30820-Communication Systems

Week 12-14 – Lecture 31-39

(Ref: Chapter 6 of text book)

SAMPLING AND ANALOG TO DIGITAL CONVERSION

Contents

- Sampling Theorem
- Pulse Code Modulation
- Differential Pulse Code Modulation (DPCM)
- Delta Modulation

Sampling Theorem

- Why do we need sampling?

- Analog signals can be digitized though sampling and quantization.
  - A/D Converter

- In A/D converter, the sampling rate must be large enough to permit the analog signal to be reconstructed from the samples with sufficient accuracy.

- The **sampling theorem** defines the basis for determining the proper (lossless) sampling rate for a given signal.

Sampling Theorem

- A consider a signal $g(t)$ whose spectrum is band limited to $B\ Hz$, that is

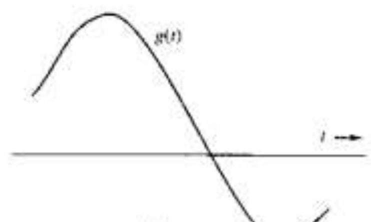
$$G(f) = 0 \qquad \text{for } |f| > B$$

- We can reconstruct the signal exactly (without any error) from its discrete time samples (samples taken uniformly at a rate of R samples per second).
  - Condition for R?
  - The condition is that $R > 2B$

- In other words, the minimum sampling frequency for perfect signal recovery is $f_s = 2B\ Hz$

Sampling Theorem

- Consider a signal $g(t)$ whose spectrum is band limited to $B\ Hz$



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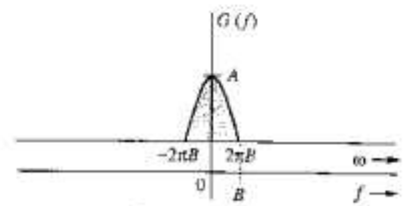
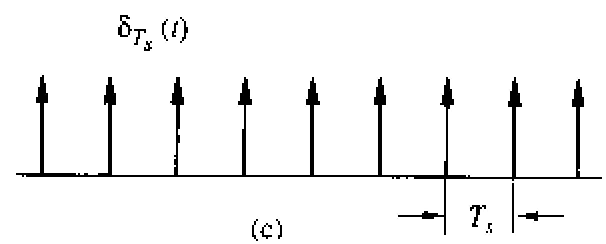


(b)

- Sampling at a rate f_s means we take f_s uniform samples per second.
- This uniform sampling can be accomplished by multiplying $g(t)$ by an impulse train $\delta_{T_s}(t)$

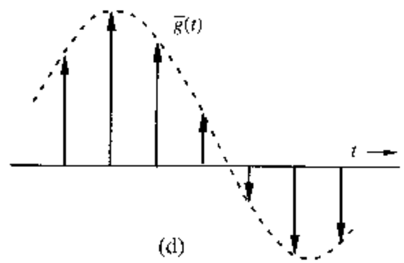


(c)

- The pulse train consist of unit impulses repeating periodically every T_s seconds.

Sampling Theorem

- This result in the sampled signal $\bar{g}(t)$ as follows



- The relationship between sampled signal $\bar{g}(t)$ and the original signal $g(t)$ is

$$\bar{g}(t) = g(t)\delta_{T_s}(t) = \sum_{n} g(nT_s)\delta(t - nT_s)$$

- Since the impulse train is a periodic signal of period T_s, it can be expressed as an exponential Fourier series i.e.,

$$\delta_{T_s}(t) = \frac{1}{T_s}\sum_{-\infty}^{\infty} e^{jnw_s t} \qquad\qquad w_s = \frac{2\pi}{T_s} = 2\pi f_s$$

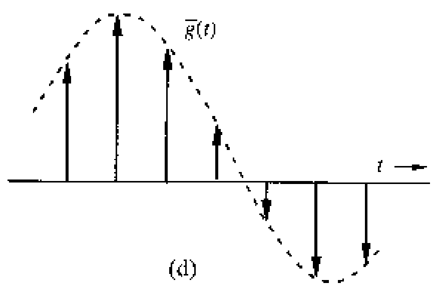
$$\bar{g}(t) = \frac{1}{T_s}\sum_{-\infty}^{\infty} g(t)e^{jnw_s t}$$

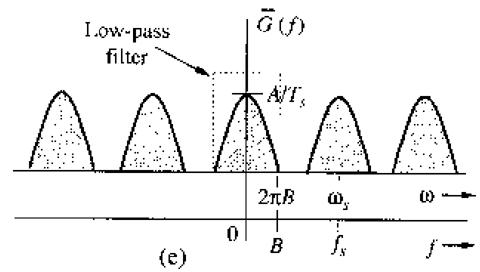
Sampling Theorem

- The Fourier transform of $\bar{g}(t)$ will be as follows

$$\bar{G}(f) = \frac{1}{T_s}\sum_{-\infty}^{\infty} G(f - nf_s)$$

- This mean that the spectrum $\bar{G}(f)$ consist of $G(f)$, scaled by a constant $\frac{1}{T_s}$, repeating periodically with period $f_s = \frac{1}{T_s}\ Hz$





- Can $g(t)$ be reconstructed from $\bar{g}(t)$ without any loss or distortion?
  - If yes then we should be able to recover $G(f)$ from $\bar{G}(f)$
  - Only possible if there is no overlap among replicas in $\bar{G}(f)$

Sampling Theorem

- It can be seen that for no overlap among replicas in $\bar{G}(f)$

$$f_s > 2B$$

- Also since $T_s = \frac{1}{f_s}$ so, $T_s < \frac{1}{2B}$
- As long as the sampling frequency f_s is greater than twice the signal bandwidth B, $\bar{G}(f)$ will have nonoverlapping repetitions of $G(f)$.
- $g(t)$ can be recovered using an ideal low pass filter of bandwidth $B\ Hz$.
- The minimum sampling rate $f_s = 2B$ required to recover $g(t)$ from $\bar{g}(t)$ is called the **Nyquist Rate** and the corresponding sampling interval $T_s = \frac{1}{2B}$ is called the **Nyquist Interval**.

Signal Reconstruction from Uniform Samples

- The process of reconstructing a continuous time signal $g(t)$ from its samples is also known as **interpolation**.
- We have already established that uniform sampling at above the Nyquist rate preserves all the signal information.
  - Passing the sampled signal through an ideal low pass filter of bandwidth $B\ Hz$ will reconstruct the message signal.
- The sampled message signal can be represented as

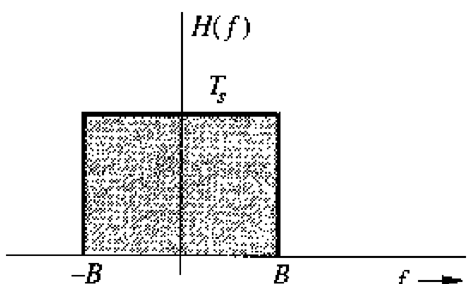
$$\bar{g}(t) = \sum_{n} g(nT_s)\delta(t - nT_s)$$

- The low pass filter of bandwidth $B\ Hz$ and gain T_s will have the following transfer function

$$H(f) = T_s \Pi\left(\frac{w}{4\pi B}\right) = T_s \Pi\left(\frac{f}{2B}\right)$$

Signal Reconstruction from Uniform Samples

- To recover the analog signal from its uniform samples, the ideal interpolation filter transfer function and impulse response is as follows



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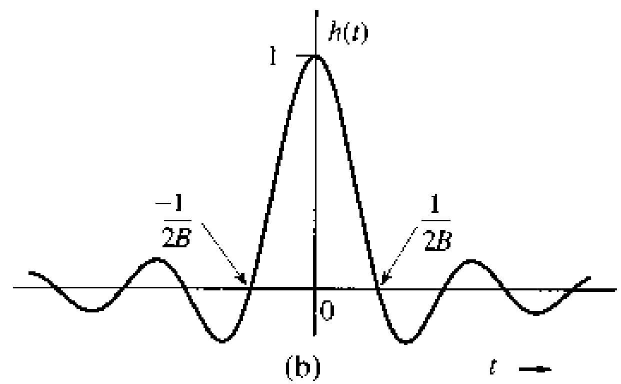


(b)

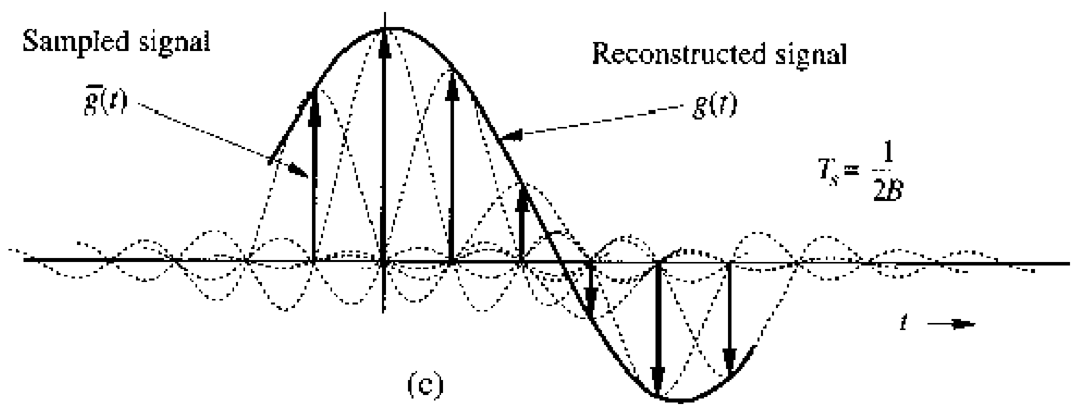
$$H(f) = T_s \Pi\left(\frac{f}{2B}\right) \quad \Leftrightarrow \quad h(t) = 2BT_s \operatorname{sinc} 2\pi Bt$$

- Assuming the use of Nyquist sampling rate i.e., $2BT_s = 1$

$$h(t) = \operatorname{sinc} 2\pi Bt$$

Signal Reconstruction from Uniform Samples

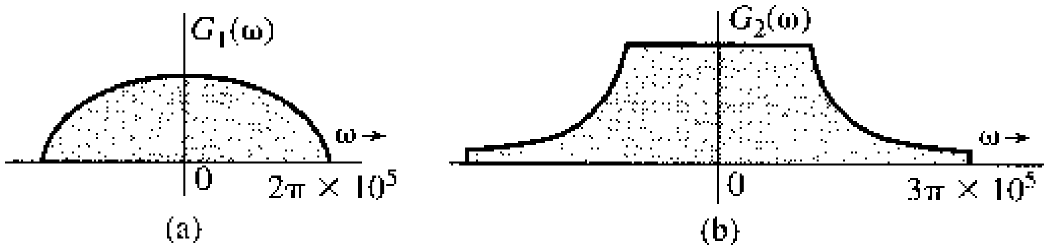
- It can be observed that $h(t) = 0$ at all Nyquist sampling instants i.e., $t = {}^{\pm n}/_{2B}$ except $t = 0$
- When the sampled signal $\bar{g}(t)$ is applied at the input of this filter, each sample in $\bar{g}(t)$, being an impulse, generate a sinc pulse of height equal to the strength of the sample.



$$g(t) = \sum_k g(kT_s)h(t - kT_s) = \sum_k g(kT_s)\,\mathrm{sinc}(2\pi Bt - k\pi)$$

Example

- Determine the Nyquist interval and sampling rate for the signal $g_1(t)$, $g_2(t)$, $g_1^2(t)$, $g_2^m(t)$ and $g_1(t)g_2(t)$ when the spectra of $G_1(w)$ and $G_2(w)$ is given as follows



- Bandwidth of $g_1(t)$ is $100KHz$ and $g_2(t)$ is $150KHz$
  - Nyquist sample rate for $g_1(t)$ is $200KHz$ and $g_2(t)$ is $300KHz$
- Bandwidth of $g_1^2(t)$ is twice the bandwidth of $g_1(t)$ i.e., $200KHz$ and $g_2^m(t)$ will have bandwidth $m \times 150KHz$
  - Nyquist sample rate for $g_1^2(t)$ is $400KHz$ and $g_2^m(t)$ is $m \times 300KHz$
- Bandwidth of $g_1(t)g_2(t)$ is $250KHz$
  - Nyquist sample rate for $g_1(t)g_2(t)$ is $500KHz$

Example

- Determine the Nyquist sampling rate and the Nyquist sampling interval for the signal
  - $\text{sin}c\ 100\pi t$
  - $\text{sinc}^2\ 100\pi t$

$$\text{sin}c\ 100\pi t \quad \Leftrightarrow \quad 0.01\Pi\left(\frac{w}{200\pi}\right)$$

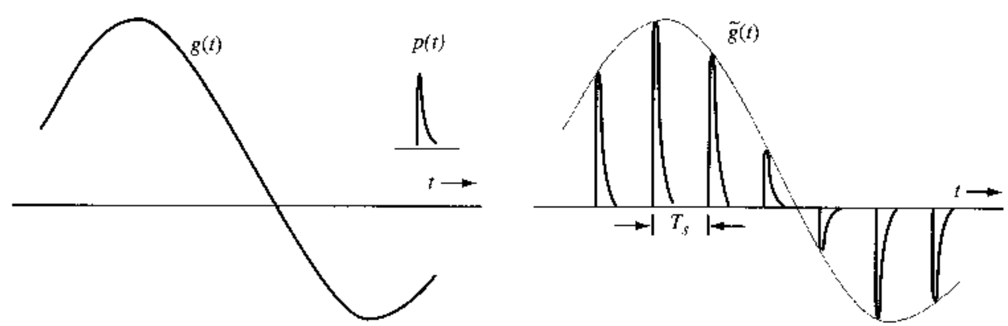
- Bandwidth is $50Hz$ and the Nyquist sample rate $100Hz$

$$\text{sinc}^2\ 100\pi t \Leftrightarrow 0.01\Delta\left(\frac{w}{400\pi}\right)$$

- Bandwidth is $100Hz$ and the Nyquist sample rate $200Hz$

Practical Signal Reconstruction (Interpolation)

- Signal reconstruction discussed earlier require an ideal low pass filter.
  - Unrealizable
  - Noncausal
  - This also lead to a infinitely long nature of sinc reconstruction pulse.
- For practical applications, we need a realizable signal reconstruction system from the uniform signal samples.
- For practical implementation, the reconstruction pulse must be easy to generate.



Practical Signal Reconstruction (Interpolation)

- We need to analyze the accuracy of the reconstructed signal when using a non-ideal interpolation pulse $p(t)$.

$$\tilde{g}(t) = \sum_{n} g(nT_s)p(t - nT_s)$$

$$\tilde{g}(t) = p(t) * \left[\sum_{n} g(nT_s)\delta(t - nT_s)\right]$$

$$\tilde{g}(t) = p(t) * \bar{g}(t)$$

- In frequency domain, the relationship between the reconstructed and the original analog signal is given as

$$\tilde{G}(f) = P(f)\frac{1}{T_s}\sum_{-\infty}^{\infty} G(f - nf_s)$$

- The reconstructed signal $\tilde{g}(t)$ using pulse $p(t)$ consist of multiple replicas of $G(f)$ shifted to nf_s and filtered by $P(f)$.

Practical Signal Reconstruction (Interpolation)

- So to fully recover the $g(t)$ from $\tilde{g}(t)$, we need further processing/filtering.
  - Such filters are often referred to as **equalizers**
- Let us denote the equalizer function as $E(f)$, so now distortionless reconstruction will require

$$G(f) = E(f)\tilde{G}(f)$$

$$G(f) = E(f)P(f)\frac{1}{T_s}\sum_{-\infty}^{\infty} G(f - nf_s)$$

- The equalizer must remove all the shifted replicas $G(f - nf_s)$ except the low pass term with $n = 0$ i.e.,

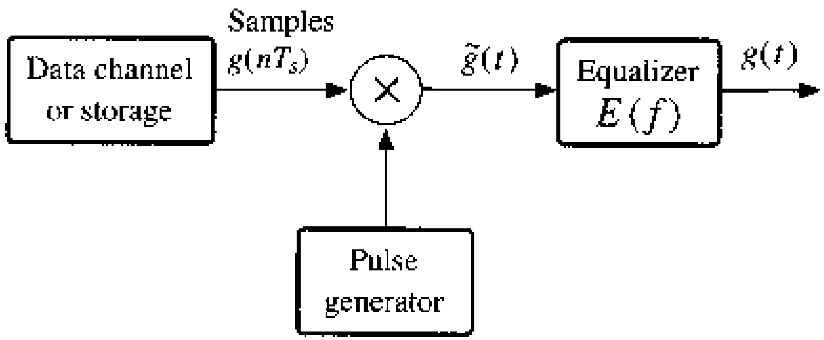
$$E(f)P(f) = 0 \qquad |f| > f_s - B$$

also

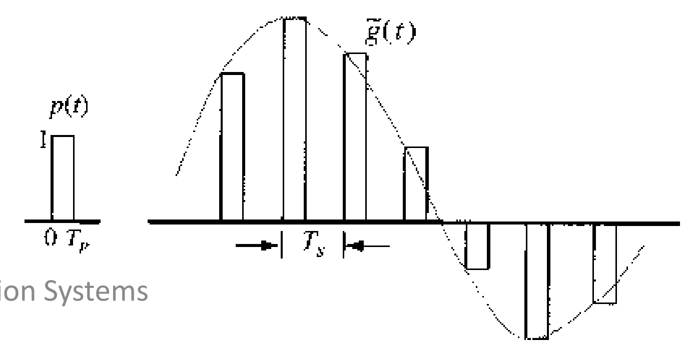
$$E(f)P(f) = T_s \qquad |f| < B$$

Practical Signal Reconstruction (Interpolation)

- The equalizer $E(f)$ must be
  - lowpass in nature to stop all frequency components above $f_s - B$
  - Inverse of $P(f)$ within the signal bandwidth of $B\ Hz$
- A practical signal reconstruction system can be represented as



- Let us consider a very simple interpolating pulse generator that generate short pules i.e.,



Practical Signal Reconstruction (Interpolation)

- The short pulse can be expressed as

$$p(t) = \prod\left(\frac{t - 0.5T_p}{T_p}\right)$$

- The reconstruction will first generate $\tilde{g}(t)$ as

$$\tilde{g}(t) = \sum_n g(nT_s)\prod\left(\frac{t - 0.5T_p - nT_s}{T_p}\right)$$

- The Fourier transform of $p(t)$ is

$$P(f) = T_p \,\mathrm{sinc}\left(\pi f T_p\right) e^{-j\pi f T_p}$$

- The equalizer frequency response should satisfy

$$E(f) = \begin{cases} T_s/P(f) & |f| \le B \\ \text{Flexible} & B < |f| < \left(\dfrac{1}{T_s} - B\right) \\ 0 & |f| \ge \left(\dfrac{1}{T_s} - B\right) \end{cases}$$

Practical Issues in Signal Sampling and Reconstruction

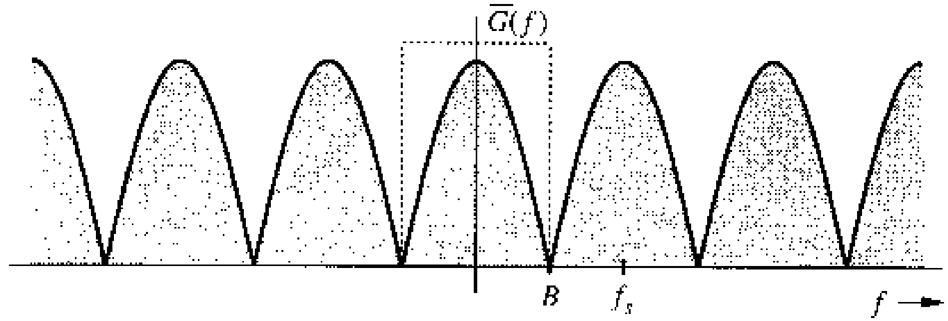
- Realizability of Reconstruction Filters

- The Treachery of Aliasing
  - Defectors Eliminated: The antialiasing Filter

- Sampling forces non-bandlimited signals to appear band limited

Realizability of Reconstruction Filters

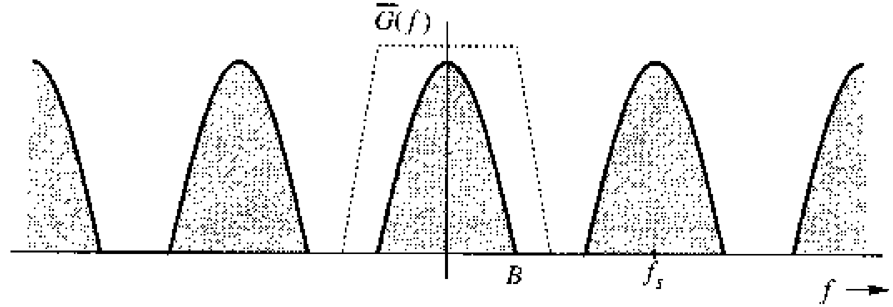
- If a signal is sampled at the Nyquist rate $f_s = 2B\ Hz$, the spectrum $\bar{G}(f)$ consists of repetitions of $G(f)$ without any gaps between successive cycles.



- To recover g(t) from $\bar{g}(t)$, we need to pass the sampled signal through an ideal low pass filter.
  - Ideal Filter is practically unrealizable
  - Solution?

Realizability of Reconstruction Filters

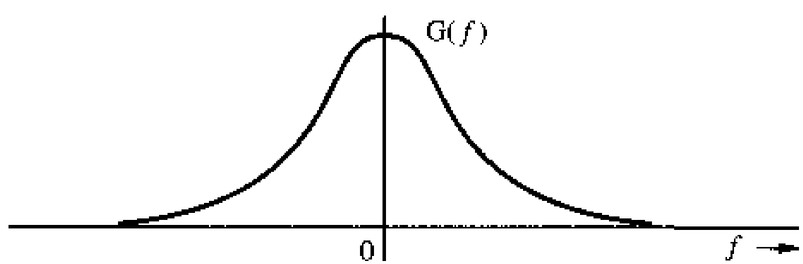
- A practical solution to this problem is to sample the signal at a rate higher than the Nyquist rate i.e., $f_s > 2B$ or $w_s > 4\pi B$
- This yield $\bar{G}(f)$, consisting of repetitions of G(f) with a finite band gap between successive cycles.



- We can recover $g(t)$ from $\bar{g}(t)$ using a low pass filter with a gradual cut-off characteristics.
- There is still an issue with this approach: Guess?
- Despite the gradual cut-off characteristics, the filter gain is required to be zero beyond the first cycle of $G(f)$.

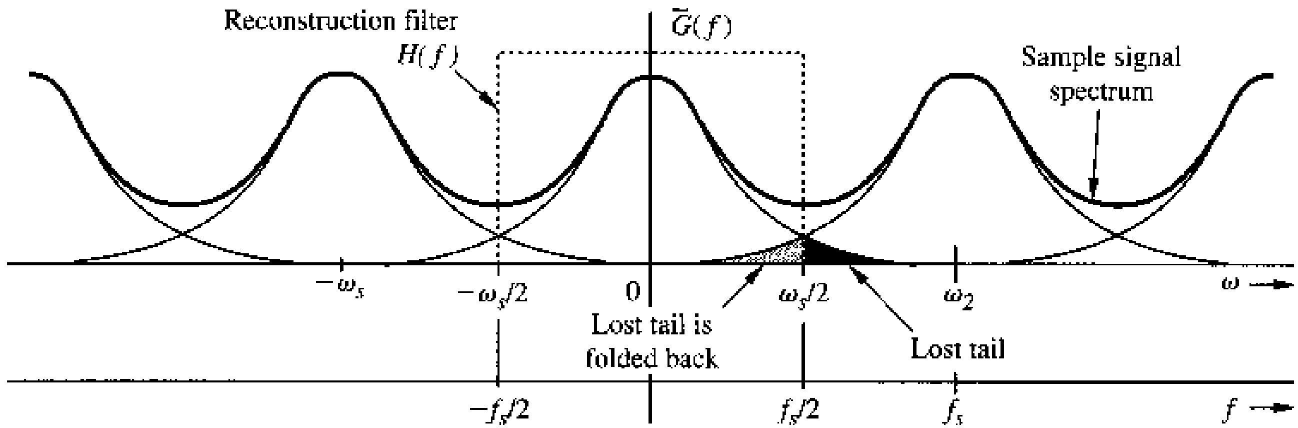
The Treachery of Aliasing

- Another practical difficulty in reconstructing the signal from its samples lies in the fundamental theory of sampling theorem.
- The sampling theorem was proved on the assumption that the signal $g(t)$ is band-limited.
- All practical signals are time limited.
  - They have finite duration or width.
  - They are non-band limited.
  - A signal cannot be time limited and band limited at the same time.



The Treachery of Aliasing

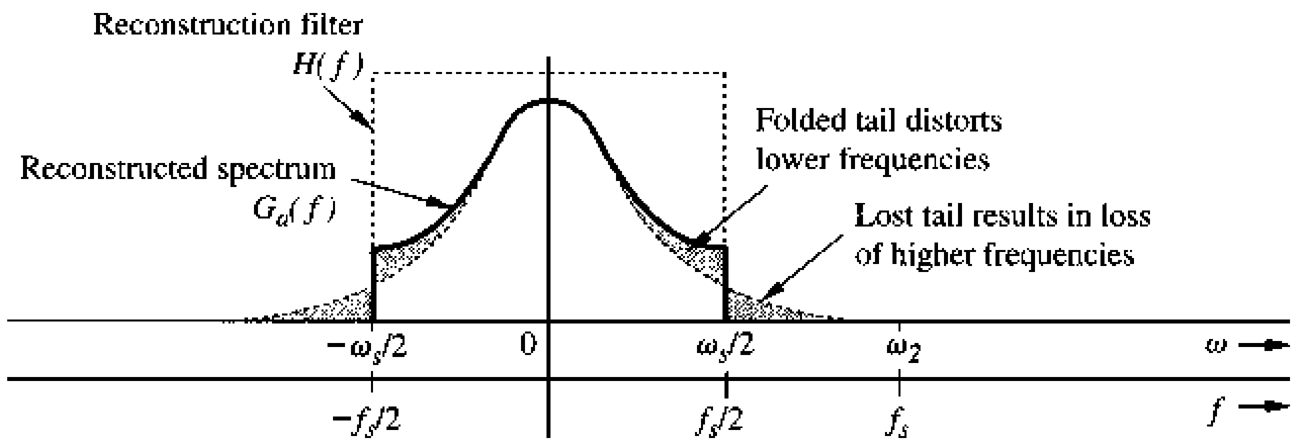
- In this case, the spectrum of $\bar{G}(f)$ will consist of **overlapping** cycles of $G(f)$ repeating every $f_s\ Hz$



- Because of non-bandlimited spectrum, the spectrum overlap is unavoidable, regardless of the sampling rate.
  - Higher sampling rates can reduce but cannot eliminate the overlaps
- Due to the overlaps, $\bar{G}(f)$ no longer has complete information of $G(f)$.

The Treachery of Aliasing

- If we pass $\bar{g}(t)$ even through an ideal low pass filter with cut-off frequency $\frac{f_s}{2}$, the output will not be $G(f)$ but as $G_a(f)$



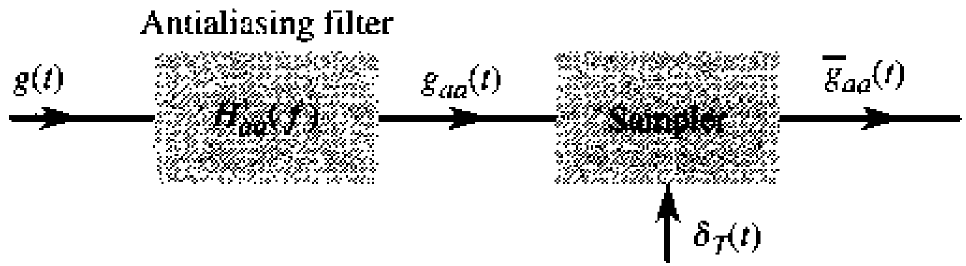
- This $G_a(f)$ is distorted version of $G(f)$ due to
  - The loss of tail of $G(f)$ beyond $|f| > {}^{f_s}/_{2}$
  - The reappearance of this tail inverted or folded back into the spectrum
- This tail inversion is known as **spectral folding** or **aliasing**.

Defectors Eliminated: The Antialiasing Filter

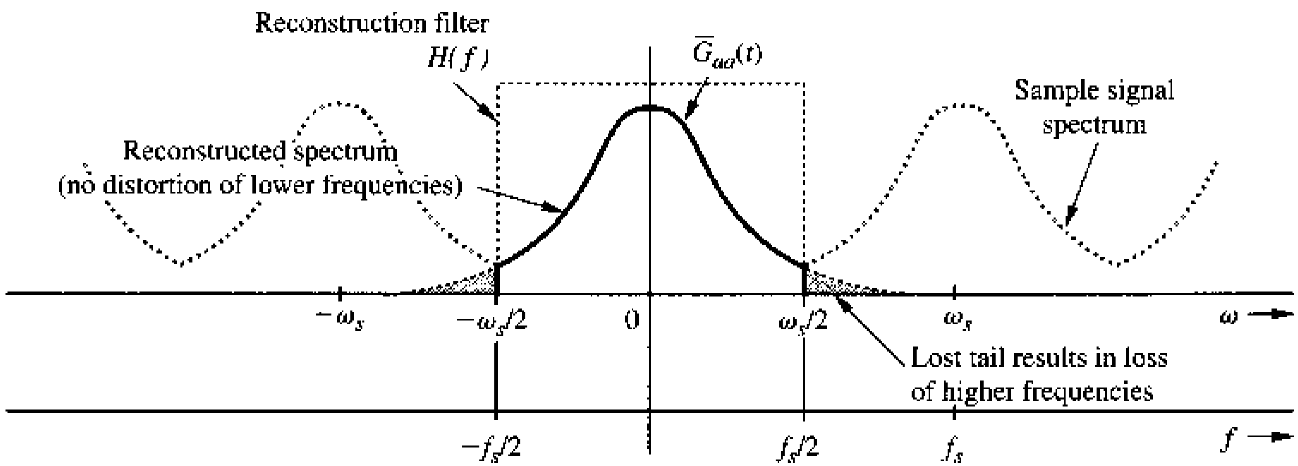
- In the process of aliasing, we are not only losing all the frequency components above the folding frequency i.e., ${f_s}/{2}\ Hz$ but these components reappear (aliased) as lower frequency components.
  - Aliasing is destroying the integrity of the frequency components below the folding frequency.
- Solution?
  - The frequency components beyond the folding frequency are suppressed from $g(t)$ before sampling.
- The higher frequencies can be suppressed using a low pass filter with cut off ${f_s}/{2}\ Hz$ known as **antialiasing filter**.

Defectors Eliminated: The Antialiasing Filter

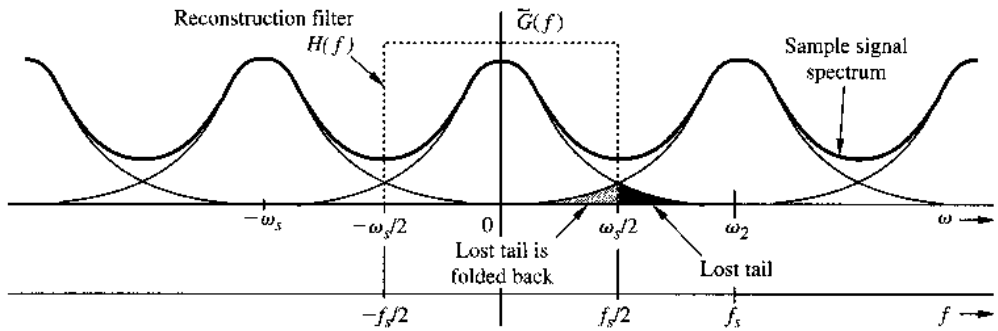
- The antialiasing filter is used before sampling



- The sampled signal spectrum and the reconstructed signal $G_{aa}(f)$ will be as follows



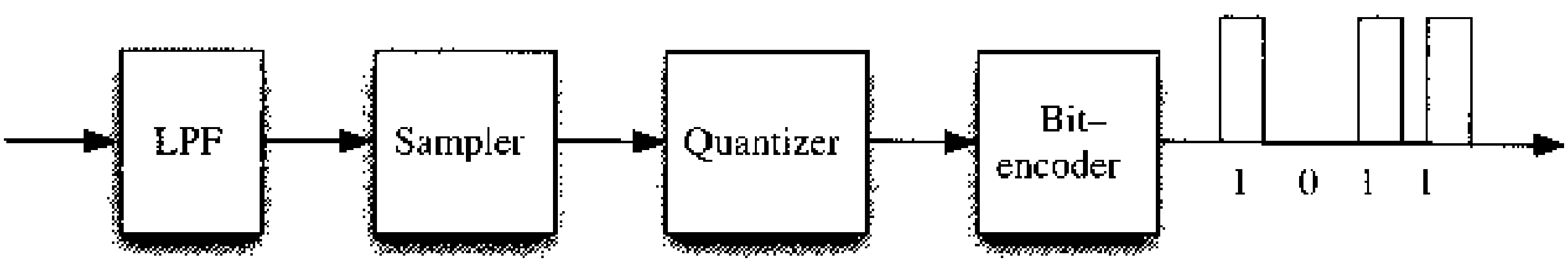
Sampling forces non-bandlimited signals to appear band limited



- The above spectrum consist of overlapping cycles of $G(f)$
  - This also mean that $\bar{g}(t)$ are sub-Nyquist samples of $g(t)$
- We can also view the spectrum as the spectrum of $G_a(f)$ repeating periodically every $f_s\ Hz$ without overlap.
  - The spectrum $G_a(f)$ is band limited to ${}^{f_s}/_{2}\ Hz$
- The sub-Nyquist samples of $g(t)$ can also be viewed as Nyquist samples of $g_a(t)$.
- Sampling a non-bandlimited signal $g(t)$ at a rate $f_s\ Hz$ make the samples appear to be Nyquist samples of some signal $g_a(t)$ band limited to ${}^{f_s}/_{2}\ Hz$

Pulse Code Modulation

- Analog signal is characterised by an amplitude that can take on any value over a continuous range.
  - It can take on an infinite number of values
- Digital signal amplitude can take on only a finite number of values.
- Pulse code modulation (PCM) is a tool for converting an analog signal into a digital signal (A/D conversion).
  - Sampling and **Quantization**

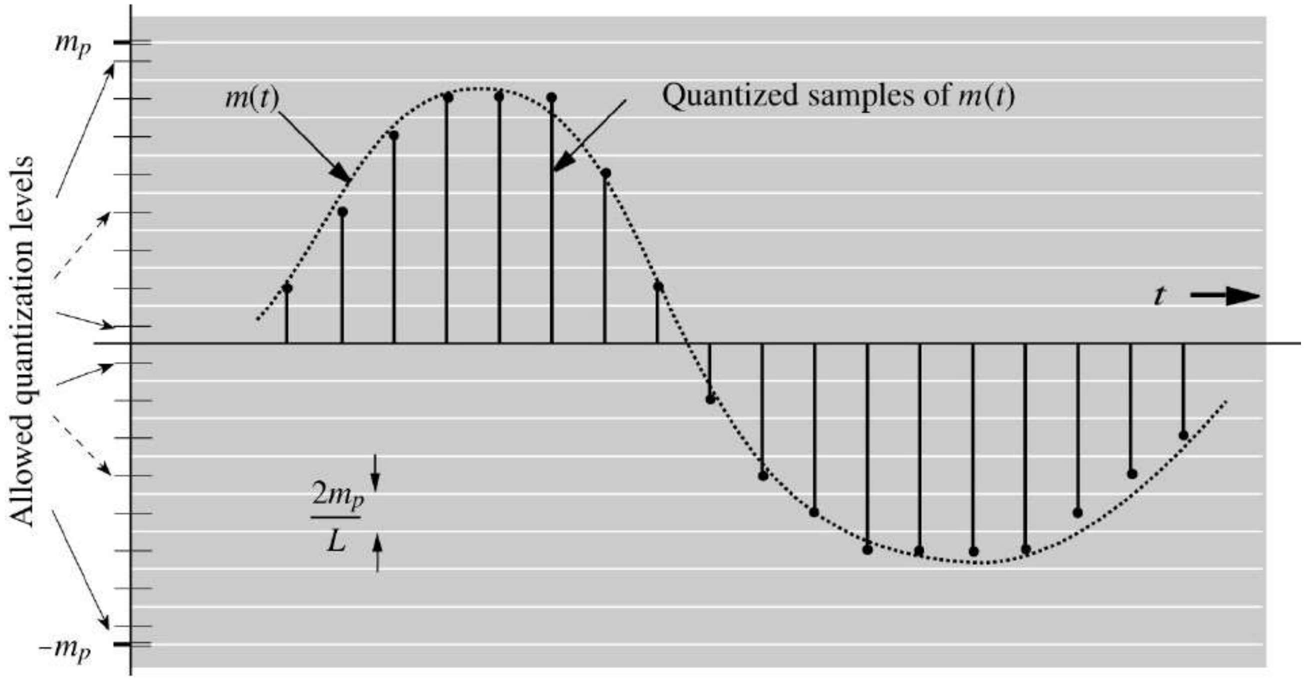


Example
(Digital Telephone)

- The audio signal bandwidth is about $15KHz$.
- For speech, subjective tests show that signal articulation (intelligibility) is not affected if all the components above $3400Hz$ are suppressed.
- Since the objective in telephone communication is intelligibility rather than high fidelity, the components above $3400Hz$ are eliminated by a low pass filter.
- The resulting bandlimited signal is sampled at a rate of $8000\ Hz$ (8000 samples per second).
  – Why higher sampling rate than Nyquist sample rate of $6.8KHz$?
- Each sample is then quantized into 256 levels $(L = 256)$.
  – 8 bits required to sample each pulse.
- Hence, a telephone signal requires $8 \times 8000 = 64K$ binary pulses per second ($64Kbps$).

Quantization

- To transmit analog signals over a digital communication link, we must discretize both time and values.



- Quantization spacing is $\frac{2m_p}{L}$; sampling interval is T, not shown in figure.

Quantization

- For quantization, we limit the amplitude of the message signal $m(t)$ to the range $(-m_p, m_p)$.
  - Note that m_p is not necessarily the peak amplitude of $m(t)$.
  - The amplitudes of $m(t)$ beyond $\pm m_p$ are simply chopped off.
- m_p is not a parameter of the signal $m(t)$; rather it is the limit of the quantizer.
- The amplitude range $\left(-m_p, m_p\right)$ is divided into L uniformly spaced intervals, each of width $\Delta v = \frac{2m_p}{L}$.
- A sample value is approximated by the midpoint of the interval in which it lies.
  - Quantization Error!

Quantization Error

- If $m(kT_s)$ is the k^{th} sample of the signal $m(t)$, and if $\hat{m}(kT_s)$ is the corresponding quantized sample, then from the interpolation formula

$$m(t) = \sum_k m(kT_s)\,\mathrm{sinc}(2\pi Bt - k\pi)$$

and

$$\hat{m}(t) = \sum_k \hat{m}(kT_s)\,\mathrm{sinc}(2\pi Bt - k\pi)$$

- Where $\hat{m}(\mathrm{t})$ is the signal reconstructed from the quantized samples.
- The distortion component $q(t)$ in the reconstructed signal is

$$q(t) = \hat{m}(t) - m(t)$$

$$q(t) = \sum_k q(kT_s)\,\mathrm{sinc}(2\pi Bt - k\pi)$$

- $q(kT_s)$ is the quantization error in the kth sample.
- The signal $q(t)$ is the undesired signal and act as noise i.e., quantization noise.

Quantization Error

- We can calculate the power or mean square value of $q(t)$ as

$$P_q = \widetilde{q^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} q^2(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\sum_k q(kT_s)\, \mathrm{sinc}(2\pi Bt - k\pi)\right]^2 dt$$

- Since the signal $\mathrm{sinc}(2\pi Bt - m\pi)$ and $\mathrm{sinc}(2\pi Bt - n\pi)$ are orthogonal i.e.,

$$\int_{-\infty}^{\infty} \mathrm{sinc}(2\pi Bt - m\pi)\, \mathrm{sinc}(2\pi Bt - n\pi)\ dt = \begin{cases} 0 & m \neq n \\ \dfrac{1}{2B} & m = n \end{cases}$$

- See Prob. 3.7-4 for proof.

Quantization Error

- As we know

$$P_q = \widetilde{q^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_k q^2(kT_s)\,\mathrm{sinc}^2(2\pi Bt - k\pi)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \sum_k q^2(kT_s) \int_{-T/2}^{T/2} \mathrm{sinc}^2(2\pi Bt - k\pi)\,dt$$

- Due to the orthogonality condition, the cross product terms will vanish and we get

$$P_q = \widetilde{q^2(t)} = \lim_{T\to\infty} \frac{1}{2BT} \sum_k q^2(kT_s)$$

- Since the sampling rate is $2B$, the total number of sample over averaging interval T is $2BT$.
  - The above relation is the average or the mean of the square of the quantization error.

Quantization Error

- Since the quantum levels are separated by $\Delta v = {}^{2m_p}/_{L}$ and the sample value is approximated by the midpoint of the subinterval (of height Δv).
  - The maximum quantization error is $\pm\, {}^{\Delta v}/_{2}$
  - The quantization error lies in the range $(-\, {}^{\Delta v}/_{2}\, , {}^{\Delta v}/_{2})$
- Assuming that the quantization error is equally likely to lie anywhere in the range $(-\, {}^{\Delta v}/_{2}, {}^{\Delta v}/_{2})$, the mean square of quantizing error $\widetilde{q^2}$ is given by

$$\widetilde{q^2} = \frac{1}{\Delta v} \int_{-\Delta v/2}^{\Delta v/2} q^2 \, dq = \frac{(\Delta v)^2}{12} = \frac{m^2{}_p}{3L^2}$$

- Hence, the quantization noise can be represented as

$$N_q = \frac{m^2{}_p}{3L^2}$$

Quantization Error

- The reconstructed signal $\hat{m}(t)$ at the receiver output is

$$\hat{m}(t) = m(t) + q(t)$$

- Can we determine the signal to noise ratio of the reconstructed signal?
- The power of the message signal $m(t)$ is $P_m = \widetilde{m^2(t)}$, then

$$S_0 = \widetilde{m^2(t)}$$

and

$$N_0 = N_q = \frac{m^2{}_p}{3L^2}$$

- Signal to noise ratio will be

$$\frac{S_0}{N_0} = 3L^2 \frac{\widetilde{m^2(t)}}{m^2{}_p}$$

Example

A signal $m(t)$ band-limited to 3 kHz is sampled at a rate $33\frac{1}{3}\%$ higher than the Nyquist rate. The maximum acceptable error in the sample amplitude (the maximum quantization error) is 0.5% of the peak amplitude m_p. The quantized samples are binary coded. Find the minimum bandwidth of a channel required to transmit the encoded binary signal. If 24 such signals are time-division-multiplexed, determine the minimum transmission bandwidth required to transmit the multiplexed signal.

The Nyquist sampling rate is $R_N = 2 \times 3000 = 6000$ Hz (samples per second). The actual sampling rate is $R_A = 6000 \times (1\frac{1}{3}) = 8000$ Hz.

The quantization step is Δv, and the maximum quantization error is $\pm\Delta v/2$.

$$\frac{\Delta v}{2} = \frac{m_p}{L} = \frac{0.5}{100} m_p \Longrightarrow L = 200$$

For binary coding, L must be a power of 2. Hence, the next higher value of L that is a power of 2 is $L = 256$.

From Eq. (6.37), we need $n = \log_2 256 = 8$ bits per sample. We require to transmit a total of $C = 8 \times 8000 = 64,000$ bit/s. Because we can transmit up to 2 bit/s per hertz of bandwidth, we require a minimum transmission bandwidth $B_T = C/2 = 32$ kHz.

The multiplexed signal has a total of $C_M = 24 \times 64,000 = 1.536$ Mbit/s, which requires a minimum of $1.536/2 = 0.768$ MHz of transmission bandwidth.

Differential Pulse Code Modulation

- PCM is not very efficient system because it generates so many bits and requires so much bandwidth to transmit.
- We can exploit the characteristics of the source signal to improve the encoding efficiency of the A/D converter.
  - Differential Pulse Code Modulation (DPCM)
- In analog messages we can make a good guess about a sample value from knowledge of past sample values.
  - High Nyquist sample rate.
- What can be done to exploit this characteristic of analog messages?
  - Instead of transmitting the sample values, we transmit the different between the successive sample values.

Differential Pulse Code Modulation

- If $m[k]$ is the k^{th} sample, instead of transmitting $m[k]$, we transmit the difference
$$d[k] = m[k] - m[k-1]$$
- At the receiver, knowing $d[k]$ and several previous sample value $m[k-1]$, we can reconstruct $m[k]$.
- How this can improve the efficiency of the A/D converter?
- The difference between successive samples is generally much smaller than the sample values.
  - The peak amplitude m_p of the transmitted values is reduced considerably.
  - This will reduce the quantization interval as $\Delta v \propto m_p$
  - This will reduce the quantization noise as $N_q \propto \Delta v^2$
  - For a given n (or transmission bandwidth), we can increase SNR
  - For a given SNR, we can reduce n (or transmission bandwidth)

Differential Pulse Code Modulation

- We can further improve this scheme by estimating (predicting) the k^{th} sample $m[k]$ from knowledge of several previous sample values i.e., $\widehat{m}[k]$.
- We now transmit the difference (prediction error) i.e.,

$$d[k] = m[k] - \widehat{m}[k]$$

- At the receiver, we determine the estimate $\widehat{m}[k]$ from the previous sample values and then generate $m[k]$ as

$$m[k] = d[k] - \widehat{m}[k]$$

- How this is an improved scheme?
  - The estimated value $\widehat{m}[k]$ will be close to $m[k]$ and their difference (prediction error), $d[k]$ will be even smaller than the difference between successive samples.
  - **Differential PCM**.
  - Previous method is a special case of DPCM i.e., $\widehat{m}[k] = m[k-1]$

Estimation of $\widehat{m}[k]$

- A message signal sample $m(t+T_s)$ can be expressed as a Taylor series expansion i.e.,

$$m(t+T_s) = m(t) + T_s\dot{m}(t) + \frac{{T_s}^2}{2!}\ddot{m}(t) + \frac{{T_s}^3}{3!}\dddot{m}(t) + \cdots$$
$$\approx m(t) + T_s\dot{m}(t) \qquad \text{for small } T_s$$

- From the knowledge of the signal and its derivatives at instant t, we can predict a future signal value at $t+T_s$
  - Even with just the first derivative we can approximate the future value
- Let the k^{th} sample of $m(t)$ be $m[k]$ i.e., putting $t = kT_s$ in $m(t)$

$$m(kT_s) = m[k] \qquad \text{and} \qquad m(kT_s \pm T_s) = m[k \pm 1]$$

- The future sample can be predicted as

$$m(kT_s + T_s) = m(kT_s) + T_s\left[\frac{m(kT_s) - m(kT_s - T_s)}{T_s}\right]$$
$$m[k+1] = 2m[k] - m[k-1]$$

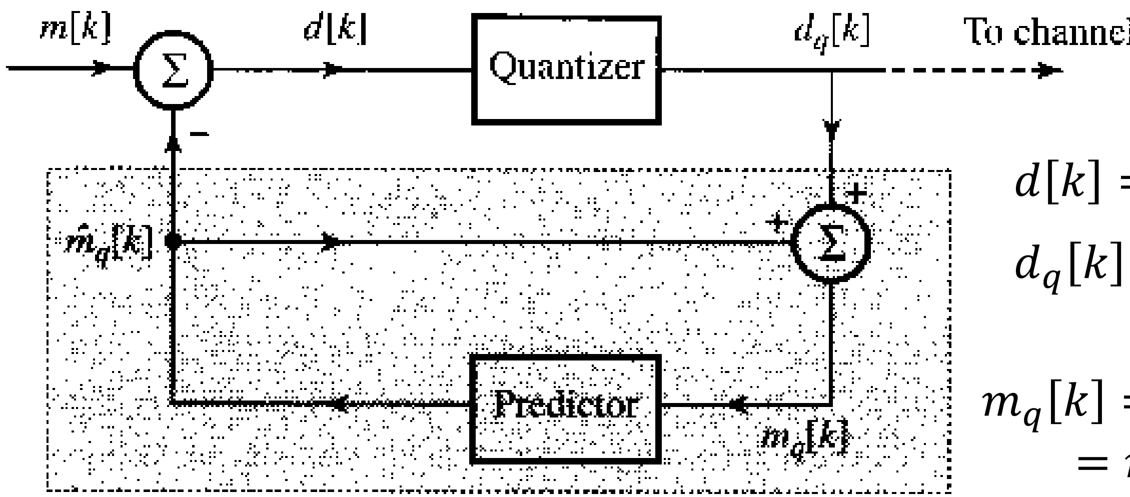
- We can predict the $(k+1)^{th}$ sample from the two previous samples.
- How the predicted value can be improved?
  - Consider more terms in the Taylor series.

Analysis of DPCM

- In DPCM, we transmit not the present sample $m[k]$, but $d[k] = m[k] - \widehat{m}[k]$. At receiver, we generate $\widehat{m}[k]$ from the past samples and add the received $d[k]$ to get $m[k]$.
- There is one difficulty associated with this scheme?
  - At receiver, instead of the past samples i.e., $m[k-1]$, $m[k-2]$, ..., we have their quantized versions i.e., $m_q[k-1], m_q[k-2]$, ...
  - We cannot determine $\widehat{m}[k]$, we can only determine $\widehat{m}_q[k]$ i.e., estimation of the quantized sample $m_q[k]$ using $m_q[k-1], m_q[k-2]$, ...
  - This will increase the error in reconstruction
- What could be a better strategy in this case?
  - Determine $\widehat{m}_q[k]$, the estimate of $m_q[k]$ (instead of $m[k]$), at the transmitter from the previous quantized samples $m_q[k-1], m_q[k-2]$, ...
  - The difference $d[k] = m[k] - \widehat{m}_q[k]$ is now transmitted
  - At receiver, we generate $\widehat{m}_q[k]$, and from the received $d[k]$, we can reconstruct $m_q[k]$

Analysis of DPCM

DPCM Transmitter

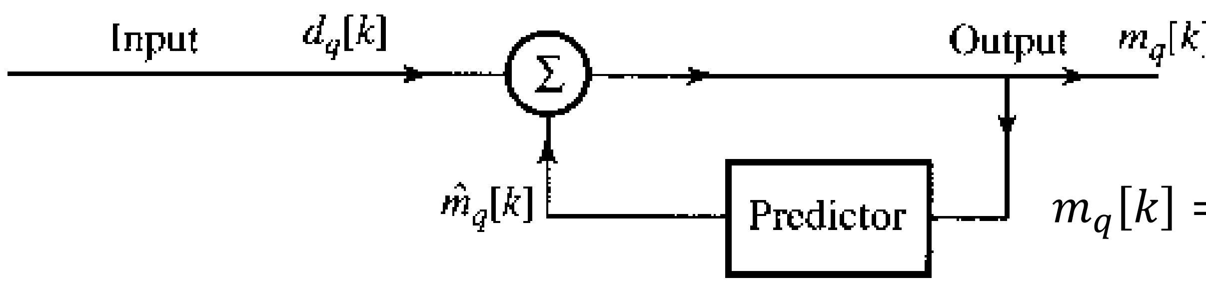


$$d[k] = m[k] - \hat{m}_q[k]$$

$$d_q[k] = d[k] + q[k]$$

$$m_q[k] = \hat{m}_q[k] + d_q[k] \\ = m[k] + q[k]$$

DPCM Receiver



$$m_q[k] = m[k] + q[k]$$

- The $q[k]$ at the receiver output is the quantization noise associated with difference signal $d[k]$, which is generally much smaller than $m[k]$.

SNR Improvement using DPCM

- Let m_p and d_p be the peak amplitudes of $m(t)$ and $d(t)$ respectively
- If we use the same value of L in both cases, the quantization step Δv in DPCM is reduced by the factor $\frac{d_p}{m_p}$
- As the quantization noise $N_p = \frac{(\Delta v)^2}{12}$, the quantization noise in case of DPCM is reduced by a factor $\left(\frac{m_p}{d_p}\right)^2$, and the SNR is increased by the same factor.
- SNR improvement in DPCM due to prediction will be

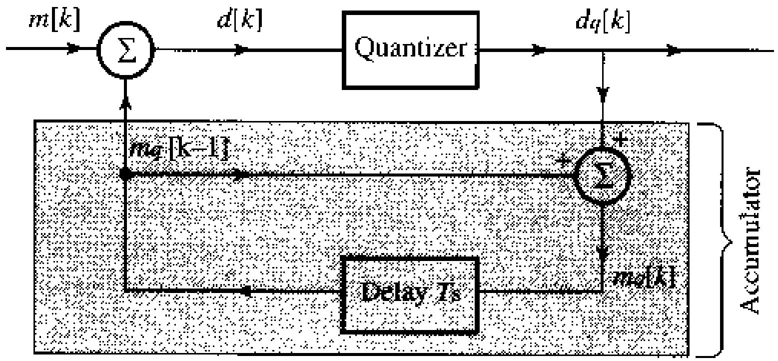
$$G_p = \frac{P_m}{P_d}$$

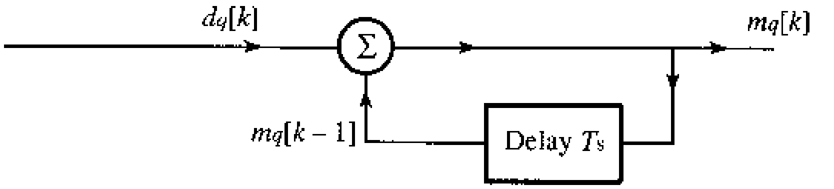
- Where P_m and P_d are the powers of $m(t)$ and $d(t)$ respectively.

Delta Modulation

- Sample correlation used in DPCM is further exploited in **delta modulation** by over sampling (typically four times the Nyquist rate) the baseband signal.
  - Small prediction error
  - Encode using 1 bit (i.e., L=2)
- Delta Modulation (DM) is basically a 1-bit DPCM.
- In DM, we use a first order predictor, which is just a time delay of T_s.
- The DM transmitter (modulator) and receiver (demodulator) are identical to DPCM with a time delay as a predictor.

Delta Modulation





$$m_q[k] = m_q[k-1] + d_q[k]$$

- So

$$m_q[k-1] = m_q[k-2] + d_q[k-1]$$

- We get

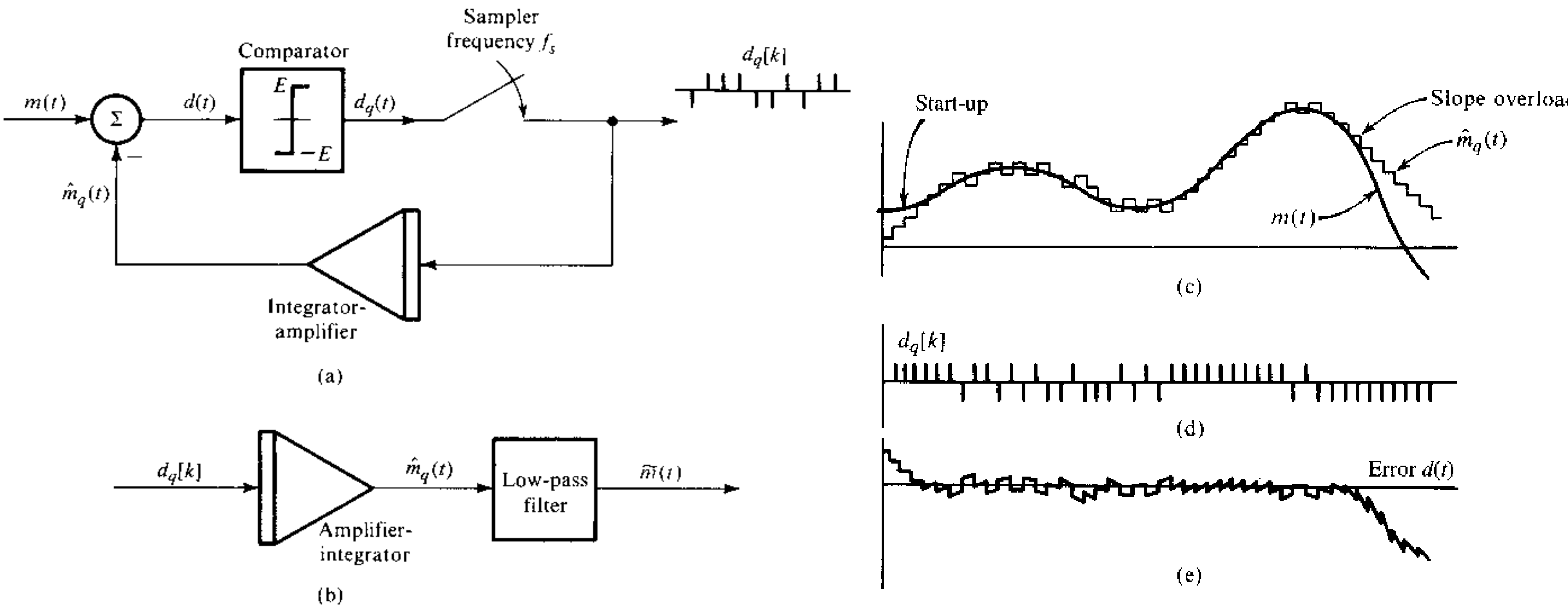
$$m_q[k] = m_q[k-2] + d_q[k-1] + d_q[k]$$

- Preceding iteratively we get

$$m_q[k] = \sum_{m=0}^{k} d_q[m]$$

- The receiver is just an accumulator (adder).
- If $d_q[k]$ is represented by impulses, then the accumulator (receiver) may be realized by an **integrator.**

Delta Modulation



- Note that in DM, the modulated signal carries information about the difference between the successive samples.
  - If difference is positive or negative, a positive or negative pulse is generated in modulated signal $d_q[k]$
  - DM carries the information about the derivative of $m(t)$
  - That's why integration of DM signal yield approximation of $m(t)$ i.e., $\widehat{m}_q(t)$