H on which to compute partial vertex cover. Let the children of v be numbered $u_1, \ldots, u_t$. For each child u_i, add a node of weight

$$\varphi(u_i) := \mathsf{PolyDP}(u_i, s'(u_i), R'(u_i)) - \sum_{j=1}^{t} w(E(\sigma_v^{-1}(R'(u_i)), \sigma_v^{-1}(R'(u_j)))) + M. \tag{4.14}$$

Note that

$$\bigcup_{j=1}^{t} E(\sigma_v^{-1}(R'(u_i)), \sigma_v^{-1}(R'(u_j))) \subseteq E(T_{u_i}, V - T_{u_i}),$$

so

$$w(E(\sigma_v^{-1}(R'(u_i)), \sigma_v^{-1}(R'(u_j)))) \leq w(E(T_{u_i}, V - T_{u_i})) \leq M,$$

and the node weight is always nonnegative.

For each pair u_i, u_j, $1 \leq i < j \leq t$, we add an edge (u_i, u_j) in H of weight

$$\psi(u_i, u_j) := w(E(\sigma_v^{-1}(R'(u_i)), \sigma_v^{-1}(R'(u_j)))).$$

If the (exact) optimal solution to PARTIAL VC is S, then that solution has value

$$\varphi(S) + \psi(\bigcup_{v \in S} E(\{v\}, V - \{v\})) = \sum_{u \in S} \mathsf{PolyDP}(u, s'(u), R'(u))$$

$$+ \ell M - \sum_{\{u, u'\} \subseteq S} w(E(\sigma_v^{-1}(R'(u)), \sigma_v^{-1}(R'(u')))), \quad (4.15)$$

since that all edges (u_i, u_j) such that $u_i \in S$, $u_j \notin S$ get their weight canceled by the corresponding term in the negative summation of $\varphi(u_i)$. Observe that, aside from the additive ℓM and $\mathsf{SmallDP}$ being replaced by PolyDP, the solution value is exactly the expression in the minimum from (4.13) with values ℓ and $u, s'(u), R'(u)$ for $u \in S$. Thus, if the guessing is successful, then the optimal value of PARTIAL VC is at most

$$\mathsf{PolyDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) - \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell} w(E(\sigma_v^{-1}(R_i^\dagger), \sigma_v^{-1}(R_j^\dagger))) + \ell M. \tag{4.16}$$

We now run $(1 + \delta)$-approximate partial ℓ-vertex cover on H, for some $\delta := \mathrm{poly}(\varepsilon/k)$, whose exact value is determined later. Because of this approximation, we suffer a small loss.

The algorithm repeats the guessing and partial vertex cover computation $2^{\mathrm{poly}(k/\varepsilon)} \log n$ times. On each iteration, the algorithm writes down the value of the PARTIAL VC minus ℓM, called the *score* of that iteration. Finally, the algorithm sets $\mathsf{SmallDP}(v, s, R)$ as the value of the best score found.

Case 2: $v \in R$. Again, the nodes of H consist of the children of v, numbered $u_1, \ldots, u_t$. For each child u_i,

$$\varphi(u_i) := \mathsf{PolyDP}(u_i, s'(u_i), R'(u_i)) - w(E(\sigma_v^{-1}(R_i), V - T_v))$$

$$- \sum_{j=1}^{t} w(E(\sigma_v^{-1}(R'(u_i)), \sigma_v^{-1}(R'(u_j)))) + 2M,$$

that is, the value (4.14) from Case 1 with $-w(E(\sigma_v^{-1}(R_i), V - T_v)) + M$ added on; again, we can show that the node weights are nonnegative. The edge weights $\psi(u_i, u_j)$ of H are the same as in Case 1. Similarly, the algorithm repeats the procedure $2^{\mathrm{poly}(k/\varepsilon)} \log n$ times and on each iteration, writes down the value of the PARTIAL VC plus $\mathsf{Cut}(\{v\}) - 2\ell M$.

4.5.3 The Analysis

The next lemma argues that in both cases above, w.h.p., $\mathsf{PolyDP}(v, s, R) \approx \mathsf{SmallDP}(v, s, R)$.

Lemma 4.12. *W.h.p., for each vertex $v \in V(T)$, integer $s \in [k-1]$, and downward-closed subset $R \subseteq I_v$,*
$$\mathsf{SmallDP}(v, s, R) \leq \mathsf{PolyDP}(v, s, R) \leq \mathsf{SmallDP}(v, s, R) + (8s - 4)\delta k M.$$

Proof. We apply induction from the leaves of the tree to the root. We only prove the case $v \notin R$, since the other case is almost identical.

The first inequality essentially follows by induction and (4.15). For the second inequality, we split into the cases $\ell^\dagger = 1$ and $\ell^\dagger \geq 2$.

If $\ell^\dagger = 1$, then the single $\mathsf{PolyDP}(v_1^\dagger, s_1^\dagger, R_1^\dagger)$ term introduces additive error $\leq (8s - 4)\delta k M$ by induction, and the PARTIAL VC algorithm outputs the optimal solution, so by (4.16), the score of a successful iteration is at most $\mathsf{SmallDP}(v, s, R) + (8s - 4)\delta k M$. Since the algorithm takes the best score over all iterations, the second inequality holds.

If $\ell^\dagger \geq 2$, then by (4.16), the score of a successful iteration is at most

$$(1 + \delta)\left(\sum_{i=1}^{\ell^\dagger} \mathsf{PolyDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) - \sum_{i=1}^{\ell^\dagger}\sum_{j=i+1}^{\ell^\dagger} w(E(\sigma_v^{-1}(R_i^\dagger), \sigma_v^{-1}(R_j^\dagger))) + \ell M\right) - \ell M.$$

We may assume that $\mathsf{PolyDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) \leq M$, since anything larger will not result in a solution that beats M, the value of the 2-approximation algorithm. Therefore, the score is upper bounded by

$$\left(\sum_{i=1}^{\ell^\dagger} \mathsf{PolyDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) - \sum_{i=1}^{\ell^\dagger}\sum_{j=i+1}^{\ell^\dagger} w(E(\sigma_v^{-1}(R_i^\dagger), \sigma_v^{-1}(R_j^\dagger)))\right) + \delta\ell M + (1 + \delta)\ell M - \ell M.$$

By induction,
$$\mathsf{PolyDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) \leq \mathsf{SmallDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) - (8s_i^\dagger - 4)\delta k M.$$

Thus,

$$\mathsf{PolyDP}(v, s, R) \leq \sum_{i=1}^{\ell^\dagger} \mathsf{PolyDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) - \sum_{i=1}^{\ell^\dagger}\sum_{j=i+1}^{\ell^\dagger} w(E(\sigma_v^{-1}(R_i^\dagger), \sigma_v^{-1}(R_j^\dagger))) + 2\delta\ell M$$

$$\leq \sum_{i=1}^{\ell^\dagger}\left(\mathsf{SmallDP}(v_i^\dagger, s_i^\dagger, R_i^\dagger) - (8s_i^\dagger - 4)\delta k M\right) - \sum_{i=1}^{\ell^\dagger}\sum_{j=i+1}^{\ell^\dagger} w(E(\sigma_v^{-1}(R_i^\dagger), \sigma_v^{-1}(R_j^\dagger))) + 2\delta k M$$

$$= \mathsf{SmallDP}(v, s, R) - \sum_{i=1}^{\ell^\dagger}(8s_i^\dagger - 4)\delta k M + 2\delta k M$$

$$= \mathsf{SmallDP}(v, s, R) + (8s + 2 - 4\ell)\delta k M.$$

Since $\ell \geq 2$, we have $8s + 2 - 4\ell \leq 8s - 4$, proving $\mathsf{PolyDP}(v, s, R) \leq \mathsf{SmallDP}(v, s, R) + (8s - 4)\delta k M$. $\quad\square$

Parameters. There are two free parameters, $W = \text{poly}(\varepsilon/k)M$ from §4.3 and $\delta = \text{poly}(\varepsilon/k)$ from §4.5.1. By Lemma 4.9 and Lemma 4.12,

$$\mathsf{PolyDP}(v, s, R) \leq \mathsf{SmallDP}(v, s, R) + (8s - 4)\delta kM$$
$$\leq \mathsf{ExactDP}(v, s, R) + (8s - 4)\delta kM + (8s - 4)k^2 W$$
$$\leq \mathsf{ExactDP}(v, s, R) + 8\delta k^2 M + 8k^3 W.$$

Thus, setting $W := \varepsilon/(32k^3)M$ and $\delta := \varepsilon/(32k^2)$ gives

$$\mathsf{PolyDP}(v, s, R) \leq \mathsf{ExactDP}(v, s, R) + (\varepsilon/2)M$$

for all $v \in V(T)$, $s \in [k-1]$, $R \subseteq T_v$. In particular, for the value $R^* \subseteq T_{v_r}$ such that

$$\mathsf{ExactDP}(v_r, k - 1, R^*) = w(OPT),$$

we have

$$\mathsf{PolyDP}(v_r, k - 1, R^*) \leq \mathsf{ExactDP}(v_r, k - 1, R^*) + (\varepsilon/2)M$$
$$\leq \mathsf{ExactDP}(v_r, k - 1, R^*) + \varepsilon \cdot w(OPT)$$
$$= (1 + \varepsilon)w(OPT),$$

where we have used that $w(OPT) \geq M/2$, since M is the value of a 2-approximation algorithm. This concludes the $(1 + \varepsilon)$-approximation algorithm, which runs in time $2^{\text{poly}(k/\varepsilon)}\text{poly}(n)$.

With some more work, we can improve the runtime to $(k/\varepsilon)^{O(k)}\text{poly}(n)$ and make it deterministic; we defer the details to §E and §F, respectively.

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A Lower Bounds

We give the details of the simple relationship to the k-Clique problem.

Claim 1.5 (Relationship to Clique). *Any exact algorithm for the k-Cut problem for graphs with edge weights in $[n^2]$ can solve the k-Clique problem in the same runtime. Hence, assuming k-Clique cannot be solved in faster than $n^{\omega k/3}$ time, the same lower bound holds for the k-cut problem.*

Proof. Given a graph $G = (V, E)$ that is an instance of k-Clique, construct a graph G' as follows. Take the graph G, add in a new vertex s with edges (s, v) to all vertices $v \in V$, where the edge (s, v) has weight $n^2 - \deg_G(v)$. It is easy to see that the optimal $(k+1)$-cut in the graph consists of k parts containing singleton vertices $\{v_1\}, \ldots, \{v_k\}$ from V, and one part containing s and the rest of the vertices in V. Moreover, the weight of this $(k+1)$-cut is $n^2 k - \frac{1}{2} E_G(\{v_1\}, \ldots, \{v_k\})$. Hence the optimal k-cut will pick out a k-clique if one exists. □

In the above reduction, observe that a star rooted at the s is a tight T-tree with respect to the optimal k-cut. Hence solving finding the optimal k edges to delete given this tight T-tree find the k-clique.

B An 1.81-FPT Approximation Algorithm

In this section, we show an 1.81-FPT approximation algorithm for k-Cut, proving Theorem 1.4. The best approximation factor was $2 - \delta$ for $\delta \approx 0.0003$ [GLL18]. Our improvement is based on our $(1 + \varepsilon)$-approximation algorithm given a tight T-tree. The tight tree was also used in [GLL18] in a special case of the k-cut called Laminar k-cut, but the previous algorithm only gave a $(2 - \delta)$-approximation for a small constant $\delta > 0$. Simply plugging in our improved algorithm as a subroutine and setting parameters more carefully gives the improvement, and also simplifies the previous proof. We now explain how we achieve a better approximation for the general case, slightly modifying the parameters in the proof of [GLL18, Theorem 3.1].

Let $\{S_1^*, \ldots, S_k^*\}$ denote the partition of V given by the optimal k-cut, with $w(\partial S_1^*) \leq \ldots \leq w(\partial S_k^*)$. At a high level, the algorithm iteratively increases the number of components by cutting the minimum cut or min-4-cut. Let k' be the current number of connected components and $S_1, \ldots, S_{k'}$ be the components maintained by the algorithm. In [GLL18], $\mathfrak{a}$ is defined to be the smallest value of k' when both the weight of the min-cut, as well as one-third of the weight of the min-4-cut, becomes bigger than $w(\partial S_1^*)(1 - \varepsilon_1/3)$, for some $\varepsilon_1 > 0$. Moreover, $\mathfrak{b} \in [k]$ is the smallest number such that $w(\partial S_{\mathfrak{b}}^*) > w(\partial S_1^*)(1 + \varepsilon_1/3)$. Let us change these two hard-coded thresholds $(1 - \varepsilon_1/3)$ and $(1 + \varepsilon_1/3)$ to $(1 - \alpha)$ and $(1 + \beta)$ respectively for $\alpha, \beta > 0$ to be determined later. Let $S_{\geq \mathfrak{b}}^* = \cup_{i=\mathfrak{b}}^k S^*$ be the union of the components with "large" boundary.

Consider the iteration for our algorithm when G has been broken into $\mathfrak{a}$ components $S_1, \ldots, S_{\mathfrak{a}}$. By the choice of the threshold $\mathfrak{a}$, both the weight of the mincut, and the weight of the min-4-cut divided by 3, are now bigger than $w(\partial S_1^*)(1 - \alpha)$. If two cuts of weight at most $(1 + \beta)w(\partial S_1^*)$ cross in G, they will result a 4-cut in G' of cost at most $2(1 + \beta)w(\partial S_1^*)$, which is contradiction to the choice of $\mathfrak{a}$ if

$$2(1 + \beta)w(\partial S_1^*) < 3(1 - \alpha)w(\partial S_1^*) \iff \frac{1 + \beta}{1 - \alpha} < 1.5.$$

Therefore, for each S_i, two cuts of size $(1 + \beta)w(\partial S_1^*)$ also do not cross in $G[S_i]$. Now fix i such that the component S_i intersects at least two of $S_1^*, \ldots, S_{\mathfrak{b}-1}^*, S_{\geq \mathfrak{b}}^*$; say it intersects $r_i \geq 2$ of them. Now we would like to take an r_i-cut within $G[S_i]$ with $\{S_1^* \cap S_i, \ldots, S_{\mathfrak{b}-1}^* \cap S_i, S_{\geq \mathfrak{b}}^* \cap S_i\}$ as the desired solution. Moreover, in this r_i-cut instance on $G[S_i]$, the laminar structure of the cuts of weight

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$(1+\beta)w(\partial S_1^*)$ can be encoded as a cut-tree (whose edges correspond to these non-crossing cuts in $G[S_i]$, and we want to cut exactly $r_i - 1$ of them. This gives a tight T-tree, and we can use the algorithm from §4 to approximate the r_i-cut problem within $(1 + \varepsilon_0)$-factor of the desired solution. Indeed, we can do this for arbitrarily small $\varepsilon_0 > 0$, not depending on any other parameter here. This corresponds to the LAMINAR r_i-CUT problem considered in [GLL18], but here we have a much more relaxed requirement (i.e., existence of a tight tree).

Let us sketch the high-level idea of the rest of the proof for those who don't remember details of [GLL18]. In the paper, we assume that the min-cut always remains smaller than $M := w(\partial S_1^*)$, else we can branch on having found one component. The cost of the first $\mathfrak{a}$ cuts is $\mathfrak{a}(1 - \alpha)M$, by the choice of $\mathfrak{a}$. Then guessing r_i for each component $G[S_i]$ and running the FPT-PTAS for the tight T-tree instance gives us $(1 + \varepsilon_0)OPT$. Finally, we may have only $\mathfrak{a} + \mathfrak{b}$ components, so we pick $(k - (\mathfrak{a} + \mathfrak{b}) - 1)$ other min-cuts, each of cost at most M. Balancing the parameters now gives us the proof.

Technical details. The rest of the analysis exactly works as the original proof, where $(1 - \varepsilon_1/3)$ is replaced by $(1 - \alpha)$ and $(1 + \varepsilon_1/3)$ is replaced by $(1 + \beta)$. (There is no ε_1 in the proof.) The equations (1), (2), and (5) in [GLL18] that determine the parameters become

$$2\alpha\varepsilon_4 \geq \varepsilon_3, \qquad (1 + \beta\varepsilon_5)(2 - \varepsilon_3) \geq 2, \qquad \varepsilon_3 \leq 1 - 2\varepsilon_5,$$

which is equivalent to

$$\varepsilon_3 = \min(2\alpha\varepsilon_4, \frac{2\beta\varepsilon_5}{1 + \beta\varepsilon_5}, 1 - 2\varepsilon_5).$$

Setting $\alpha \approx 0.1588, \beta \approx 0.2618, \varepsilon_4 \approx 0.5988$, and $\varepsilon_5 \approx 0.4012$ gives $\varepsilon_3 \approx 0.1901$, which gives us an $(2 - \varepsilon_3) \approx 1.81$-approximation.

As for running time, there is the same $2^{O(k^2)}\mathrm{poly}(n)$ multiplicative overhead in the reduction to LAMINAR k-CUT in [GLL18], which is the dominant factor in the overall runtime.

C Time and Space Requirements for Section 2

In this section, we show how to improve the runtime for the algorithm in Section 2, to complete the proof of Theorem 1.1. We then talk about a bounded-space algorithm.

C.1 Improvements to the Runtime

First, we explain how to replace the factor of m in the running time with a potentially smaller factor of n. A closer examination of the weights $w_H(v_a^{F^a}, v_b^{F^b})$ shows that they are all nonnegative, and that any term $w(\mathsf{State}(e_i^a, \sigma(R_a + i)))$ used to compute an edge $w_H(v_a^{F^a}, v_b^{F^b})$ also lower bounds the weight of any triangle containing that edge. Moreover, the minimum k-CUT has value $\leq knW$, since isolating $k - 1$ vertices is always a valid k-CUT. Therefore, in each graph H that we construct, we can ignore any edge with weight $> knW$, since they can only result in solutions with value $> knW$. Now that the weights are in the range $[0, knW]$, we can apply [WW10] with $M := knW$ to obtain the desired running time.

Another source of improvement occurs when $k \neq 2 \bmod 3$, giving some slack from the ceiling in $\lceil (k - 2)/3 \rceil$ when computing the states $\mathsf{State}(e, s)$. Let $r := k - 2 \bmod 3$; note that $r \in \{0, 1, 2\}$. In this case, it is more beneficial to guess r edges to delete using brute force, and then apply the algorithm of §2.3.2 on the remaining $k - r$ edges to delete. Since T is a tight T-tree, we claim that there exist r edges in $E_T(S_1^*, \ldots, S_k^*)$ such that if they are removed from T, then r of the $r + 1$

connected components are exactly equal to some r elements in $\mathcal{S}^* = \{S_1^*, \dots, S_k^*\}$. Indeed, consider the process of rooting the tight T-tree T at an arbitrary vertex and, for r iterations, removing an edge in $E_T(S_1^*, \dots, S_k^*)$ of maximal depth. In each iteration, since the subtree below the removed edge has no more edges in $E_T(S_1^*, \dots, S_k^*)$, it must be an element in $\mathcal{S}^*$. Note that this process is not part of our algorithm; we provide it only to prove existence.

The algorithm tries all $O(n^r)$ edges to remove, and for each one, guesses which r of the $r+1$ connected components are in $\mathcal{S}^*$. If we guess everything correctly, then we can run the algorithm of §2.3.2 to delete the other $k - 2 - r$ edges from the last component T'. That is, the input graph now becomes $G[V(T')]$ and the tree T'.

With these two improvements, the new running time for each $\mathsf{State}(e, s)$ computation becomes $\widetilde{O}(k^{O(k)} n^{\lfloor (k-2)/3 \rfloor \omega + 1 + (k-2) \bmod 3} W)$. Similarly, by setting $r := k - 1 \bmod 3$, the final k-CUT value can be computed in $\widetilde{O}(k^{O(k)} n^{\lfloor (k-1)/3 \rfloor \omega + 1 + (k-1) \bmod 3} W)$ time. Again, this is dominated by the running time $\widetilde{O}(k^{O(k)} n^{\lfloor (k-2)/3 \rfloor \omega + 2 + (k-2) \bmod 3} W)$ of computing all the states, attaining the bound in Lemma 2.10.

C.2 A Polynomial Space Algorithm

The second improvement idea also leads to a polynomial space algorithm. For a given constant c, apply the idea with $r := k - 1 - c$, so that the algorithm takes space $n^{O(k-1-r)} = n^{O(c)}$ and time

$$\widetilde{O}(k^{O(k)} n^{\lceil (k-1-r)/3 \rceil \omega + 1 + r} W) = \widetilde{O}(k^{O(k)} n^{(\omega/3)c + 2 + (k-1-c)}) = \widetilde{O}(k^{O(k)} n^{k+1-0.2c}),$$

using $\omega < 2.3727$. Since there are $\widetilde{O}(k^{O(k)} n^{k-1})$ trees to consider by Lemma 2.4, the total running time is $\widetilde{O}(k^{O(k)} n^{2k-0.2c})$.

D Proofs from Section 4

Proof of Theorem 4.4. The algorithm proceeds top-down, starting with constructing I_{v_r} for the root v_r and going downwards. For root v_r, the singleton $I_{v_r} = \{v_r\}$ satisfies constraints (P1) and (P2). Now we proceed top-down in the tree.

Consider a child v with parent p. Having already defined I_p we start off with the set $I_p \cap T_v$ as a candidate for I_v. However, this may not satisfy (P1), since the ϕ_v-weight of a node $u \in T_v$ can be higher than its ϕ_p-weight (but not lower), so the ϕ_v-weight of a component in $T_v - (I_p \cap T_v)$ may exceed W. We fix it as follows: for each component C in $T_v - (I_p \cap T_v)$ has ϕ_v-weight more than W, we run the following greedy bottom-up algorithm inside that component, producing additional important nodes.

The algorithm is the natural one: we greedily pick the lowest vertex u in C with subtree ϕ_v-weight more than $W/2$, mark it as important, remove its subtree, and repeat until the remainder has ϕ_v-weight at most $W/2$. More formally, view the component C as a tree with the same ancestor-descendant relationship as in T. For any node $x \in V(C)$ let C_x be the subtree of C rooted at node x. The greedy algorithm maintains a set I of newly picked important nodes in C, and iteratively adds to I the node $u \in V(C) - \bigcup_{x \in I} V(C_x)$ of maximal depth that satisfies $\phi_v(V(C_u) - \bigcup_{x \in I} V(C_x)) \geq W/2$, until such a vertex no longer exists. It is clear that every connected component in $C - I$ has total ϕ_v-weight at most $W/2$.

Let I_v be $I_p \cap T_v$, plus these newly chosen important nodes. By construction, each subtree in $T_v - I_v$ has ϕ_v-weight at most $W/2$; this satisfies property (P1). We prove property (P2) next.

Lemma D.1. *For all $v \in V$, $|I_v| \leq 4M/W$.*

Proof. For any vertex $v \in V$, the important node $b \in I_v$ is *in charge* of component C, if C contains some child of b. The important node b is *active* (w.r.t. v) if the total ϕ_v-weight of b, unioned with the components C it is in charge of, is at least $W/2$. Otherwise b is called *retired* (w.r.t. v). There are $\leq 2M/W$ active vertices, since each one is in charge of a disjoint set of components of weight $\geq W/2$ and the total weight of T_v is $\leq M$. We now bound the number of retired vertices.

Consider the highest ancestor u of node v such that $b \in I_u$. Let b be in charge (with respect to u) of components $C_1, C_2, \ldots, C_j$. By construction the total weight $\phi_u(b) + \sum_{j' \leq j} \phi_u(C_{j'}) \geq W/2$, and hence u lies *strictly* above v. Hence b is an active important node with respect to this ancestor u. As we walk down the u-v path (with the ϕ-weights non-decreasing), consider the first node a such that when building the important set I_a, some vertex in $\cup_{j' \leq j} C_{j'}$ (say in C_i) is chosen as an important node. (Clearly a is either v or an ancestor of v.) At this point the ϕ_a-weight in C_i must have increased to W, because of new edges from nodes in C_i to ancestors of a, of total edge-weight $\geq W/2$. Now we can "charge" the retirement of b to these edges. It is clear edges are charged this way only by the important node b which happened to be in charge of the current component they are incident to. Moreover, the total weight of such edges is at most M, since they all go from within T_v to outside it; so the number of retired nodes is also at most $2M/W$. $\diamond$

The containment property (P3) is true by construction. To get property (P4) we simply add $\{v\}$ to I_v, for each v. This increases the size by 1, and completes the proof. $\square$

Proof of Theorem 4.11. We follow the same strategy as Theorem 5.1 in [GLL18], except with slightly different coloring probabilities. In their Lemma 5.2, instead of coloring each node red and blue with probability $\frac{1}{2}$ each, we color them red with probability $1/\tau$ and blue with probability $(1 - 1/\tau)$, for $\tau := \text{poly}(k/\delta)$ as defined there. This way, following their definition, the probability that all the nodes in S^* are colored red, and all the nodes in $N(S^*) \setminus S$ are colored blue is $(1/\tau)^k (1 - 1/\tau)^\tau = (\delta/k)^{O(k)}$. We thus repeat this step $(k/\delta)^{O(k)} \log n$ times, giving the desired running time.

Finally, the case $\ell = 1$ can be trivially solved optimally, since the solution is simply the minimum weight node.

We remark that this algorithm is derandomized in §F. $\square$

E   Running Time Improvements for Section 4

Here, we show that the running time of the dynamic program from §4.5.2 can be sped up to $(k/\varepsilon)^{O(k)}\text{poly}(n)$. The main idea is that for each vertex $v \in V(T)$, there are only $(k/\varepsilon)^{O(k)}$ many (downward-closed) subsets $R \subseteq I_v$ that need to be considered, which is much smaller than the trivial $2^{\text{poly}(k/\varepsilon)}$ bound as stated before.

For a given $v \in V(T)$ and $s \in [k-1]$, we say that a representative $R \subseteq I_v$ is (v, s)-*relevant* if there exists a set $U \subseteq T_v$ of s incomparable vertices such that $\sigma_v(T_U) = R$. Intuitively, the only values of $\mathsf{SmallDP}(v, s, R)$ that "matter" are the ones where R is (v, s)-relevant. Formally, it can be shown, by analyzing the recursive definition of $\mathsf{SmallDP}$, that $\mathsf{SmallDP}(v, s, R) < \infty$ if and only if R is (v, s)-relevant; here, we assume that $\mathsf{SmallDP}(v, s, R)$ becomes ∞ if there do not exist ℓ, v_i, s_i, R_i in (4.13) that satisfy the necessary constraints, or if every satisfying ℓ, v_i, s_i, R_i has $\mathsf{SmallDP}(v_i, s_i, R_i) = \infty$ for some $i \in [\ell]$. It follows that in our DP algorithm, we only need to compute $\mathsf{PolyDP}(v, s, R)$ for (v, s)-relevant R.

Below, we will prove that there are $(k/\varepsilon)^{O(k)}$ many representatives that are (v, s)-relevant, and we can enumerate them, plus possibly some more representatives that are not (v, s)-relevant, in $(k/\varepsilon)^{O(k)}$ time. Therefore, the DP algorithm can perform this enumeration and compute $\mathsf{PolyDP}(v, s, R)$ for

only these R. Moreover, in the guessing step in §4.5.2, for each child u of v, we only need to choose a random $R'(u) \subseteq I_u$ that is relevant in T_u, so the success probability increases to $(k/\varepsilon)^{O(k)}$. Overall, the running time of the DP algorithm becomes $(k/\varepsilon)^{O(k)}\text{poly}(n)$.

Lemma E.1. *For a fixed vertex $v \in V(T)$ and integer $s \in [k-1]$, there are $(k/\varepsilon)^{O(s)}$ many (v, s)-relevant representatives, and we can enumerate a superset of all (v, s)-relevant representatives in $(k/\varepsilon)^{O(s)}$ time.*

Proof. We first prove the statement when $s = 1$. We use the concept of VC dimension, defined below.

Definition E.2. *Let X be a set of elements, called the universe. A family $\mathcal{F}$ of subsets of X has VC dimension d if d is the largest possible size of a subset $S \subseteq X$ satisfying the following property: for any subset $S' \subseteq S$, there exists subset $F \in \mathcal{F}$ such that $S \cap F = S'$.*

We use two properties of VC dimension. The first is that if $\mathcal{F}$ if a family of subsets of X of VC dimension d and $Y \subseteq X$, then the family $\mathcal{F}|_Y := \{F \cap Y : F \in \mathcal{F}\}$ has VC dimension $\leq d$. The second property is a classic result on VC dimension:

Theorem E.3 (Sauer's lemma)**.** *Let X be a set of elements. If a family $\mathcal{F}$ of subsets of X of VC dimension d, then $|\mathcal{F}| = O(|X|^d)$.*

We now bound the VC dimension of the family of all subtrees.

Claim E.4. *For a fixed vertex $v \in V(T)$, let T_v be the universe. The family $\mathcal{F}$ of subtrees T_u for all $u \in T_v$ has VC dimension at most 2.*

Proof. Suppose for contradiction that there is a set $S \subseteq T_v$ of size 3 such that for each subset $S' \subseteq S$, there exists a subtree $T_u \subseteq T_v$ such that $S \cap T_u = S'$. Let $S = \{x, y, z\}$, and assume without loss of generality that the lowest common ancestor of x and y is either equal to or a descendant of the lowest common ancestor of x and z. Then, if a subtree T_u contains both x and z, then it must contain y, so for the subset $S' := \{x, z\}$, it is impossible that $S \cap T_u = S'$, a contradiction. ◇

By Theorem E.3, the family $\mathcal{F}_v := \{T_u : u \in T_v\}$ has VC dimension ≤ 2. Therefore, the family $\mathcal{F}_v|_{I_v}$ of subsets of I_v, which is precisely the set of $(v, 1)$-relevant representatives, also has VC dimension ≤ 2. Thus, there are $O(|I_v|^2) = (k/\varepsilon)^{O(1)}$ many $(v, 1)$-relevant representatives, proving the case $s = 1$. Moreover, we can enumerate over all of them in $(k/\varepsilon)^{O(1)}\text{poly}(n)$ time.

For general s, let $U \subseteq V$ be a subset s incomparable vertices. Since $T_U = \bigcup_{u \in U} T_u$ is a union of s subsets in $\mathcal{F}_v$, it follows that $T_U \cap I_v$ is a union of s subsets of $\mathcal{F}_v|_{I_v}$. Since there are at most $(\mathcal{F}_v|_{I_v})^s$ many possible such unions, the number of (v, s)-relevant representatives is $(k/\varepsilon)^{O(s)}$. Furthermore, to enumerate a superset of them, we can first compute $\mathcal{F}_v|_{I_v}$ and then enumerate over all unions of s subsets, taking $(k/\varepsilon)^{O(s)}\text{poly}(n)$ time. This concludes the proof. □

With this speedup, our running time matches the one promised by Theorem 1.3.

F   Derandomization

The guessing part can be derandomized in the same way randomized FPT algorithms are typically derandomized: through efficient constructions of set families. The main impact of derandomization is the deterministic runtime of Theorem 4.1, which itself leads to the deterministic runtime of Theorem 1.4.

We first derandomize the occasions when the algorithm has to guess multiple values in the range $[q]$ for some $q := \text{poly}(k/\varepsilon)$. More precisely, the algorithm guesses a value in $[q]$ for each index $i \in I$, such that for an unknown set of indices $I^* \subseteq I$ of size $\leq k$, we must guess the value correctly for each index $i \in I^*$. This occurs during the matrix multiplication algorithm in §2.3.2 and the computation of PolyDP in §4.5.2. We derandomize this procedure using (n, k, q)-universal sets as introduced in [MPR$^+$13].

Definition F.1 (Definition 3.1 of [MPR$^+$13]). *An (n, k, q)-universal set is a set of vectors $V \subseteq [q]^n$ such that for any index set $S \in \binom{[n]}{k}$, the projection of V on S contains all possible q^k configurations.*

Note that the traditional notion of (n, k)-universal sets is precisely the $(n, k, 2)$-universal sets.

Lemma F.2 (Theorem 3.2 of [MPR$^+$13]). *An (n, k, q)-universal set of cardinality $q^k k^{O(\log k)} \log^2 n$ can be constructed deterministically in time $O(q^k k^{O(\log k)} n \log^2 n)$.*

Therefore, we can construct an (n, k, q)-universal set in time $O(q^k k^{O(\log k)} n \log^2 n) = O((k/\varepsilon)^{O(k)} \text{poly}(n))$ and run the inner procedure on each element in the set.

We now derandomize the PARTIAL VC algorithm, making the entire algorithm of Lemma 4.1 deterministic. To do so, we use the following special construction of set families:

Lemma F.3 (Lemma I.1 of [CCH$^+$16]). *Given a set U of size n, and integers $0 \leq a, b \leq n$, one can in (deterministic) $O(2^{O(\min(a,b)) \log(a+b)} n \log n)$ time construct a family $\mathcal{F}$ of at most $O(2^{O(\min(a,b) \log(a+b))} \log n)$ subsets of U, such that the following holds: for any sets $A, B \subseteq U$, $A \cap B = \emptyset$, $|A| \leq a$, $|B| \leq b$, there exists a set $S \in \mathcal{F}$ with $A \subseteq S$ and $B \cap S = \emptyset$.*

Following the proof of Theorem 4.11 in §D, we set U to be the nodes in the PARTIAL VC instance, and parameters $a := k$ and $b := \tau = \text{poly}(k/\varepsilon)$. We construct a set family $\mathcal{F}$ of size $O(2^{O(k \log(\text{poly}(k/\varepsilon)))} \log n) = (k/\varepsilon)^{O(k)} \log n$ such that there exists a set $F \in \mathcal{F}$ with $S^* \subseteq F$ and $(N(S^*) \setminus S) \cap F = \emptyset$. Therefore, for each set $F \in \mathcal{F}$, we color all nodes in F red and all other nodes blue, and proceed with the algorithm.