Assignment 3

Due: 10 October, 2025 11:55 PM

Instructions

- Ensure that all answers include complete and detailed workings for full credit.
- Collaboration between students is permitted, but copying answers is strictly prohibited.
- Any assistance received for the completion of this assignment must be clearly indicated.
- Assignments must be submitted on LMS prior to the specified deadline.
- Late submissions will not be accepted under any circumstances.
- Submit **both** the .pdf and the .tex file of your solution in a zipped folder named according to the convention `CS210_A3_RollNumber`.

Graded Problems

Question 1 [5 points]

Let A and B be finite sets with $|A| = 5$ and $|B| = 3$.

(a) How many functions exist from $A \to B$? (2)

(b) How many are surjective? (2)

(c) How many are not surjective? (1)

Question 2 [5 points]

Determine whether each of the following functions is **injective** and find its **range**.

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \cos(x)$ (1)

(b) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = (-1)^x$ (1)

(c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{x}{1 + x^2}$ (1)

(d) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \lfloor x \rfloor$ (1)

(e) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \ln(x^2 + 1)$ (1)

Question 3 [8 points]

Let f, g be functions defined by

$$f : \left(-\frac{1}{2}, \infty\right) \to \mathbb{R}, \quad f(x) = \ln(2x + 1),$$

$$g : \mathbb{R} \setminus \{-1\} \to \mathbb{R}, \quad g(x) = \frac{1}{x + 1}.$$

(a) Compute $(f \circ g) \circ (g \circ f)$ and simplify as much as possible. (3)

(b) Determine the domain of $(f \circ g) \circ (g \circ f)$ carefully, taking into account all restrictions from f and g. (2)

(c) Determine whether this composition is injective and/or surjective. Justify your answer for full credit. (2)

Question 4 [7 points]

(a) Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show, using the definition of an inverse function, that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. (5)

(b) Given $f = \{(1, a), (2, b), (3, c), (4, d)\}$ and $g = \{(a, 3), (b, 1), (c, 4), (d, 2)\}$, find the following:

(i) f^{-1} (1)

(ii) g^{-1} (1)

Question 5 [10 points]

Your TA Uzair is building an AI chatbot for Discrete Mathematics. There are exactly n verified students and exactly n private help threads (one thread per slot). Let A be the finite set of students and B be the finite set of threads, with $|A| = |B| = n$. When a student types `/ask`, the bot assigns that student to one specific thread. We can define a function $f : A \to B$ by letting $f(a)$ be the thread assigned to student $a \in A$. Prove that f is injective **if and only if** f is surjective. (10)

Question 6 [15 points]

Your TA Haroon is designing a simple processor with n general-purpose registers $R_1, R_2, \ldots, R_n$ and n instructions $I_1, I_2, \ldots, I_n$. Each instruction writes to exactly one register. Let

$$f : \{I_1, I_2, \ldots, I_n\} \to \{R_1, R_2, \ldots, R_n\}$$

be the assignment of instructions to registers.

Constraints:

- No instruction writes to the register with the same index, i.e. $f(I_i) \neq R_i$ for all i.
- At least m instructions must write to registers with index $<$ their instruction index (i.e. $f(I_i) = R_j$ with $j < i$).

(a) For $n = 4$ and $m = 2$, how many valid assignments f satisfy the constraints? Show your reasoning. (3)

(b) For n in the general case, argue whether the set of all valid assignments is always:

(i) a function, (3)

(ii) a one-to-one function (injective), (3)

(iii) an onto function (surjective). (3)

Give examples (or counterexamples) for small n to support your answers above.

(c) Can f be injective under these restrictions? Justify your answer. (3)

Ungraded Problems

Question 1

Determine whether each of the following functions is **injective** and find its **range**.

(a) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = 3x - 4$

(b) $f : \mathbb{N} \to \mathbb{N},\ f(x) = x^2$

(c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = |x| + 1$

(d) $f : [0, \infty) \to \mathbb{R},\ f(x) = \sqrt{x^2 + 1}$

(e) $f : [0, \pi] \to \mathbb{R},\ f(x) = \cos(x)$

Question 2

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 - x + 1, \quad g(x) = \sqrt{x + 2}.$$

Compute the following compositions and their domains:

(a) $f \circ g$

(b) $g \circ f$

(c) $f \circ f$

(d) $g \circ g$

Question 3

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. List all functions $f : A \to B$ that are bijections. Then compute the inverses of all these bijections explicitly.

Question 4

Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Question 5

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Find a function $f : A \implies B$ which is either injective or surjective, but not both.

Question 6

Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = 2x + 1$

(b) $f(x) = x^2 + 1$

(c) $f(x) = x^3$

(d) $\dfrac{x^2 + 1}{x^2 + 2}$

Question 7

Determine whether f is a function from $\mathbb{Z}$ to R if:

(a) $f(n) = \pm n$.

(b) $f(n) = \sqrt{(n^2 + 1)}$

(c) $f(n) = \frac{1}{n^2 - 4}$.

Question 8

Your TA Umer finds himself trapped in a mysterious labyrinth and wants to escape. The labyrinth has n rooms, numbered 1 through n, and each room has exactly one door leading into another room (possibly back to itself).

We can represent the layout of the labyrinth by a function

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$$

where $f(i)$ is the room you enter when leaving room i.

(a) For $n = 5$, how many bijections (no room repeats as a destination) are possible?

(b) If f is injective but not surjective, how many rooms **must** be unreachable?

(c) Suppose f represents a "trapdoor" maze where some rooms loop back to themselves ($f(x) = x$). How many functions can exist with exactly two such fixed points?

Question 9

For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, determine whether f is invertible, and, if so, find f^{-1}.

(a) $f = \{(x, y) \mid 2x + 3y = 7\}$

(b) $f = \{(x, y) \mid ax + by = c,\ b \neq 0\}$

(c) $f = \{(x, y) \mid y = x^3\}$

(d) $f = \{(x, y) \mid y = x^4 + x\}$

Question 10

Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be two functions. Explain why the following are functions:

(a) $h : \mathbb{Z} \to \mathbb{Z}$ defined as $h(x) = f(g(x))$ when $g(x)$ is an onto function.

(b) $h : \mathbb{Z} \to \mathbb{Z}$ defined as $h(x) = f(x) + g(x)$.

(c) $h : \mathbb{Z} \to \mathbb{Z}$ defined as $h(x) = f(x) \cdot g(x)$.