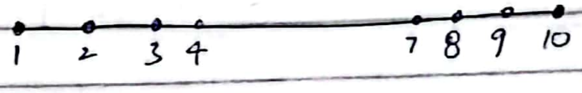
# DM HW 4

211-6260 Zain Al Aboldin

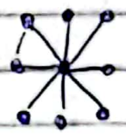
## Q1




(A)

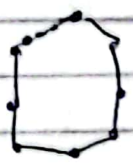
$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & & & & 0 & 1 & 0 \end{pmatrix}$$

(B)



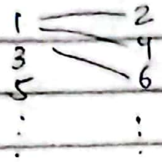
$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

(c)



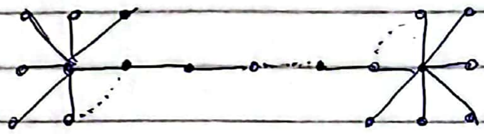
$$\begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 & 1 \\ 1 & 0 & 1 & \cdots & & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

(D)




$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & 1 & \cdots & & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & & 1 & 0 \end{pmatrix}$$

(F)



1 to n n+1 to m m+1 to p.

$$\begin{array}{c|c} & \begin{array}{cccccccccccc} & & n & & & & m & & & & & \end{array} \\ \begin{array}{c} \\ \\ n \\ \\ \\ \\ \\ r \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & \vdots & \vdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & \cdots 0 & 0 & 1 & 0 & 0 & 0 & 0 & & 0 \\ 0 & \vdots & \vdots & 0 & 1 & 0 & 0 & \vdots & \vdots & \vdots & & \vdots \\ 0 & \vdots & \vdots & 0 & 0 & 1 & 0 & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & \cdots 0 & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & & 1 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 1 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & & 0 \end{pmatrix} \end{array}$$

Q#2

$G(V, E)$ is unweighted and undirected graph.

$N_v = |V|$ $\quad$ $N_e = |E|$ $\quad$ $A$ = adjacency Matrix.

We are given that.

$[A^k]_{ij}$ is $\Rightarrow$ $k^{th}$ power of adjacency matrix.
$ij$ is number walks of length $k$.

A) Base Case $k = 1$:

$A$ being adjacency matrix of graph $G$.
$[A^k]_{ij}$ is walk length $k$ from $i$ to $j$ vertex.
so $[A^k]_{ij}$, $k = 1$ means $\{i, j\} \in E$ and
$E$ are edges of $G$. and $[A^1]_{ij}$ represents walklength of 1 from $i$ to $j$ with direct edges between them.

Graph $G$ $V_1$ as $i$ vertex to $V_2$ $j$ vertex and edge $\{i, j\} \in E$ then $k = 1$ representing walk length of 1. Hence $k = 1$ is true.

B) Inductive Step

For $n = k - 1$;
$[A^k]_{ij}$ holds for $k = 1$

we know

$$A^k = A^{k-1} \times A$$

$$A^k{}_{ij} = \sum_{p=1}^{k-1} A_{ip}{}^{k-1} \times A_{pj}$$

where $p$ is some vertex. case:

$i - \{P, i\} \in G \rightarrow A_{Pi} = 1$
$ii - \{P, i\} \notin G \rightarrow A_{Pi} = 0.$

so in particular all walks from i-j of length k are from some vertex p. therefore

$$A_{ij}^{k} = \sum_{p=1}^{k-1} A_{ip}^{k-1} \times A_{pj}$$

represents total i-j walks holds true for. k-1.

$$A^2 = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$ 2 means 2 step path.

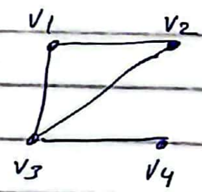
c) Generalization for Digraphs : Yes, the matrix multiplication result also applies for Digraphs where $[A^k]_{ij}$ gives the number of i-j walks of length k in graph.

D) $|A^2|_{ij} = d_i$ , $(A^3)_{ij}/2 = \# \Delta$.

As $\frac{tr(A^2)}{2} = N_e \Rightarrow$ no of edges

$$\frac{tr(A^3)}{6} = \# \Delta$$

Suppose we take graph $G \Rightarrow$




$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now Computing $A^2$ and $A^3$ ~~computation~~

$$A^2 = A \cdot A$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

As $Ne = \frac{tr(A2)}{2}$ $\qquad tr(A^2) = 2+2+2+2 \Rightarrow 8$

$Ne = 8/2 \Rightarrow 4.$

We Can see G has 4 edges.

Now Similarly $A^3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 3 & 3 & 2 & 2 \\ 1 & 1 & 3 & 0 \end{bmatrix}$

We know $\# \Delta = \frac{tr(A^3)}{6} \Rightarrow \frac{2+2+2+0}{6} \Rightarrow 1$

we see that 1 triangle exists in G

Hence proved.

E) Checking connectivity, $S = A + A^2 + A^3$ would provide full matrix of non-zero values, showing multiple paths of varying lengths connecting every pair. If any vertex is not connected, it will have 0 value in $S$ matrix as no walk $i$-$j$ exists.