Assignment 2

Due: 26 September, 2025 11:55 PM

Instructions

- Ensure that all answers include complete and detailed workings for full credit.
- Collaboration between students is permitted, but copying answers is strictly prohibited.
- Any assistance received for the completion of this assignment must be clearly indicated.
- Assignments must be submitted on LMS prior to the specified deadline.
- Late submissions will not be accepted under any circumstances.
- Submit **both** the .pdf and the .tex file of your solution in a zipped folder named according to the convention `CS210_A2_RollNumber`.

Graded Problems

Question 1 [8 marks]

Construct a logical statement using quantifiers for each of the given English statements. You may need to use more than one quantifier. Unless indicated otherwise, you may assume all names to be constants (that is, they would not require a predicate to be defined or are pre-defined as part of a predicate). (0.5×16)

(a) **Variables:** $o \in$ Offices
Predicates: $W(o)$: Izzah waited outside o for at least one hour.
Statement: Izzah waited outside some professor's office for over an hour.

(b) **Variables:** $i \in$ Images, $c \in$ Cats
Predicates: $U(i, c)$: i shows c in an ushanka; $P(i)$: Hassaan uses i as a profile picture.
Statement: Hassaan uses an image of a cat in an ushanka as a profile picture.

(c) **Variables:** $c \in$ Cats
Predicates: $M(c)$: Khadeeja met cat c; $F(c)$: c is friendly.
Statement: Not every cat Khadeeja met was friendly.

(d) **Variables:** $c \in$ Clocks
Predicates: $B(c)$: Ayma built clock c; $W(c)$: c works correctly.
Statement: Every clock Ayma built works correctly.

(e) **Variables:** $s \in$ Students, $t \in$ Topics
Predicates: $G(s, t)$: Abdullah agrees with student s on topic t.
Statement: Abdullah agrees with at least one student on every topic.

(f) **Variables:** $s \in$ Students, $c \in$ Cups
Predicates: $S(c)$: c is a Stanley cup; $O(c)$: Hania owns c; $A(s, c)$: s admires c.
Statement: Every student admires at least one of Hania's Stanley cups.

(g) **Variables:** $c \in$ Classes, $i \in$ Items
Predicates: $F(i, c)$: Haroon forgets item i in class c.
Statement: Haroon forgets at least one item every class.

(h) **Variables:** $i \in$ Items, $d \in$ Days
Predicates: $G(i)$: i is green; $H(i, d)$: Vania has i on day d.
Statement: Vania has at least one green item every day.

(i) **Variables:** $t \in$ TAs, $v \in$ VideoGames
Predicates: $P(t, v)$: TA t played video game v.
Statement: Every TA but Daniyal has played a video game.

(j) **Variables:** $q \in$ Questions, $t \in$ Topics
Predicates: $C(q, t)$: q covers topic t; $P(q)$: q was approved.
Statement: Every question Shanzay added to this assignment for at least one of the topics covered was approved.

(k) **Variables:** $d \in$ Days, $j \in$ Juices, $c \in$ Classes
Predicates: $T(d)$: Tayyab attends class on d; $B(j, c, d)$: Tayyab brings j to c on d.
Statement: Every day Tayyab attends class, he brings some kind of juice.

(l) **Variables:** $c \in$ Classes, $s \in$ Students, $b \in$ Bands
Predicates: $E(s, c)$: s is enrolled in class c with Rija; $L(s, b)$: s listens to band b.
Statement: Rija has found no student in any of her classes who listens to either Starset or Spiritbox.

(m) **Variables:** $h \in$ OfficeHours, $s \in$ Students, $q \in$ Questions
Predicates: $H(h)$: h is an office hours session held by Uzair; $A(s, q, h)$: s asks q in h.
Statement: In every office hours session Uzair holds, at least one student shows up with at least one question.

(n) **Variables:** $i \in$ ICPs, $p \in$ Problems, $s \in$ Students
Predicates: $M(i)$: Meesum checked i; $O(i, p)$: i contains p; $R(s, p)$: s got p right.
Statement: Out of all the ICPs Meesum checked, there was one problem no one got right.

(o) **Variables:** $f \in$ FootballTeams, $c \in$ CricketTeams
Predicates: $C(c)$: Umer supports c; $F(f)$: Umer supports f.
Statement: Umer supports only one cricket team, but he likes every football team except Tottenham Hotspur.

(p) **Variables:** $s \in$ Students, $c \in$ Concepts
Predicates: $U(s, c)$: s tries to understand c; $P(s)$: the professors prefer s.
Statement: The professors prefer every student who tries to understand a concept over those who don't.

Question 2 [5 Marks]

Determine whether each of these statements is true or false. Explain your reasoning.

(a) $\emptyset \in \{0\}$ (1)

(b) $\emptyset \subset \{0\}$ (1)

(c) $\forall x \in \mathbb{Z}\ (x^2 > 0)$ (1)

(d) $\exists x \in \mathbb{Z}\ (x^2 = 2)$ (1)

(e) $(X \times Y) \times Z = X \times Y \times Z$ (1)

Question 3 [5 Marks]

Show using set identities that $(P \cap Q \cap R \cap \overline{Y}) \cup (\overline{P} \cap R) \cup (\overline{Q} \cap R) \cup (R \cap Y) = R$.

Question 4 [10 Marks]

Prove the following using both set identities and membership tables:

(a) $(X \setminus Y) \setminus Z = X \setminus (Y \cup Z)$ (5)

(b) $\overline{(\overline{X \cap Y}) \cup (X \cap \overline{Z})} = X \cap Y \cap Z$ (5)

Question 5 [3 marks]

Let $D = \{p, q\}$, $E = \{7, 8\}$, and $F = \{r, s, t\}$. List all the elements in the following Cartesian products.

a. $D \times E \times F$ (1)

b. $E \times E \times D$ (1)

c. $D \times D \times D$ (1)

Question 6 [3 marks]

Write down all elements of A, B and C, where (1 × 3)

$$A = \mathcal{P}(\mathcal{P}(\varnothing)), \quad B = \mathcal{P}(A), \quad C = \mathcal{P}(B).$$

Question 7 [10 marks]

(a) Is $(A \cup B) \setminus A = B$ **generally** a valid claim? Why or why not? (3)

(b) Zaina is making two playlists of videos explaining some concepts. **Playlist A** centers on core Discrete Math concepts taken from **Channel A**, while **Playlist B** covers Narrative Essays taken from **Channel B**. Looking at the final products, she claims: $(A \cup B) \setminus A = B$.

Is Zaina correct in this specific context? Justify your answer. (3)

(c) Abeera divides her friends into three sets:

- A: friends who like cats,
- B: friends who like coffee,
- C: friends who like studying late at night.

She notices that:

$$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B \cup C) \setminus (A \cap B \cap C).$$

Is this always true? Prove or provide a counterexample, and remember to explain your view. You may also assume any standard set or make up your own. (4)

Question 8 [6 marks]

Let U be the universal set. For subsets $A, B \subseteq U$, consider the statement:

$$\forall x \in U,\ (x \in A \to x \in B).$$

(a) Your friend Mutaal has not entirely understood the statement. Translate it into plain English and explain it using an example. You may also assume any standard set or make up your own. (3)

(b) Show it is equivalent to $A \subseteq B$. (3)

Ungraded Problems

Question 1

Write 5 true logical statements using nested quantifiers about the given Sudoku puzzle. Each statement must use at least two quantifiers (e.g $\forall, \exists$). At least 1 statement must include negation ($\neg$). Also provide an English explanation of what each statement means.

For example, you may have something of the form:

$$\exists r\, \exists c\, \exists n\; C(r, c, n) \wedge (n = 8)$$

where $C(r, c, n)$ is a predicate defined such that the cell at row r and column c has the number n. You are required to produce unique statements: simply replacing the number in the sample statement will not count.

9						1		
	2							9
			2		9			
5			1	6	8	9		
1	9	8	5	2	3	4	7	6
	3		7	9	4			1
			9				8	
	4	9			1			
		6					9	

Question 2

Find $\bigcup_{i=1}^{\infty} P_i$ and $\bigcap_{i=1}^{\infty} P_i$ if for every positive integer i for the following:

1. $P_i = \{i, i+1, i+2, \dots\}$
2. $P_i = \{-i, -i+1, \dots, -1, 0, 1, \dots, i-1, i\}$
3. $P_i = \{0, i\}$
4. $P_i = (0, i)$, the set of real numbers such that $0 < x < i$
5. $P_i = \{-i, i\}$

Question 3

Prove the distributive law for sets:

$$A \cap (B \cup C) \ = \ (A \cap B) \cup (A \cap C).$$

You should try both set identities (algebraic manipulation) and a membership-table.

Question 4

Verify that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

for the following sets:

(a) $A = \{a, c, e, f, h\},\ B = \{c, d, e, f\},\ C = \{a, b, c, f\}$

(b) $A = \{1, 3, 5\},\ B = \{2, 3, 5, 6\},\ C = \{1, 5, 6, 7\}$

Question 5

Let

$$X = \{(p, 4), (q, 2), (r, 3)\}, \quad Y = \{(p, 2), (q, 5), (s, 1)\}.$$

Find:

(a) $X \cup Y$

(b) $X \cap Y$

(c) $X - Y$

(d) $Y - X$

(e) $X + Y$

Question 6

Prove or disprove each of the following for sets $A, B \subseteq U$.

(a) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

(b) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

Question 7

Let A, B and C be any sets. Show using set identities that

$$\overline{(A \cup C) \cap B} = \overline{B} \cup (\overline{C} \cap \overline{A}).$$

Question 8

Let A, B and C be any sets. Show using set identities that

$$\overline{\overline{(A \cup B) \cap C} \cup \overline{B}} = B \cap C$$

Question 9

Suppose A, B, and C are sets, and $C \neq \emptyset$. Prove that if $A \times C = B \times C$, then $A = B$.

Question 10

Let A, B, C, D be non-empty sets.

(a) Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$.

(b) What happens to the above result if any of the sets A, B, C, or D is empty?

Question 11

Let X, Y, and Z be three sets such that Y and Z have the same cardinality. Define:

$$A = \mathcal{P}(X \times Y), \quad B = \mathcal{P}(X) \times \mathcal{P}(Z).$$

Compare the cardinalities of A and B. In which cases are they equal, and in which cases are they not?