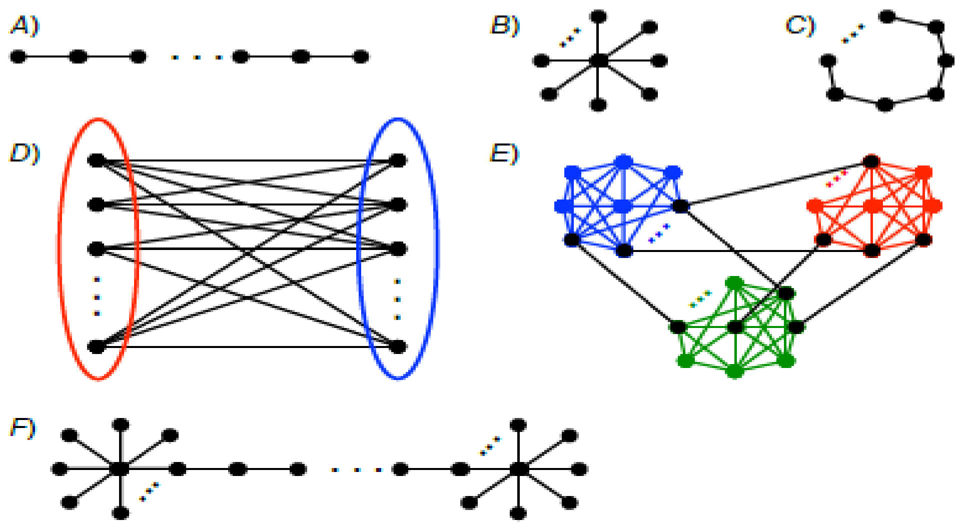
QUESTION 1:

Adjacency matrix structure. Sketch (qualitatively) the sparsity pattern of the adjacency matrices associated with the following network graphs.



QUESTION 2:

- **2) Interpretation of the powers of an adjacency matrix.** Consider an undirected and unweighted network graph $G(V, E)$, with order $N_v := |V|$, size $N_e := |E|$, and adjacency matrix $\mathbf{A}$. Let matrix $\mathbf{A}^k$ be the k-th power of $\mathbf{A}$, with i, j-th entry denoted as $[\mathbf{A}^k]_{ij}$. In parts *A)-B)* of this problem, we will show that $[\mathbf{A}^k]_{ij}$ yields the number of $i-j$ walks of length k in G. To that end, we will argue by mathematical induction.

  A) Base case: Show that the claim holds true for $k = 1$.

  B) Inductive step: Supposing the claim is true for $k-1$, i.e., $[\mathbf{A}^{k-1}]_{ij}$ yields the number of $i-j$ walks of length $k-1$ in G; show it also holds for k. (Hint: Notice that $\mathbf{A}^k = \mathbf{A}^{k-1}\mathbf{A}$ and expand $[\mathbf{A}^k]_{ij}$ using the definition of matrix multiplication.)

  C) Does the result generalize for digraphs?

  D) Argue that $[\mathbf{A}^2]_{ii} = d_i$ and $[\mathbf{A}^3]_{ii}/2 = \#\Delta_i$, i.e., the degree of vertex i and the number of triangles vertex i is part of, respectively. Hence or otherwise, conclude that $\mathrm{tr}(\mathbf{A}^2)/2 = N_e$ and $\mathrm{tr}(\mathbf{A}^3)/6 = \#\Delta$ in G, where $\mathrm{tr}(\cdot)$ stands for the matrix trace operator.

  E) Explain how you would use the information in the matrix $\mathbf{S} = \sum_{k=1}^{N_v-1} \mathbf{A}^k$ to check whether G is a connected graph.

QUESTION 3:

3) A few properties of the Laplacian. Consider an undirected and unweighted network graph $G(V, E)$, with order $N_v := |V|$, size $N_e := |E|$, and adjacency matrix $\mathbf{A}$. Let $\mathbf{D} = \text{diag}(d_1, \ldots, d_{N_v})$ be the degree matrix and $\mathbf{L} := \mathbf{D} - \mathbf{A}$ the Laplacian of G.

A) Verify that $\mathbf{1} := [1, \ldots, 1]^\top$ is an eigenvector of $\mathbf{L}$ with associated eigenvalue 0.

B) Despite G being an undirected graph, consider assigning an arbitrary 'virtual' orientation to each edge in E, i.e., for each edge pick one of its incident vertices as the 'head' and the other as the 'tail'. Given these assignments, consider the *signed* incidence matrix $\tilde{\mathbf{B}} \in \{-1, 0, 1\}^{N_v \times N_e}$ with i, j-th entry given by

$$[\tilde{\mathbf{B}}]_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is incident to edge } j \text{ as a tail} \\ -1, & \text{if vertex } i \text{ is incident to edge } j \text{ as a head} \\ 0, & \text{otherwise} \end{cases}.$$