30820-Communication Systems

Week 4-5 – Lecture 10-15
(Ref: Chapter 3 of text book)

ANALYSIS AND TRANSMISSION OF SIGNALS

Contents

Introduction

- We electrical engineers think of signals in terms of their spectral content.
- We have studied the spectral representation of periodic signals i.e., Fourier Series
- We now extend this spectral representation to the case of aperiodic signals.
  - Fourier Integrals

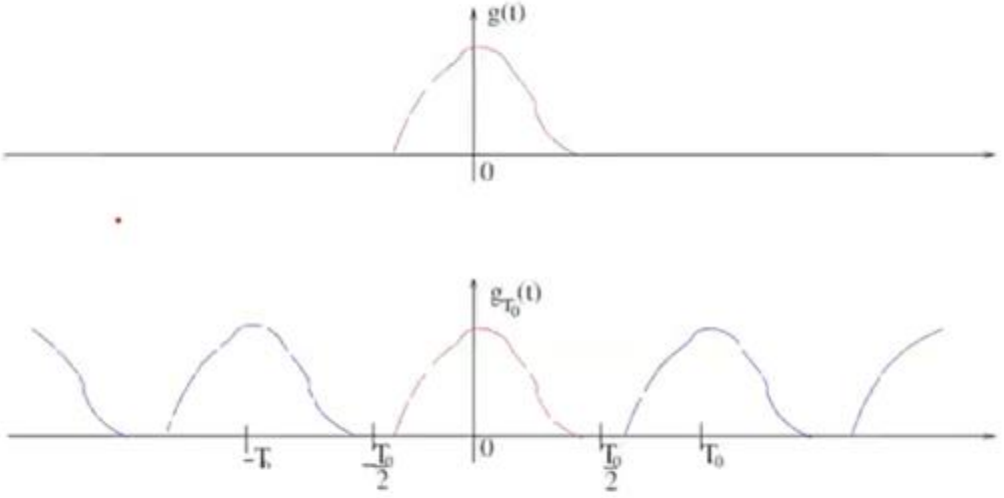
Aperiodic signal representation

- Signals encountered in real-life are often non-periodic, and more importantly, they are samples of random (stochastic) processes, such as speech signals, imagery, etc.

- The signals we use in communication systems are aperiodic (non-periodic) in nature. In order to study their behavior in frequency-domain we follow a simple procedure:
  - First we form a periodic extension.
  - Next we study the Fourier series for this new signal set.
  - Finally, we consider the limiting case when the period is allowed to become infinity, which is equivalent to saying let the signal become aperiodic.

Aperiodic signal representation

- We have an aperiodic signal $g(t)$ and we consider a periodic version $g_{T_0}(t)$ of such signal obtained by repeating $g(t)$ every T_0 seconds.



The periodic signal $g_{T_0}(t)$

- The periodic signal $g_{T_0}(t)$can be expressed in terms of $g(t)$ as follows:

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$$

Notice that, if we let $T_0 \to \infty$, we have that

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t)$$

The Fourier representation of $g_{T_0}(t)$

- The signal $g_{T_0}(t)$ is periodic, so it can be represented in terms of its Fourier series.
- The basic intuition here is that the Fourier series of $g_{T_0}(t)$ will also represent $g(t)$ in the limit for $T_0 \to \infty$.
- The exponential Fourier series of $g_{T_0}(t)$ is

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jnw_0 t}$$

Where

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-jnw_0 t}\, dt$$

and

$$w_0 = \frac{2\pi}{T_0}$$

The Fourier representation of $g_{T_0}(t)$

- Integrating $g_{T_0}(t)$ over $({}^{-T_0}/_2, {}^{T_0}/_2)$ is the same as integrating $g(t)$ over $(-\infty, \infty)$. So we can write

$$D_n = \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)e^{-jnw_0 t}\,dt$$

- If we define a function

$$G(w) = \int_{-\infty}^{\infty} g(t)e^{-jwt}\,dt$$

then we can write the Fourier coefficients D_n as follows

$$D_n = \frac{1}{T_0}G(nw_0)$$

Computing the $\lim_{T_0 \to \infty} g_{T_0}(t)$

- Thus $g_{T_0}(t)$ can be expressed as

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jnw_0 t} = \sum_{n=-\infty}^{\infty} \frac{G(nw_0)}{T_0} e^{jnw_0 t}$$

- Assuming $\frac{1}{T_0} = \frac{\Delta w}{2\pi}$, we get

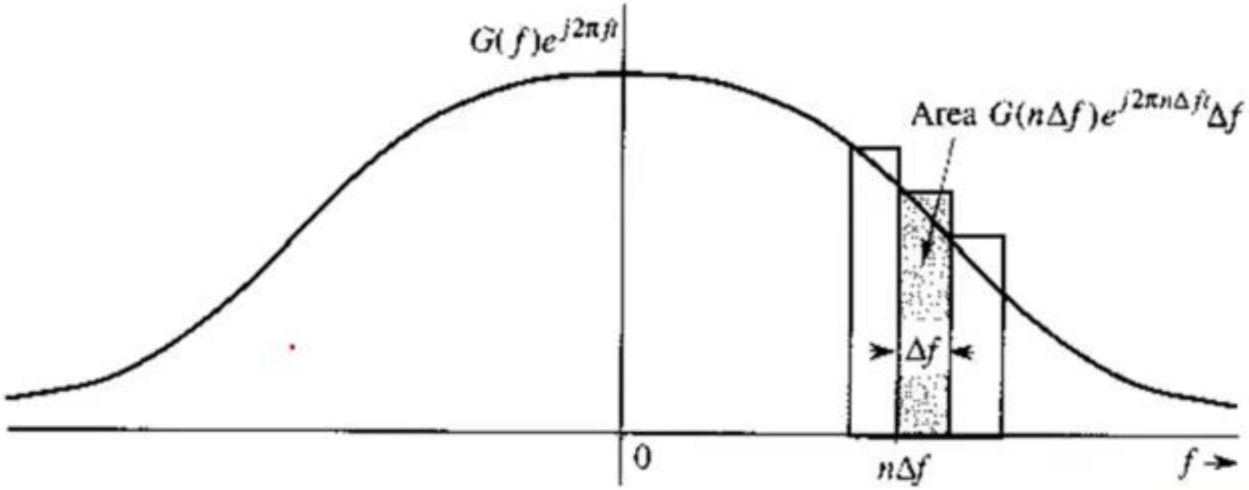
$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta w)\Delta w}{2\pi} e^{j(n\Delta w)t}$$

- In the limit for $T_0 \to \infty$, $\Delta w \to 0$ and $g_{T_0}(t) \to g(t)$
- We thus get

$$g(t) = \lim_{T_0 \to \infty} g_{T_0}(t) = \lim_{\Delta w \to 0} \sum_{n=-\infty}^{\infty} \frac{G(n\Delta w)\Delta w}{2\pi} e^{j(n\Delta w)t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(w) e^{jwt} dw$$

The Fourier series becomes the Fourier integral in the limit



Fourier Transform and Inverse Fourier Transform

- The spectral representation $G(w)$ of $g(t)$, that is, from

$$G(w) = \int_{-\infty}^{\infty} g(t)\, e^{-jwt} dt$$

We can obtain $g(t)$ back by computing

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(w) e^{jwt}\, dw$$

Fourier Transform of $g(t)$

$G(w) = \int_{-\infty}^{\infty} g(t)\, e^{-jwt} dt$

Inverse Fourier Transform

- $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(w) e^{jwt}\, dw$

Fourier Transform relationship

- $g(t) \Leftrightarrow G(w)$

Conjugate Symmetric Property

If $g(t)$ is a real function of t, then

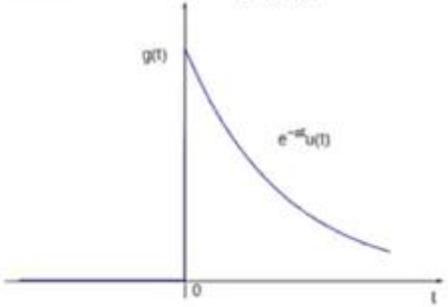
$$G(-f) = G^{*}(f)$$

Therefore,

$$|G(-f)| = |G(f)|$$
$$\theta_g(-f) = -\theta_g(f)$$

Example

- Find the Fourier transform of $g(t) = e^{-at}u(t)$

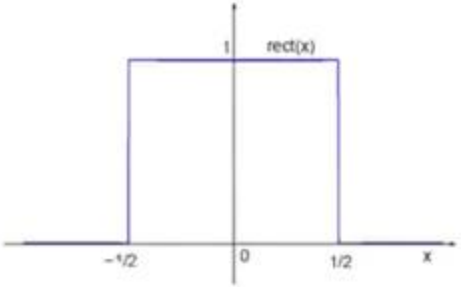


$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}dt \\
&= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}
\end{aligned}
$$

Since $|e^{-j\omega t}| = 1$, we have that $\lim_{t\to\infty} e^{-at}e^{-j\omega t} = 0$. Therefore:

Transforms of Some Useful Functions

- **The Unit Gate Function:** A square pulse with height 1, and with unit width, centered at origin is called unite gate function.

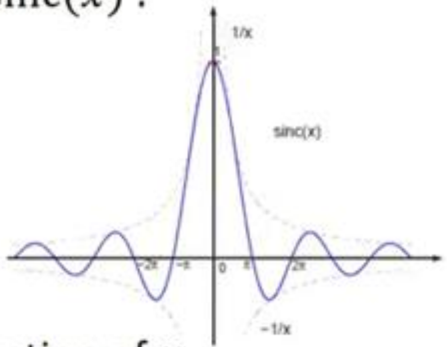


- The unit gate function $rect(x)$ is defined as:

$$rect(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| < 1/2 \end{cases}$$

Transforms of Some Useful Functions

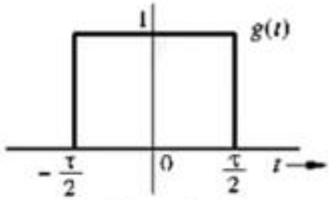
- **Sinc Function:** The function $\frac{\sin x}{x}$ is the "sine over argument" function denoted by $\mathrm{sinc}(x)$.



- $\mathrm{sinc}(x)$ is an even function of x.
- $\mathrm{sinc}(x) = 0$ when $\sin(x) = 0$ and $\mathrm{x} \neq 0$
- Using L'Hopital's rule, we find that $\mathrm{sinc}(0) = 1$
- $\mathrm{sinc}(x)$ is the product of an oscillating signal $\sin(x)$ and a monotonically decreasing function $^1/_x$

Example

- Find the Fourier transform of $g(t) = rect(^t/_\tau)$.



$$G(f) = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j2\pi ft}\, dt$$

Since $\Pi(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$\begin{aligned} G(f) &= \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}\, dt \\ &= -\frac{1}{j2\pi f}(e^{-j\pi f\tau} - e^{j\pi f\tau}) = \frac{2\sin(\pi f\tau)}{2\pi f} \\ &= \tau\frac{\sin(\pi f\tau)}{(\pi f\tau)} = \tau \ \mathrm{sinc}\ (\pi f\tau) \end{aligned}$$

Properties of the Fourier Transform

- Fourier Transform Table
  - Time-Frequency Duality
- Symmetry of Fourier transformation
- Time and frequency shifting property
- Convolution
- Time differentiation and time integration

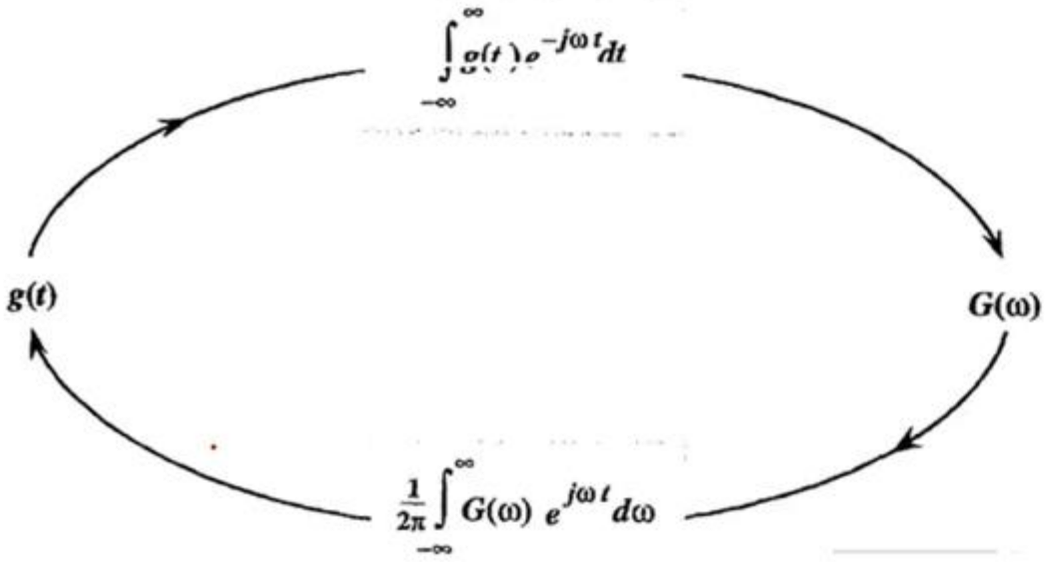
Fourier Transform Table

	$g(t)$	$G(f)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+(2\pi f)^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j2\pi f)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	

Fourier Transform Table

12	$\operatorname{sgn} t$	$\dfrac{2}{j2\pi f}$	
13	$\cos 2\pi f_0 t\, u(t)$	$\dfrac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\dfrac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\, u(t)$	$\dfrac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\dfrac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\, u(t)$	$\dfrac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
16	$e^{-at}\cos 2\pi f_0 t\, u(t)$	$\dfrac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
17	$\Pi\left(\dfrac{t}{\tau}\right)$	$\tau \operatorname{sinc}(\pi f \tau)$	
18	$2B \operatorname{sinc}(2\pi Bt)$	$\Pi\left(\dfrac{f}{2B}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\pi f \tau}{2}\right)$	
20	$B \operatorname{sinc}^2(\pi Bt)$	$\Delta\left(\dfrac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	$f_0=\frac{1}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\, e^{-2(\sigma\pi f)^2}$	

Time-Frequency Duality



* Note: If we consider $G(f)$ then we don't consider $1/2\pi$

Symmetry Property

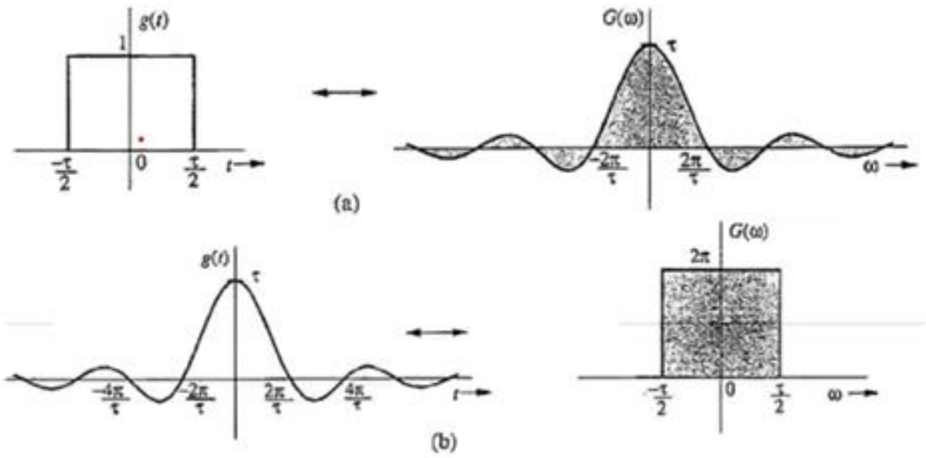
- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(w)$$

Then,

$$G(t) \Leftrightarrow 2\pi g(-w)$$

Example



Scaling Property

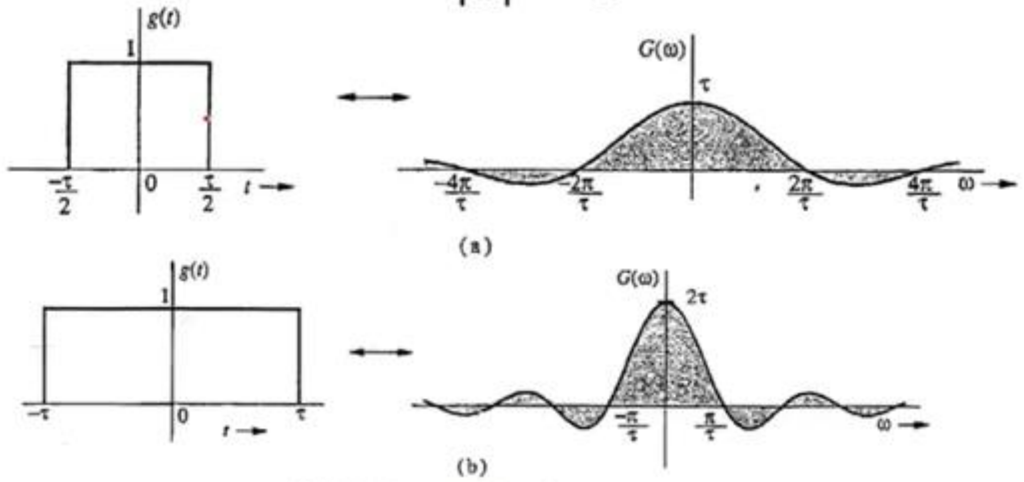
- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(w)$$

Then,

$$g(at) \Leftrightarrow \frac{1}{|a|} G\left(\frac{w}{a}\right)$$

Example



Time-Shifting Property

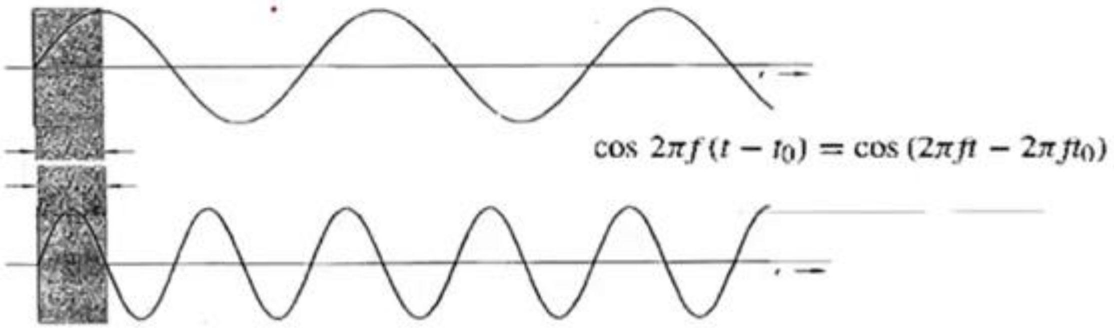
- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(w)$$

- Time shifting introduces phase shift

$$g(t - t_0) \Leftrightarrow G(w)e^{-jwt_0}$$

Example



Frequency-Shifting Property

- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(w)$$

- Exponential Multiplication introduces frequency shift

$$g(t)e^{jw_0t} \Leftrightarrow G(w - w_0)$$

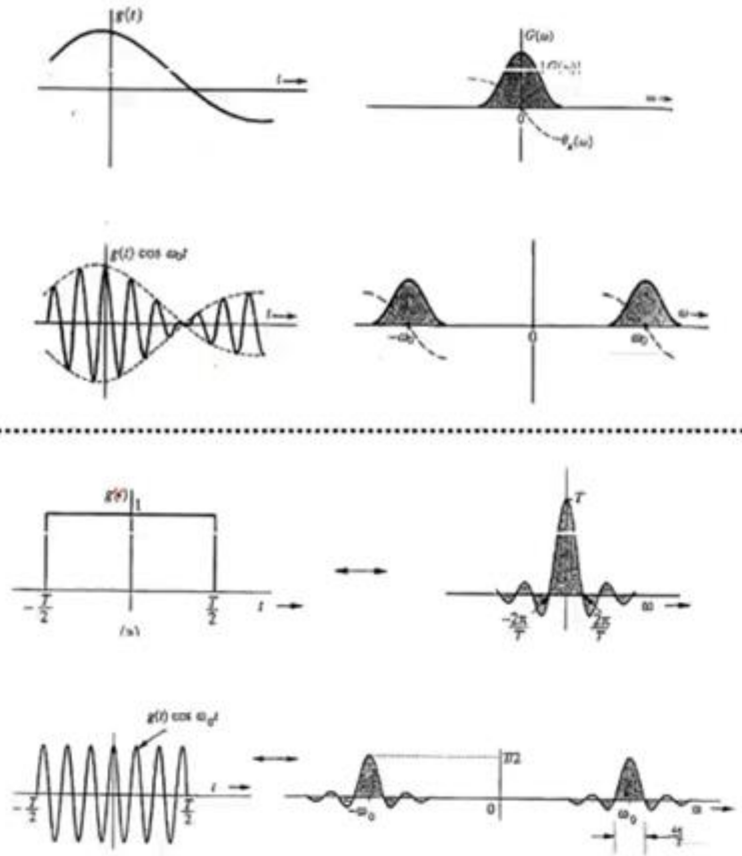
or

$$g(t)e^{-jw_0t} \Leftrightarrow G(w + w_0)$$

- Cosine multiplication leads to

$$g(t)\cos w_0t = \frac{1}{2}\left[g(t)e^{jw_0t} + g(t)e^{-jw_0t}\right]$$
$$= \frac{1}{2}[G(w - w_0) + G(w + w_0)]$$

Frequency-Shifting Property



Convolution

- The convolution of two functions $g(t)$ and $w(t)$ is given as

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau)w(t-\tau)d\tau$$

- Consider two waveforms

$$g_1(t) \Leftrightarrow G_1(w) \text{ and } g_2(t) \Leftrightarrow G_2(w)$$

- Convolution in time domain

$$g_1(t) * g_2(t) \Leftrightarrow G_1(w)G_2(w)$$

- Convolution in frequency domain

$$g_1(t)g_2(t) \Leftrightarrow \frac{1}{2\pi} G_1(w) * G_2(w)$$

Time Differentiation and Time Integration

- Consider the Fourier transform relationship

$$g(t) \Leftrightarrow G(w)$$

- The following relationship exists for integration

$$\int_{-\infty}^{t} g(\tau)d\tau \Leftrightarrow \frac{G(w)}{jw} + \pi G(0)\delta(w)$$

- The following relationship exists for differentiation

$$\frac{dg}{dt} \Leftrightarrow jwG(w) \qquad \frac{d^n g}{dt^n} \Leftrightarrow (jw)^n G(w)$$

Important Fourier Transform Operation

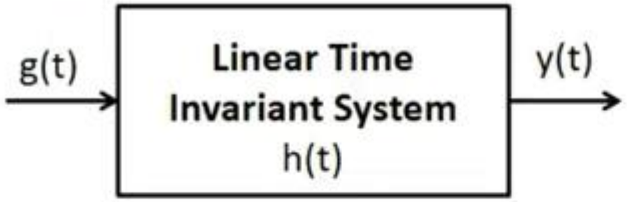
Operation	$g(t)$	$G(f)$
Superposition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Scalar multiplication	$kg(t)$	$kG(f)$
Duality	$G(t)$	$g(-f)$
Time scaling	$g(at)$	$\frac{1}{\lvert a\rvert} G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$G(f)e^{-j2\pi f t_0}$
Frequency shifting	$g(t)e^{j2\pi f_0 t}$	$G(f - f_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Frequency convolution	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x)\,dx$	$\frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$

Signal Transmission over a Linear System

- To introduce linear systems
- To introduce convolution
- To examine signal transmission through a linear system
- To introduce signal distortion during transmission
- To give examples of real and ideal filters

Linear System

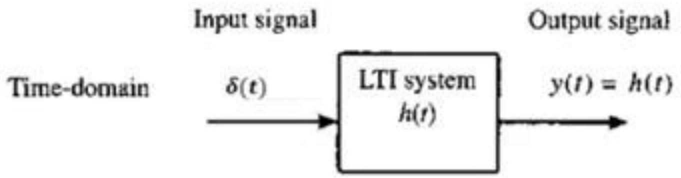
- A system is a black box that converts an input signal $g(t)$ in an output signal $y(t)$.



- Assume the output of a signal $g_1(t)$ is $y_1(t)$ and the output of $g_2(t)$ is $y_2(t)$.
  - The system is linear if the output of $g_1(t)$ + $g_2(t)$ is $y_1(t)$+$y_2(t)$.
  - A system is time invariant if its properties do not change with the time. That is, if the response to $g(t)$ is $y(t)$, then the response to $g(t - t_0)$ is going to be $y(t - t_0)$.

Linear Time Invariant (LTI) System

- Consider a linear time invariant (LTI) system. Assume the input signal is a Dirac delta function δ(t).
  - The output will be the impulse response of the system.



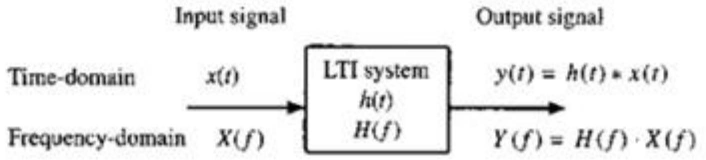
- $h(t)$ is called the "unit impulse response" function.
- With $h(t)$, we can relate the input to its output signal through the convolution formula:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Frequency Response of LTI systems

- If $x(t) \Leftrightarrow X(w)$ and $h(t) \Leftrightarrow H(w)$ then the convolution reduces to a product in Fourier domain

$$y(t) = h(t) * x(t) \;\Leftrightarrow Y(w) = H(w)X(w)$$



- $H(w)$ is called the "system transfer function" or the "system frequency response" or the "spectral response".

$$Y(w) = H(w)X(w)$$
$$|Y(w)|e^{j\theta_y(w)} = |H(w)|e^{j\theta_h(w)}|X(w)|e^{j\theta_x(w)}$$
$$|Y(w)|e^{j\theta_y(w)} = |H(w)||X(w)|e^{j[\theta_h(w)+\theta_x(w)]}$$

So,

$$|Y(w)| = |H(w)||X(w)|$$
$$\theta_y(w) = \theta_h(w) + \theta_x(w)$$

Distortionless Transmission

- Transmission is said to be distortionless if the input and the output have identical wave shapes with a multiplicative constant.
  - A delayed output that retains the input waveform is also considered distortionless.

- Given an input signal $x(t)$, the output differs from the input only by a multiplying constant and a finite time delay

$$y(t) = k.x(t - t_d)$$

- The Fourier transform of this equation yields

$$Y(f) = kX(f)e^{-j2\pi f t_d}$$

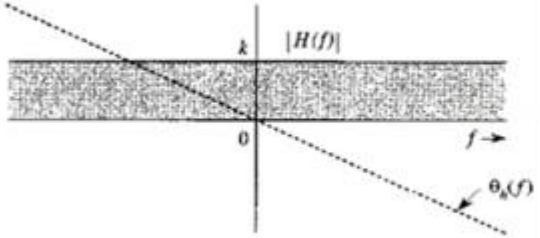
Distortionless Transmission

- As we know that $Y(f) = H(f)X(f)$
- The transfer function of a distortionless transmission system is

$$H(f) = ke^{-j2\pi f t_d}$$

We can write,

$$|H(f)| = k$$
$$\theta_h(f) = -2\pi f t_d$$



- The amplitude response $|H(f)|$ of a distortionless transmission system must be a constant and the phase response $\theta_h(f)$ must be a linear function of f going through the origin at $f = 0$.

Ideal and Practical Filters

- **Filter:** An electronic device or mathematical algorithm to modify the signals.

- In communications, filters are used for separating an information bearing signal from unwanted contaminations such as interference, noise and distortion products.

  – Low-pass filter (LPF)
  – High-pass filter (HPF)
  – Bandpass filter (BPF)
  – Bandstop filter (BSF)

Ideal and Practical Filters

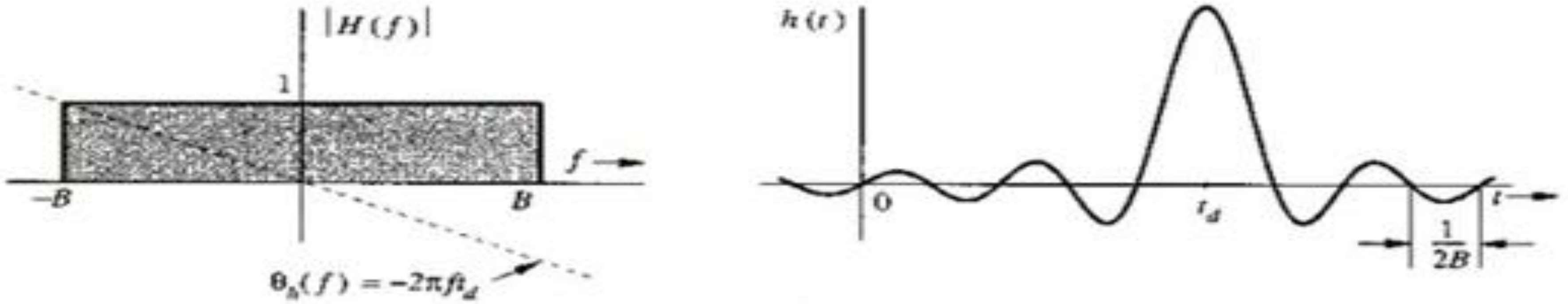
- Ideal filters allow distortionless transmission of a certain band of frequencies and suppression of all the remaining frequencies.

- Practical filters have long tails, complex impulse response, non-fixed bandwidth, and complex transfer function expression.

- For simplicity, we often use ideal filter in our deduction. Which has sharp stop band in frequency domain, and accurate bandwidth.

Ideal Low Pass Filter

- The ideal low pass filter, allows all components below $f = B\ Hz$ to pass without distortion and suppresses all components above $f = B\ Hz$



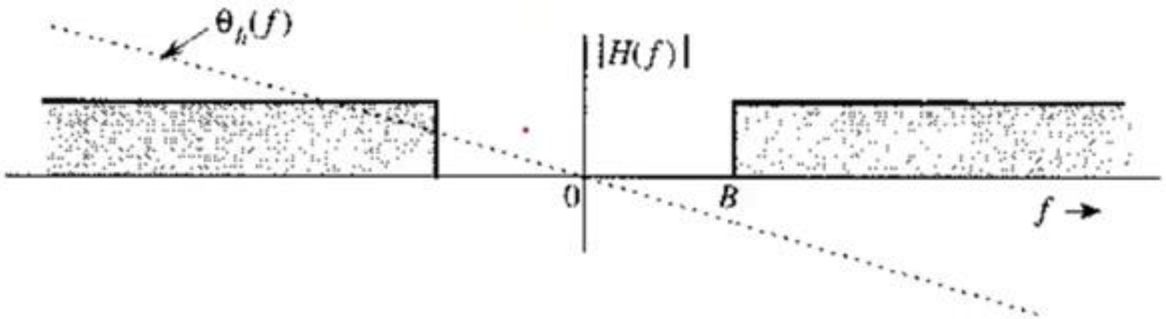
- The ideal low pass filter response can be expressed as

$$H(f) = \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_d}$$

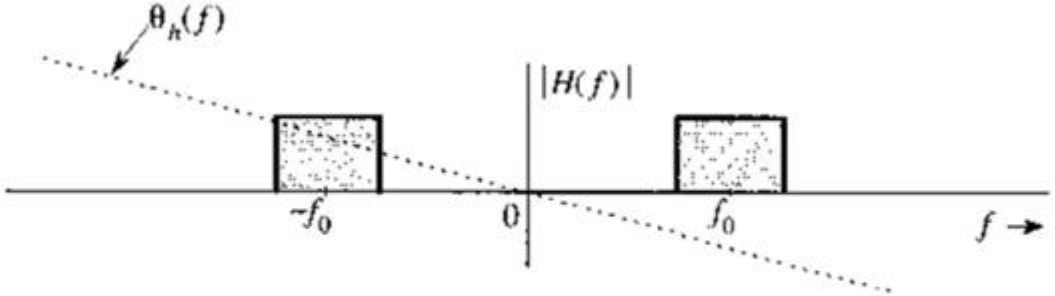
- The ideal low pass filter impulse response will be

$$h(t) = \mathcal{F}^{-1}\left[\Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_d}\right]$$
$$= 2B\,\text{sinc}[2\pi B(t - t_d)]$$

Ideal High-Pass and Band-Pass filters



(a) High Pass Filter

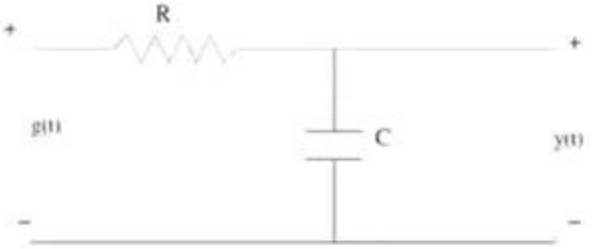


(b) Band Pass Filter

Practical Filters

- The filters in the previous examples are ideal filters.
- They are not realizable since their unit impulse responses are everlasting (think of the sinc function).
- Physically realizable filter impulse response $h(t) = 0$ for $t < 0$.
- Therefore, we can only obtain approximated version of the ideal low-pass, high-pass and band-pass filters.

Example of a linear system: RC circuit



$$H(w) = \frac{1/_{jwC}}{R + 1/_{jwC}} = \frac{1}{1 + jwRC} = \frac{a}{a + jw}$$

where,

$$a = \frac{1}{RC}$$

and,

$$|H(w)| = \frac{a}{\sqrt{a^2 + w^2}} \Rightarrow |H(0)| = 1,\ \lim_{w \to \infty} |H(w)| = 0$$

Signal Distortion over a Communication Channel

- Linear Distortion
- Non-Linear Distortion
- Distortion caused by multipath effects
- Fading channels

Linear Distortion

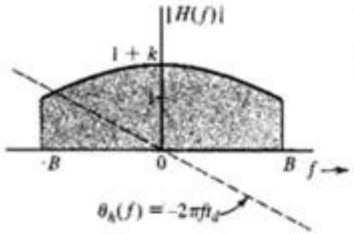
- Caused due to channel's non-ideal characteristics of either the magnitude or phase or both.
- For a time limited pulse, spreading or "dispersion" will occur if either the amplitude response or the phase response or both are non ideal.
- For TDM, it causes interference in adjacent channels (cross talk).
- For FDM, it causes dispersion in each multiplexed signal which will distort the spectrum of each signal, but no interference, since each signal occupies a separate channel.

Example

- A low pass filter transfer function $H(f)$ is given by

$$H(f) = \begin{cases} (1 + k\cos 2\pi f T)e^{-2\pi f t_d} & |f| < B \\ 0 & |f| > B \end{cases}$$

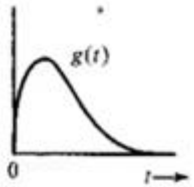
A pulse $g(t)$ band-limited to $B\ Hz$ is applied at the input of the filter. Find the output $y(t)$.

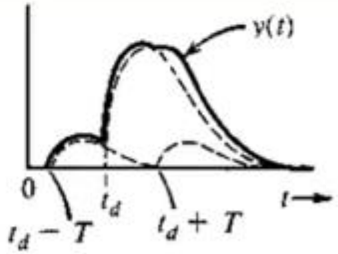


$$\begin{aligned} Y(f) &= G(f)H(f) \\ &= G(f) \cdot \Pi\left(\frac{f}{2B}\right)(1 + k\cos 2\pi f T)e^{-j2\pi f t_d} \\ &= G(f)e^{-j2\pi f t_d} + k\,[G(f)\cos 2\pi f T]\ e^{-j2\pi f t_d} \end{aligned}$$

$$\Updownarrow$$

$$y(t) = g(t - t_d) + \frac{k}{2}[g(t - t_d - T) + g(t - t_d + T)]$$





Nonlinear Distortion

- Nonlinear distortion is caused by larger signal amplitudes.
- Changes a band limited frequency spectrum $B\ Hz$ to $kB\ Hz$.
- In case of nonlinear channels, input g and output y are related as a function expanded in Maclaurin series

$$y = f(g)$$

$$y(t) = a_0 + a_1 g(t) + a_2 g^2(t) + a_3 g^3(t) + \cdots + a_k g^k(t) + \cdots$$

- In broadcast communication, high power amplifiers are desirable, but they are non-linear.
- Linear distortion causes interference among signals within the same channel.
- Spectral dispersion due to nonlinear distortion causes interference among signals using different frequency channels.
  - TDM faces no threat from it.
  - FDM, faces serious interference problems due to this spectral dispersion.

Example

The input $x(t)$ and the output $y(t)$ of a certain nonlinear channel are related as

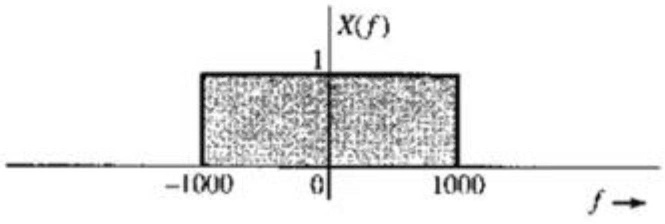
$$y(t) = x(t) + 0.000158x^2(t)$$

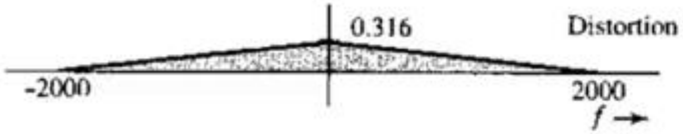
- Find the output signal $y(t)$ and its spectrum $Y(f)$ if the input signal is $x(t) = 2000\text{sinc}(2000\pi t)$.

Example (Contd)

- Verify that the bandwidth of the output signal is twice that of the input signal.

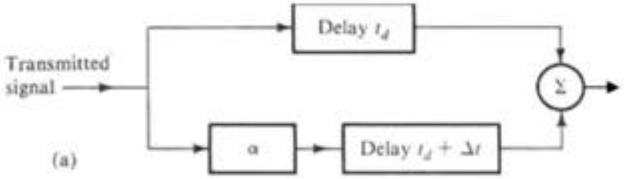
$$Y(f) = \Pi\left(\frac{f}{2000}\right) + 0.316\,\Delta\left(\frac{f}{4000}\right)$$



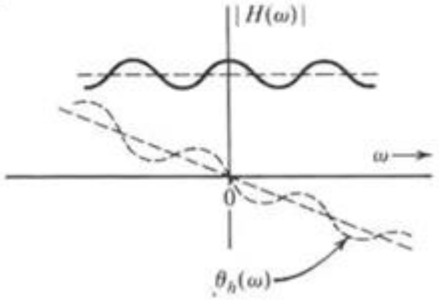


Distortion due to multipath effects

- In radio links, the signal can be received by direct path between the transmission and the receiving antenna and also by reflection from nearby objects.
- Similar behavior observed for ionosphere.



$$
\begin{aligned}
H(\omega) &= e^{-j\omega t_d} + \alpha e^{-j\omega(t_d+\Delta t)} \\
&= e^{-j\omega t_d}(1 + \alpha e^{-j\omega \Delta t}) \\
&= e^{-j\omega t_d}(1 + \alpha \cos \omega \Delta t - j\alpha \sin \omega \Delta t) \\
&= \underbrace{\sqrt{1 + \alpha^2 + 2\alpha \cos \omega \Delta t}}_{|H(\omega)|}\; e^{-j\underbrace{\left[\omega t_d + \tan^{-1} \frac{\alpha \sin \omega \Delta t}{1 + \alpha \cos \omega \Delta t}\right]}_{\theta_h(\omega)}}
\end{aligned}
$$



Fading Channels

- Practically channel characteristics vary with time because of periodic and random changes in the propagation characteristics of the medium, causing random attenuation of the signal. Also termed as "fading"
- Can be reduced by "Automatic Gain Control" (AGC).
- Fading may be strongly frequency dependent where different frequency components are affected unequally.
  - Such fading is called frequency-selective fading.
  - Multipath propagation can cause frequency-selective fading.

Energy/Power Signals and Energy/Power Spectral Density

- To introduce Energy spectral density (ESD)
- Input and Output Energy Spectral Densities
- To introduce Power spectral density (PSD)
- Input and Output Power Spectral Densities

Signal Energy: Parseval's Theorem

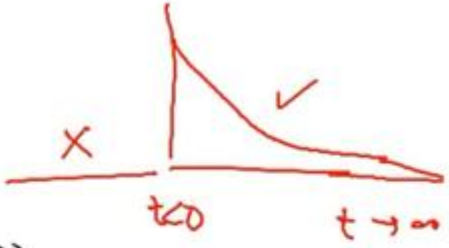
- Consider an energy signal $g(t)$, Parseval's Theorem states that

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(w)|^2 dw$$

Proof:

$$\begin{aligned}
E_g &= \int_{-\infty}^{\infty} g(t) g^*(t) dt = \int_{-\infty}^{\infty} g(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) e^{-j\omega t} d\omega \right] dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \left[\int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \right] d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) G^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega
\end{aligned}$$

Example



- Consider the signal $g(t) = e^{-at}u(t)\ (a > 0)$
- Its energy is

$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}$$

- We now determine E_g using the signal spectrum $G(w)$ given by

$$\boxed{G(w) = \left|\frac{1}{jw + a}\right|^2}$$ FT Table.

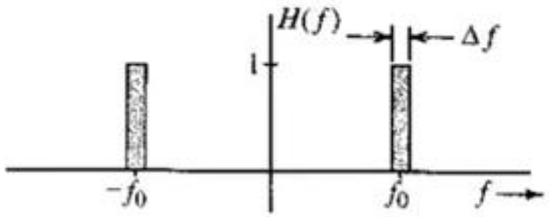
- It follows

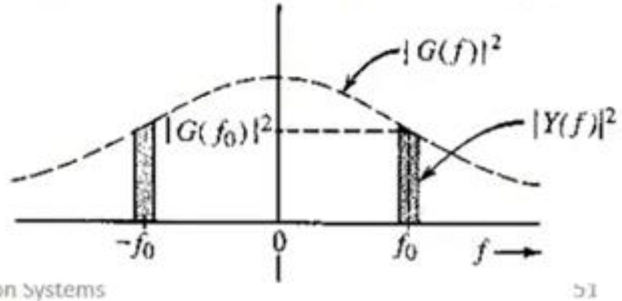
$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty}|G(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2 + a^2}d\omega = \frac{1}{2\pi a}[tan^{-1}\frac{\omega}{a}]_{-\infty}^{\infty} = \frac{1}{2a}$$

- Which verifies Parseval's theorem.

Energy Spectral Density

- Parseval's theorem can be interpreted to mean that the energy of a signal $g(t)$ is the result of energies contributed by all spectral components of a signal $g(t)$.
- The contribution of a spectral component of frequency f is proportional to $|G(f)|^2$.
- Therefore, we can interpret $|G(f)|^2$ as the energy per unit bandwidth of the spectral components of $g(t)$ centered at frequency f.
- In other words, $|G(f)|^2$ is the energy spectral density of $g(t)$.





Energy Spectral Density (continued)

- The energy spectral density (ESD) $\psi(w)$ is thus defined as

$$\psi_g(f) = |G(f)|^2$$

and

$$E_g = \int_{-\infty}^{\infty} \psi_g(f) df$$

Thus, the ESD of the signal $g(t) = e^{-at}u(t)$ of the previous example is

$$\psi_g(f) = |G(f)|^2 = \frac{1}{(2\pi f)^2 + a^2}$$

Essential Bandwidth of a signal

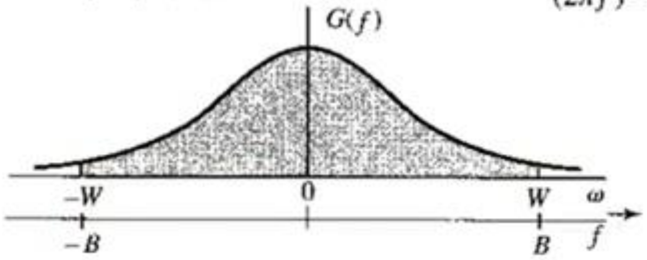
- The spectra of most signals extend to infinity.
- But since energy of practical signal is finite, signal spectrum $\rightarrow$ 0, as frequency $\rightarrow \infty$.
- Most of the signal energy is contained in a certain band of $B\ Hz$, we can suppress the spectrum beyond $B\ Hz$ with little effect on shape or energy.
- The bandwidth B is called the essential bandwidth of the signal
- The criterion for suppressing B depends on the error tolerance in a particular application
- For example, we may say that select B to be that bandwidth that contains 95% of the signal energy.

Example

- Determine the essential Bandwidth W (rad/sec) of the following signal if the essential band is required to contain 95% of the signal energy.

$$g(t) = e^{-at}u(t) \ (a > 0)$$

$$G(f) = \frac{1}{j2\pi f + a} \qquad |G(f)^2| = \frac{1}{(2\pi f)^2 + a^2}$$



$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}$$

$$\frac{0.95}{2a} = \int_{-W/2\pi}^{W/2\pi} \frac{df}{(2\pi f)^2 + a^2}$$

$$= \frac{1}{2\pi a} \tan^{-1} \frac{2\pi f}{a} \Big|_{-W/2\pi}^{W/2\pi} = \frac{1}{\pi a} \tan^{-1} \frac{W}{a}$$

$$\frac{0.95\pi}{2} = \tan^{-1} \frac{W}{a} \Longrightarrow W = 12.7\, a \text{ rad/s}$$

In terms of hertz, the essential bandwidth is

$$\boxed{B = \frac{W}{2\pi} = 2.02\, a \quad \text{Hz}}$$